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The HyperKähler Geometry Associated to Wolf Spaces.

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Sunto. – *Sia G un grupo di Lie compatto e semplice. Sia $\mathcal{O}_{\min}$ la più piccola orbita nilpotente non-banale nell'algebra di Lie complessa $\mathfrak{g}^{\mathbb{C}}$. Si presenta una costruzione diretta di teoria di Lie delle metriche iperKahler su $\mathcal{O}_{\min}$ con potenziale Kahleriano G-invariante e compatibili con la forma simplettica complessa di Kostant-Kirillov-Souriau. In particolare si ottengono le metriche iperKahler dei fibrati associati sugli spazi di Wolf (spazi simmetrici quaternionali a curvatura scalare positiva).*

1. – Introduction.

One of the glories of homogeneous geometry is Cartan's classification of the compact Riemannian symmetric spaces [5, 6]. Many manifolds that play a central rôle in geometry are symmetric and it is fascinating to look for patterns in the presentations G/H. One obvious family is provided by the sphere $S^n = SO(n+1)/SO(n)$, complex projective space $\mathbb{C}P(n) = U(n+1)/(U(n)\,U(1))$, quaternionic projective space $\mathbb{H}P(n) = Sp(n+1)/(Sp(n)\,Sp(1))$ and the Cayley projective plane $F_4/Spin(9)$. Another consists of the Hermitian symmetric spaces: these are of the form $G/(U(1)\,L)$ (see [4]). However, the most surprising is the family of quaternionic symmetric spaces $W(G) := G/(SP(1)\,K)$, which has the feature that there is precisely one example for each compact simple simply-connected Lie group G. The manifolds in this last family have become known as Wolf spaces following [14]. Alekseevsky [1] proved that they are the only homogeneous positive quaternionic Kähler manifolds (cf. [2]).

Wolf showed that the quaternionic symmetric spaces may be constructed by choosing a highest root α for $\mathfrak{g}^{\mathbb{C}}$. The corresponding root vector E_α is a nilpotent element in $\mathfrak{g}^{\mathbb{C}}$. In [13] it was shown that there is a fibration of the nilpotent adjoint orbit $\mathcal{O}_{\min} = G^{\mathbb{C}} \cdot E_\alpha$ over the Wolf space $W(G)$.

Nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}^{\mathbb{C}}$ have a rich and interesting geometry. Firstly, they are complex submanifolds of $\mathfrak{g}^{\mathbb{C}}$ with respect to the natural complex structure I. Secondly, the construction of Kirillov, Kostant and Souriau endows them with a $G^{\mathbb{C}}$-invariant complex symplectic form ω_c. It is natural to ask whether one can find a metric making the orbit hyperKähler, i.e., can one find a Riemannian metric g on $\mathcal{O}$, such that the real and imaginary parts of ω_c are

Kähler forms with respect to complex structures J and K satisfying $IJ = K$. By identifying $\mathcal{O}$ with a moduli space of solutions to Nahm's equations, Kronheimer [12] showed that there is indeed such a hyperKähler metric on $\mathcal{O}$. This hyperKähler structure is invariant under the compact group G, and has the important additional property that it admits [13] a hyperKähler potential ϱ: a function that is simultaneously a Kähler potential with respect to I, J and K. Using ϱ, one can define an action of $\mathbb{H}^*$ on $\mathcal{O}$ such that the quotient is a quaternionic Kähler manifold. It is in this way that one may obtain the Wolf space $W(G)$ from $\mathcal{O}_{\min}$. In contrast to the semi-simple case [3], currently one does not know how many invariant hyperKähler metrics a given nilpotent orbit admits.

The aim of this paper is to study the hyperKähler geometry of $\mathcal{O}_{\min}$ in an elementary way. We look for all hyperKähler metrics on $\mathcal{O}_{\min}$ with a G-invariant Kähler potential and which are compatible with the complex symplectic structure. Note that we do not restrict our attention to metrics with hyperKähler potentials. We derive a simple formula for the a priori unknown complex structure J. The orbit $\mathcal{O}_{\min}$ is particularly straight-forward to study in this way, since G acts with orbits of codimension one. This means that the metrics we obtain are already known, they are covered by the classification [7], but it is interesting to see how these metrics can be constructed directly from their potentials. In agreement with the classification, the hyperKähler structure is found to be unique, unless $\mathfrak{g} = \mathfrak{su}(2)$, in which case one obtains a one-dimensional family of metrics, the Eguchi-Hanson metrics.

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2. – Definitions.

On the simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, let $\langle \cdot, \cdot \rangle$ be the *negative* of the Killing form and let σ be a real structure giving a compact real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. An element X of $\mathfrak{g}^{\mathbb{C}}$ is said to be nilpotent if $(\mathrm{ad}_X)^k = 0$ for some integer k. Let $\mathcal{O}$ be the orbit of a nilpotent element X under the adjoint action of $G^{\mathbb{C}}$. At $X \in \mathcal{O}$, the vector field generated by A in $\mathfrak{g}^{\mathbb{C}}$ is $\xi_A = [A, X]$. Using the Jacobi identity it is easy to see that these vector fields satisfy $[\xi_A, \xi_B] = \xi_{-[A, B]}$, for $A, B \in \mathfrak{g}^{\mathbb{C}}$. The orbit $\mathcal{O}$ is a complex submanifold of the complex vector space $\mathfrak{g}^{\mathbb{C}}$ and so has a complex structure I given by $I\xi_A = i\xi_A = \xi_{iA}$.

On a hyperKähler manifold M with complex structures I, J and K and metric g, we define Kähler two-forms by $\omega_I(X, Y) = g(X, IY)$, etc., for tangent vectors X and Y. The condition that a function $\varrho : M \to \mathbb{R}$ be a Kähler potential for I is

$$\omega_I = -i\partial_I\overline{\partial}_I\varrho = -\, id\overline{\partial}_I\varrho = -\frac{i}{2}d(d-iId)\,\varrho = -\frac{1}{2}dI\,d\varrho\,. \tag{2.1}$$

On the orbit $\mathcal{O}$, the complex symplectic form of Kirillov, Kostant and Souriau is given by $\omega_c(\xi_A, \xi_B)_X = \langle X, [A, B]\rangle = -\langle \xi_A B\rangle$.

We will be looking for hyperKähler structures with Kähler potential ϱ and such that $\omega_c = \omega_J + i\omega_K$. This will be done by computing the Riemann metric g defined by ϱ via (2.1) and then using this to determine an endomorphism J of $T_X\mathcal{O}$ via $\omega_J = g(\cdot, J\cdot)$. The constraints on ϱ will come from the two conditions that g is positive definite and that $J^2 = -1$.

3. – Highest roots and minimal orbits.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$. Fix a system of roots Δ with positive roots Δ_+. We write $\mathfrak{g}_\beta$ for the root space of $\beta\in\Delta$. Choose a Cartan basis $\{E_\beta, H_\beta, F_\beta \colon \beta\in\Delta_+\}$, which we may assume is compatible with the real structure σ, in the sense that $\sigma(E_\beta) = -F_\beta$ and $\sigma(H_\beta) = -H_\beta$. One important property of the Cartan basis is that for each β, $\mathrm{Span}_{\mathbb{C}}, \{E_\beta, H_\beta, F_\beta\}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has Cartan basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

The irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ are the symmetric powers $S^k = S^k\mathbb{C}^2$ of the fundamental representation $S^1 = \mathbb{C}^2$. The representation S^k has dimension $k+1$ and E, H and F act as

$$\varphi_E = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & k \\ & & & & 0 \end{bmatrix}, \quad \varphi_H = \begin{bmatrix} k & & & & \\ & k-2 & & & \\ & & \ddots & & \\ & & & 2-k & \\ & & & & -k \end{bmatrix} \tag{3.2}$$

$$\text{and } \varphi_F = \begin{bmatrix} 0 & & & & \\ k & 0 & & & \\ & \ddots & \ddots & & \\ & & 2 & 0 & \\ & & & 1 & 0 \end{bmatrix}$$

respectively. In particular, $(\varphi_E)^{k+1}=0$ and $(\varphi_E)^k$ has rank one, with image the k-eigenspace of φ_H.

Let $\alpha\in\Delta_+$ be a highest root; this is characterised by the condition $[E_\alpha, E_\beta]=0$ for all $\beta\in\Delta_+$. We define $\mathcal{O}_{\min}$ to be the adjoint orbit of E_α under the action of $G^{\mathbb{C}}$. Define $\mathfrak{sl}(2,\mathbb{C})_\alpha := \mathrm{Span}_{\mathbb{C}}\{E_\alpha, H_\alpha, F_\alpha\}$.

PROPOSITION 3.1. – (i) *Under the action of* $\mathfrak{sl}(2,\mathbb{C})_\alpha$ *the Lie algebra* $\mathfrak{g}^{\mathbb{C}}$ *decomposes as*

$$\mathfrak{g}^{\mathbb{C}}\cong \mathfrak{sl}(2,\mathbb{C})_\alpha\oplus\mathfrak{k}^{\mathbb{C}}\oplus(V\otimes S^1),$$

where $\mathfrak{k}^{\mathbb{C}}$ *is the centraliser of* $\mathfrak{sl}(2,\mathbb{C})$, V *is a* $\mathfrak{k}^{\mathbb{C}}$*-module.*

(ii) *The action of the compact group* G *on the nilpotent orbit* $\mathcal{O}_{\min}$ *has cohomogeneity one.*

PROOF. – (i) Consider the action of $\mathrm{ad}\,E_\alpha$ on $\mathfrak{g}^{\mathbb{C}}$. For $\beta\in\Delta_+$, we have $[E_\alpha, F_\beta]\in\mathfrak{g}_{\alpha-\beta}$. If $\beta\neq\alpha$, then we have two cases: (a) if $\alpha-\beta$ is not a root then $\mathfrak{g}_{\alpha-\beta}=\{0\}$ and $[E_\alpha,F_\beta]=0$; (b) if $\alpha-\beta$ is a root, then the condition that α is a highest root implies $\alpha-\beta\in\Delta_+$, since otherwise $\alpha-\beta=-\gamma$ for some $\gamma\in\Delta_+$ and then $[E_\alpha,E_\gamma]$ is non-zero, which for a highest root α is impossible. We therefore have that $(\mathrm{ad}\,E_\alpha)^2$ is zero on the complement of $\mathfrak{sl}(2,\mathbb{C})_\alpha$ and the decomposition follows.

(ii) At E_α the tangent space to $\mathcal{O}_{\min}$ is

$$\mathrm{ad}_{E_\alpha}\mathfrak{g}^{\mathbb{C}}=\mathrm{Span}_{\mathbb{C}}\{E_\alpha, H_\alpha\}+\mathrm{Span}_{\mathbb{C}}\{E_{\alpha-\beta}\colon \beta\in\Delta_+\}.$$

The real Lie algebra $\mathfrak{g}$ is the real span of $\{E_\beta-F_\beta,\ iH_\beta,\ i(E_\beta+F_\beta)\}$. Thus the tangent space $\mathrm{ad}_{E_\alpha}\mathfrak{g}$ to the G-orbit is

$$\mathrm{Span}_{\mathbb{R}}\{iE_\alpha, H_\alpha, iH_\alpha\}+\mathrm{Span}_{\mathbb{R}}\{E_{\alpha-\beta}, iE_{\alpha-\beta}\colon \beta\in\Delta_+\}$$

and we see that it has codimension one in $T_{E_\alpha}\mathcal{O}_{\min}$, the complement being $\mathbb{R}E_\alpha$. As G is compact, this implies that G acts with cohomogeneity one. ∎

As in [8], it is possible to use this result to show that $\mathcal{O}_{\min}$ is minimal with respect to the partial order on nilpotent orbits given by inclusions of closures. This explains the name $\mathcal{O}_{\min}$, but will not be needed in the subsequent discussion.

4. – Kähler potentials in cohomogeneity one.

Let $\varrho:\mathcal{O}_{\min}\to\mathbb{R}$ be a smooth function invariant under the action of the compact group G. The group G acts with cohomogeneity one, and the function $\eta(X)=\|X\|^2=\langle X,\sigma X\rangle$ is G-invariant and distinguishes orbits of G. We may therefore assume that ϱ is just a function of η, i.e., $\varrho = \varrho(\eta)$.

We wish to consider ϱ as a Kähler potential for the complex manifold $(\mathcal{O}_{\min}, I)$. The corresponding Kähler form is given by (2.1):

$$\omega_I = -\frac{1}{2} d(\varrho' I d\eta) = -\frac{1}{2}\varrho' \, dI\, d\eta - \frac{1}{2}\varrho'' \, d\eta \wedge I d\eta\,, \tag{4.1}$$

where $\varrho' = d\varrho/d\eta$, etc.

LEMMA 4.1. – *The Kähler form defined by* $\varrho(\eta)$ *is*

$$\omega_I(\xi_A, \xi_B) = 2\ \mathrm{Im}\,(\varrho'\langle \xi_A, \sigma\xi_B\rangle + \varrho''\langle \xi_A, \sigma X\rangle\langle \sigma\xi_B, X\rangle)\,. \tag{4.2}$$

PROOF. – The exterior derivative of η is

$$d\eta(\xi_A)_X = \langle [A, X], \sigma X\rangle + \langle X, \sigma[A, X]\rangle = 2\,\mathrm{Re}\,\langle \xi_A, \sigma X\rangle \tag{4.3}$$

so $Id\eta(\xi_A)_X = 2\ \mathrm{Im}\,\langle \xi_A, \sigma X\rangle$ and hence

$$(d\eta \wedge Id\eta)(\xi_A, \xi_B) = -4\ \mathrm{Im}\,(\langle \xi_A, \sigma X\rangle\langle \sigma\xi_B, X\rangle)\,.$$

Using the Jacobi identity we find that the exterior derivative of $Id\eta$ is given by

$$\begin{aligned} dI\,d\eta(\xi_A, \xi_B)_X &= \xi_A(Id\eta(\xi_B)) - \xi_B(Id\eta(\xi_A)) - Id\eta([\xi_A, \xi_B]) \\ &= 2\ \mathrm{Im}\,\langle \xi_B, \sigma\xi_A\rangle + 2\ \mathrm{Im}\,\langle [B,[A, X]], \sigma X\rangle \\ &\quad -2\ \mathrm{Im}\,\langle \xi_A, \sigma\xi_B\rangle - 2\ \mathrm{Im}\,\langle [A,[B, X]], \sigma X\rangle \\ &\quad +2\ \mathrm{Im}\,\langle [[A, B], X], \sigma X\rangle \\ &= -4\ \mathrm{Im}\,\langle \xi_A, \sigma\xi_B\rangle . \end{aligned}$$

Putting these expressions into (4.1) gives the result. ■

Using the relation $g(\xi_A, \xi_B) = \omega_I(I\xi_A, \xi_B)$, we can now obtain the induced metric on $\mathcal{O}_{\min}$. In general, this metric will be indefinite; the signature may be determined by considering $\mathrm{Span}_{\mathbb{R}}\{X, \sigma X\}$ and its orthogonal complement with respect to the Killing form.

PROPOSITION 4.1. – *The pseudo-Kähler metric defined by* $\varrho(\eta)$ *is*

$$g(\xi_A, \xi_B) = 2\ \mathrm{Re}\,(\varrho'\langle \xi_A, \sigma\xi_B\rangle + \varrho''\langle \xi_A, \sigma X\rangle\langle \sigma\xi_B, X\rangle)\,. \tag{4.4}$$

This is positive definite if and only if $\varrho' > \max\{0, -\eta\varrho''\}$. ■

5. – HyperKähler metrics.

Given a function $\varrho(\eta)$ on $\mathcal{O}_{\min}$ we have obtained a metric g. Let us assume that g is non-degenerate. Using the definition of ω_c and its splitting into real and imaginary parts, we get endomorphisms J and K of $T_X\mathcal{O}_{\min}$ via

$$g(\xi_A, \xi_B) = \omega_J(J\xi_A, \xi_B) = -\operatorname{Re}\langle J\xi_A, B\rangle,$$

etc. This implies that

$$J_X\xi_A = -2\varrho'[X, \sigma\xi_A] - 2\varrho''\langle\sigma\xi_A, X\rangle[X, \sigma X]. \tag{5.1}$$

and $K = IJ$. Note that (5.1) implies $JI = -K$.

Suppose that $J^2 = -1$ and g is positive definite. Then we have I, J and K satisfying the quaternion identities, and with ω_I, ω_J and ω_K closed two-forms. By a result of Hitchin [10], this implies that I, J and K are integrable and that g is a hyperKähler metric.

PROPOSITION 5.1. – *The nilpotent orbit of $\mathfrak{sl}(2, \mathbb{C})$ has a one-parameter family of hyperKähler metrics with $SU(2)$-invariant Kähler potential and compatible with the Kostant-Kirillov-Souriau complex symplectic form ω_c.*

PROOF. – The algebra $\mathfrak{sl}(2, \mathbb{C})$ has only one nilpotent orbit $\mathcal{O} = \mathcal{O}_{\min}$ and this has real dimension 4. Using the action of $SU(2)$ we may assume that $X = tE$, where $t > 0$ and E is given by (3.1). Then $T_X\mathcal{O}$ is spanned by H and E. We have $J_XH = -4\varrho' tE$ and $J_XE = 2t(\varrho' + \eta\varrho'')H$, which implies $J^2 = -\mathrm{Id}$ if and only if $8t^2(\varrho'^2 + \eta\varrho'\varrho'') = 1$. Now $\eta(E) = 4$, so we get the following ordinary differential equation for ϱ:

$$2(\eta\varrho'^2 + \eta^2\varrho'\varrho'') = 1\,.$$

The left-hand side of this equation is $(\eta^2\varrho'^2)'$, so $\varrho' = \sqrt{\eta + c}/\eta$, for some real constant c. For this to be defined for all positive η, we need $c \geqslant 0$. Now $\varrho'' = -(\eta + 2c)/(2\eta^2\sqrt{\eta + c})$, so the metric is

$$g(\xi_A, \xi_B) = \frac{1}{\eta^2\sqrt{\eta + c}}\operatorname{Re}\left(2\eta(\eta + c)\langle\xi_A, \sigma\xi_B\rangle - (\eta + 2c)\langle\xi_A, \sigma X\rangle\langle\sigma\xi_B, X\rangle\right), \tag{5.2}$$

which is positive definite. ∎

This hyperKähler metric is of course well-known. We put it in standard form as follows. Using (4.3), we find $(\partial/\partial\eta) = E/(8t)$ at $X = tE$. An $SU(2)$-invariant basis of $T_X\mathcal{O}$ is now given by $\{\partial/\partial\eta, \xi_{s_1}, \xi_{s_2}, \xi_{s_3}\}$, where

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad s_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad s_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

This basis is orthogonal with respect to (5.2) and in terms of the dual basis of one-forms is $\{d\eta, \sigma_1, \sigma_2, \sigma_3\}$, g is

$$\frac{1}{4\eta^2\varrho'}d\eta^2 + \eta\varrho'(\sigma_1^2+\sigma_2^2) + \frac{1}{\varrho'}\sigma_3^2.$$

Substituting $\eta = (r/2)^4 - c$, we get

$$g = W^{-1}dr^2 + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2+W\sigma_3^2),$$

with $W = 1 - 16c/r^4$, which are the Eguchi-Hanson metrics [9].

THEOREM 5.1. – *For* $\mathfrak{g}^{\mathbb{C}} \neq \mathfrak{sl}(2,\mathbb{C})$, *the minimal nilpotent orbit* $\mathcal{O}_{\min}$ *admits a unique hyperKähler metric with G-invariant Kähler potential compatible with the complex symplectic form* ω_c.

PROOF. – Let α be a highest root. Using the action of G, we may assume that $X = tE_\alpha$, for some $t>0$. On $\xi_A \in \mathfrak{sl}(2,\mathbb{C})_\alpha$, the condition $J^2 = -\mathrm{Id}$ gives $8t^2(\varrho'^2 + \eta\varrho'\varrho'') = 1$, as in Proposition 5.1. Putting $\lambda^2 = \eta(E_\alpha)$, we have $t^2 = \eta(X)/\lambda^2$ and hence $\varrho' = \sqrt{\lambda^2\eta + c}/2\eta$. Now for ξ_A Killing-orthogonal to $\mathfrak{sl}(2,\mathbb{C})$, we have

$$J\xi_A = -2\varrho'[X, \sigma\xi_A] = -2t\varrho'[E_\alpha, \sigma\xi_A]$$

and hence

$$J^2\xi_A = -(4\eta\varrho'^2/\lambda^2)\,\mathrm{ad}_{E_\alpha}\mathrm{ad}_{F_\alpha}\xi_A = -\left(1+\frac{c}{\lambda^2\eta}\right)\mathrm{ad}_{E_\alpha}\mathrm{ad}_{F_\alpha}\xi_A.$$

As η is not constant, the condition $J^2 = -\mathrm{Id}$ implies $c=0$ and we have a unique hyperKähler metric. ■

The proof enables us to write down J explicitly for $\mathcal{O}_{\min}$ in $\mathfrak{g}^{\mathbb{C}} \neq \mathfrak{sl}(2,\mathbb{C})$:

$$J_X\xi_A = -\frac{\lambda}{2\eta^{3/2}}(2\eta[X,\sigma\xi_A] - \langle\sigma\xi_A, X\rangle[X,\sigma X]).$$

The number λ^2 is a constant depending only on the Lie algebra $\mathfrak{g}^{\mathbb{C}}$, with values $2n$ ($\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(n-1,\mathbb{C})$, $\mathfrak{so}(n+2,\mathbb{C})$), 8 ($G_2$), 18 ($F_4$), 24 ($E_6$), 36 ($E_7$), 70 ($E_8$).

REMARK 5.1. – *Theorem* 5.1 *only assumes that* ϱ *is a Kähler potential. However, the uniqueness result implies that this potential is in fact* *hyperKähler* (*cf.* [13]). *This corresponds to Proposition* 5.1, *where* ϱ *is a hyperKähler potential only when* $c = 0$.

Finally, let us observe that the form of the potential determines the nilpotent orbit.

PROPOSITION 5.2. – *If a nilpotent orbit* $\mathcal{O}$ *has a Kähler potential* ϱ *that is only a function of* $\eta = \|X\|^2$ *and which defines a hyperKähler structure compatible with* ω_c, *then* $\mathcal{O}$ *is a minimal nilpotent orbit.*

PROOF. – Choose $X \in \mathcal{O}$, such that $\mathrm{Span}_{\mathbb{C}}\{X, \sigma X, [X, \sigma X]\}$ is a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; this is always possible by a result of Borel (cf. [11]). Let $X = tE$, for $t > 0$, and write $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{m}$. The proofs of Proposition 5.1 and Theorem 5.1 imply that $\varrho' = \lambda\eta^{-1/2}/2$ and $J^2\xi_A = -\mathrm{ad}_E \mathrm{ad}_F \xi_A$ on $\mathfrak{m}$. Let S^k, $k > 0$, be an irreducible $\mathfrak{sl}(2, \mathbb{C})$-summand of $\mathfrak{m}$. Then ad_E and ad_F act via the matrices φ_E and φ_F of (3.2), so $\mathrm{ad}_E \mathrm{ad}_F$ acts as a diagonal matrix with entries $k, 2(k-1), 3(k-2), \ldots, (k-1)2, k$ and 0. As ξ_A is in the image of ad_E, in order to have $J^2\xi_A = -\xi_A$, we need all the non-zero eigenvalues of $\mathrm{ad}_E \mathrm{ad}_F$ to be 1. This forces $k = 1$.

Let $\mathfrak{g}(i)$ be the i-eigenspace of ad_H on $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$ is a parabolic subalgebra, so we may choose a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ lying in $\mathfrak{p}$ and a root system such that the positive root spaces are also in $\mathfrak{p}$. The discussion above shows that ad_E is zero on all these positive root spaces, and so E is a highest root vector. Therefore $\mathcal{O} = \mathcal{O}_{\min}$. ■

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