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Function Space Topologies Deriving from Hypertopologies and Networks (*).

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Sunto. – *In un progetto di generalizzazione delle classiche topologie di tipo «set-open» di Arens-Dugundji introduciamo un metodo generale per produrre topologie in spazi di funzioni mediante l'uso di ipertopologie. Siano X, Y spazi topologici e $C(X, Y)$ l'insieme delle funzioni continue da X verso Y. Fissato un «network» α nel dominio X ed una topologia τ nell'iperspazio $CL(Y)$ del codominio Y si genera una topologia τ_α in $C(X, Y)$ richiedendo che una rete $\{f_\lambda\}$ di $C(X, Y)$ converge in τ_α ad $f \in C(X, Y)$ se e solo se la rete $\{\overline{f_\lambda(A)}\}$ converge in τ ad $\overline{f(A)}$ per ogni elemento A in α. Quando Y é metrizzabile acquisiamo prima interessanti proprietà individuali delle topologie determinate in $C(X, Y)$ mediante la procedura descritta da una metrica di Hausdorff nell'iperspazio $CL(Y)$ di Y indotta a sua volta da una metrica compatibile con Y e poi focalizziamo la nostra attenzione sulle proprietà del loro estremo superiore che è indotto in $C(X, Y)$ dalla ipertopologia localmente finita.*

Introduction.

Let X, Y be topological spaces and $C(X, Y)$ the set of all continuous functions from X to Y. It can be shown that a net $\{f_\lambda\}$ in $C(X, Y)$ converges to $f \in C(X, Y)$ in the compact-open topology iff for each compact subset K of X the net $\{f_\lambda(K)\}$ in the hyperspace $CL(Y)$ of the codomain space Y converges to $f(K)$ in the Vietoris topology. Also, when $Y = \mathbb{R}$ equipped with the euclidean distance, a net $\{f_\lambda\}$ in $C(X, Y)$ converges to $f \in C(X, Y)$ in the bounded-open topology, [7], iff for each functionally bounded subset B of X the net $\{\overline{f_\lambda(B)}\}$ in $CL(Y)$ converges to $\overline{f(B)}$ again in the Vietoris topology. Looking at the compact-open topology and bounded-open topology in this new perspective reveals an interplay between set-open topologies on $C(X, Y)$ and Vietoris topology on $CL(Y)$. Indeed, the compact-open topology and the bounded-open topology are both a particular case in a more general result: *when α is a network containing all singletons the net $\{f_\lambda\}$ in $C(X, Y)$ converges to $f \in C(X, Y)$ in the α-open topology,* as defined in (1), *iff for each member A in α the net $\{\overline{f_\lambda(A)}\}$ in*

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$CL(Y)$ converges to $\overline{f(A)}$ in the Vietoris topology. This suggests, in the fruitful and well-tested method to construct from known hypertopologies new function space topologies *via some natural way*, to consider for any network α in X the topology $\tau_{\alpha, \text{loc-fin}}$ in $C(X, Y)$ induced from the locally finite hypertopology on $CL(Y)$, which is a Vietoris-type topology [2, 10], denoted in [2] $\tau_{\text{loc-fin}}$. Briefly: $\{f_\lambda\}$ $\tau_{\alpha, \text{loc-fin}}$*-converges to f in $C(X, Y)$ iff $\{\overline{f_\lambda(A)}\}$* $\tau_{\text{loc-fin}}$*-converges to $\overline{f(A)}$ in $CL(Y)$ for each member A in α*. And, when Y is metrized by a metric d, the topology $\tau_{\alpha, d}$ induced from the Hausdorff metric hypertopology $\tau_H(d)$ determined in $CL(Y)$ from d. Then $\{f_\lambda\}$ $\tau_{\alpha, d}$*-converges to f in $C(X, Y)$ iff $\{\overline{f_\lambda(A)}\}$ $\tau_H(d)$-converges to $\overline{f(A)}$ in $CL(Y)$ for each member A in α.*

Any topology $\tau_{\alpha, d}$ is weaker than $\tau_{\alpha, \text{loc-fin}}$. Both kinds of topologies admit other equivalent and simple descriptions. Any topology $\tau_{\alpha, d}$ is uniformizable, metrizable when α is countable and usually weaker than the topology of uniform convergence on members of α determined by the same metric d. When the domain is metric too, $\tau_{\alpha, d}$-convergence is usally different from Hausdorff and Attouch-Wets convergences. A topology $\tau_{\alpha, d}$, depending essentially on the choice of the metric d in Y, is an hybrid object attached to X and Y, halh-topological, half-uniform in nature. But the supremum of $\{\tau_{\alpha, d}$: d runs over all compatible metrics on $Y\}$ is a topological character of X and Y since it coincides with $\tau_{\alpha, \text{loc-fin}}$. Further, $\tau_{\alpha, \text{loc-fin}}$ is uniformizable when Y is uniformizable and α is normal w.r.t. Y, i.e. any $\overline{f(A)}$, $f \in C(X, Y)$, $A \in \alpha$, can be functionally separated from any disjoint closed set in Y. All old and new examined cases suggest a more general context. Any network α and any topology τ in $CL(Y)$ induce jointly a topology τ_α in $C(X, Y)$ by requiring:

(*) *A net $\{f_\lambda\}$ in $C(X, Y)$ τ_α-converges to $f \in C(X, Y)$ iff the net $\{\overline{f_\lambda(A)}\}$ τ-converges to $\overline{f(A)}$ in $CL(Y)$ for each member $A \in \alpha$.*

For arbitrary networks in X uniformizable hypertopologies generate by the procedure (*) uniformizable function space topologies.

We thank prof. S. Naimpally who suggested to study further generalizations of set-open topologies especially the locally finite case.

1. – Preliminaries and generalities.

Let X, Y be topological T_1 spaces and $C(X, Y)$ the set of all continuous functions from X to Y. A collection $\alpha = \{A\}$ of subsets of X is a *network* if for any point x in X and any open set U containing x there exists A in α such that $x \in A \subset U$. McCoy-Ntantu [8] in passing from compact networks to more general ones considered a modification of Arens-Dugundji's original definition of set-open topologies, [1], introducing as subbase for a *new α-open topology* the collection of all sets:

$$[A : V] = \{f \in C(X, Y) : \overline{f(A)} \subset V\} \tag{1}$$

when A is in α and V is open in Y.

Recall that the Vietoris topology on the hyperspace $CL(Y)$ of Y, the set of all non-empty closed subsets of Y, admits as basic open nhbds all sets of the type:

$$[V_1, \ldots, V_n] = \{E \in CL(Y) : E \subset \bigcup_{i=1}^{n} V_i,\ E \cap V_i \neq \phi,\ \forall_i = 1, \ldots, n\} \tag{2}$$

where $V_1, \ldots, V_n$ are open in Y. The Vietoris topology can be splitted into a *miss part* or *plus part* generated from the sets:

$$V^+ = \{E \in CL(Y) : E \subset V\} \tag{3}$$

where V is open in Y and into a *hit part* or *minus part* generated from the sets:

$$V^- = \{E \in CL(Y) : E \cap V \neq \phi\} \tag{4}$$

where again V is open in Y.

It becomes evident that:

LEMMA 1.1. – *A net $\{f_\lambda\}$ converges to f in the α-open topology on $C(X, Y)$, as defined in* (1), *iff $\{\overline{f_\lambda(A)}\}$ converges to $\overline{f(A)}$ in Vietoris plus in $CL(Y)$ for each member A in α.* ■

PROPOSITION 1.2. – *If α contains all singletons, then $\{f_\lambda\}$ converges to f in the α-open topology, as defined in* (1), *iff $\{\overline{f_\lambda(A)}\}$ converges to $\overline{f(A)}$ in the Vietoris topology of $CL(Y)$ for each member A in α.*

PROOF. – It is enough to show that if $A \in \alpha$ and $\{\overline{f_\lambda(A)}\}$ converges to $\overline{f(A)}$ in Vietoris plus then it converges also in Vietoris minus. Suppose V is an open set in Y and $\overline{f(A)} \in V^-$. Then there exists $x \in A$ with $f(x) \in V$. Consider $[x : V]$. Thus $f \in [x : V]$. So eventually $f_\lambda \in [x : V]$, that means eventually $f_\lambda \in V^-$. ■

Motivated from prop. 1.2 *from now on* we suppose any network closed, containing all singletons and will denote the α-open topology, defined in (1), as $\tau_{\alpha, V}$. The above described interaction between set-open topologies and Vietoris hypertopology suggests a new general method to produce from known hypertopologies new function space topologies.

THEOREM 1.3. – *A network α in X and a topology τ in $CL(Y)$ induce in $C(X, Y)$ a natural convergence τ_α, which topologizes $C(X, Y)$, by requiring:* (*) $\{f_\lambda\}$ *τ_α-converges to f in $C(X, Y)$ iff $\{\overline{f_\lambda(A)}\}$ τ-converges to $\overline{f(A)}$ in $CL(Y)$ for each member A in α.*

PROOF. – It is straightforward to show that τ_α is of topological nature. ■

From now on we refer to the above procedure as (*).

THEOREM 1.4. – *A topology τ_α on $C(X, Y)$ induced via* (*) *from an arbitrary network α in X and a uniformizable* (*completely regular*) *topology τ on $CL(Y)$, is itself uniformizable* (*completely regular*).

PROOF. – Let $\boldsymbol{U} = \{U\}$ a uniformity compatible with τ. Put:

$$(A, U) = \{(f, g) : (\overline{f(A)}, \overline{g(A)}) \in U\} \tag{5}$$

where $A \in \alpha$, $U \in \boldsymbol{U}$. Then the collection $\{(A, U) : A \in \alpha, U \in \boldsymbol{U}\}$ is a subbase for a uniformity on $C(X, Y)$ which induces τ_α. ■

We, now, focus our attention on those function space topologies deriving from the locally finite hypertopology and, when Y is metrizable, from Hausdorff metric hypertopologies.

Let $\mathcal{U} = \{U_i : i \in I\}$ be a locally finite family of open sets in Y. Denote:

$$\mathcal{U}^- = \{E \in CL(Y) : E \cap U_i \neq \phi, \ \forall i \in I\}.$$

Recall that the locally finite hypertopology $\tau_{\text{loc-fin}}$, [2], is generated from the sets V^+, as defined in (3), where V is open in Y, and $\mathcal{U}^-$, where $\mathcal{U}$ runs over all open locally finite families in Y.

When Y is metrizable and d metrizes it, then the hyperset 2^Y of Y, the set of all non-empty subsets of Y, is equipped with the following generalized pseudometric:

$$d_H(A, B) = \begin{cases} \max\{\sup\limits_{a \in A} d(a, B), \sup\limits_{b \in B} d(b, A)\}, & \text{when it exists.} \\ \infty, & \text{otherwise.} \end{cases}$$

It is easy to prove that $d_H(A, B) = d_H(\overline{A}, \overline{B})$ for each $A, B \in 2^Y$. The restriction of d_H to $CL(Y)$ is a *metric* universally known as *the Hausdorff metric.*

Beer and others proved in [2] that when Y is metrizable the locally finite hypertopology $\tau_{\text{loc-fin}}$ on $CL(Y)$ agrees with the supremum of all Hausdorff metric hypertopologies induced in $CL(Y)$ from compatible metrics on Y.

When Y is completely regular, any compatible uniformity $\boldsymbol{U} = \{U\}$ determines in $CL(Y)$ the Hausdorff uniformity, whose a base is the collection of sets:

$$\underline{U} = \{(A, B) \in CL(Y) \times CL(Y) : A \subset U[B] \text{ and } B \subset U[A]\} \tag{7}$$

where $U \in \boldsymbol{U}$. Any Hausdorff hyperuniformity induces in turn a hypertopology $\tau_H(\boldsymbol{U})$. Naimpally and Sharma in [10] proved that the Hausdorff hypertopology $\tau_H(\mathit{fine})$ induced from the finest uniformity is weaker than the locally finite hypertopology $\tau_{\text{loc-fin}}$ and they agree iff Y is normal.

2. – Topologies via (*) from Hausdorff metric hypertopologies.

Suppose Y is metrizable and d is a compatible metric. Let α be an arbitrary network in X. For each $A \in \alpha$, put:

$$\widehat{d}_A(f, g) = d_H(\overline{f(A)}, \overline{g(A)}) \quad f, g \in C(X, Y). \tag{8}$$

LEMMA 2.5. – *For each $A \in \alpha$, $\widehat{d}_A$ is a pseudometric on $C(X, Y)$.* ■

Denote $\tau_{\alpha, d} = \sup\{\tau(\widehat{d}_A) : A \in \alpha\}$, where $\tau(\widehat{d}_A)$ is the topology induced from $\widehat{d}_A$.

PROPOSITION 2.6. – *For any network α and any metric d compatible with Y, the topology $\tau_{\alpha, d}$ is uniformizable and then completely regular.*

PROOF. – $\tau_{\alpha, d}$ is induced in $C(X, Y)$ from the uniformity supremum of all pseudometrizable uniformities generated by the gage-base $\{\widehat{d}_A : A \in \alpha\}$ which has as subbase the collection of all sets:

$$(A, \varepsilon) = \{(f, g) : \widehat{d}_A(f, g) < \varepsilon\} \tag{9}$$

when $A \in \alpha$ and $\varepsilon > 0$. ■

LEMMA 2.7. – *$\tau_{\alpha, d}$ is induced via procedure (*) from the Hausdorff hypertopology $\tau_{H(d)}$ (7), jointly with the network α.* ■

PROPOSITION 2.8. – *The embedding $e : Y \to C(X, Y)$ defined by identifying any point y in Y with the relative constant function c_y on X is a uniform isomorphism, when Y is equipped with the natural metric uniformity $\boldsymbol{U}(d)$ and $C(X, Y)$ is uniformized from $\sup\{\boldsymbol{U}(\widehat{d}_A) : A \in \alpha\}$.*

PROOF. – It is enough to observe that the trace of any subbasic diagonal nhbd (A, ε), see (9), $A \in \alpha$, $\varepsilon > 0$ is just the e-image of the diagonal nhbd determined in $\boldsymbol{U}(d)$ from ε. ■

In comparison of $\tau_{\alpha,d}$'s deriving from different compatible metrics on Y we need to introduce a notion of network especially related to $C(X, Y)$. We say that a network α is *Y-compact* iff $\overline{f(A)}$ is compact in Y for each $f \in C(X, Y)$ and $A \in \alpha$. When $Y = \mathbb{R}$, Y-compactness flats in $\mathbb{R}$-boundedness or relatively pseudocompactness, [6].

THEOREM 2.9. – *If d, ϱ are uniformly equivalent metrics on Y or α is Y-compact, then $\tau_{\alpha,d} = \tau_{\alpha,\varrho}$.*

PROOF. – It follows from two well-known facts. Two compatible metrics on Y uniformly equivalent give the same Hausdorff hypertopology on $CL(Y)$ and any two Hausdorff hypertopologies agree on compact sets since they all agree with Vietoris hypertopology. ■

But usually for different compatible metrics d, ϱ, $\tau_{\alpha,d}$ is different from $\tau_{\alpha,\varrho}$.

EXAMPLE 1. – Let $X = Y = \mathbb{R}^+$. Suppose $d(x, y) = |x - y|$ and $\varrho(x, y) = |1/x - 1/y|$, $x, y \in X$ and α contains the set P of all even integers. Consider the collection $\Im$ of all continuous functions which take an odd integer value for some even integer. Then the identity map i belongs to the closure of $\Im$ in $\tau_{\alpha,\varrho}$ but not in $\tau_{\alpha,d}$. Indeed, the $\tau_{\alpha,d}$-subbasic nhbd of the map i, $(P, 1)_d(i)$ does not intersect $\Im$. On the other side, when $(A, \varepsilon)_\varrho(i)$ is any $\tau_{\alpha,\varrho}$-subbasic nhbd of the map i, we meet two cases: all even integers belong to A or not. In the former case, we choose an odd integer n in such a way that if $x \geqslant n$ and $y \geqslant n$ then $\varrho(x, y) < \varepsilon$. Next, by putting $g(x) = x$, $x \leqslant n$ and $g(x) = 2x - n$, $x \geqslant n$, we construct a function g which belongs to $\Im$ and $(A, \varepsilon)_\varrho(i)$. In the latter one, we choose an even integer n not belonging to A and then construct a continuous function h with the property $h(x) = x$, $x \in A$ and $h(n) = 1$. Naturally $h \in (A, \varepsilon)_\varrho(i)$ but furthermore $h \in \Im$. ■

By the way, $\tau_{\alpha,d}$ is stronger than the point-open topology and more when α contains all compact sets is stronger than the compact-open topology.

THEOREM 2.10. – *Let α be any network in X and d any compatible metric on Y. The topology of uniform convergence on members of α induced from d is stronger than $\tau_{\alpha,d}$.*

PROOF. – The result follows from comparison of uniformities. The topology of uniform convergence on α can be uniformized by using the subbase of diagonal nhbds of the type:

$$[A, \varepsilon] = \{(f, g) : d(f(x), g(x)) < \varepsilon, \ \forall x \in A\} \tag{10}$$

where $A \in \alpha$, and $\varepsilon > 0$. On the other hand, $\tau_{\alpha,d}$ is uniformized from the

$\sup\{U(\widehat{d}_A): A \in \alpha\}$ which admits as subbasic nhbds the sets (A, ε), as defined in (9), $A \in \alpha$, $\varepsilon > 0$. It is evident that $(A, \varepsilon) \supset [A, \varepsilon]$, $A \in \alpha$, $\varepsilon > 0$. ■

The above topologies are generally different as we will see later, but:

THEOREM 2.11. – *If α is Y-compact and hereditarily closed, then $\tau_{\alpha, d}$ agrees with the topology of uniform convergence on α.*

PROOF. – In this case any $\overline{f(A)}$, $A \in \alpha$, $f \in C(X, Y)$ is compact in Y, but on compacta the Hausdorff hypertopology induced from d coincides with Vietoris topology. Thus from prop. 1.2 and from generalized Arens theorem, [6], the result follows. ■

It derives immediately:

COROLLARY 2.12. – *When α contains all Y-compacta, $\tau_{\alpha, d}$ sits in between the topology of uniform convergence on Y-compacta and uniform convergence on α.*

3. – Comparison.

When $(X, d_1), (Y, d_2)$ are both metric spaces and $d = d_1 \times d_2$ is the box-metric in $X \times Y$, $C(X, Y)$ can be equipped with the Attouch-Wets topology $\tau_{AW(d)}$ via the natural identification of functions with their graphs, [3].

Let $X = l_2$ equipped with the Hilbert distance $d_1(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n \in N} (x_n - y_n)^2}$ and $Y = \mathbb{R}$ with the euclidean distance $d_2(x, y) = |x - y|$.

THEOREM 3.13. – *Let α be the network in l_2 consisting of all compacta plus all closed spheres centered at* 0 *and more the entire l_2. Then τ_{α, d_2} in $C(l_2, \mathbb{R})$ is different from the Attouch-Wets topology induced in $C(l_2, \mathbb{R})$ from d_1, d_2.*

PROOF. – Consider in l_2 the natural system $\{e_h : h \in N^+\}$, $e_h = \{x_n\}$, $x_n = \begin{cases} 0, & n \neq h \\ 1, & n = h \end{cases}$ and the uniformly discrete collection of closed balls B_h centered at e_h and having all radius ε, $\varepsilon < \sqrt{2}/2$.

Define $f: l_2 \to \mathbb{R}$ by putting: $f(x) = \begin{cases} d_1(x, e_n), & \text{if } x \in B_n \text{ for some } n. \\ \varepsilon, & \text{otherwise}. \end{cases}$

and for each $n \in N^+$, $f_n: l_2 \to \mathbb{R}$, $f_n(x) = \begin{cases} f(x), & \text{if } x \in \bigcup_{h=1}^{n} B_h. \\ \varepsilon, & \text{otherwise}. \end{cases}$

All functions f_n, f are continuous. The sequence $\{f_n\}$ τ_{α, d_2}-converges to f but does not converge to it in the Attouch-Wets topology.

Choose $x_0 = \mathbf{0} \in l_2$, $y_0 = 0 \in \mathbb{R}$ and take $k \in N^+$ such that $1/k < \varepsilon$. Then all $(e_n, f(e_n))$ are in the d-ball centered at $(\mathbf{0}, 0)$ and radius k. But for each $n, h \in N^+$ and each point x whose d_1-distance from e_{n+h} is less than $1/k$, $f_n(x) = \varepsilon$ and $f(e_{n+h}) = 0$. So $d_2(f(e_{n+h}), f_n(x)) = \varepsilon > 1/k$.

Now, observe that any compact set can meet at most a finite number of B_n. So $\{f_n\}$ is eventually constant on compacta. Thus $\{f_n\}$ uniformly converges on compacta. Next, consider a closed ball B centered at $\mathbf{0}$ and for each $n, m \in N^+$, $n \neq m$ the isometry of l_2, $g_{n,m}$, defined by:

$$g_{n,m}(\{x_h\}) = \{y_h\}, \text{ where } y_h = \begin{cases} x_h, & h \neq n,\, h \neq m \\ x_m, & h = n \\ x_n, & h = m\,. \end{cases}$$

Any $g_{n,m}$ fixes $\mathbf{0}$, then B and interchanges e_n with e_m. More, $g_{n,m}(B \cap B_n) = B \cap B_m$ and $f(B \cap B_n) = f(B \cap B_m)$. From the above considerations the ball B contains all B_n or none of them or intersects all of them in a uniform way. In any case $f(B) = f_n(B)$ for each $n \in N^+$. The same for $f(l_2)$ and $f_n(l_2)$. And the result definitively follows. ∎

COROLLARY 3.14. – *Let α be a network in X and d a metric in Y. Then the topology of uniform convergence on α is usually strictly finer than $\tau_{\alpha, d}$.*

PROOF. – The example studied in the theorem 3.13 works. The sequence $\{f_n\}$ cannot converge to f uniformly (on α), otherwise trivially that would imply uniform convergence on bounded sets and then Attouch-Wets convergence [4]. ∎

Again suppose X, Y as before but now $C(X, Y)$ equipped with the Hausdorff convergence in the box-metric, [5]. The following theorems state the different nature of the Hausdorff convergence and $\tau_{\alpha, d}$.

THEOREM 3.15. – *Let $X = \mathbb{R}^- \cup N$, $Y = \mathbb{R}$, both metrized by the euclidean distance d. Suppose α consists of all compacta plus the entire space X. Then the Hausdorff convergence in $C(X, Y)$ is different from $\tau_{\alpha, d}$ one.*

PROOF. – Put for each $n \in N^+$ and $x \in X$, $f_n(x) = (x + 1/n)^2$ and $f(x) = x^2$. Then $\{f_n\}$ does not converge in Hausdorff but $\tau_{\alpha, d}$-converges to f. When $h \in X$, $h \geq 1$ and $\varepsilon = 1$ we find only the same h to a distance less than 1 from h, but for no fixed $n \in N^+$ $d(f(h), f_n(h)) = 1/n^2 + 2(h/n)$ can be less than 1 for any $h \geq 1$.

On the other side $\{f_n\}$ is uniformly convergent to f on compacta. Next, if $h \in X$, $h \geq 0$ then $(-h - 1/n) \in X$, $d(f(h), f_n(-h - 1/n)) = 0$ and $d(f_n(h),$

$f(-h-1/n))=0$ for each $n \in N^+$. More, if $h \in X$, $h \leqslant -1$, then $(h-1/n) \in X$ and $(h+1/n) \in X$, $d(f(h), f_n(h-1/n))=0$ and $d(f_n(h), f(h+1/n))=0$ for each $n \in N^+$. Finally, for each $\varepsilon > 0$ there exists $\nu \in N^+$ such that if $n > \nu$ and $x \in [-1, 0]$, $d(f_n(x), f(x)) < \varepsilon$. Therefore, for each $\varepsilon > 0$ any integer $\bar{\nu}$ greater than ν and $1/\varepsilon$ works. ■

THEOREM 3.16. – *The Hausdorff convergence related to a box-metric $d_1 \times d_2$ does not imply τ_{α, d_2}-convergence also when the entire space X is in the network α.*

PROOF. – Let $X = Y = \mathbb{R}$, $d(x, y) = |x - y|$. Suppose α contains X and a uniformly discrete set, to simplify N. If $f_n(x) = (x + 1/n)^2$ and $f(x) = x^2$, $\forall x \in X$, $\forall n \in N^+$, then $\{f_n\}$, as Naimpally proved, Hausdorff converges to f. But on the other hand for each $n, m \in N^+$, $d(f(m), f_n(N)) = 2m/n + 1/n^2$. So $d_H(f(N), f_n(N)) = +\infty$ for each n.

4. – $\tau_{\alpha, \text{loc-fin}}$.

Let X, Y be T_1 topological spaces and α a closed network in X. The typical set-open topology on $C(X, Y)$ [1], has as subbase the collection $\{[A, B]: A \in \alpha, B \text{ open in } Y\}$, where $[A, B] = \{f \in C(X, Y): f(A) \subset B\}$. To generalize the compact-open topology to real-valued non continuous functions, to balance the disadvantage A is compact but $f(A)$ is not compact McCoy-Ntantu, [8], considered a modification by introducing as subbase the collection of all sets of the type: $[A : B] = \{f \in \mathbb{R}^X: \overline{f(A)} \subset B\}$, where $A \in \alpha$ and B is open in $\mathbb{R}$, already considered in preliminaries . The using of closures reveals as a right option. It creates a really new class, but anyway containing all most used ones, of set-open topologies. And furthermore, in producing via procedure (*) function space topologies from hypertopologies allows to work in a confortable setting such as the hyperspace instead of the hyperset of the codomain which is a too big extension of it.

Now we enlarge open sets in locally finite families of open sets again using closures, supporting later this option with an example contained in the last theorem.

Suppose $\mathcal{U} = \{U_i: i \in I\}$ is a family of subsets in Y and $A \in \alpha$. Set:

$$[A : \mathcal{U}] = \{f \in C(X, Y): \overline{f(A)} \subset \cup\, \mathcal{U} \text{ and } \overline{f(A)} \cap U_i \neq \phi,\ \forall_i \in I\}. \tag{11}$$

The collection $\{[A : \mathcal{U}]: A \in \alpha$ and $\mathcal{U}$ running over all locally finite open families of $Y\}$ is a subbase for a topology on $C(X, Y)$ which we will denote $\tau_{\alpha, \text{loc-fin}}$.

Observe that if $f \in [A : \mathcal{U}]$ and $\overline{f(A)}$ is compact, then $\mathcal{U}$ must be finite.

More, if the cardinality of A is less than the cardinality of $\mathcal{U}$, then $[A : \mathcal{U}]$ is empty.

If we denote, for A and $\mathcal{U}$ as above, $[A : \mathcal{U}]^+ = \{f \in C(X, Y) : \overline{f(A)} \subset \cup \mathcal{U}\}$ and $[A : \mathcal{U}]^- = \{f \in C(X, Y) : \overline{f(A)} \cap U_i \neq \phi, \ \forall_i \in I\}$ after realizing that $[A : \mathcal{U}]^+ = [A : \cup \mathcal{U}]$, see (1) , we can deduce:

LEMMA 4.17. – *$\tau_{\alpha, \text{loc-fin}}$ can be splitted into two parts, a miss part generated from $\{[A : V] : A \in \alpha, V$ open in $Y\}$, (1), that is $\tau_{\alpha, V}$, and a hit part generated from $\{[A : \mathcal{U}]^- : A \in \alpha, \mathcal{U}$ locally finite open family in $Y\}$.* ■

And, when Y is regular:

LEMMA 4.18. – *$\tau_{\alpha, \text{loc-fin}}$ is generated from $\{[A : \mathcal{U}] : A \in \alpha, \mathcal{U}$ discrete open family in $Y\}$.*

PROOF. – Recall that if you pick $x_i \in U_i$ where $\{U_i : i \in I\}$ is a locally finite open family, then you can find a discrete open family $\{V_i : i \in I\}$ such that $x_i \in V_i \subset U_i$. ■

LEMMA 4.19. – *$\tau_{\alpha, \text{loc-fin}}$ is generated by procedure (*) from α jointly with the locally finite hypertopology.* ■

Usually $\tau_{\alpha, \text{loc-fin}}$ is different from $\tau_{\alpha, V}$ as the following example shows.

EXAMPLE 2. – Let $X = l_2$, $Y = \mathbb{R}$, α done from all closed balls centered at $\mathbf{0}$. Consider in $\mathbb{R}$ the locally finite open family $\mathcal{U} = \{]1/(n+1), 1/n[: n \in N^+\}$. Let B the unitary ball in l_2. Then $[B : \mathcal{U}]^-$ is a subbasic $\tau_{\alpha, \text{loc-fin}}$ nhbd of the usual Hilbert norm $\| \|_2$ defined in l_2 by $\|\{x_n\}\|_2 = \sqrt{\sum_{n \in N} x_n^2}$ which is trivially a continuous function from l_2 to the reals. Well, none of $\tau_{\alpha, V}$-nhbds of the norm of the type $[B_1 : V_1] \cap \ldots \cap [B_n : V_n]$, with $B_1, \ldots, B_n$ balls of α, $V_1, \ldots, V_n$ open sets in $\mathbb{R}$, can be contained in $[B : \mathcal{U}]^-$. Say r the minimum of B_i's radius. Shrink any ball B_i and B in a ball whose radius is less than $1/n$, for some $n > 2$, and less than r, just multiplying for a convenient coefficient any element of l_2. And then operate with the norm, so obtaining a continuous function $g : l_2 \to \mathbb{R}$, which on any ball B_i takes values in the respective V_i, but, for which $\overline{g(B)}$ does not intersect $]1/n, 1/(n-1)[$ in $\mathcal{U}$. ■

Of course, when α is Y-compact or Y is feebly compact, any locally finite open family is finite ,$\tau_{\alpha, \text{loc-fin}}$ and $\tau_{\alpha, V}$ are indistinguishable.

Vice versa their coincidence can force compactness of α.

THEOREM 4.20. – *Let $X = \mathbb{R}^n$, $Y = \mathbb{R}$ and α done from closed connected subsets in $\mathbb{R}^n$. Then $\tau_{\alpha, V} = \tau_{\alpha, \text{loc-fin}}$ iff α is compact.*

PROOF. – Firstly consider $n=1$, since the general case can be reduced to the one-dimensional one. Suppose α contains a not compact, then unbounded, interval A. If $\{a_n\}$ is a uniformly discrete sequence in A, $\mathcal{U} = \{]a_n - 1/n, a_n + 1/n[: n \in N^+\}$ is a locally finite open collection whose union is not bounded in $\mathbb{R}$. Then $[A : \mathcal{U}]^-$ contains the identity map i of $\mathbb{R}$ but none of the sets $[A_1 : V_1] \cap \ldots \cap [A_n : V_n]$ which i belongs to, with $A_1, \ldots, A_n \in \alpha$ and $V_1, \ldots V_n$ open in $\mathbb{R}$, is contained in $[A : \mathcal{U}]^-$. When $\bigcup_{i=1}^{n} A_i$ is bounded, that is compact, it is enough to pick a continuous bounded function g which coincides with the identity map on the $\bigcup_{i=1}^{n} A_i$. Otherwise, order the indices in such a way when $i = 1, \ldots, h$, $f(A_i)$ is bounded, when $i = h+1, \ldots k$, A_i is bounded below but not above, when $i = k+1, \ldots, r$, A_i is bounded above but not below. Observing that A_i, $i = h+1, \ldots, r$, is an half-line, let $m_1 \in \cap\{A_i : i = k+1, \ldots, r\}$ and less than any element in $\bigcup_{i=1}^{k} A_i$ and $m_2 \in \cap\{A_i : i = h+1, \ldots, k\}$ and greater than any element in $\cup\{A_i : i = 1, \ldots, h, k+1, \ldots, r)$. Shrink the half-line $(-\infty, m_1]$ to the interval $]m_1 - \varepsilon, m_1]$ and the half-line $[m_2, +\infty)$ to the interval $[m_1, m_1 + \varepsilon[$ for a convenient $\varepsilon > 0$ and glue with the restriction of the identity map on $[m_1, m_2]$, so obtaining a continuous function which works.

The general case. When $A \in \alpha$ is not compact, it must be unbounded. Thus at least one of its natural projections $\pi(A)$ is unbounded. Also its closure $\overline{\pi(A)}$ is unbounded but connected. Furthermore $\pi \in [A : \mathcal{U}]$ is equivalent to say $i \in [\overline{\pi(A)} : \mathcal{U}]$ and $\pi \in [A_1 : V_1] \cap \ldots \cap [A_n : V_n]$ to $i \in [\overline{\pi(A_1)} : V_1] \cap \ldots \cap [\overline{\pi(A_n)} : V_n]$. And so the result follows from one-dimensional case. ∎

Suppose α, β are networks in X. We say that α *refines* β when any element in α can be covered by a finite union of elements in β.

PROPOSITION 4.21. – *If α refines β and β is hereditarily closed, then* $\tau_{\alpha, \text{loc-fin}} \subset \tau_{\beta, \text{loc-fin}}$.

PROOF. – Let $A \in \alpha$, $\mathcal{U} = \{U_i : i \in I\}$ a locally finite open family in Y and $f \in [A : \mathcal{U}]$. Suppose $A \subset \bigcup_{h=1}^{n} B_h$, $B_h \in \beta$ and $A \cap B_h \neq \phi$, $\forall h = 1, \ldots, n$. Denote by $\mathcal{V}_h = \{U_i : U_i \in \mathcal{U}$ and $U_i \cap f(A \cap B_h) \neq \phi\}$. Then $\cup \mathcal{V}_h = \mathcal{U}$. More $[A \cap B_1 : \mathcal{V}_1] \cap \ldots \cap [A \cap B_n : \mathcal{V}_n]$ is a $\tau_{\alpha, \text{loc-fin}}$-nhbd of f all contained in $[A : \mathcal{U}]$. ∎

PROPOSITION 4.22. – *If X is completely regular, Y contains a non trivial path and $\tau_{\alpha, \text{loc-fin}} \subset \tau_{\beta, \text{loc-fin}}$, then α refines β.*

PROOF. – See analogous in McCoy-Ntantu [8]. ■

In the above conditions:

COROLLARY 4.23. – *$\tau_{\alpha,\,\text{loc-fin}}$ coincides with the point-open topology iff α is the finite network.* ■

COROLLARY 4.24. – *$\tau_{\alpha,\,\text{loc-fin}}$ coincides with the compact-open topology iff α is the compact network.* ■

PROPOSITION 4.25. – *The codomain space Y embeds as a closed subspace in $C(X, Y)$ equipped with $\tau_{\alpha,\,\text{loc-fin}}$.*

PROOF. – Suppose Y identified with the set of all constant functions on X. Observe that $e(Y) \cap [A : \mathcal{U}] = \phi$, when $\mathcal{U}$ is not finite and $e(Y) \cap [A : V] = e(V)$ for each A and each open set in Y. Then, more, $e(Y)$ is closed since it is closed in the point-open topology which is naturally contained in $\tau_{\alpha,\,\text{loc-fin}}$. ■

PROPOSITION 4.26. – *The codomain space Y is Hausdorff iff $\tau_{\alpha,\,\text{loc-fin}}$ is Hausdorff.*

PROOF. – It follows from proposition 4.25 and Hausdorffness of the point-open topology which is weaker than $\tau_{\alpha,\,\text{loc-fin}}$. ■

To acquire stronger separation properties for $\tau_{\alpha,\,\text{loc-fin}}$ we need to use networks especially close to Y. We say that a network α in X is *regular w.r.t. Y* iff when $\overline{f(A)} \subset V$, $f \in C(X, Y)$, $A \in \alpha$, V is open in Y, then there exists an open set U in Y such that $\overline{f(A)} \subset U \subset \overline{U} \subset V$. We say that a network α in X is *normal w.r.t. Y* iff when $\overline{f(A)} \subset V$, then there exists a Urysohn function which separates $\overline{f(A)}$ from $Y - V$. Naturally normality of a network w.r.t. Y implies regularity w.r.t. Y.

When Y is completely regular and $\boldsymbol{U}$ is a compatible uniformity, the Hausdorff hypertopology $\tau_H(\boldsymbol{U})$ jointly with a network α generates via procedure (*) a completely regular function space topology $\tau_\alpha(\boldsymbol{U})$. Denote $\tau_{\alpha,\,\text{fine}}$ that one induced from the finest uniformity which, as is known, is generated from all open normal coverings of Y.

THEOREM 4.27.

(i) *Y regular and α regular w.r.t. Y implies $\tau_{\alpha,\,\text{loc-fin}}$ is regular.*

(ii) *Y completely regular (uniformizable) and α normal w.r.t. Y implies $\tau_{\alpha,\,\text{loc-fin}}$ is completely regular (uniformizable).*

PROOF. – (i) Let $f \in [A : B]$, $A \in \alpha$, B open in Y. Then $\overline{f(A)} \subset V \subset \overline{V} \subset B$, for some open set V in Y. Since the closure of $[A : V]$ in $\tau_{\alpha, V}$ is contained in $[A : \overline{V}]$, then $Cl[A : V]$ in $\tau_{\alpha, \text{loc-fin}}$ is in $[A : \overline{V}]$ as well. So $f \in Cl[A : V] \subset [A : \overline{V}] \subset [A : B]$. Next, $f \in [A : \mathcal{U}]^-$, where $\mathcal{U} = \{U_i : i \in I\}$ is a locally finite open family in Y. For each $i \in I$ select $x_i \in \overline{f(A)} \cap U_i$ and an open set V_i such that $x_i \in V_i \subset \overline{V}_i \subset U_i$. Put $\mathcal{V} = \{V_i : i \in I\}$. Then $f \in [A : \mathcal{V}]^-$. Since $Cl[A : \mathcal{V}]^-$ in $\tau_{\alpha, \text{loc-fin}}$ is contained in $[A : \overline{\mathcal{V}}]^-$ it follows that $f \in [A : \mathcal{V}]^- \subset Cl[A : \mathcal{V}]^- \subset [A : \overline{\mathcal{V}}]^- \subset [A : \mathcal{U}]^-$.

(ii) We show that $\tau_{\alpha, \text{loc-fin}} = \tau_{\alpha, \text{fine}}$. Let us suppose $A \in \alpha$, $\mathcal{U} = \{U_i : \in I\}$ is a discrete open family in Y and $f \in [A : \mathcal{U}]$. Then $\overline{f(A)} \subset \cup \{U_i : i \in I\}$ and $\overline{f(A)} \cap U_i \neq \phi$, $\forall i \in I$. The hypothesis assures the open set covering $\mathcal{R}_1 = \{Y - \overline{f(A)}, \bigcup_{i \in I} U_i\}$ is normal. Next pick $x_i \in \overline{f(A)} \cap U_i$ and choose a Urysohn function g_i such that $g_i(x_i) = 1$ and $g_i(Y - U_i) = 0$. Since $Y - g_i^{-1}(0) \subset U_i$, $\{Y - g_i^{-1}(0) : i \in I\}$ is a discrete family and the cozero-set covering $\mathcal{R}_2 = \{Y - \cup_{i \in I} g_i^{-1}(1), Y - g_i^{-1}(0) : i \in I\}$ is normal. So $\mathcal{R}_1$, $\mathcal{R}_2$ both belong to the fine uniformity of Y. Consider the diagonal nhbds U_1, U_2 determined from $\mathcal{R}_1$, $\mathcal{R}_2$ respectively. If g is in $(A, \underline{U_1})(f)$, see (5), (7), then $\overline{g(A)} \subset \cup_{i \in I} U_i$ and furthermore if g is in $(A, \underline{U_2})(f)$, then $\overline{g(A)} \cap Y - g_i^{-1}(0) \neq \phi$, for each $i \in I$. It follows that $(A, \underline{U_1})(f) \cap (A, \underline{U_2})(f) \subset [A : \mathcal{U}]$; that means $\tau_{\alpha, \text{loc-fin}} \subset \tau_{\alpha, \text{fine}}$. Vice versa, suppose $A \in \alpha$, $\mathcal{U} = \{U_i : i \in I\}$ is an open normal covering in Y, U is its natural diagonal nbhd. Then $\mathcal{U}$ admits an open locally finite refinement $\mathcal{V} = \{V_j : j \in J\}$. Consider $\mathcal{W} = \{V_j \in \mathcal{V} : Vj \cap \overline{f(A)} \neq \phi\}$. Then $[A : \mathcal{W}]$ is a $\tau_{\alpha, \text{loc-fin}}$-nhbd of f contained in $(A, \underline{U})(f)$. If $g \in [A : \mathcal{W}]$, any $y \in \overline{g(A)}$ belongs to some $V_j \in \mathcal{W}$ which in turn has to contain some point of $\overline{f(A)}$ and has to be contained in some $U_i \in \mathcal{U}$. The same happens when f substitues g. And the result definitively follows. ■

COROLLARY 4.28. – *When Y is normal, for any network α in X, $\tau_{\alpha, \text{loc-fin}} = \tau_{\alpha, \text{fine}}$.*

THEOREM 4.29. – *When Y is metrizable, $\tau_{\alpha, \text{loc-fin}}$ is the supremum of all $\tau_{\alpha, d}$'s where d runs over all compatible metrics on Y.*

PROOF. – It follows from results in [2] and procedure (*). ■

In conclusion we consider the locally finite Arens-Dugundji generalization, call it $c_{\alpha, \text{loc-fin}}$, which admits as subbase

$\{[A, \mathcal{U}] : A \in \alpha,\ \mathcal{U} = \{U_i, i = 1, \ldots, n\}$ is a locally finite open family in $Y\}$

where $[A, \mathcal{U}] = \{f \in C(X, Y) : f(A) \subset \cup\, \mathcal{U}$ and $f(A) \cap U_i \neq \phi,\ \forall i \in I\}$.

It is very easy to see that $c_{\alpha, \text{loc-fin}}$ and $\tau_{\alpha, \text{loc-fin}}$ have the same *hit* part.

We show that:

THEOREM 4.30. – *When Y is metrizable, $\tau_{\alpha, \text{loc-fin}}$ is weaker than $c_{\alpha, \text{loc-fin}}$ but generally different.*

PROOF. – By Beer and others results any $\tau_{\alpha, d}$ is weaker than $c_{\alpha, \text{loc-fin}}$ since in the definition of $\tau_{\alpha, d}$'s we use Hausdorff pseudometrics for which the closures have no influence. Since $\tau_{\alpha, \text{loc-fin}}$ is the supremum of all $\tau_{\alpha, d}$'s, then, $\tau_{\alpha, \text{loc-fin}}$ is weaker than $c_{\alpha, \text{loc-fin}}$. In the following example they are different. Let $X = \{(x, y) \in \mathbb{R}^2 \colon x > 0, y > 0\}$, $Y = \mathbb{R}$ and $\alpha = CL(X)$. Consider $C = \{(x, 1/x) \colon x > 0\}$ and the projection π on the x-axis. Then $\pi \in [C, \mathbb{R}^+]$ but no $[C_1 \colon \mathcal{U}_1] \cap \ldots \cap [C_n \colon \mathcal{U}_n]$ containing π can be contained in $[C, \mathbb{R}^+]$, with $C_1 \ldots, C_n$ closed in X and $\mathcal{U}_1, \ldots, \mathcal{U}_n$ locally finite open families in $\mathbb{R}$. Set $\mathcal{U}_i = \{U_{ij} \colon j \in J_i\}$, $i = 1, \ldots, n$ and pick $x_{ij} \in \overline{\pi(C_i)} \cap U_{ij}$. Observe that $E = \{x_{ij} \colon j \in J_i, i = 1, \ldots, n\}$ is discrete.

Suppose the C_i's are indexed in such a way $0 \in \cup \mathcal{U}_i$ if $i = 1, \ldots, h$ and $0 \notin \cup \mathcal{U}_i$ if $i = h + 1, \ldots, n$. Then for a convenient $\varepsilon > 0, [0, \varepsilon] \subseteq \cup \mathcal{U}_i$, $i = 1, \ldots, h$ and $[0, \varepsilon] \cap \pi(C_i) = \phi$, $i = h + 1, \ldots, n$. Naturally E cannot contain $]0, \varepsilon[$. Select in $]0, \varepsilon[$ a point x_0 and a nhbd $]x_1, x_2[\subset]0, \varepsilon[$ of x_0 which does not intersect E. Denote $S_1 =]x_1, x_0] \times [0, \infty)$, $S_2 = [x_0, x_2[\times [0, \infty)$ and $S =]x_1, x_2[\times [0, \infty)$. Define the following continuous partial map g, $g|_{C_i} = \pi$ if $i = h+1, \ldots, n$; $g|_{C_i - S} = \pi$, $i = 1, \ldots, h$; $g|_{\overline{S}_1} = h_1 \circ \pi$ with $h_1(x) = \dfrac{x_1}{x_1 - x_0}(x - x_0)$; $g|_{\overline{S}_2} = h_2 \circ \pi$ with $h_2(x) = \dfrac{x_2}{x_2 - x_0}(x - x_0)$. After glueing on $\bigcup_{i=1}^{n} C_i \cup \overline{S}$ extend continuously over all X by Tietze's extension theorem. Call again g the extension. Then $g \notin [C, \mathbb{R}^+]$ since $g(x_0, 1/x_0) = 0 \notin \mathbb{R}^+$, but g is in $\bigcap_{i=1}^{n} [C_i \colon \mathcal{U}_i]$. ■

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