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(Finite) Presentations of the Albert-Frank-Shalev Lie Algebras.

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Sunto. – *In questo lavoro vengono studiate le algebre di Albert-Frank-Shalev. Queste sono algebre di Lie modulari di dimensione infinita, ottenute da un* loop *di certe algebre semplici di dimensione finita. Si dimostra che le algebre di Albert-Frank-Shalev sono unicamente determinate, a meno di elementi centrali o secondo centrali, da un certo quoziente finito-dimensionale. Tale risultato si ottiene dando la presentazione finita di un'algebra il cui quoziente sul secondo centro (infinito-dimensionale) è isomorfo alle algebre di Albert-Frank-Shalev.*

1. – Introduction.

Aner Shalev [Sha94] has shown that over any field of prime characteristic there are countably many (pairwise non-isomorphic) infinite-dimensional graded Lie algebras of maximal class that are insoluble. To construct these algebras, Shalev exploits the fact that certain modular simple Lie algebras originally constructed by Albert and Frank admit a (non singular) outer derivation, and constructs its examples using a twisted loop algebra construction. The algebras of Albert and Frank [AF55] are nowadays known as Hamiltonian nongraded Lie algebras [Kos96].

Caranti, Mattarei and Newman [CMN97] have shown that over any field $\boldsymbol{F}$ of prime characteristic there are $|\boldsymbol{F}|^{\aleph_0}$ algebras of maximal class, thus uncountably many. These algebras can be classified (see [CN99] and [Jur99]), and the purpose of this paper is to provide part of the classification for odd primes. We show that the algebras constructed by Shalev can be recognised by a certain finite-dimensional quotient. These algebras are called Albert-Frank-Shalev or AFS.

Our work can be conveniently formulated in terms of (finite) presentations. The main result is the following.

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THEOREM 1. – *For every* AFS *algebra L there is a finitely presented algebra M such that the quotient of M by its second centre is isomorphic to L.*

The algebra M is defined by a certain finite number of relations that determine the initial segment of its homogeneous components. For the explicit construction of the algebra M and the proof that its second centre is infinite-dimensional see [C99]. Note that the fact that the second centre of M is infinite-dimensional implies, by a result of B. H. Neumann (see for example [Rob82], Theorem 2.2.3), that L itself is not finitely presented.

Our methods are direct and elementary, and rely mainly on commutator expansions using the generalized Jacobi identity. We use several time Lucas' Theorem to evaluate binomial coefficients modulo a prime. The ANU p-quotient program [HNO95] has been invaluable in gathering the computational evidence that guided our calculations (see [CMN97] for details of this approach).

In Section 2 we introduce some preliminary notions about graded Lie algebras of maximal class. We mainly cite some results from [CMN97] and [CN99], and we clearly state and explain the main result of the work. In Section 3 we introduce some notations, while in Section 4 we describe the finite presentation of the algebra M and we use a diagram to explain the role of AFS Lie algebras in the classification of graded Lie algebras of maximal class. Section 5 contains the computations proving that the algebra AFS is finitely presented. Finally in Section 6 and Section 7 we describe the second central elements of the algebra M, respectively for p odd prime and $p=2$.

2. – Preliminaries.

A finite-dimensional graded Lie algebra

$$L = \bigoplus_{i=1}^{n-1} L_i,$$

generated by L_1, is said to be of *maximal class* if it has dimension n and nilpotency class $n-1$. Equivalently, $\dim(L_1)=2$ and $\dim(L_i)=1$ for $1<i<n$. Note that the condition that L is generated by L_1 is equivalent to $[L_i L_1] = L_{i+1}$ for $i \geqslant 1$.

This definition can be naturally extended to cover the infinite-dimensional case. A graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i$$

generated by L_1, is of *maximal class* if every finite-dimensional graded factor of it is so in the above sense. In other words, $\dim(L_1)=2$ and $\dim(L_i) \leqslant 1$ for

$i > 1$. Clearly, L is infinite-dimensional if and only if $\dim(L_i) = 1$ for all $i > 1$.

A p-group (finite or profinite) is of maximal class if the graded Lie algebra associated to it with respect to the lower central series is of maximal class. In the theory of p-groups studied by Blackburn [Bla58], an important role is played by the so-called two-step centralizers. In the graded Lie algebras context, the *two-step centralizers* are the subspaces C_i of L_1 centralizing the homogeneous components L_i:

$$C_i = C_{L_1}(L_i) \qquad \text{for } i > 1 .$$

It is useful to put formally $C_1 = C_2$.

In [CMN97], Theorem 3.2, it is proved that graded Lie algebras of maximal class are determined by their sequence of two-step centralizers $(C_i)_{i>1}$. Moreover if $C_2 = C_{i-1} \neq C_i$ and the algebra has class at least $i+2$, then $C_{i+1} = C_2$. Thus it is natural to give the following

DEFINITION 1. – A constituent *of a graded Lie algebra of maximal class L is a subsequence $(C_i, \ldots, C_j)$ of the sequence of two-step centralizers such that $C_2 = C_i = C_{i+1} = \ldots = C_{j-1} \neq C_j$ with either $i = 1$ or $C_{i-1} \neq C_2$.*

Note that this is the definition given in [CN99]. It is slightly different from the one previously used in [CMN97]. Practically in the more recent definition the length of every constituent is increased by one.

Following [CN99] we also give the following

DEFINITION 2. – *Let v be a nonzero homogeneous element of L. Then v is said to be* at the end *of the constituent $(C_i, \ldots, C_j)$ if $v \in L_j$, while it is said to be* at the beginning *of the constituent if $v \in L_{i-1}$.*

Note that in particular if $C_2 = \boldsymbol{F}y$ and v is an element at the end or at the beginning of a constituent, then $[vy] \neq 0$.

In this work we consider some problems related to the *algebras of Albert-Frank-Shalev*. These are graded Lie algebras of maximal class obtained as *(twisted) loop algebras* of the simple algebras studied by Albert and Frank [AF55]. These algebras depend on three integer parameters a, b, n with $0 < a < b \leqslant n$, and on a prime p, and are denoted by $\mathrm{AF}(a, b, n, p)$. Note that the original choice of the parameters was $0 \leqslant a < b < n$. It is easy to see the equivalence of the two notations (see also [CN99]). Indeed the algebras $\mathrm{AF}(0, b, n, p)$ and $\mathrm{AF}(b, n, n, p)$ have the same constituent lengths (see [CMN97]), and thus they are isomorphic. The choice $a > 0$ allows us to use a more uniform notation.

The (twisted) loop process permits to move from the finite-dimensional algebras $\mathrm{AF}(a, b, n, p)$ to the infinite-dimensional $\mathrm{AFS}(a, b, n, p)$.

In the scheme of Kostrikin [Kos96], the Lie algebra $\mathrm{AF}(a, b, n, p)$ can also be seen as the *Hamiltonian* algebra $H(2;(n_1, n_2), \omega_2)$, where $n_2 = b - a$ and $n_1 + n_2 = n$.

In [CMN] it is proved that $\mathrm{AFS}(a, b, n, p)$ has the following sequence of constituent lengths

$$2q, q^{r-2},(2q-1, q^{r-2},(2q, q^{r-2})^{t-1})^{\infty}$$

where

$$q = p^{a}, \qquad r = p^{b-a}, \qquad t = p^{n-b}. \tag{1}$$

Note that we used our formal assumption $C_1 = C_2$. Saying that an algebra has constituent lengths $\gamma^{s},(\alpha^{c}, \beta^{d})^{\infty}$, we mean that it has s constituents of length γ followed by c constituents of length α and d constituents of length β, then again c of length α and d of length β cyclically.

From now on we will fix q, r and t as in (1), calling q the *parameter* of the algebra. Note that $q, r \geqslant p$, while it is possible to have $t = 1$. For this reason in particular $\mathrm{AFS}(b, n, n, p)$ has no constituents of length $2q$ besides the very first one.

In [CMN97], Proposition 5.6, it is proved that in an infinite-dimensional graded Lie algebra of maximal class with parameter q the length l of any constituent is bounded by $q \leqslant l \leqslant 2q$. Hence it is natural to call respectively *short* and *long* the constituents of length q and $2q$. All other constituents are called *intermediate*. Note that in an AFS algebra the only intermediate constituents are those of length $2q-1$.

We claim that the algebra AFS is determined by a large enough finite quotient. Let $L = \mathrm{AFS}(a, b, n, p) = \langle x, y \rangle$. Consider a base R^{m} of its homogeneous relations of weight at most m in x, y. Since L is of maximal class the set R^{m} has cardinality at most $m-2$; the set R^{m} describes the initial segment of the pattern of constituents of L, up to L_m. Let $M = \langle x, y : R^{m} \rangle$. We claim that for a suitable choice of m the quotient of M by its second centre $Z_2(M)$ is isomorphic to the algebra L.

THEOREM 2. – *Let* $L = \mathrm{AFS}(a, b, n, p)$ *and* $m = (t+1)rq + 2q - 1$, *where* q, r *and* t *are as in* (1), *and* p *is any prime. Let* R^{m} *be a base of the homogeneous relations of* L *of weight at most* m *in* x, y.

Then $M = \langle x, y : R^{m} \rangle$ *is a finitely presented, infinite-dimensional Lie algebra such that* $L \cong M/Z_2(M)$.

Note that we supposed that M starts as a graded Lie algebra of maximal class up to the homogeneous component of weight m. In Section 4 we describe

R^m precisely, and we also see that we do not need all the $m-2$ relators but that it is enough to consider a subset $R \subset R^m$. Note also that to prove Theorem 1 only the computations up to Section 5 are needed. Indeed at the end of Section 5 we can already say that the finitely presented algebra M is such that $M/Z_2(M) \cong L$. In Sections 6 and 7 a complete description of $Z_2(M)$ is given.

Most of the computations are based on the *generalized Jacobi identity* or its analogue suggested in [CN99] that we will use without specific mention:

$$\begin{aligned}
[w\,[y\,\underbrace{x\dots x}_{\lambda}]] &= \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}[w\,\underbrace{x\dots x}_{i}\,y\,\underbrace{x\dots x}_{\lambda-i}] \\
&= (-1)^{\lambda}\sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}[w\,\underbrace{x\dots x}_{\lambda-i}\,y\,\underbrace{x\dots x}_{i}] \\
&\approx \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}[w\,\underbrace{x\dots x}_{\lambda-i}\,y\,\underbrace{x\dots x}_{i}],
\end{aligned}$$

where with the notation $a \approx b$ we mean $a = \pm b$.

To evaluate binomial coefficients modulo p we use the following

LUCAS' THEOREM [Luc78] 1. – *Let p be a prime and a and b be two non-negative integers with p-adic decomposition*

$$\begin{aligned}
a &= a_k p^k + a_{k-1}p^{k-1} + \dots + a_1 p + a_0, \\
b &= b_k p^k + b_{k-1}p^{k-1} + \dots + b_1 p + b_0,
\end{aligned}$$

where $0 \leqslant a_i,\ b_i < p$. *Then*

$$\binom{a}{b} \equiv \binom{a_k}{b_k}\binom{a_{k-1}}{b_{k-1}}\dots\binom{a_1}{b_1}\binom{a_0}{b_0} \quad \bmod p\,.$$

3. – Notation.

Suppose $M = \bigoplus_{i=1}^{\infty} M_i$ is a graded Lie algebra such that $M/Z_2(M)$ is a graded Lie algebra of maximal class. We define a *constituent* of M as a constituent of $M/Z_2(M)$.

Following [CN99] we introduce the element $z = x + y$. Indeed if v is an element centralized by x (resp. y), we obtain the same result commuting v with y

(resp. x) or with z:

$$[vz] = [vx] + [vy] = [vy] \quad (\text{resp.}\ [vx])\,.$$

To simplify notation, we write $[y\ x^n\ y] = [y\ \underbrace{x \dots x}_{n}\ y]$, or

$$[y\ x^{2q-2}\ y\ (x^{q-1}\ y)^n] = [y\ \underbrace{x \dots x}_{2q-2}\ y\ \underbrace{\underbrace{x \dots x}_{q-1}\ y \dots\dots \underbrace{x \dots x}_{q-1}\ y}_{n}]\,,$$

and similarly for more complex formulas. We also write $[\dots x]$ (or $[\dots y]$, or $[\dots z]$) when all the components of a commutator except the last one are clear from the context.

We denote by ϑ_1 the element $[y\ x^{2q-1}\ y\ (x^{q-1}\ y)^{r-2}]$. Inductively we let

$$\vartheta_{i+1} = [\vartheta_i\ x^{2q-2}\ y\ (x^{q-1}\ y)^{r-2}\,(x^{2q-1}\ y\ (x^{q-1}\ y)^{r-2})^{t-1}]\,.$$

Note that the element ϑ_i has weight $1 + rq + (i-1)(p^n - 1)$. For $n > b$ we also introduce the elements

$$u_{j,\,i} = [\vartheta_i\ x^{2q-2}\ y\ (x^{q-1}\ y)^{r-2}\,(x^{2q-1}\ y\ (x^{q-1}\ y)^{r-2})^j]$$

for $0 \leqslant j \leqslant t-1$. Note that $u_{t-1,\,i} = \vartheta_{i+1}$.

4. – The finite presentation.

We have seen in Section 2 that $M = \langle x, y : R^m \rangle$, where the set R^m consists of the homogeneous relations of weight at most $m = (t+1)\,rq + 2q - 1$ defining $L = \mathrm{AFS}(a, b, n, p)$.

In other words we know that M starts as a graded Lie algebra of maximal class with initial segment of constituent lengths

$$2q,\ q^{r-2},\ 2q-1,\ q^{r-2}, (2q,\ q^{r-2})^{t-1},\ 2q-1\,, \tag{2}$$

and $C_i \in \{\boldsymbol{F}x, \boldsymbol{F}y\}$ for every i, $2 \leqslant i \leqslant m$.

Apparently in R^m we need one relation for every homogeneous component M_i with $2 \leqslant i < m$. Indeed all M_i are one-dimensional and either $[M_i x]$ or $[M_i y]$ vanishes. Hence we have to specify whether $C_i = \boldsymbol{F}x$ or $C_i = \boldsymbol{F}y$, for $2 \leqslant i < m$.

Actually we will see that giving all the $m-2$ relations is redundant. We claim that, using the notation introduced in Section 3, it is enough to consider the subset $R \subset R^m$ defined in the following way:

$$R = R_1 \cup R_2 \cup R_3 \cup R_4\,,$$

with

$$
\begin{aligned}
&R_1 = \{[y\ x^{2i+1}\ y],\ \text{ for } 0 \leqslant i \leqslant q-2,\ [y\ x^{2q}]\}\,, \\
&R_2 = \{[y\ x^{2q-1}\ y\ (x^{q-1}\ y)^m\ x^q],\ \text{ for } 0 \leqslant m \leqslant r-3\}\,, \\
&R_3 = \{[\vartheta_1\, x^{2q-p^s-1}\ y],\ \text{ for } 1 \leqslant s \leqslant a,\ [\vartheta_1\, x^{2q-1}]\}\,, \\
&R_4 = \{[u_{j,1}\, x^{2q-2}\ y],\ \text{ for } 0 \leqslant j \leqslant t-2,\ [\vartheta_2\, x^{2q-1}]\}\,.
\end{aligned} \tag{3}
$$

Note that the relations involving the elements $u_{j,1}$ are given only for $n > b$.

We will prove that defining R as above, the Lie algebra $M = \langle x, y : R \rangle$ has initial segment of constituent lengths (2), with the exception of the central elements

$$[\vartheta_1\, x^{2q-2}\ y\ (x^{q-1}\ y)^{r-p^s-1}\ x^q],\ \text{ for } 1 \leqslant s \leqslant b-a-1\,,$$

$$[\vartheta_1\, x^{2q-2}\ y\ (x^{q-1}\ y)^{r-2}\ x^{2q-p^s-1}\ y],\ \text{ for } 1 \leqslant s \leqslant a\,.$$

Note that to have $\langle x, y : R \rangle = \langle x, y : R^m \rangle$, we should require these additional relators, while without them $\langle x, y : R^m \rangle$ is isomorphic to the quotient of $\langle x, y : R \rangle$ modulo its centre. Since anyway we are interested in $M/Z_2(M)$, we can skip requiring them.

Finally we can express the main result of this work in the following way

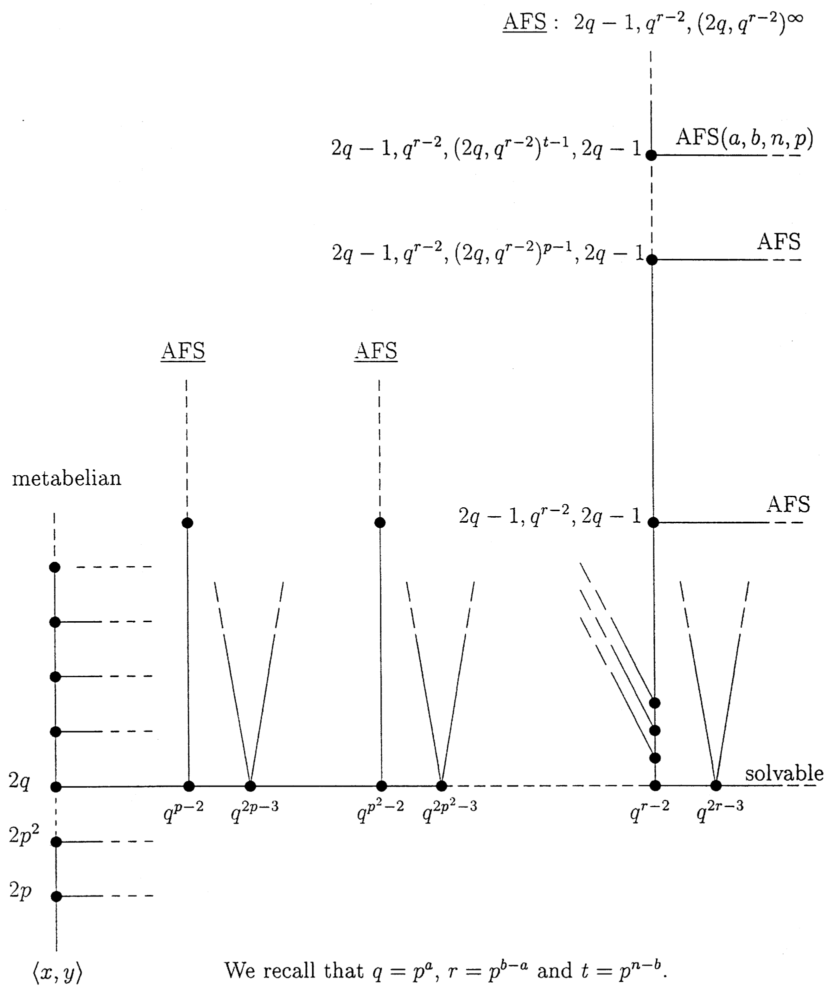
THEOREM 3. – *Let $L = \mathrm{AFS}(a, b, n, p)$ and $M = \langle x, y : R \rangle$, where $R = R_1 \cup R_2 \cup R_3 \cup R_4$ with the sets R_i defined as in* (3). *Then M is a finitely presented, infinite-dimensional Lie algebra such that $L \cong M/Z_2(M)$.*

We now try to explain the meaning of the sets $R_1 \ldots R_4$ also in view of the results of [CN99]. The part of this work that we use can be outlined in the following picture that represents the tree of the infinite-dimensional graded Lie algebras of maximal class. See the following description for more details. We refer to [CN99], [CMN97] for the underlying general theory.

In a Lie algebra of maximal class $L = \langle x, y \rangle$ with two two-step centralizers, given a homogeneous component L_i, either $[L_i x]$ or $[L_i y]$ vanish. The points of the picture where two lines depart represent the homogeneous components $L_i = \boldsymbol{F} v$ such that both the relations $[vx]$ and $[vy]$ lead to different infinite-dimensional Lie algebras of maximal class. Let us denote by L and L' the two algebras obtained. It is clear that

$$L \Big/ \bigoplus_{j>i} L_j \cong L' \Big/ \bigoplus_{j>i} L_j'\,.$$

In any of these points we have to give a relation to specify which one of the two algebras we choose.



The relations in R_1 fix the parameter q of the algebra. Indeed in [CMN97], Theorem 5.5, it is proved that the first constituent of an infinite-dimensional non-metabelian graded Lie algebra of maximal class over a field of characteristic p is twice a power of the prime p. The value $q=p^h$ of this power is called the parameter of the algebra.

Looking at the metabelian branch of the picture, the relations in R_1 say that we move on the principal branch of the metabelian Lie algebra up to the homogeneous component of weight $2q$. At this point, labelled $2q$, the relator $[yx^{2q}]$ make us move in the direction of the horizontal branch labelled *solvable.*

Moreover in [CMN97], Property 5.6, it is shown that if the parameter of the algebra is $q = p^h$, then the only possible constituent lengths are $2q$ and $2q - p^s$ with $s \in \{0, 1, \dots, h\}$.

Theorem 8.2 of [CMN97] states that a Lie algebra with all short constituents besides the first one is solvable. Moreover if the first non-short constituent beside the first one is long, the algebra is inflated [CN99]. If we are not in one of these cases, then in [CN99], Property 5.3, 5.5, it is proved that between the first constituent and the next non-short one there are $r - 2$ short constituents, where r is a power of p. The value of r is fixed by the relations in R_2.

Looking at the picture, the relations in R_2 imply that we ignore all the alternative on the solvable branch up to the point labelled q^{r-2}. At this point, in corrispondence of the homogeneous component of weight $(r+1)q$, is fixed the next non-short constituent, and we start moving in the direction of the inverse limit $\underline{\mathrm{AFS}}$ (see [CN99]).

Note also that by R_3 the non-short constituent has length $2q - 1$. This means that we proceed in the $\underline{\mathrm{AFS}}$ direction avoiding the inflated algebras. These are represented in the picture by the oblique lines departing from the points over the one labelled q^{r-2}.

Finally in [CN99], Section 7, it is shown that in a Lie algebra of maximal class with initial segment of constituents $2q$, q^{r-2}, $2q-1$, the sequence of constituent lengths consists of repetitions of the patterns $2q$, q^{r-2} or $2q-1$, q^{r-2}. For this reason, as we will see better later, we can avoid many relations. Moreover in [CN99], Section 8, it is proved that if an algebra starts as above and it has another long constituent, then the initial segment of constituents has to be $2q$, q^{r-2}, $2q-1$, q^{r-2}, $(2q, q^{r-2})^{t-1}$, $2q-1$, with t power of the prime p (possibly 1). The value of t is determined by the relations in R_4.

Looking at the picture the relations in R_4 fix the last point labelled $2q-1$, q^{r-2}, $(2q, q^{r-2})^{t-1}$, $2q-1$ where we finally choose our specific algebra $\mathrm{AFS}(a, b, n, p)$.

We will prove that from then on there are no more infinite branches departing from the one we are moving on.

We claim that the homogeneous components of $M = \langle x, y : R^m \rangle$ have dimention at most two, and that $M/Z_2(M) \cong L$.

More precisely we claim that for every $i \geqslant 2$ between $[\vartheta_i\, x^{2q-2}]$ and $[\vartheta_{i+1}\, x^{2q-2}]$, in M there can be only the following extra central elements:

(4a) $\qquad [\vartheta_i\, x^{2q-1}]$, possibly second central for $n = b$,

(4b) $\qquad [\vartheta_i\, x^{2q-1}\, y]$, for $n = b$, and $i \equiv 2 \mod p$, (4c) $[\vartheta_i\, x^{2q-2}\, y\ (x^{q-1}\, y)^{r-p^s-1}\, x^q]$, for $1 \leqslant s \leqslant b-a-1$,

(4d) $[\vartheta_i\, x^{2q-2}\, y\ (x^{q-1}\, y)^{r-2}\, x^{2q-p^s-1}\, y]$, for $1 \leqslant s \leqslant a$,

(4e) $[u_{0,\, i}\, x^{2q-2}\, y]$, possibly second central,

(4f) $[u_{0,\, i}\, x^{2q-2}\, yx]$, for $i \equiv 1 \mod p$,

(4g) $[u_{t-p^s,\, i}\, x^{2q-2}\, y]$, for $1 \leqslant s \leqslant n-b-1$.

Note that the elements (4c) and (4d) occur also for $i=1$ as we noticed above, while by the defining relations the elements (4a) occur only for $i \geqslant 3$. Moreover the elements in (4e), (4f) and (4g) exist only for $n > b$. The central elements (4d) correspond to $[\vartheta_{i+1}\, x^{2q-p^s-1}\, y]$ for $n=b$, and to $[u_{0,\, i}\, x^{2q-p^s-1}\, y]$ for $n > b$.

Note also that in a complete period of the algebra, that is in a section of p^n-1 homogeneous components, there are n central elements, and possibly one second-central element.

5. – The proof.

5.1. *The first constituent of M has length* $2q$.

We prove that the relations in R_1 are enough to say that the first constituent of M has length $2q$.

We first claim that $[y\ x^{2i}\ y]=0$ for $0 \leqslant i \leqslant q-1$. We proceed by induction, the starting point being a trivial property of Lie algebras. Supposing it true up to $i \leqslant q-2$ we find:

$$0 = [[y\ x^{i+1}][y\ x^{i+1}]] = \sum_{j=0}^{i+1} (-1)^j \binom{i+1}{j} \cdot [y\ x^{i+1+j}\ y\ x^{i+1-j}].$$

By the inductive hypothesis and the defining relations it follows that the elements $[y\ x^{2i+2}\ y]$ vanish.

Hence the relations $[y\ x^{2i+1}\ y]=0$ say that the first constituent has length greater or equal to $2q$. By the additional relator $[y\ x^{2q}]$ the length has to be exactly $2q$, and $C_{2q} = \boldsymbol{F}x$.

5.2. *Using the first constituent* $2q$.

The fact that the first constituent has length $2q$ implies that the only possible constituent lengths in M are of the form $2q$ or $2q-p^s$ with $s \in \{0, 1, \ldots, a\}$.

This is a well known property of Lie algebras of maximal class, and the computations to prove it in M are substantially the same of [CMN97], Proposition 5.6. We have just to be careful to the possible additional central or second central elements. For example when in [CMN97] an element $v \in L_i$ is considered and it is supposed $C_i = \boldsymbol{F}y$, it follows that $[vy] = 0$. Since in M the element $[vy]$ can be second central we can only say that $[vyzz] = 0$. In any case the only proof that we have to give in a slightly different way is the following.

Let w be a nonzero element of M_i and let $C_i = \boldsymbol{F}y$, then $[wxyy] = 0$. Indeed we can compute

$$0 = [w[yxy]] = 2[wyxy] - [wxyy] - [wyyx] = -[wxyy].$$

Note that at this point the relators in $R_1 \cup R_2 \cup R_3$ are enough to say that the initial segment of constituent lengths of M is

$$2q,\ q^{r-2},\ 2q-1\ .$$

Moreover $C_i \in \{\boldsymbol{F}x, \boldsymbol{F}y\}$ for $2 \leqslant i \leqslant (r+4)\,q$.

5.3. *Using the first segment* $2q,\ q^{r-2}$.

In this part we consider two different situations.

Suppose first that in M there is a segment of constituent lengths

$$\dots,\ q^{r-2},\ 2q\ .$$

Then such segment is followed by another sequence q^{r-2}, and the centralizers belong to the set $\{\boldsymbol{F}x, \boldsymbol{F}y\}$.

Let v be an element at the beginning of the long constituent of the segment. By 5.2 we have only to show that

$$[vy\ x^{2q-1}\ y\ (x^{q-1}\ y)^m\ x^q] = 0$$

for every $0 \leqslant m < r-2$. We consider the element w at the beginning of the (p^s-1)-th short constituent of the segment before the long one.

We let $m = dp^s - 2$ with $d \not\equiv 0 \bmod p$. Since $m < r-2$, it follows that $d < r/p^s$. Suppose now $m + p^s - 1 \geqslant r - 2$, then $d > (r/p^s) - 1$ in contrast with our previous result. As a consequence we can use the defining relation $[y\ z^\lambda\ x] = 0$ with $\lambda = (m + p^s + 2)\,q - 1 = (d+1)\,p^s q - 1$. We find

$$\begin{aligned}
0 &= [w[y\ z^\lambda\ x]] = [w[y\ z^\lambda]\ \ x] \\
&= \left(\sum_{i=0}^{m+p^s+1} (-1)^{iq} \binom{\lambda}{iq} - (-1)^{p^s} \binom{\lambda}{p^s q} \right) \cdot [w \dots x] \\
&= \binom{dp^s + p^s - 1}{p^s} \cdot [w \dots x] \\
&= d \cdot [w \dots x].
\end{aligned}$$

Since $d \not\equiv 0$ it follows that $[vy\ x^{2q-1}\ y\ (x^{q-1}\ y)^m\ x^q] = 0$.

Suppose now that in M there is a segment of constituent lengths

$$\ldots, q^{r-2}, 2q-1 .$$

In this case we can prove that such a segment is followed by another sequence q^{r-2}, but with the exception of the central elements

$$[vy\ x^{2q-2}\ y\ (x^{q-1}\ y)^{r-p^s-1}\ x^q],$$

where v is an element at the beginning of the intermediate constituent of the segment. Also in this case the centralizers belong to the set $\{\boldsymbol{F}x, \boldsymbol{F}y\}$.

As above we have just to consider the elements

$$[vy\ x^{2q-2}\ y\ (x^{q-1}\ y)^m\ x^q]$$

with $0 \leqslant m < r-2$. We claim that if $m \neq r-p^s-1$ for some $1 \leqslant s \leqslant b-a-1$ they vanish, while in the other cases they are central.

To prove the first part of the claim we let $m = dp^s - 1$ with $d \not\equiv 0 \bmod p$. Since $p^s - 1 < r-2$ we can consider the element w such that $[wx]$ is at the beginning of the (p^s-1)-th short constituent before the intermediate one of the segment. We also want to use the defining relation $[yz^{(m+p^s)q+q-1}\ y\ x^q] = 0$, hence we have to verify that $m + p^s - 1 \leqslant r-3$. Note that $m = dp^s - 1 < r-2$ and thus $d < (r/p^s)$. Suppose now $m + p^s - 1 = (d+1)\,p^s - 2 \geqslant r-2$. Then $d \geqslant (r/p^s) - 1$, and by the previous observations the only possibility is $d = (r/p^s) - 1$, that is $m = r - p^s - 1$. Finally for $m \neq r - p^s - 1$ we let $\lambda = (m+p^s)\,q + q - 1$, and we compute

$$\begin{aligned}
0 &= [w[yz^{\lambda}\ y\ x^q]] = [w[y\ z^{\lambda}]\ y\ x^q] \\
&\approx \sum_{i=1}^{m} (-1)^{iq-1} \binom{\lambda}{iq-1} \cdot [[w \ldots y]\, x^q] \\
&\quad + \sum_{i=m+2}^{m+p^s+1} (-1)^{iq-2} \binom{\lambda}{iq-2} \cdot [[w \ldots y]\ x^q] \\
&= \sum_{i=0}^{m-1} (-1)^{i} \binom{m+p^s}{i} \cdot [[w \ldots y]\ x^q] \\
&\quad + \sum_{i=m+1}^{m+p^s} (-1)^{i} \binom{m+p^s}{i} \cdot [[w \ldots y]\ x^q] \\
&\approx \binom{m+p^s}{m} \cdot [w \ldots y\ x^q] \\
&= d \cdot [vy\ x^{2q-2}\ y\ (x^{q-1}\ y)^m\ x^q].
\end{aligned}$$

Hence the first part of the claim is proved.

We now show that for $m = r - p^s - 1$ the elements

$$[vy \ x^{2q-2} \ y \ (x^{q-1} \ y)^m \ x^q]$$

are central. In view of Subsection 5.2 we have only to prove that they are centralized by x. We use the defining relation $[y \ z^\lambda \ x] = 0$ with $\lambda = (m+3)q - 1$ obtaining:

$$\begin{aligned} 0 &= [v[y \ z^\lambda \ x]] = [v[y \ z^\lambda] \ x] \\ &\approx [\dots yx] + \sum_{i=1}^{m+1} (-1)^{iq} \binom{\lambda}{iq} \cdot [\dots zx] + (-1)^\lambda \cdot [\dots zx] \\ &= [\dots yx] + \left(-1 + \sum_{i=1}^{m+1} (-1)^i \binom{m+2}{i}\right) \cdot [\dots zx] \\ &= -[vy \ x^{2q-2} \ y \ (x^{q-1} \ y)^m \ x^{q+1}]. \end{aligned}$$

5.4. *Using the initial segment* $2q$, q^{r-2}, $2q-1$.

We now consider the possible constituents of length $2q - p^s$ for some $1 \leqslant s \leqslant a$. We distinguish two cases.

Let first v be an element at the end of a segment of constituent lengths

$$\dots, 2q, q^{r-2}.$$

We claim that $[vy \ x^{2q-p^s-1} \ y] = 0$ for every $1 \leqslant s \leqslant a$. Let w be an element at the beginning of the long constituent of the segment. We use the defining relation $[y \ z^\lambda \ y] = 0$, with $\lambda = rq + 2q - p^s - 1$. We find

$$\begin{aligned} 0 &= [w[y \ z^\lambda] \ y] - [wy \ [y \ z^\lambda \ y]] \\ &= [w \dots y] + \sum_{i=2}^{r} (-1)^{iq} \binom{\lambda}{iq} \cdot [w \dots y] \\ &\quad - \sum_{i=2}^{r} (-1)^{iq-1} \binom{\lambda}{iq-1} \cdot [w \dots z] - (-1)^\lambda \cdot [w \dots y] \\ &= \left(2 + \sum_{i=2}^{r} (-1)^i \binom{\lambda}{iq}\right) \cdot [w \dots y] - \sum_{i=1}^{r-1} (-1)^i \binom{\lambda}{iq+q-1} \cdot [w \dots z]. \end{aligned}$$

To evaluate the binomial coefficients we have to manage separately the value $p^s = q$, but in both cases

$$\sum_{i=2}^{r} (-1)^i \binom{\lambda}{iq} = (-1)^r = -1, \qquad \text{while} \qquad \sum_{i=1}^{r-1} (-1)^i \binom{\lambda}{iq+q-1} = 0\,.$$

Hence from the previous computation it follows that $[vy\ \ x^{2q-p^s-1}\ y]=0$.

When v is an element at the end of a segment of constituent lengths

$$\ldots, 2q-1,\ q^{r-2},$$

we can only prove that $[vy\ \ x^{2q-p^s-1}\ y]$ is central. By Subsection 5.2 it is enough to show that it is centralized by x. We consider the element w at the beginning of the intermediate constituent of the segment, and we use the defining relation $[y\ \ z^\lambda\ y]=0$ with $\lambda = rq+2q-p^s-1$. We find

$$\begin{aligned} 0 &= [w[y\ \ z^\lambda\ y]] = -[wy[y\ \ z^\lambda]] \\ &= -\sum_{i=2}^{r}(-1)^{iq-2}\binom{\lambda}{iq-2}\cdot[[w\ldots z]\,z]-(-1)^{\lambda-1}\lambda\cdot[[w\ldots y]\,z]. \end{aligned}$$

As above we have to distinguish the value $p^s=q$ to evaluate the binomial coefficient. In both cases

$$\sum_{i=2}^{r}(-1)^i\binom{\lambda}{iq-2}=0\,,$$

and from the previous formula we obtain

$$0=[w\ldots yx]=[vy\ \ x^{2q-p^s-1}\ yx].$$

Note that by all the previous properties and by using the whole set of relators R^m, we can finally say that the initial segment of constituent lengths of M is

$$2q,\ q^{r-2},\ 2q-1,\ q^{r-2},(2q,\ q^{r-2})^{t-1},\ 2q-1\,,$$

with the exception of the central elements $(4c)$ and $(4d)$ for $i=1$. Moreover $C_i\in\{\boldsymbol{F}x,\boldsymbol{F}y\}$ for $2\leqslant i\leqslant (t+1)\,rq+2q-1$.

5.5. *Using the initial segment* $2q,\ q^{r-2},\ 2q-1,\ q^{r-2},(2q,\ q^{r-2})^{t-1},\ 2q-1$.

At this point we can complete the proof.

We first observe that if v is an element such that both $[v\ \ x^{2q-1}\ y]$ and $[v\ \ x^{2q-2}\ yx]$ are different from zero, then

$$[v\ \ x^{2q-1}\ yy]=0=[v\ \ x^{2q-2}\ yxy].$$

Both computations are immediate using respectively the defining relations $[y\ \ x^q\ y]=0$ and $[y\ \ x\ \ y]=0$.

In the following three proofs we will always use the last defining relation

$$[\vartheta_2\ x^{2q-1}]=[y\ \ z^\lambda\ x]=0$$

with $\lambda = ((t+1)\,r+2)\,q-3$. Note that in the case $q=p=2$, when $t=1$ the value of λ is $(t+1)\,rq+q-1=2rq+q-1$, hence in this case they have to be worked out separately.

Let us suppose first v is an element at the end of a segment of constituent lengths

$$\ldots, 2q,\, q^{r-2},\, 2q-1,\, q^{r-2}, (2q,\, q^{r-2})^{t-1}.$$

We claim that $[vy\ x^{2q-1}\ y]=0$.

To prove it we let w be a element at the beginning of the first constituent of the segment. If $q>2$ we find

$$\begin{aligned}
0 &= [wy[y\ z^{\lambda}\ x]] = [wy[y\ z^{\lambda}]\,x] - [wyx[y\ z^{\lambda}]] \\
&= -\sum_{j=1}^{t}\sum_{i=1}^{r-1}(-1)^{(jr+i)\,q+q-3}\binom{\lambda}{(jr+i)\,q+q-3}\cdot[wy\ldots z] \\
&\qquad +(-1)^{\lambda}\cdot[wy\ldots yx]-(-1)^{\lambda-1}\lambda\cdot[wy\ldots yx]-(-1)^{\lambda}\cdot[wy\ldots y] \\
&= 2\cdot[wy\ldots yx]+[wy\ldots y]+\sum_{j=1}^{t}(-1)^{j}\binom{t+1}{j}\cdot[wy\ldots z] \\
&= -[vy\ x^{2q-1}\ y].
\end{aligned}$$

In the case $q=p=2$ computing the same bracket we obtain

$$\begin{aligned}
0 &= [wy[y\ z^{\lambda}\ x]] = [wy[y\ z^{\lambda}]x] - [wyx[y\ z^{\lambda}]] \\
&= \left(\sum_{i=1}^{r-1}\binom{\lambda}{iq+q-1}+\sum_{j=1}^{t}\sum_{i=1}^{r-1}\binom{\lambda}{(jr+i)\,q+q-2}\right)\cdot[wy\ldots zx]+[wy\ldots yx] \\
&\quad +\left(\sum_{i=1}^{r-1}\binom{\lambda}{iq+q-2}+\sum_{j=1}^{t}\sum_{i=1}^{r-1}\binom{\lambda}{(jr+i)\,q+q-3}\right)\cdot[wy\ldots zz] \\
&\quad +\lambda[wy\ldots yx]+[wy\ldots yx] \\
&= \left(\sum_{i=1}^{r-1}\binom{(t+1)\,r}{i}+\sum_{j=1}^{t}\sum_{i=1}^{r-1}\binom{(t+1)\,r}{jr+i}\right)\cdot[wy\ldots zx] \\
&\quad +\left(\sum_{i=1}^{r-1}\binom{(t+1)\,r}{i}+\sum_{j=1}^{t}\sum_{i=1}^{r-1}\binom{(t+1)\,r}{jr+i-1}\right)\cdot[wy\ldots zz]+[wy\ldots y] \\
&= [wy\ldots y]+\sum_{j=1}^{t}\binom{t+1}{j}[wy\ldots zz] \\
&= [vy\ x^{2q-1}\ y].
\end{aligned}$$

Note that by this property and Subsection 5.2 it follows that for $n > b$ the elements $[\vartheta_i\, x^{2q-1}]$ are central.

We now suppose v is an element at the end of a sequence of constituent lengths

$$\ldots, 2q-1,\, q^{r-2}, (2q,\, q^{r-2})^{t-1},\, 2q-1,\, q^{r-2}.$$

In this case it follows that

$$\begin{cases} [vy \;\; x^{2q-1}\; yx] = 0 & \text{if } n = b \\ [vy \;\; x^{2q-2}\; yxx] = 0 & \text{if } n > b\,. \end{cases}$$

To prove it we consider an element w at the beginning of the first constituent of the segment. If $q > 2$ we find

$$\begin{aligned}
0 &= [wy[y \;\; z^{\lambda} \;\; x]] = [wy[y \;\; z^{\lambda}]x] - [wyx[y \;\; z^{\lambda}]] \\
&\approx [[w \ldots y]x] - \lambda \cdot [[w \ldots x]x] + \lambda \cdot [[w \ldots y]x] - \binom{\lambda}{2} \cdot [[w \ldots y]xx] \\
&\quad - \left(\sum_{i=2}^{r} (-1)^{iq+1} \binom{\lambda}{iq+1} + \sum_{j=1}^{t} \sum_{i=2}^{r} (-1)^{(jr+i)q} \binom{\lambda}{(jr+i)\,q} \right) \cdot [[w \ldots z]z] \\
&= -\, 2 \cdot [[w \ldots y]x] - 3 \cdot [[w \ldots y]xx] \\
&\quad + \left(-\binom{t+1}{1} + 3\binom{t+1}{1} + \sum_{j=1}^{t} (-1)^{j} \binom{t+1}{j+1} \right) \cdot [[w \ldots z]z] \\
&= -\, 2 \cdot [[w \ldots y]\, x] - 3 \cdot [[w \ldots y]\, xx] \\
&\quad + \left(2\binom{t+1}{1} - \sum_{j=2}^{t+1} (-1)^{j} \binom{t+1}{j} \right) \cdot [w \ldots zz]\,.
\end{aligned}$$

We have now to distinguish two cases. For $t = 1$ we have

$$0 = -\, 2 \cdot [[w \ldots y]x] - 3 \cdot [[w \ldots y]\, xx] + 3 \cdot [[w \ldots z]\, z] = [vy \;\; x^{2q-1}\; yx],$$

while for $t \geqslant p$ we find

$$0 = -2 \cdot [[w \ldots y]\, x] - 3 \cdot [[w \ldots y]\, xx] + 2 \cdot [[w \ldots z]\, z] = -[vy \;\; x^{2q-2}\; yxx]\,,$$

as claimed.

As above we now compute the same commutator in the case $q = p = 2$:

$$\begin{aligned}
0 &= [wy[y \;\; z^{\lambda} \;\; x]] = [wy[y \;\; z^{\lambda}]\, x] - [wyx[y \;\; z^{\lambda}]] \\
&= \left(\sum_{j=0}^{t-1} \sum_{i=1}^{r-1} \binom{\lambda}{(jr+i)\,q + q - 2} + \sum_{i=1}^{r-1} \binom{\lambda}{(tr+i)\,q + q - 3} \right) \cdot [w \ldots zzx] \\
&\quad + \lambda [w \ldots yzx] + [w \ldots zyx]
\end{aligned}$$

$$+\left(\sum_{j=0}^{t-1}\sum_{i=1}^{r-1}\binom{\lambda}{(jr+i)\,q+q-3}+\sum_{i=1}^{r-1}\binom{\lambda}{(tr+i)\,q+q-4}\right)\cdot[w\ldots zzz]$$

$$+\binom{\lambda}{\lambda-2}[w\ldots yzz]+\lambda[w\ldots zyz]$$

$$=\left(\binom{t+1}{t}+\sum_{j=0}^{t-1}\binom{t+1}{j}+\binom{t+1}{t}\right)[w\ldots zzx]+[w\ldots yxx]$$

$$=\sum_{j=0}^{t-1}\binom{t+1}{j}[w\ldots zzx]+[w\ldots yxx].$$

For $t=1$ we obtain

$$[w\ldots zzx]+[w\ldots yxx]=[vy\ \ x^{2q-1}\ yx]=0\,,$$

while for $t>1$ we find

$$[w\ldots yxx]=[vy\ \ x^{2q-2}\ yxx]=0\,.$$

Let us consider now the elements $[\vartheta_i\, x^{2q-1}\, y]$ for $n=b$, and $[u_{0,\,i}\, x^{2q-2}\, yx]$ for $n>b$. Using the previous computation and the observations at the beginning of Subsection 5.5 it follows that they are central when they do not vanish.

Actually we can prove that already $[\vartheta_i\, x^{2q-1}]$ and $[u_{0,\,i}\, x^{2q-2}\, yx]$ can be central. More precisely we claim that $[\vartheta_i\, x^{2q-1}\, y]=0$ for every $i\not\equiv 2 \bmod p$ in the case $n=b$, and that $[u_{0,\,i}\, x^{2q-2}\, yx]=0$ for every $i\not\equiv 1 \bmod p$ in the case $n>b$. The computations to prove this claim are quite long so we postpone them at next session.

In the last of this series of computations we let v be an element at the end of the segment of constituent lengths

$$\ldots,(2q,\, q^{r-2})^{t-1},\, 2q-1,\, q^{r-2},(2q,\, q^{r-2})^{j} \tag{5}$$

with $j>0$. Then $[vy\ \ x^{2q-2}\ yx]=0$.

To prove it we take the element w at the beginning of the $(t-j)$-th long constituent preceding the intermediate one. If $q>2$ we find

$$0=[wy[y\ \ z^{\lambda}\ x]]$$

$$=[wy[y\ \ z^{\lambda}]\,x]-[wyx\ \ [y\ \ z^{\lambda}]]$$

$$\approx[[w\ldots y]\,x]-[[w\ldots y]]+\lambda\cdot[[w\ldots y]\,x]$$

$$-\sum_{h=0}^{j}\sum_{i=2}^{r}(-1)^{(hr+i)q}\binom{\lambda}{(hr+i)\,q}\cdot[[w\ldots z]\,z]$$

$$= -[[w\dots y]] - 2\cdot[[w\dots y]\,x] - \sum_{h=0}^{j}(-1)^{h+1}\binom{t+1}{h+1}\cdot[[w\dots z]\,z]$$

$$= -[w\dots y] - 2\cdot[w\dots yx] + [w\dots zz]$$

$$= -[vy\ \ x^{2q-2}\ yx].$$

If $q=2$ we obtain:

$$0 = [wy[y\ \ z^{\lambda}\ x]]$$

$$= [wy[y\ \ z^{\lambda}]\,x] - [wyx\ \ [y\ \ z^{\lambda}]]$$

$$= \left(\sum_{h=0}^{t-j-1}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-1} + \sum_{h=t-j}^{t}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-2}\right)\cdot[w\dots zx]$$

$$+[w\dots yx]$$

$$+\left(\sum_{h=0}^{t-j-1}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-2} + \sum_{h=t-j}^{t}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-3}\right)\cdot[w\dots zz]$$

$$+\lambda[w\dots yz] + [w\dots y]$$

$$= \sum_{h=t-j}^{t}\binom{t+1}{h}[w\dots zz] + [w\dots y]$$

$$= [w\dots zz] + [w\dots y]$$

$$= [vy\ \ x^{2q-2}\ yx].$$

By this property it follows that in the case $n > b$ the elements $[u_{j,\,i}\ x^{2q-2}\ y]$ are central for every $1 \leqslant j \leqslant t-2$. Actually we can prove that most of these elements vanish. As above we let v be an element at the end of a segment of constituent lengths (5). We claim that $[vy\ \ x^{2q-2}\ y] = 0$ if $j \neq t - p^{s}$ for some $1 \leqslant s \leqslant n-b$.

For $q>2$ we distinguish two cases. Let us suppose first j is odd. We consider the element w at the beginning of the intermediate constituent of the segment (5). Note that $[wy] = \vartheta_{i}$. Moreover since $j>0$ we can use the defining relation $[y\ \ z^{\lambda}\ y] = 0$ with $\lambda = ((j+1)\,r+2)\,q-3$. We find

$$0 = [w[y\ \ z^{\lambda}\ y]] = [w[y\ \ z^{\lambda}]\,y] - [wy[y\ \ z^{\lambda}]]$$

$$= [wy\dots y] - (-1)^{\lambda}\cdot[wy\dots y]$$

$$= 2\cdot[vy\ \ (2q-2)\ \ y].$$

We now let $j = dp^{s}$ with $d \not\equiv 0 \bmod p$ and even. Since $p^{s} \leqslant j \leqslant t-2$, we can take the element w at the beginning of the p^s-th long constituent before the intermediate one of (5). To use the defining relation $[y\ z^\lambda\ y] = 0$ with $\lambda = (((d+1)\,p^s+1)\,r+1)\,q+q-3$ we have to require $(d+1)\,p^s \leqslant t-1$ and thus $j \neq t-p^s$. In these cases we obtain

$$
\begin{aligned}
0 &= [w[y\ z^\lambda\ y]] = [w[y\ z^\lambda]y] - [wy\ [y\ z^\lambda]] \\
&= \left(1-(-1)^\lambda + \sum_{h=0}^{p^s-1}\sum_{i=2}^{r}(-1)^{(hr+i)\,q}\binom{\lambda}{(hr+i)\,q}\right)\cdot[w\ldots y] \\
&= \left(2+\sum_{h=0}^{p^s-1}(-1)^h\sum_{i=2}^{r}(-1)^i\binom{(dp^s+p^s+1)\,r+}{hr+i}\right)\cdot[w\ldots y] \\
&= \left(2+\sum_{h=1}^{p^s}(-1)^h\binom{(d+1)\,p^s+1}{h}\right)\cdot[w\ldots y] \\
&= -\,d\cdot[vy\ x^{2q-2}\ y].
\end{aligned}
$$

Since $d \not\equiv 0$ our claim follows.

When $q=p=2$ we let again $j=dp^s$ with $d \not\equiv 0 \bmod p$, and we use the same defining relation considered in the last computation. Note that in this case it is more convenient to write the value of λ as $\lambda = ((d+1)\,p^s+1)\,rq+q-1$. We obtain

$$
\begin{aligned}
0 &= [w[y\ z^\lambda\ y]] = [w[y\ z^\lambda]\,y] - [wy\ [y\ z^\lambda]] \\
&= \left(1+\sum_{h=0}^{p^s-1}\sum_{i=2}^{r}\binom{\lambda}{(hr+i)\,q}+\sum_{h=p^s}^{p^s+j}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-1}\right)\cdot[w\ldots y] \\
&\quad+\left(\sum_{h=0}^{p^s-1}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-1}+\sum_{h=p^s}^{p^s+j}\sum_{i=1}^{r-1}\binom{\lambda}{(hr+i)\,q+q-2}\right)\cdot[w\ldots z] \\
&\quad+[w\ldots y] \\
&= \sum_{h=0}^{p^s-1}\binom{(d+1)\,p^s+1}{h+1}[w\ldots y] \\
&= d\cdot[vy\ x^{2q-2}\ y].
\end{aligned}
$$

Finally we proved that for $j \neq t-p^s$ the elements $[u_{j,\,i}\,x^{2q-2}\ y]$ vanish, while $[u_{t-p^s,\,i}\,x^{2q-2}\ y]$ with $1 \leqslant s \leqslant n-b-1$ are central and $[u_{0.\,i}\,x^{2q-2}\ y]$ second central.

6. – Second central elements (odd primes).

6.1. *The case* $n = b$.

In the previous section we claimed that in the case $b = n$ the elements $[\vartheta_i\, x^{2q-1}\, y]$ vanish when $i \not\equiv 2 \bmod p$, and we proved that in any case they are central. We now prove the claim.

First of all we introduce the elements γ_i such that $[\gamma_i xy] = \vartheta_i$. Note that these elements have weight equivalent to zero modulo $p^n - 1$, the dimension of the finite algebra AF. For this reason and because of the periodical structure of M, commuting γ_i with any element $w = [\vartheta_j z^\lambda]$, we obtain an element in the homogeneous component generated by an analogous element $w' = [\vartheta_{j+i} z^\lambda]$. As a consequence, if w is such that, for instance, $[wx] = 0$, then also $[w\gamma_i x] = [w' x] = 0$. We will often use the following property that results from the previous observation. If $[wxy] = 0 = [wyx]$, then $[w[\gamma_i xy]] = 0$. We can see a use of this property in the following example. Let w be an element such that $[wz^{q+1}] = [wxy\ \ x^{q-1}]$. We find

$$[w[[\gamma_i xy]\, z^{q-1}]] = [w[\gamma_i xy] z^{q-1}] + (-1)^1 \binom{q-1}{1} \cdot [wz[\gamma_i xy]\, z^{q-2}]$$

$$= [w[\gamma_i x]\, yz^{q-1}] - [wzy[\gamma_i x]\, z^{q-2}]$$

$$= [w[\gamma_i x]\, y\ \ x^{q-1}] - [wxy[\gamma_i x]\, x^{q-2}].$$

It is clear that to proceed we have to know the behavior of $[\gamma_i x]$.

To simplify notation we let $v_{k,i} = [\vartheta_i\, x^{2q-2}\, (y\ \ x^{q-1})^k\,]$. Note that $v_{r-2,i} = \vartheta_{i+1}$.

We claim that:

(6*a*) $\qquad [\gamma_i\, x\ \ \gamma_1] = [\gamma_{i+1}\, x]$,

(6*b*) $\qquad [\vartheta_i\, x^{2q-3}\, [\gamma_j\, x]] = -\, [\vartheta_{i+j}\, x^{2q-2}]$,

(6*c*) $\qquad [\vartheta_i\, x^{2q-2}\, [\gamma_j\, x]] = -\, [\vartheta_{i+j}\, x^{2q-1}]$,

(6*d*) $\qquad [\vartheta_i\, x^{2q-2}\, y[\gamma_i x]] = -\, [\vartheta_{i+j}\, x^{2q-2}\, yx] - j \cdot [\vartheta_{i+j}\, x^{2q-1}\, y]$,

(6*e*) $\qquad [v_{k,i} y\ \ x^{q-2}\, [\gamma_j x]] = -\, v_{k+1,\, i+j} \quad$ for $0 \leqslant k \leqslant r-3$

(6*f*) $\qquad [v_{k,i} y\ \ [\gamma_j x]] = -\, [v_{k,\, i+j}\, yx] \quad$ for $1 \leqslant k \leqslant r-2$,

for every i, j.

The element $[\vartheta_i\, x^{2q-1}\, y]$ has weight $i(p^n - 1) + 2q + 2$ that is even, thus the Lie bracket between an element of weight

$$\frac{1}{2}\cdot(i(p^n-1)+2q+2)$$

and itself reaches the homogeneous component generated by $[\vartheta_i\, x^{2q-2}\, yx]$ and possibly by $[\vartheta_i\, x^{2q-1}\, y]$. To find such element we have to distinguish the case i even from i odd.

Suppose first $i=2j$ even. Using $(6b)$, $(6c)$ and $(6d)$ we find:

$$\begin{aligned}
0 &= [[\vartheta_j\, x^{q-1}][\vartheta_j\, x^{q-1}]] \\
&= [\vartheta_j\, x^{q-1}[[\gamma_j xy]\ \ x^{q-1}] \\
&= [\vartheta_j\ x^{2q-3}[\gamma_j\ x]\ \ yx]+[\vartheta_j\ x^{2q-2}[\gamma_j\ x]\ \ y]-[\vartheta_j\ x^{2q-2}\ \ y[\gamma_j\ x]] \qquad (7)\\
&= -\,[\vartheta_{2j}\ x^{2q-2}\ \ yx]-[\vartheta_{2j}\ x^{2q-1}\ \ y]+[\vartheta_{2j}\ x^{2q-2}\ \ yx]+j[\vartheta_{2j}\ x^{2q-1}\ \ y] \\
&= (j-1)\cdot[\vartheta_{2j}\, x^{2q-1}\, y].
\end{aligned}$$

Substituting the value $i=2j$, it follows that

$$(i-2)\cdot[\vartheta_i\, x^{2q-1}\, y]=0$$

as we claimed.

Suppose now $i=2j+1$ odd. In this case we let $2k=r-3$, and compute:

$$\begin{aligned}
0 &= [[v_{k,j}y\ \ x^{(q-1)/2}][v_{k,j}y\ \ x^{(q-1)/2}]] \\
&\approx ((q-1)/2)\cdot[v_{k,j}y\ \ x^{q-2}\,[v_{k,j}y]\,x]-[v_{k+1,j}\,[v_{j,k}y]].
\end{aligned}$$

Note that $[v_{k,j}y]=[[\gamma_j xy]z^\lambda]$, with $\lambda=(k+1)q+q-1$. Hence by the previous computation, we can consider the identity

$$2\cdot[v_{k+1,j}\,[[\gamma_j xy]z^\lambda]]+[v_{k,j}y\ \ x^{q-2}\,[[\gamma_j xy]\,z^\lambda]\,x]=0\,.$$

We find

$$\begin{aligned}
0 = 2&\sum_{i=0}^{k}(-1)^{iq}\binom{\lambda}{iq}\cdot[v_{k+i+1,j}[\gamma_j xy]\dots zz] \\
&+2\sum_{i=1}^{k}(-1)^{iq-1}\binom{\lambda}{iq-1}\cdot[v_{k+i,j}y\ \ x^{q-2}\,[\gamma_j xy]\dots zz] \\
&+2(-1)^\lambda\cdot[\vartheta_{j+1}\,x^{2q-2}\,[\gamma_j xy]]+2(-1)^{\lambda-1}\lambda\cdot[\vartheta_{j+1}\,x^{2q-3}\,[\gamma_j xy]\,z]
\end{aligned}$$

$$+\sum_{i=0}^{k}(-1)^{iq+1}\binom{\lambda}{iq+1}\cdot[v_{k+i+1,j}[\gamma_j xy]\ldots zx]$$

$$+\sum_{i=0}^{k}(-1)^{iq}\binom{\lambda}{iq}\cdot[v_{k+i,j}y\ \ x^{q-2}\,[\gamma_j xy]\ldots zx]$$

$$+(-1)^{\lambda}\cdot[\vartheta_{j+1}\,x^{2q-3}\,[\gamma_j xy]\,x]$$

$$=-2\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{i}\cdot[v_{k+i+1,j}y[\gamma_j x]\ldots zz]$$

$$+2\sum_{i=1}^{k}(-1)^{i-1}\binom{k+1}{i-1}\cdot[v_{k+i,j}y\ \ x^{q-2}\,[\gamma_j x]\,y\ldots zz]$$

$$+2(-1)^{k+1}\cdot[\vartheta_{j+1}\,x^{2q-2}\,[\gamma_j x]\,y]-2(-1)^{k+1}\cdot[\vartheta_{j+1}\,x^{2q-2}\,y[\gamma_j x]]$$

$$+2(-1)^{k+1}\lambda\cdot[\vartheta_{j+1}\,x^{2q-3}\,[\gamma_j x]\,yx]$$

$$+\sum_{i=0}^{k}(-1)^{i+1}\binom{k+1}{i}\cdot[v_{k+i+1,j}y[\gamma_j x]\ldots yx]$$

$$+\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{i}\cdot[v_{k+i,j}y\ \ x^{q-2}\,[\gamma_j x]\,y\ldots yx]$$

$$+(-1)^{\lambda}\cdot[\vartheta_{j+1}\,x^{2q-3}\,[\gamma_j x]\,yx].$$

We now use formulas $(6b)\ldots(6f)$ to obtain

$$0=\left(2\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{i}-2\sum_{i=0}^{k-1}(-1)^{i}\binom{k+1}{i}\right)\cdot[\ldots zz]$$

$$+\left(\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{i}-\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{i}\right)\cdot[\ldots zx]$$

$$+2(-1)^{k}\cdot[\ldots xy]-2(-1)^{k}\cdot[\ldots yx]-2(-1)^{k}j\cdot[\ldots xy]+3(-1)^{k}\cdot[\ldots yx]$$

$$=2(-1)^{k}(k+1)\cdot[\ldots zz]+2(-1)^{k}(1-j)\cdot[\ldots xy]+(-1)^{k}\cdot[\ldots yx]$$

$$\approx(2k+4-2j)\cdot[\vartheta_{2j+1}\,x^{2q-1}\,y]+(2k+3)\cdot[\vartheta_{2j+1}\,x^{2q-2}\,yx].$$

Substituting the values $i=2j+1$ and $2k=r-3$ we finally obtain our claim:

$$(i-2)\cdot[\vartheta_i\,x^{2q-1}\,y]=0\,.$$

We now prove formulas (6a)…(6f). Note that $\gamma_1 = [y \ z^\lambda]$ with $\lambda = p^n - 2 = (r-1)\,q + q - 2$. From now on we will always consider this value for λ.

Formula (6a) is a direct computation:

$$[\gamma_i x \ \gamma_1] = [\gamma_i x \ [yz^\lambda]] = [\gamma_i xyz^\lambda] = [\gamma_{i+1} xy].$$

To prove the other formulas proceed by induction on the index of the elements γ_j. To prove (6b) we start evaluating

$$[\vartheta_i \, x^{2q-3} \ \gamma_1] = \sum_{k=0}^{r-2} (-1)^{kq+1} \binom{\lambda}{kq+1} \cdot [\vartheta_i \ \dots] = -2 \cdot [\vartheta_{i+1} \, x^{2q-3}].$$

Analogously

$$[\vartheta_i \, x^{2q-2} \ \gamma_1] = \sum_{k=0}^{r-2} (-1)^{kq} \binom{\lambda}{kq} \cdot [\vartheta_i \ \dots] = -[\vartheta_{i+1} \, x^{2q-2}].$$

Using these two formulas we obtain

$$[\vartheta_i \, x^{2q-3} \, [\gamma_1 x]] = [\vartheta_i \, x^{2q-3} \ \gamma_1 x] - [\vartheta_i \, x^{2q-2} \ \gamma_1] = -[\vartheta_{i+1} \, x^{2q-2}],$$

and (6b) is verified for $j = 1$. Supposing it true up to j and using (6a) we easily obtain it for $j+1$:

$$\begin{aligned}[\vartheta_i \, x^{2q-3} \, [\gamma_{j+1} x]] &= [\vartheta_i \, x^{2q-3} \, [\gamma_j x] \, \gamma_1]] - [\vartheta_i \, x^{2q-3} \ \gamma_1 [\gamma_j x]] \\ &= -\,[\vartheta_{i+j+1} \, x^{2q-2}].\end{aligned}$$

To compute formula (6c) we can observe that $[\gamma_j xx] = 0$. We find

$$0 = [\vartheta_i \, x^{2q-3} [\gamma_j xx]] = [\vartheta_i \, x^{2q-3} \, [\gamma_j x] \, x] - [\vartheta_i \, x^{2q-2} \, [\gamma_j x]],$$

and (6c) is a direct consequence of (6b).

As above, to compute formula (6d) we first evaluate

$$[\vartheta_i \, x^{2q-2} \ y \ \gamma_1] = (-1)^\lambda \cdot [\vartheta_i \ \dots y] = -[\vartheta_{i+1} \, x^{2q-2} \ y].$$

Also

$$\begin{aligned}[\vartheta_i \, x^{2q-2} \ yx\gamma_1] = \sum_{k=0}^{r-3} (-1)^{kq+q-2} &\binom{\lambda}{kq+q-2} \cdot [\vartheta_i \ \dots zz] \\ &+ (-1)^{\lambda-1} \lambda \cdot [\vartheta_i \ \dots yx] + (-1)^\lambda \cdot [\vartheta_i \ \dots y] \\ &= 2 \cdot [\vartheta_{i+1} \, x^{2q-2} \ zz] - 2 \cdot [\vartheta_{i+1} \, x^{2q-2} \ yx] - [\vartheta_{i+1} \, x^{2q-1} \ y] \\ &= [\vartheta_{i+1} \, x^{2q-1} \ y].\end{aligned}$$

Note that we obtain the same result also in the case $r = p = 2$ or when the first summatory vanishes. It follows that

$$[\vartheta_i\, x^{2q-2}\, y[\gamma_1 x]] = -[\vartheta_{i+1}\, x^{2q-2}\, yx] - [\vartheta_{i+1}\, x^{2q-1}\, y],$$

and ($6d$) is verified for $j=1$. By induction we suppose it true up to j. Using formula ($6a$) we find

$$\begin{aligned}[\vartheta_i\, x^{2q-2}\, y[\gamma_{j+1} x]] &= [\vartheta_i\, x^{2q-2}\, y\ \ [\gamma_j x]\,\gamma_1] - [\vartheta_i\, x^{2q-2}\, y\gamma_1[\gamma_j x]] \\ &= -\,[\vartheta_{i+j}\, x^{2q-2}\, yx\gamma_1] - j\cdot[\vartheta_{i+j}\, x^{2q-1}\, y\gamma_1] \\ &\quad + [\vartheta_{i+1}\, x^{2q-2}\, y[\gamma_j x]] \\ &= -\,[\vartheta_{i+j+1}\, x^{2q-2}\, yx] - (j+1)\cdot[\vartheta_{i+j+1}\, x^{2q-1}\, y].\end{aligned}$$

As in the previous cases to compute formula ($6e$) we start from

$$\begin{aligned}[v_{k,\,i}\, y\gamma_1] &= [v_{k,\,i}\, y\ \ [y\ \ z^{\lambda}]] \\ &= \sum_{j=r-1-k}^{r-1} (-1)^{jq+q-2}\binom{\lambda}{jq+q-2}\cdot[v\ldots] \\ &= -\sum_{j=r-1-k}^{r-1} (-1)^{j}\binom{r-1}{j}\cdot[v\ldots] \\ &= -\,(k+1)\cdot[v_{k,\,i+1}\, y].\end{aligned}$$

Note that we ignored the possible central elements $[v_{k-1,\,i+1}\, y\ \ x^{q}]$, because anyway they vanish in the final computation.

In a similar way we find

$$\begin{aligned}[v_{k,\,i}\, yx\gamma_1] &= [v_{k,\,i}\, yx[y\ \ z^{\lambda}]] \\ &= \sum_{j=0}^{r-3-k} (-1)^{jq+q-2}\binom{\lambda}{jq+q-2}\cdot[v\ldots] \\ &\quad + \sum_{j=r-1-k}^{r-1} (-1)^{jq+q-3}\binom{\lambda}{jq+q-3}\cdot[v\ldots] \\ &= \left(-\sum_{j=0}^{r-3-k}(-1)^{j}\binom{r-1}{j} - 2\sum_{j=r-1-k}^{r-1}(-1)^{j}\binom{r-1}{j}\right)\cdot[v\ldots] \\ &= (-(r-2-k) - 2(k+1))\cdot[v\ldots] \\ &= -k\cdot[v_{k,\,i+1}\, yx].\end{aligned}$$

Note that this formula is valid also for $k=r-2$ or for $q=p=2$ when respectively the first or the second summatory vanishes. We can now compute

$$[v_{k,i}y[\gamma_1 x]] = [v_{k,i}y\gamma_1 x] - [v_{k,i}yx\gamma_1] = -[v_{k,i+1}yx]$$

and (6*e*) is verified for $j=1$. Supposing it true up to j we obtain

$$[v_{k,i}y[\gamma_{j+1}x]] = [v_{k,i}y[\gamma_j x]\ \ \gamma_1] - [v_{k,i}y\gamma_1[\gamma_j x]] = -[v_{k,i+j+1}yx].$$

Finally to compute formula (6*f*) we can observe that

$$\begin{aligned} 0 &= [v_{k,i}[\gamma_j xy\ \ x^{q-1}\ y]] \\ &= -[v_{k,i}y[\gamma_j xy\ \ x^{q-1}]] \\ &= -[v_{k,i}y\ \ x^{q-2}[\gamma_j xy]x] - [v_{k+1,i}[\gamma_j xy]] \\ &\approx [v_{k,i}y\ \ x^{q-2}\ \ [\gamma_j x]\,yx] - [v_{k+1,i}y[\gamma_j x]]. \end{aligned}$$

It follows that

$$[v_{k+1,i}y[\gamma_j x]] = [v_{k,i}y\ \ x^{q-2}[\gamma_j x]\,yx]$$

for every $k \geqslant 0$, and formula (6f) is a direct consequence of (6*e*).

6.2. *The case $n > b$.*

In Section 5 we saw that in the case $n > b$ the elements $[u_{0,i}\,x^{2q-2}\,y]$ can be second central in M. More precisely we proved that $[u_{0,i}\,x^{2q-2}\,yx]$ is central for every i, but we claimed that already

$$[u_{0,i}\,x^{2q-2}\,yx] = 0$$

when $i \not\equiv 1$ modulo p.

As above, to prove the claim we introduce the elements γ_i of weight zero modulo p^n-1. In this case we let γ_i be the element such that $[\gamma_i y] = u_{t-2,i}$. Indeed as in the case $n=b$ commuting any element $w = [\vartheta_j z^\lambda]$ with γ_j we obtain an element in the homogeneous component generated by an analogous element $w' = [\vartheta_{j+i}z^\lambda]$. It follows that if w is an element such that $[wy]=0$, then also $[w[\gamma_j y]] = 0$. The use of this property is easier than in the case $n = b$. Let us consider the following example. Let w be an element such that $[wz^q] = [wyx^q - 1]$. Then

$$[w[[\gamma_i y]\,z^{q-1}]] = [w[\gamma_i y]\,z^{q-1}] = [w[\gamma_i y]\ \ x^q - 1].$$

Hence in this case we have to study the behavior of $[\gamma_i y]$.

We keep for the elements $v_{k,i}$ the same notation as in the case $n = b$.

We claim that the elements γ_j verify the following properties:

(8a) $$[\gamma_{i+1} y] = -[\gamma_i y \gamma_1],$$

(8b) $$[v_{k,i}[\gamma_j y]] = [v_{k,i+j} y] \quad \text{for } 0 \leqslant k \leqslant r-2.$$

(8c) $$[u_{0,i}\, x^{2q-2}[\gamma_j y]] = [u_{0,i+j}\, x^{2q-2}\, y],$$

(8d) $$[u_{0,i}\, x^{2q-1}[\gamma_j y]] = [u_{0,i+j}\, x^{2q-1}\, y] + j\cdot[u_{0,i+j}\, x^{2q-2}\, yx],$$

for every $i, j \geqslant 1$. Note that in formula (8b) we ignored the possible elements $[\vartheta_{i+j}\, x^{2q-1}]$ or $[v_{k,i+j}\, x^q]$ as they are central and thus useless in the proof of the claim.

We now consider the homogeneous component generated by $[u_{0,i+1}\, x^{2q-1}\, y]$ and possibly by the central element $[u_{0,i+1}\, x^{2q-2}\, yx]$. Since this component has weight $1 + i(p^n - 1) + 2rq - 1 + 2q$, to reach it we can compute the Lie bracket between two elements of weight

$$\frac{1}{2}\cdot(1 + i(p^n - 1) + 2rq - 1 + 2q).$$

We have to distinguish the case i even from i odd.

Suppose first $i = 2j$, even. We compute the Lie bracket between $[\vartheta_{j+1}\, x^q - 1]$ and itself. Note that $[\vartheta_{j+1}\, x^q - 1] = [[\gamma_j y]z^\lambda]$ with $\lambda = rq + q - 1$. Using (8b), (8c) and (8d) we obtain

(9)
$$\begin{aligned}
0 &= [\vartheta_{j+1}\, x^q - 1[[\gamma_j y]\, z^\lambda]] \\
&= \sum_{k=0}^{r-2} (-1)^{kq+q-1} \binom{\lambda}{kq+q-1} \cdot [v_{k,j+1}[\gamma_j y] \ldots z] \\
&\quad + (-1)^{\lambda-1} \lambda \cdot [u_{0,j+1}\, x^{2q-2}\, [\gamma_j y]\, z] + (-1)^\lambda \cdot [u_{0,j+1}\, x^{2q-1}\, [\gamma_j y]] \\
&= [\vartheta_{j+1}\, x^{2q-2}\, [\gamma_j y] \ldots z] - [u_{0,j+1}\, x^{2q-2}\, [\gamma_j y] z] - [u_{0,j+1}\, x^{2q-1}\, [\gamma_j y]] \\
&= -j \cdot [u_{0,2j+1}\, x^{2q-2}\, yx].
\end{aligned}$$

Substituting the value $i = 2j$ we finally obtain

$$(i-1)\cdot[u_{0,i}\, x^{2q-2}\, yx] = 0$$

as we claimed.

Suppose now $i = 2j - 1$ odd. We let $2k = t - 3$, $2m = r - 3$ and $2h = q - 3$, and compute the Lie bracket between

$$w = [u_{k,j}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^m\, x^{h+1}]$$

and itself. Note that $w = [\gamma_{j-1} y \ z^{\lambda}]$ with $\lambda = ((k+2)\,r + m + 2)\,q + h$. We find

$$0 = [ww] = [w[\gamma_{j-1} y \ z^{\lambda}]]$$

$$\approx [u_{0,\,j+1}\, x^{2q-1}\,[\gamma_{j-1} y]] - \lambda \cdot [u_{0,\,j+1}\, x^{2q-2}[\gamma_{j-1} y]\, z]$$

$$+ \sum_{i=2}^{r} (-1)^{iq} \binom{\lambda}{iq} \cdot [v_{r-i,\,j+1}[\gamma_{j-1} y] \ldots zz].$$

Using formulas (8b), (8c) and (8d) we obtain

$$0 = [u_{0,\,2j} \ldots xy] + (j-1) \cdot [u_{0,\,2j} \ldots yx] - (h-1) \cdot [u_{0,\,2j} \ldots yx]$$

$$+ \sum_{i=2}^{r} (-1)^{i} \binom{(k+2)\,r + m + 2}{i} \cdot [u_{0,\,2j} \ldots zz]$$

$$= [u_{0,\,2j} \ldots xy] + (j-h) \cdot [u_{0,\,2j} \ldots yx]$$

$$+ \left(\sum_{i=2}^{r} (-1)^{i} \binom{m+2}{i} + (-1)^{r} (k+2) \right) \cdot [u_{0,\,2j} \ldots zz]$$

$$= [u_{0,\,2j} \ldots xy] + (j-h) \cdot [u_{0,\,2j} \ldots yx] + (m-k-1) \cdot [u_{0,\,2j} \ldots zz].$$

Substituting the original values of j, h, k, and m we finally obtain

$$0 = 2 \cdot [u_{0,\,i+1} \ldots xy] + (i+2) \cdot [u_{0,\,i+1} \ldots yx] - 2 \cdot [u_{0,\,i+1} \ldots zz]$$

$$= i \cdot [u_{0,\,i+1}\, x^{2q-2}\, yx].$$

It follows that $(i-1) \cdot [u_{0,\,i}\, x^{2q-2}\, yx] = 0$ as we claimed.

We now prove formulas (8a) … (8d) proceeding by induction on the index of the elements γ_j. Since $\gamma_1 = [y \ z^{p^n - 2}]$ from now on we let $\lambda = p^n - 2 = ((t-1)\,r + r - 1)\,q + q - 2$.

Formula (8a) is a direct computation:

$$[\gamma_i y \gamma_1] = [\gamma_i y \ [y \ z^{\lambda}]]$$

$$= \sum_{h=1}^{t-1} \sum_{i=2}^{r} (-1)^{(hr+i)\,q-2} \binom{\lambda}{(hr+i)\,q-2} \cdot [\gamma_i \ldots z]$$

$$= \sum_{h=1}^{t-1} (-1)^{h} \binom{t-1}{h} \cdot [\gamma_i \ldots z]$$

$$= -[\gamma_{i+1} y].$$

To prove formula (8b) we start computing

$$
\begin{aligned}
[v_{k,\,i}\gamma_1] &= [v_{k,\,i}[yz^\lambda]]\\
&= \left(\sum_{i=0}^{r-2-k}(-1)^{iq}\binom{\lambda}{iq}+\sum_{j=1}^{t-1}\sum_{i=-k}^{r-2-k}(-1)^{(jr+i)\,q}\binom{\lambda}{(jr+i)\,q}\right)\cdot[\ldots z]\\
&= \sum_{i=0}^{r-2-k}(-1)^i\binom{(t-1)\,r+r-1}{i}\cdot[\ldots z]\\
&\quad+\sum_{j=1}^{t-1}(-1)^j\sum_{i=-k}^{r-2-k}(-1)^i\binom{(t-1)\,r+r-1}{jr+i}\cdot[\ldots z]\\
&= \sum_{j=0}^{t-1}(-1)^j\binom{t-1}{j}\sum_{i=0}^{r-2-k}(-1)^i\binom{r-1}{i}\cdot[\ldots z]\\
&\quad+\sum_{j=1}^{t-1}(-1)^j\binom{t-1}{j-1}\sum_{i=-k}^{-1}(-1)^i\binom{r-1}{r+i}\cdot[\ldots z]\\
&= -\,k\cdot v_{k,\,i+1}.
\end{aligned}
$$

Analogously we find

$$
\begin{aligned}
[v_{k,\,i}y\gamma_1] &= \sum_{i=-k}^{0}(-1)^{(tr+i)\,q-2}\binom{\lambda}{(tr+i)\,q-2}\cdot[\ldots z]\\
&= \sum_{i=-k-1}^{-1}(-1)^i\binom{r-1}{r+i}\cdot[\ldots z]\\
&= -\,(k+1)\cdot[v_{k,\,i+1}y].
\end{aligned}
$$

Putting together the last two formulas we obtain (8b) for $j=1$. We now suppose it valid up to j and using (8a) we obtain it for $j+1$:

$$[v_{k,\,i}[\gamma_{j+1}y]] = -[v_{k,\,i}[\gamma_j y]\,\gamma_1]+[v_{k,\,i}\gamma_1[\gamma_j y]] = [v_{k,\,i+j+1}y].$$

To compute formula (8c), as above we start computing

$$
\begin{aligned}
[u_{0,\,i}\,x^{2q-2}\,\gamma_1] &= [u_{0,\,i}\,x^{2q-2}\,[y\ \ z^\lambda]]\\
&= \sum_{h=0}^{t-2}\sum_{i=0}^{r-2}(-1)^{(hr+i)\,q+1}\binom{\lambda}{(hr+i)\,q+1}\cdot[\ldots z]\\
&\quad+\sum_{i=0}^{r-2}(-1)^{((t-1)\,r+i)\,q}\binom{\lambda}{((t-1)\,r+i)\,q}\cdot[\ldots z]
\end{aligned}
$$

$$= 2 \sum_{h=0}^{t-2} (-1)^h \binom{t-1}{h} \sum_{i=0}^{r-2} (-1)^i \binom{r-1}{i} \cdot [\dots z]$$

$$+ \sum_{i=0}^{r-2} (-1)^i \binom{r-1}{i} \cdot [\dots z]$$

$$= \left(-2 \sum_{h=0}^{t-2} (-1)^h \binom{t-1}{h} - 1 \right) \cdot [\dots z]$$

$$= [u_{0,\, i+1}\, x^{2q-2}].$$

Since $[u_{0,\, i}\, x^{2q-2}\, y]$ is at most second central, we easily obtain formula (8b) for $j = 1$:

$$[u_{0,\, i}\, x^{2q-2}\, [\gamma_1 y]] = [u_{0,\, i}\, x^{2q-2}\, \gamma_1 y] = [u_{0,\, i+1}\, x^{2q-2}\, y].$$

Supposing it true up to j and using again formula (8a) we find

$$[u_{0,\, i}\, x^{2q-2}\, [\gamma_{j+1} y]] = -[u_{0,\, i}\, x^{2q-2}\, [\gamma_j y]\ \ \gamma_1] + [u_{0,\, i}\, x^{2q-2}\, \gamma_1\, [\gamma_j y]]$$

$$= [u_{0,\, i+j+1}\, x^{2q-2}\, y].$$

Finally, to compute formula (8d) we first observe that

$$0 = [u_{0,\, i}\, x^{2q-2}\, [\gamma_1 x]] = [u_{0,\, i}\, x^{2q-2}\, \gamma_1 x] - [u_{0,\, i}\, x^{2q-1}\, \gamma_1].$$

Hence by the computations in (8c) we find

$$[u_{0,\, i}\, x^{2q-1}\, \gamma_1 y] = [u_{0,\, i}\, x^{2q-2}\, \gamma_1 xy] = [u_{0,\, i+1}\, x^{2q-1}\, y].$$

We now compute

$$[u_{0,\, i}\, x^{2q-1}\, y\gamma_1] = \sum_{i=0}^{r-2} (-1)^{((t-1)r+i)q-2} \binom{\lambda}{((t-1)\, r + i)\, q - 2} \cdot [u_{0,\, i} \dots zz]$$

$$+ (-1)^{\lambda - 1} \lambda \cdot [u_{0,\, i} \dots yx] + (-1)^{\lambda} \cdot [\dots zy]$$

$$= \left(-1 - \sum_{i=0}^{r-3} (-1)^i \binom{r-1}{i} \right) \cdot [\dots zz]$$

$$-2 \cdot [\dots yx] - [\dots xy]$$

$$= [\dots zz] - 2 \cdot [\dots yx] - [\dots xy]$$

$$= -[u_{0,\, i+1}\, x^{2q-2}\, yx].$$

Hence we obtain (8d) for $j=1$:

$$[u_{0,\,i+1}\, x^{2q-1}\, [\gamma_1 y]] = [u_{0,\,i+1}\, x^{2q-1}\, y] + [u_{0,\,i+1}\, x^{2q-2}\, yx]\,.$$

Supposing (8d) true up to j we finally obtain

$$\begin{aligned}[u_{0,\,i}\, x^{2q-1}\, [\gamma_{j+1} y]] &= -[u_{0,\,i}\, x^{2q-1}\, [\gamma_j y]\ \ \gamma_1] + [u_{0,\,i}\, x^{2q-1}\, \gamma_1\, [\gamma_j y]] \\ &= -[u_{0,\,i+j}\, x^{2q-1}\, y\ \ \gamma_1] - j\cdot[u_{0,\,i+j}\, x^{2q-2}\, yx\ \ \gamma_1] \\ &\quad + [u_{0,\,i+1}\, x^{2q-1}\, [\gamma_j y]] \\ &= [u_{0,\,i+j+1}\, x^{2q-1}\, y] + (j+1)\cdot[u_{0,\,i+j+1}\, x^{2q-2}\, yx]\,.\end{aligned}$$

7. – The differences in the case $p=2$.

When we consider a field of characteristic 2 there are some differences. We have seen that the computations of Section 5 are valid also for $p=2$. Hence we already know that $\mathrm{AFS}(a,\,b,\,n,\,2)$ is (second-central)-by-finitely presented. Actually we can be more precise. We claim that between ϑ_{2i} and ϑ_{2i+2} we have only the following extra central elements for every $i \geqslant 2$:

$[\vartheta_{2i}\, x^{2q-1}]$, possibly second central for $n=b$,

$[\vartheta_{2i}\, x^{2q-1}\, y]$, for $n=b$, and $i=2k+1$,

$[\vartheta_{2i}\, x^{2q-2}\, y\ \ (x^{q-1}\, y)^{r-p^s-1}\, x^q]$, for $1 \leqslant s \leqslant b-a-1$,

$[\vartheta_{2i}\, x^{2q-2}\, y\ \ (x^{q-1}\, y)^{r-2}\, x^{2q-p^s-1}\, y]$, for $1 \leqslant s \leqslant a$,

$[u_{0,\,2i+1}\, x^{2q-2}\, y]$, possibly second central,

$[u_{0,\,2i+1}\, x^{2q-2}\, yx]$, for $i=2k$.

$[u_{t-p^s,\,2i+1}\, x^{2q-2}\, y]$, for $1 \leqslant s \leqslant n-b-1$.

Note that in characteristic 2 the algebra that we obtain has n central elements in twice the period of the algebra, from ϑ_{2i} to ϑ_{2i+2}.

By the results of Section 5 we have only to prove that

(10a) $\quad [\vartheta_{2i}\, x^{2q-1}\, y] = 0$, for $n=b$, and i even number,

(10b) $\quad [u_{t-p^s,\,2i}\, x^{2q-2}\, y] = 0$, for $1 \leqslant s \leqslant n-b-1$.

(10c) $\quad [\vartheta_{2i+1}\, x^{2q-1}] = 0$, (10$d$) $[\vartheta_{2i+1}\, x^{2q-2}\, y\ (x^{q-1}\, y)^{r-p^s-1}\, x^q]$, for $1 \leqslant s \leqslant b-a-1$,

(10e) $[\vartheta_{2i+1}\, x^{2q-2}\, y\, (x^{q-1}\, y)^{r-2}\, x^{2q-p^s-1}\, y]$, for $1 \leqslant s \leqslant b$,

(10f) $[u_{0,\,2i+1}\, x^{2q-2}\, yx]$, for i odd number.

To prove it we distinguish the case $n=b$ from $n>b$.

7.1. *The case* $n=b$.

In this case we have just to prove (10a), (10c), (10d) and (10e). Note that (10e) is equivalent to

$$[\vartheta_{2i}\, x^{2q-p^s-1}\, y]=0\,.$$

We proceed as in Subsection 6.1. In particular we keep the same notation for the elements γ_i and $v_{k,\,i}$. We also use formulas (6a), …, (6f) of Subsection 6.1, valid also for $p=2$. Moreover we prove the following formulas:

(11a) $[\vartheta_i\, x^{2q-p^s-2}[\gamma_j x]] = [\vartheta_{i+j}\, x^{2q-p^s-1}]$,

(11b) $[v_{h,\,i},\, v_{k,\,j}] = [v_{k+h-r+2,\,i+j+1}\, x]$, for $h+k \geqslant r-2$.

Formula (11a) can be proved in the same way as formulas (6a), …, (6f) by induction on j. We remind that $\gamma_1 = [yz^\lambda]$ with $\lambda = p^n-2 = (r-1)\,q+q-2$. We first compute

$$\begin{aligned}[\vartheta_i\, x^{2q-p^s-2}\, \gamma_1] &= [\vartheta_i\, x^{2q-p^s-2}[y\ z^\lambda]]\\ &= \sum_{j=0}^{r-2}\binom{\lambda}{jq+p^s}[\vartheta_{i+1}\, x^{2q-p^s-2}]\\ &= [\vartheta_{i+1}\, x^{2q-p^s-2}].\end{aligned}$$

Analogously we find

$$[\vartheta_i\, x^{2q-p^s-1}\, \gamma_1] = \sum_{j=0}^{r-2}\binom{\lambda}{jq+p^s-1}[\vartheta_{i+1}\, x^{2q-p^s-1}] = 0\,.$$

It follows that (11a) is verified for $j=1$. Supposing it true up to j and using (6a) we find

$$\begin{aligned}[\vartheta_i\, x^{2q-p^s-2}\, [\gamma_{j+1} x]] &= [\vartheta_i\, x^{2q-p^s-2}[\gamma_j x \gamma_1]]\\ &= [\vartheta_i\, x^{2q-p^s-2}\, [\gamma_j x]\, \gamma_1] + [\vartheta_i\, x^{2q-p^s-2}\, \gamma_1 [\gamma_j x]]\\ &= [\vartheta_{i+j}\, x^{2q-p^s-1}\, \gamma_1] + [\vartheta_{i+1}\, x^{2q-p^s-2}\, [\gamma_j x]]\\ &= [\vartheta_{i+j+1}\, x^{2q-p^s-1}].\end{aligned}$$

To prove formula (11b) we first observe that $[v_{h,\,i},\, v_{k,\,j}] = [v_{h,\,i}[[\gamma_j xy]z^\lambda]$

with $\lambda = (k+1)\,q + q - 1$. If $h + k = r - 2$, using (6b) and (6f) we find

$$[v_{h,\,i},\, v_{k,\,j}] = \sum_{l=0}^{k} \binom{\lambda}{lq} [v_{h+l,\,i}\, y[\gamma_j x] \ldots z] + [\vartheta_{i+1}\, x^{2q-3}[\gamma_j x]\, y]$$

$$= \sum_{l=0}^{k} \binom{k+1}{q} [\vartheta_{i+j+1}\, x^{2q-2}\, z] + [\vartheta_{i+j+1}\, x^{2q-2}\, y]$$

$$= [\vartheta_{i+j+1}\, x^{2q-2}\, z] + [\vartheta_{i+j+1}\, x^{2q-2}\, y]$$

$$= [\vartheta_{i+j+1}\, x^{2q-1}] = [v_{0,\, i+j+1} x]\,.$$

Analogously if $h + k > r - 2$ we obtain

$$[v_{h,\,i},\, v_{k,\,j}] = \sum_{l=0}^{r-h-2} \binom{\lambda}{lq} [v_{h+l,\,i}\, y[\gamma_j x] \ldots z]$$

$$+ \binom{\lambda}{(r-h-1)\,q + q - 2} [\vartheta_{i+1}\, x^{2q-3}[\gamma_j x]\, y \ldots z]$$

$$+ \sum_{l=r-h}^{k} \binom{\lambda}{lq + q - 2} [v_{l+h-r,\, i+1}\, y \;\; x^{q-2}[\gamma_j x]\, y \ldots z]$$

$$+ [v_{h+k-r+1,\, i+1}\, y \;\; x^{q-2}[\gamma_j x]\, y]$$

$$= \sum_{l=0}^{k} \binom{(k+1)\,q}{lq} [v_{h+k-r+2,\, i+j+1}\, z]$$

$$+ [v_{h+k-r+1,\, i+j+1}\, y \;\; x^{q-1}\, y]$$

$$= [v_{h+k-r+2,\, i+j+1}\, z] + [v_{h+k-r+2,\, i+j+1}\, y]$$

$$= [v_{h+k-r+2,\, i+j+1}\, x]\,.$$

Formulas (10c) and (10d) are now a direct consequence of (11b). We let $2k = r - 2$ and compute

$$0 = [v_{k,\,i},\, v_{k,\,i}] = [v_{2k-r+2,\, 2i+1} x] = [v_{0,\, 2i+1} x] = [\vartheta_{2i+1}\, x^{2q-1}]$$

$$0 = [v_{r-p^{s-1}-1,\, i},\, v_{r-p^{s-1}-1,\, i}] = [v_{r-p^s,\, 2i+1} x] = [\vartheta_{2i+1}\, x^{2q-2}\, (y \;\; x^{q-1})^{r-p^s} x]\,.$$

To prove (10e) we use formula (11a):

$$0 = [[\vartheta_i\, x^{q-p^{s-1}-1}][\vartheta_i\, x^{q-p^{s-1}-1}]] = [\vartheta_i\, x^{q-p^{s-1}-1}[[\gamma_i xy] \;\; x^{q-p^{s-1}-1}]]$$

$$= [\vartheta_i\, x^{2q-p^s-2}[[\gamma_i x]\, y]] = [\vartheta_{2i}\, x^{2q-p^s-1}\, y]\,.$$

Finally to prove (10*a*) we proceed as in (7), using equations (6*b*), (6*c*) and (6*d*). We obtain that $(i+1)\cdot[\vartheta_{2i}\, x^{2q-i}\, y]$, and thus for i even the elements $[\vartheta_{2i}\, x^{2q-i}\, y]$ vanishes.

7.2. *The case $n > b$.*

In this case we have to prove formulas (10*b*), …, (10*f*). As above we proceed as in Subsection 6.2, keeping the same notation for the elements γ_i and $v_{k,i}$, and using formulas (8*a*), …, (8*d*). We also need the following equations:

(12*a*) $[u_{t-2,i}\, x^{2q-1}(y\ \ x^{q-1})^k[v_j y]] = [u_{t-2,i+j}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^k]$, for $1 \leqslant k \leqslant r-2$

(12*b*) $[u_{t-p^s,i}\, x^{2q-2}[\gamma_j y]] = [u_{t-p^s,i+j}\, x^{2q-2}\, y]$, for $1 \leqslant s \leqslant b$.

Note that in (12*a*) we ignored the possible elements $[u_{t-2,\,i+j}\, x^{2q-1}\,(y\ \ x^{q-1})^k x]$ because they are central and useless in the final computation.

To prove (12*a*) and (12*b*) we proceed as in Section 6.2. We remind that $\gamma_1 = [y\ \ z^\lambda]$ with $\lambda = p^n - 2 = ((t-1)\,r + r - 1)\,q + q - 2$.

We start computing

$$[u_{t-2,\,i}\, x^{2q-1}\,(y\ \ x^{q-1})^k\gamma_1] = [u_{t-2,\,i}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^k[y\ \ z^\lambda]]$$

$$= \sum_{l=0}^{r-2-k}\binom{\lambda}{lq}[u_{t-2,\,i+j}\, x^{2q-1}\,(y\ \ x^{q-1})^k]$$

$$= (k+1)\cdot[u_{t-2,\,i+j}\, x^{2q-1}\,(y\ \ x^{q-1})^k].$$

Analogously we can compute

$$[u_{t-2,\,i}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^k\gamma_1]$$

$$= \sum_{j=1}^{t-1}\sum_{l=-k}^{r-k-2}\binom{\lambda}{(jr+l)\,q-2}[u_{t-2,\,i}\ldots] + \sum_{l=-k}^{0}\binom{\lambda}{(tr+l)\,q-2}[u_{t-2,\,i}\ldots]$$

$$= \sum_{j=1}^{t-1}\sum_{l=-k-1}^{r-k-3}\binom{(t-1)\,r+r-1}{jr+l}[u_{t-2,i}\ldots] + \sum_{l=-k-1}^{-1}\binom{(t-1)\,r+r-1}{(t-1)\,r+r+l}[u_{t-2,i}\ldots]$$

$$= \sum_{j=0}^{t-1}\binom{t-1}{j}\sum_{l=-k-1}^{-1}\binom{r-1}{r+l}[u_{t-2,\,i+1}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^k]$$

$$+ \sum_{j=1}^{t-1}\binom{t-1}{j}\sum_{l=0}^{r-k-3}\binom{r-1}{l}[u_{t-2,\,i+1}\, x^{2q-1}\, y\ \ (x^{q-1}\, y)^k].$$

Note that for $k = r - 2$ the second summatory does not exist, while for $k < r - 2$ its coefficient has value k. Since the first addendum vanishes, in any case it follows that

$$[u_{t-2,\,i}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k \gamma_1] = k \cdot [u_{t-2,\,i+1}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k].$$

Hence we obtain (12a) for $j = 1$:

$$\begin{aligned}
&[u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k [\gamma_1 y]] \\
&= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k \gamma_1 y] + [u_{t-2,\,i}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k\, \gamma_1] \\
&= [u_{t-2,\,i+1}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k].
\end{aligned}$$

Supposing inductively formula (12a) verified up to j and using (8a) we can compute it for $j + 1$:

$$\begin{aligned}
&[u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k [\gamma_{j+1} y]] = [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k [\gamma_j y v_1]] \\
&= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k [\gamma_j y] \gamma_1] + [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^k\, \gamma_1 [\gamma_j y]] \\
&= [u_{t-2,\,i+j}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k\, \gamma_1] + (k+1) \cdot [u_{t-2,\,i+1}\, x^{2q-1}\, (y\ x^{q-1})^k [\gamma_j y]] \\
&= [u_{t-2,\,i+j+1}\, x^{2q-1}\, y\ (x^{q-1}\, y)^k].
\end{aligned}$$

To prove formula (12b), as above we start computing

$$\begin{aligned}
[u_{t-p^s,\,i}\, x^{2q-2}\, \gamma_1] &= [u_{t-p^s,\,i}\, x^{2q-2}\, [y\ z^\lambda]] \\
&= \sum_{l=p^s-1}^{t-1} \sum_{h=0}^{r-2} \binom{\lambda}{lr+h} [u_{t-p^s,\,i+1}\, x^{2q-2}] \\
&= \sum_{l=p^s-1}^{t-1} \binom{t-1}{l} [u_{t-p^s,\,i+1}\, x^{2q-2}] \\
&= [u_{t-p^s,\,i+1}\, x^{2q-2}].
\end{aligned}$$

Observing that $[u_{t-p^s,\,i}\, x^{2q-2}\, y] = 0$ we obtain (12b) for $j = 1$:

$$[u_{t-p^s,\,i}\, x^{2q-2}\, [\gamma_1 y]] = [u_{t-p^s,\,i}\, x^{2q-2}\, \gamma_1 y] = [u_{t-p^s,\,i+1}\, x^{2q-2}\, y].$$

Note that we obtain a central element. We now suppose inductively (12b) verified up to j and compute it for $j + 1$. Using (8a) we easily find

$$\begin{aligned}
[u_{t-p^s,\,i}\, x^{2q-2}\, [\gamma_{j+1} y]] &= [u_{t-p^s,\,i}\, x^{2q-2}\, [\gamma_j y]\, \gamma_1] + [u_{t-p^s,\,i}\, x^{2q-2}\, \gamma_1 [\gamma_j y]] \\
&= [u_{t-p^s,\,i+1}\, x^{2q-2}\, [\gamma_j y]] \\
&= [u_{t-p^s,\,i+j+1}\, x^{2q-2}\, y].
\end{aligned}$$

To prove (10*b*) we use formula (12*b*). Indeed we can compute

$$\begin{aligned}
0 &= [u_{t-p^{s-1}-1,\,i}\, x^{q-1}\, [u_{t-p^{s-1}-1,\,i}\, x^{q-1}]] \\
&= [u_{t-p^{s-1}-1,\,i}\, x^{q-1}\, [[\gamma_{i-1} y]\, z^{(t-p^{s-1}+1)rq+q-2}]] \\
&= [u_{t-p^{s-1}-1,\,i}\, x^{q-1}\, z^{(t-p^{s-1}+1)rq+q-2}\, [\gamma_{i-1} y]] \\
&= [u_{t-p^{s},\,i+1}\, x^{2q-2}\, [\gamma_{i-1} y]] \\
&= [u_{t-p^{s},\,2i}\, x^{2q-2}\, y].
\end{aligned}$$

Analogously (10*c*) and (10*d*) are consequences of formula (12*a*). In the first proof we let $2k = r$ and compute

$$\begin{aligned}
0 &= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{k-1} [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{k-1}]] \\
&= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{k-1} [u_{t-2,\,i}\, x^{2q-1}\, y\ (x^{q-1}\, y)^{k-2}]\ x^{q-1}] \\
&= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{k-1} [\gamma_i y z^{kq}]\ x^{q-1}] \\
&= \sum_{l=0}^{k-1} \binom{kq}{lq} [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{k+l-1} [\gamma_i y] \dots x^{q-1}] \\
&= \sum_{l=0}^{k-1} \binom{k}{l} [\vartheta_{2i+1}\, x^{2q-1}] \\
&= [\vartheta_{2i+1}\, x^{2q-1}].
\end{aligned}$$

Similarly in the second proof we compute

$$\begin{aligned}
0 &= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{r-p^{s-1}-1} [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{r-p^{s-1}-1}]] \\
&= [u_{t-2,i}\, x^{2q-1}\, (y\ x^{q-1})^{r-p^{s-1}-1} [u_{t-2,i}\, x^{2q-1}\, y\ (x^{q-1}\, y)^{r-p^{s-1}-2}]\ x^{q-1}] \\
&= [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{r-p^{s-1}-1} [\gamma_i y z^{(r-p^{s-1})q}]\ x^{q-1}] \\
&= \sum_{l=0}^{p^{s-1}-1} \binom{(r-p^{s-1})\,q}{lq} [u_{t-2,\,i}\, x^{2q-1}\, (y\ x^{q-1})^{r-p^{s-1}-1+l} [\gamma_i y] \dots x^{q-1}] \\
&= \sum_{l=0}^{p^{s-1}-1} \binom{r-p^{s-1}}{l} [\vartheta_{2i+1}\, x^{2q-2}\, y\ (x^{q-1}\, y)^{r-p^{s}-1}\, x^{q}] \\
&= [\vartheta_{2i+1}\, x^{2q-2}\, y\ (x^{q-1}\, y)^{r-p^{s}-1}\, x^{q}].
\end{aligned}$$

To prove (10e) we use formula (8b):

$$
\begin{aligned}
0 &= [\vartheta_{i+1}\, x^{q-p^{s-1}-1}[\vartheta_{i+1}\, x^{q-p^{s-1}-1}]] \\
&= [\vartheta_{i+1}\, x^{q-p^{s-1}-1}[\gamma_i y\ z^{rq-2}\ y\ x^{q-p^{s-1}-1}]] \\
&= [\vartheta_{i+1}\, x^{2q-p^s-2}[\gamma_i y\ z^{rq-2}]\, y] \\
&= \sum_{l=0}^{r-2} \binom{(r-1)\,q+q-2}{lq+p^s} [\vartheta_{i+1}\, x^{2q-2}\,(y\ x^{q-1})^l\,[\gamma_i y] \dots y] \\
&= \sum_{l=0}^{r-2} \binom{(r-1)\,q+q-2}{lq+p^s} [u_{0,\,2i+1}\, x^{2q-p^s-1}\ y]\,.
\end{aligned}
$$

To evaluate the summatory we have to distinguish the case $p^s = q$ from $p^s < q$, but in both cases we find

$$\sum_{l=0}^{r-2} \binom{(r-1)\,q+q-2}{lq+p^s} = 1\,.$$

It follows that

$$[\vartheta_{2i+1}\, x^{2q-2}\ y\ \ (x^{q-1}\ y)^{r-2}\ \ x^{2q-p^s-1}\ y] = 0\,.$$

Finally to prove (10f) we proceed as in (9), using formulas (8b), (8c) and (8d). We compute $[\vartheta_{i+1}\, x^{q-1}[\vartheta_{i+1}\, x^{q-1}]]$ and obtain that $i \cdot [u_{0,\,2i+1}\, x^{2q-2}\, yx] = 0$. It follows that for i odd number the element $[u_{0,\,2i+1}\, x^{2q-2}\, yx]$ vanishes.

8. – Computations.

The ANU p-Quotient Program [HNO95] and GAP [S$^+$95] were very useful to discover the structures of the involved algebras and to see how to perform computations in full generality.

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