Bollettino Unione Matematica Italiana

L. A. Kurdachenko, J. Otal

Simple modules over CC-groups and monolithic just non-CC-groups

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **4-B** (2001), n.2, p. 381–390.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_2_381_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 381-390

Simple Modules over CC-Groups and Monolithic Just non-CC-Groups.

L. A. Kurdachenko - J. Otal

Sunto. – *In questo lavoro studiamo i non CC-gruppi G monolitici con tutti i quozienti propri CC-gruppi, che hanno sottogruppi abeliani normali non banali.*

1. – Introduction.

Let G be a group. If $1 \neq H \trianglelefteq G$, then G/H is called *a proper factor-group* of G. Let $\boldsymbol{X}$ be a class a of groups. Then a group G is said to be *a just non-$\boldsymbol{X}$-group if G is not an* $\boldsymbol{X}$-group but every proper factor-group of G lies in $\boldsymbol{X}$. The structure of just non-$\boldsymbol{X}$-groups has been considered for the following choices of $\boldsymbol{X}$: For $\boldsymbol{X}$ = abelian groups by M. F. Newman [14, 15]; for $\boldsymbol{X}$ = finite groups by D. McCarthy [12, 13] and J. S. Wilson [22]; for $\boldsymbol{X}$ = polycyclic groups by J. R. J. Groves [6] and D. J. S. Robinson and J. S. Wilson [19]; for $\boldsymbol{X}$ = Chernikov groups by S. Franciosi and F. de Giovanni [3]. More recently, D. J. S. Robinson and Z. Zhang [20] introduce the important case $\boldsymbol{X}$ = finite-by-abelian groups. It is a well-known result due to B. H. Neumann the characterization of finite-by-abelian groups as groups with boundedly finite conjugacy classes (*BFC*-groups) so that the above paper draws our attention to conditions on the conjugacy classes. We recall that, if $\boldsymbol{Y}$ is a class of groups, *an* $\boldsymbol{Y}C$-group is a group G with conjugacy classes in $\boldsymbol{Y}$, that is, $G/C_G(\langle x\rangle^G) \in \boldsymbol{Y}$ for all $x \in G$. The next step is done by S. Franciosi, F. de Giovanni and L. A. Kurdachenko [6], which consider just non-$\boldsymbol{F}C$-groups. Therefore, it is natural to extend these ideas by considering the next generalization of the class of $\boldsymbol{F}C$-groups: the class of $\boldsymbol{C}C$-groups ([16]). As in other papers, we are writing FC or CC instead of $\boldsymbol{F}C$ or $\boldsymbol{C}C$.

In the study of just non-$\boldsymbol{X}$-groups and to avoid simple groups, it is very usual to allow they contain non-trivial abelian normal subgroups. Thus this paper investigates the structure of *just non-$\boldsymbol{C}C$* groups G with $Fitt(G) \neq 1$. In this study two cases appear at first: $FC(G) \neq 1$ and $FC(G) = 1$. The first case has been described in [10]. In the second case $A = Fitt(G)$ *is abelian* and either A *is a torsion-free abelian group* or A is an elementary abelian p-group, for some prime p ([10]). As usual A can be viewed as a $\mathbb{Z}H$-module where $H = G/A$ is a CC-group. Let M be the intersection of all non-trivial G-invariant subgroups of G. Obviously $M = 1$ or $M \neq 1$. In the latter, M is called *the monolith of* G and G is said to be *monolithic*. Further, in this case, we note that the monolith is a simple $\mathbb{Z}H$-module. Thus, to study *monolithic just non-CC-groups*, we need to describe first simple modules over CC-groups. Both topics are the goals of this paper.

Throughout, our notation is standard and follows [17]. Some of our results, depend on the so called *0−rank* of a group. We recall that *the group G is said to have finite 0-rank* $r_0(G) = r$ if G has a subnormal series of finite length in which r factors are infinite cyclic and the remainders periodic; otherwise, G is said to have *infinite 0-rank*.

2. – Some elementary results.

The following result is very useful.

(2.1). LEMMA. – *Let G be a just non-CC-group. If $1 \neq N$, $M \trianglelefteq G$ then $N \cap M \neq 1$.*

PROOF. – If $N \cap M = 1$, by Remak theorem, G embeds in $(G/N) \times (G/M)$. ■

Let $\mathbf{X}$ be a class of groups. The class $\mathbf{X}$ is said to be *a formation of groups* if the following two conditions are satisfied:

(i) If $G \in \mathbf{X}$ and $H \trianglelefteq G$, then $G/H \in \mathbf{X}$.

(ii) If H_1, $H_2 \trianglelefteq G$ and G/H_1, $G/H_2 \in \mathbf{X}$, then $G/(H_1 \cap H_2) \in \mathbf{X}$.

Given a formation $\mathbf{X}$ and a group G, *the* $\mathbf{X}C$-center of G is the characteristic subgroup $\mathbf{X}C(G)$ of G consisting of all $x \in G$ such that $G/C_G(\langle x\rangle^G) \in \mathbf{X}$. A subset $H \subseteq G$ is said to be $\mathbf{X}C$-central in G if $H \subseteq \mathbf{X}C(G)$. This clearly generalizes the ordinary centrality allowing us to identify the $\mathbf{X}C$-groups G as the groups G which satisfy $\mathbf{X}C(G) = G$. Given a group G, starting from the $\mathbf{X}C$-center we may construct *the* (*upper*) *$\mathbf{X}C$-central series of G* as follows. This series is denoted by $\{\mathbf{X}C_\alpha(G)\}$ and defined by transfinite induction by $\mathbf{X}C_0(G) = 1$ and $\mathbf{X}C_{\alpha+1}(G)/\mathbf{X}C_\alpha(G) = \mathbf{X}C(G/\mathbf{X}C_\alpha(G))$. The last term of this series is $\mathbf{X}C_\infty(G)$, *the $\mathbf{X}C$-hypercenter of G*. G is said to be *$\mathbf{X}C$-hypercentral* if $\mathbf{X}C_\infty(G) = G$. Note that these notions extend those of the center and the hypercenter of a group as well as other centers just as the FC-center or the FC-hypercenter. Similar concepts were already introduced in [11].

(2.2). LEMMA. – *Let $\mathbf{X}$ be a formation of groups. If P is the $\mathbf{X}C$-center of G an H is a non-trivial G-invariant subgroup of the $\mathbf{X}C$-hypercenter Q of G, then $H \cap P \neq 1$.*

PROOF. – Let us denote by $1 = C_0 \leqslant C_1 \leqslant \ldots C_\alpha \leqslant C_{\alpha+1} \leqslant \ldots C_\gamma = Q$ the upper $\boldsymbol{X}C$-central series of G. By construction, there must exist an ordinal α such that $H \cap C_\alpha \neq 1$. If β be the least ordinal with this property, then β cannot be limit, so $\beta - 1$ exists and therefore $H \cap C_{\beta-1} = 1$. If $1 \neq x \in H \cap C_\beta$ and $N = \langle x \rangle^G$, then $xC_{\beta-1}$ is *$\boldsymbol{X}C$-central* and so $G/C_G(NC_{\beta-1}/C_{\beta-1}) \in \boldsymbol{X}$.

Let $g \in C_G(NC_{\beta-1}/C_{\beta-1})$; then $[g, N] \leq C_{\beta-1}$. On the other hand $[g, N] \leq H$, since $H \trianglelefteq G$. Hence $[g, N] \leq H \cap C_{\beta-1} = 1$, so $g \in C_G(N)$. It follows that $C_G(NC_{\beta-1}/C_{\beta-1}) \leq C_G(N)$ and $G/C_G(N) \in \boldsymbol{X}$ and therefore $x \in \boldsymbol{X}C(G)$. ■

(2.3). LEMMA. – *Let $\boldsymbol{X}$ be a formation such that $\boldsymbol{X}(\boldsymbol{X} \cap \boldsymbol{F}) = \boldsymbol{X}$. Let G be $\boldsymbol{F}C$ and $\boldsymbol{X}C$-hypercentral group, $\mathbb{F}$ be a field and $B \leq A$ $\mathbb{F}G$-modules satisfying*:

(i) *B and A/B are simple $\mathbb{F}G$-modules.*

(ii) *$G/C_G(A/B) \in \boldsymbol{X}$ but $G/C_G(B) \notin \boldsymbol{X}$.*

Then there exists an $\mathbb{F}G$-submodule C of A such that $A = B \oplus C$.

PROOF. – We may assume that $C_G(A) = 1$. Since $\boldsymbol{X}$ is a formation and $G/C_G(A) \notin \boldsymbol{X}$, we have that $G \notin \boldsymbol{X}$. This and $G/C_G(A/B) \in \boldsymbol{X}$ give that $C_G(A/B) \neq 1$. Since G is both $\boldsymbol{F}C$ and $\boldsymbol{X}C$-hypercentral, then G is $(X \cap \boldsymbol{F})$ C-hypercentral, so that, if Q is the $(X \cap \boldsymbol{F})$ C-center of G, lemma (2.2) yields that $Q \cap C_G(A/B) \neq 1$. Take a non-trivial element of this intersection, $1 \neq x \in Q \cap C_G(A/B)$. Thus $\langle x \rangle^G$ is central-by-finite and $G/C_G(\langle x \rangle^G) \in (\boldsymbol{X} \cap \boldsymbol{F})$. Put $K = \langle x \rangle^G$ and $H = C_G(K)$.

We consider the family $\Xi = \{E \leq_{\mathbb{F}H} A \,|\, E \nleq B\}$. Trivially, $A \in \Xi$, so $\Xi \neq \emptyset$. By [21, Theorem A], A is an artinian $\mathbb{F}H$-module and hence Ξ has a minimal element, say M. Let $y \in K$. Note that $M(y-1) \cong_{\mathbb{F}H} M/C_M(y)$. Since $[H, K] = 1$, we have that $M(y-1)$ and $C_M(y)$ are $\mathbb{F}H$-modules. Since $y \in C_G(A/B)$, we also have that $M(y-1) \leq B$. Therefore, by [23, lemma], $B = \oplus_{t \in T} Ut$, where U is a simple $\mathbb{F}H$-module and T is a finite subset of G. We claim that $H/C_H(U) \notin \boldsymbol{X}$; for, otherwise every $H/C_H(Ut) \in \boldsymbol{X}$ and, since $\boldsymbol{X}$ is a formation, then $H/C_H(B) \in \boldsymbol{X}$. Since $G/H \in (\boldsymbol{X} \cap \boldsymbol{F})$ and $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{X} \cap \boldsymbol{F})$, we have $G/C_G(B) \in \boldsymbol{X}(\boldsymbol{X} \cap \boldsymbol{F}) = \boldsymbol{X}$, contradicting (ii). Thus $H/C_H(U) \notin \boldsymbol{X}$, as claimed.

Given $y \in K$, suppose that $C_M(y) \leq B$. Then $M/C_M(y)$ has a $\mathbb{F}H$-composition series of finite length whose factors are $\boldsymbol{X}C$-central. On the other hand, since $M(y-1) \cong_{\mathbb{F}H} M/C_M(y) \leq B$, $M(y-1)$ can be written as $M(y-1) = \oplus_{t \in T_1} Ut$, where $T_1 \subseteq T$, what indicates that $M(y-1)$ cannot have $\boldsymbol{X}C$-central factors in an $\mathbb{F}H$-composition series, since $H/C_H(U) \notin X$. Hence this contradiction gives that $C_M(y) \in \Xi$, and, by the election of M, it follows that $C_M(y) = M$ for every $y \in K$. This yields that $M \leq C_A(K)$. In particular, $C_A(K) \nleq B$. Since $K \trianglelefteq G$, then $C_A(K)$ is really an $\mathbb{F}G$-submodule of A, so that, by the irreducibility of B, we have that either $B \leq C_A(K)$ or $B \cap C_A(K) = 0$. In the first case, by the irreducibility of A/B, $A = C_A(K)$, and so $K \leq C_G(A) = 1$, which contradicts lemma (2.1). Therefore, putting $C = C_A(K)$, we have that $A = B \oplus C$, as required. ■

(2.4). Corollary. – *Let G be a monolithic just non-CC-group with $FC(G) = 1$ and $Fitt(G) \neq 1$. Then $Fitt(G)$ is the monolith of G.*

Proof. – Let $A = Fitt(G)$ be. By [10, corollary to theorem 1], either A is elementary abelian or A is torsion-free abelian. In any case, A is a $\mathbb{Z}H$-module, where $H = G/A$ is a CC-group. Let M be the monolith of G; then $M \leqslant A$ is a simple submodule. We assume that there exists some $a \in A \backslash M$ and put $D/M = \langle a \rangle^G M/M$.

Suppose first that A is p-elementary, p a prime. In this case A becomes an $\mathbb{F}H$-module, where $\mathbb{F} = \mathbb{F}_p$ is the field with p elements, and D/M is finite. It follows that D/M contains a simple $\mathbb{F}H$-submodule E/M. By lemma (2.3), there exists an $\mathbb{F}H$-submodule C such that $E = M \oplus C$, which contradicts lemma (2.1).

Assume now that A is torsion-free; since M is a simple $\mathbb{Z}H$-module, it is also divisible, and in this case D/M is infinite cyclic. Put $E = D \otimes_{\mathbb{Z}} \mathbb{Q}$ and note that $M \otimes_{\mathbb{Z}} \mathbb{Q} = M$. Also $C_H(D/M) = C_H(E/M)$, so that $H/C_H(E/M)$ is finite. Again lemma (2.3) gives that $E = M \oplus C$, for some C. Put $C_1 = C \cap D$, By construction $C_1 \neq 1$ but $C \cap M = 1$, a new contradiction to lemma (2.1). ■

3. – The structure of simple modules over CC-groups.

Many papers on group theory, especially on groups with finiteness conditions, contain the following question: Given a group G and a minimal normal subgroup A of G, study the factor group $H = G/C_G(A)$. If A is abelian, we may consider A as a simple $\mathbb{Z}H$-module and with this approach some results have obtained. For example, when G is finite, these modules have been described ([2, chapter B.10]). R. Baer [1] has considered simple modules over locally finite groups, obtaining that A is elementary. Similar results have been obtained by P. Hall [7] for polycyclic groups and generalizations of these ones by D. I. Zaitsev in studying locally (polycyclic-by-finite) groups of finite 0-rank. Other examples are simple modules over Chernikov groups for metabelian groups with **Min-n** (B. Hartley and D. McDougall [8]), over abelian and nilpotent groups (D. J. S. Robinson and Z. Zhang [20]) or over locally soluble FC-groups (S. Franciosi, F. de Giovanni and L. A. Kurdachenko [4]). Here we are generalizing the latter, because we will obtain a description of simple modules over a CC-group. This description will contain the study of the CC-group itself as well as the existence of a simple and faithful module over a CC-group.

(3.1). Lemma. – *Let $\mathbb{F}$ be a field, G a group and A an infinite simple $\mathbb{F}G$-module such that $C_G(A) = 1$. If p is a prime and P is a G-invariant elementary abelian p-subgroup of $FC(G)$, then $char\,\mathbb{F} \neq p$ and there exists a $J \leqslant P$ such that $|P/J| = p$ and $J_G = 1$.*

PROOF. – Proceed as in the proof of [4, lemma 8.2]. ■

Given a group G, we recall that *the socle of* G is the subgroup $Soc(G)$ generated by its minimal normal subgroups and so this is a direct product of some of them. The product of the abelian ones is $Soc_a(G)$. A related subgroup is one given by the following.

(3.2). LEMMA. – *A CC-group G contains a normal subgroup Q such that*:

(i) *Q is a direct product of finite minimal and infinite cyclic normal subgroups.*

(ii) *If* $1 \neq H \trianglelefteq G$, *then* $Q \cap H \neq 1$.

PROOF. – Let $Q_0 = Soc(FC(G))$, so that Q_0 is a direct product of finite minimal normal subgroups of G. Suppose that $Q_0 \cap H = 1$, where $1 \neq H \trianglelefteq G$. If $1 \neq x \in H$ and $X = \langle x \rangle^G$, since $Q \cap X = 1$, then X is infinite cyclic and $Q_1 = Q_0 \times X$ is a direct product. Proceeding in this way and by transfinite induction, we construct a subgroup Q satisfying (i) and (ii). ■

A subgroup Q satisfying the above properties is said to be *a quasi-socle of* G. Lemma (3.2) assures that a CC-group G has at least a quasi-socle Q and, by construction, $Soc_a(G) \leqslant Soc(G) \leqslant Q$. However, this subgroup is not necessarily unique, and, in general, its existence is doubtful.

(3.3). THEOREM. – *Let G be a CC-group and let A be a simple* $\mathbb{Z}G$*-module such that* $C_G(A) = 1$.

(i) *There is a* $J \leqslant Soc_a(G)$ *such that* $Soc_a(G)/J$ *is locally cyclic and* $J_G = 1$.

(ii) *If A is an elementary abelian p-group, then* $Soc_a(G)$ *is a* p'*-group.*

(iii) *If G has finite* 0*-rank,* $Soc(G)$ *is a quasi-socle of G* (*in particular* $FC(G)$ *is periodic*) *and the underlying additive group of A is elementary.*

PROOF. – Before starting, it is worth noting that A can be viewed as an $\mathbb{F}G$-module, where $\mathbb{F}$ is a field. For, if A is torsion-free, being simple, it is also radicable and hence $\mathbb{F} = \mathbb{Q}$. If A has elements of finite order, again by simplicity, there must exist a prime p such that A is an elementary abelian p-group and then $\mathbb{F} = \mathbb{F}_p$.

Let Q be the Sylow q-subgroup of $Soc_a(G)$, q a prime. By lemma (3.2), there exists $J_q \leqslant Q$ such that $|Q: J_q| = q$ and $(J_q)_G = 1$. If $J = \times J_q$, where the product runs all over the primes occurring in $Soc_a(G)$, then $Soc_a(G)/J$ is locally cyclic and $J_G = 1$, which gives (i). (ii) follows from lemma (3.2) as well.

(ii) Since a CC-group is FC-hypercentral and finitely generated FC-hypercentral groups are nilpotent-by-finite, an FC-hypercentral group is locally (polycyclic-by-finite). If G has finite 0-rank, by [24, corollary of theorem 2.3], then A is an elementary abelian p-group, for some prime p: $pA = 0$. Assume that $Soc(G)$ is not a quasi-socle of G; this means that G must contain a non-trivial normal infinite cyclic subgroup C. Put $H = C_G(C)$ so that $|G: H| \leqslant 2$. In this case A contains a simple $\mathbb{F}_p H$-module B such that $A = B \times Bx$, where $x \in G \setminus H$. If $C \cap C_H(B) \neq 1$, then $C \cap C_H(Bx) \neq 1$ as well. Since C is infinite cyclic, then $C \cap C_H(A) \neq 1$ while $C_G(A) = 1$. This contradiction shows that $C \cap C_H(B) = 1$. Thus C embeds in $\zeta(H/C_H(B))$ and this contradicts the results of [24]. ■

Theorem (3.3) raises the natural question of examining its converse, that is, of deciding if the conditions of the above result are also sufficient. To study this, we shall need the following auxiliary result.

(3.4). LEMMA. – *Let* $\mathbb{F}$ *be a field,* $S = \times_{\lambda \in \Lambda} S_\lambda$, *where each* S_λ *is a finite non-abelian simple group. Then, there exists a simple* $\mathbb{F}S$*-module* A *such that* $C_S(A) = 1$.

PROOF. – Let γ be the type of Λ. By transfinite induction, we are going to show that S has an ascending normal series $1 = C_0 \leqslant C_1 \leqslant \dots C_\alpha \leqslant C_{\alpha+1} \leqslant \dots C_\gamma = S$ such that, if $\alpha \geqslant 0$, $C_{\alpha+1} = C_\alpha \times S_{\alpha+1}$, where $S_{\alpha+1} \simeq S_{\lambda_\alpha}$, for some $\lambda_\alpha \in \Lambda$ and C_α admits a simple KC_α-module A_α with $C_{C_\alpha}(A_\alpha) = 1$.

Let $\gamma = 1$. In this case S is finite non-abelian and simple and it is know that there exists a simple and faithful $\mathbb{F}S$-module: see [2, lemma B.10.2].

Suppose that $\gamma = \delta + 1$ is not limit, then there exists a simple $\mathbb{F}C_\delta$-module D with $C_{C_\delta}(D) = 1$. Define $C_{\delta+1} = C_\delta \times S_{\delta+1}$. There is a simple $\mathbb{F}S_{\delta+1}$-module B with $C_{S_{\delta+1}}(B) = 1$. Form $U = D \otimes_{\mathbb{F}} B$, so that U is a $\mathbb{F}(C_\delta \times S_{\delta+1})$-module (see [2, corollary B.1.12]). There exists $X \subseteq S_{\delta+1}$ such that $U = \times_{x \in X} Dx$. If A is a composition $\mathbb{F}C_{\delta+1}$-factor of U and $L = C_{C_{\delta+1}}(A)$, since A is a direct factor of U, then $L \cap C_\delta = 1$. Similarly, $L \cap S_{\delta+1} = 1$, and so $L \leqslant \zeta(C_{\delta+1}) = 1$. Hence $C_{C_{\delta+1}}(A) = 1$.

If γ is limit, by induction, for each $\alpha < \gamma$, there exists a simple $\mathbb{F}C_\alpha$-module A_α such that $C_{C_\alpha}(A_\alpha) = 1$. We note that the above paragraph shows that $A_\alpha \leqslant A_{\alpha+1}$, for every α and therefore we may consider the (injective) limit A of the A_α. We write A as an union $A = \cup_{\alpha < \gamma} \overline{A_\alpha}$, where $\overline{A_\alpha} \simeq A_\alpha$, and consider it as a $\mathbb{F}C_\gamma$-module, where C_γ is the limit of the C_α, $\alpha < \gamma$. Let E be a non-zero $\mathbb{F}C_\gamma$-submodule of A. Then there is a least ordinal δ such that $E \cap \overline{A_\delta} \neq 0$. It is clear that we may view $\overline{A_\delta}$ as a simple $\mathbb{F}\overline{C_\delta}$-module, so that $E \cap \overline{A_\delta} = \overline{A_\delta}$, that is, $\overline{A_\delta} \leqslant E$. Trivially δ is not limit, so $\delta + 1 < \gamma$. In this case, we may consider the $\mathbb{F}\overline{C_{\delta+1}}$-module $\overline{A_{\delta+1}}$. Obviously $\overline{A_{\delta+1}} = \times_{z \in Z} \overline{A_\delta} z$, for some subset $Z \subseteq \overline{C_{\delta+1}}$. Since E is a $\mathbb{F}C_\delta$-module, the inclusion $A_\gamma \leqslant E$ gives $\overline{A_\delta} z \leqslant E$, for every $z \leqslant Z$. This means that $\overline{A_{\delta+1}} \leqslant E$.

The usual induction gives that $\overline{A_\alpha} \leqslant E$, for every $\alpha < \gamma$, so that $E = A$ and it turns out that A is a simple $\mathbb{F}C_\gamma$-module. Call $L = C_{C_\gamma}(A)$. For every $\alpha < \gamma$, $L \cap C_\alpha \leqslant C_{C_\alpha}(A_\alpha) = 1$. Then $L = 1$ and the construction has just carried out. ■

To use the above lemma, it is convenient to consider two cases, depending on the finiteness of the 0-rank of the group in consideration.

(3.5). THEOREM. – *Let G be a CC-group of finite 0-rank satisfying the following*:

(i) *$Soc(G)$ is a quasi socle of G.*

(ii) *There exists $J \leqslant Soc_a(G)$ such that $Soc_a(G)/J$ is locally cyclic and $J_G = 1$.*

Then, if $p \notin \pi(Soc_a(G))$, there exists a simple $\mathbb{F}_p G$-module such that $C_G(A) = 1$.

PROOF. – Put $S = Soc(G)$ and $R = Soc_a(G)$. We may write $S = R \times T$, where T is a direct product of G-invariant finite non-abelian simple groups. There exists a simple $\mathbb{F}_p R$-module B such that $C_R(B) = J$(see [20, corollary 4.12], for example). By lemma (3.4) there exists a simple $\mathbb{F}_p T$-module C such that $C_T(C) = 1$. Form $U = B \otimes_{\mathbb{F}_p} C$ viewed as an $\mathbb{F}_p(R \times T)$-module ([2, corollary B.1.12]). Let E be a composition $\mathbb{F}_p S$-factor of U; then E is a simple $\mathbb{F}_p S$ module. Let $L = C_S(E)$ be. Since U is a semisimple $\mathbb{F}_p T$-module, then $E = \oplus_{x \in X} Cx$, for some $X \subseteq S$. Therefore $L \cap T = C_T(C) = 1$, that is, $L \leqslant C_S(T)$. Since T is a direct product of finite non-abelian simple groups, then $C_S(T) = R$, that is, $L \leqslant R$.

Similarly $E = \oplus_{y \in Y} By$, for some $Y \subseteq S$ and it follows that $L = C_R(B) = J$. Let $V = E \otimes_{\mathbb{F}_p S} \mathbb{F}_p G$ be and let A be a composition $\mathbb{F}_p G$-factor of V. Again we have $A = \times_{z \in Z} Ez$, for some $Z \subseteq G$ and clearly A is a simple $\mathbb{F}_p G$-module. If $g \in G$, then $C_S(A) \leqslant C_S(Eg) \leqslant g^{-1} C_S(A)\, g$ and hence $C_S(A) \leqslant J_G = 1$. By (i), $C_S(A) = 1$ gives $C_G(A) = 1$, completing the proof. ■

(3.6). THEOREM. – *Let G be a CC-group of infinite 0-rank and suppose that $Soc_a(G)$ contains a subgroup J such that $Soc_a(G)/J$ is locally cyclic and $J_G = 1$. Then*

(i) *There exists a simple $\mathbb{Z}H$-module A, the additive group of which is torsion-free such that $C_G(A) = 1$.*

(ii) *If $p \notin \pi(Soc_a(G))$, there exists a simple $\mathbb{F}_p G$-module A such that $C_G(A) = 1$.*

PROOF. – We will follow the arguments given in the proof of theorem (3.5).

Let Q be a quasi socle of G and write $R = Soc_a(G)$. In this case $Q = R \times T \times U$, where T is as in the above result and U is a direct product of G-invariant infinite cyclic subgroups. As U is torsion-free, $[U, H] = 1$ and so $U \leqslant \zeta(G)$. Let Z be the hypercenter of G. Then we may decompose $R = R_1 \times R_2$, where $R_1 = R \cap Z$ and R_2 is an G-invariant subgroup such that $R_2 \cap Z = 1$. On the other hand, by [5, theorem 3.2], G/Z is a periodic group and so $r_0(G) = r_0(Z)$. By Maltsev's theorem (see [17, theorem 6.36]), Z contains a maximal abelian subgroup V of Z with $r_0(V)$ infinite. Since $R_1 U$ is a central subgroup of Z, it follows that $R_1 U \leqslant V$. Since J is core-free, $J \cap Z = 1$. Let W be a periodic subgroup of V maximal under $R_1 \cap W = 1$. Then the torsion subgroup of V/W is a locally cyclic group and $\pi(R_1) = \pi(V/W)$. Moreover, $[Z, TR_2] = 1$, and in particular $V_1 = V \times R_2$ is abelian. A verbatim repetition of the arguments given in [20, proposition 4.13] allows us to construct a simple $\mathbb{Z}V_1$-module B whose underlying additive group is torsion-free (divisible) group and satisfies $C_{V_1}(B) = W \times J$. By lemma (3.4), there is a simple $\mathbb{Q}T$-module C with $C_T(C) = 1$. Put $B_1 = B \otimes_{\mathbb{Q}} C$ and think of U as a $\mathbb{Q}(V_1 \times T)$-module. If $T_1 = V_1 \times T$, then $Q \leqslant T_1$, and a composition $\mathbb{Q}T_1$-factor of B_1 is a simple $\mathbb{Q}T_1$-module and we may write $B_1 = \times_{y \in Y} By$ and $E = \times_{y \in Y_1} By$, for some $Y_1 \subseteq Y \subseteq R$. It follows that E is a simple $\mathbb{Q}T_1$-module and $C_{S_1}(E) = W \times J$.

If $\overline{E} = E \times_{\mathbb{Z}T_1} \mathbb{Z}G$, take A to be a composition $\mathbb{Z}G$-factor of $\overline{E}$. As usual, $\overline{E} = \times_{x \in X} Ex$, $A = \times_{x \in X_1} Ex$ and $X_1 \subseteq X \subseteq G$. Let $L = C_G(A)$ be and suppose that $L \neq 1$. Thus $L \cap Q = C_Q(A) \neq 1$ and $L \cap Q \leqslant W \times J$. Thus $L \cap Q \leqslant (W \times J)_G = \overline{W}$. We have that $\overline{W} \cap Z \leqslant (W \times J) \cap Z = J(J \times Z) = W$. Similarly, $\overline{W} \cap Z \leqslant W_G$. If $W_G \neq 1$, then $W_G \cap \zeta(G) \neq 1$, contradicting the election of W. Therefore $W_G = 1$ and so $\overline{W} \cap Z = 1$. If $\overline{W} \neq 1$, then $\overline{W}$ contains a finite minimal normal subgroup M of G. Since $\overline{W} \cap Z = 1$, then $C_G(M) \neq G$. On the other hand, $\overline{W} \cap R_2 \leqslant (W \times J) \cap R_2 = W$, which implies $\overline{W} \cap R_2 = 1$. Since $M \leqslant R_1$, the quotient MR_2/JR_2 embeds in ZR_2/R_2 and hence, if $g \in G$ and $x \in M$, $[g, x] \in R_2$. It follows that $[g, x] \in M \cap R_2 = 1$, which gives that $C_G(M) = G$, a contradiction. Thus $\overline{W} = 1$, $L \cap Q = 1$ and so $L = 1$.

The proof of (ii) is similar and we omit. ■

4. – The structure of monolithic just non-*CC*-groups.

(4.1). LEMMA. – *Let G be a just non-CC-group with $A = Fitt(G) \neq 1$ and $FC(G) = 1$. Then $A = C_G(A)$.*

PROOF. – Since $C = C_G(A)$ centralizes A, the factor-group C/A has no non-trivial abelian normal subgroups. If $x \in C$, put $X/A = \langle x \rangle^G A/A$. By definition, X/A is Chernikov-by-cyclic ([16]) and so the above yields that X/A is finite. By Schur's theorem (see [17, theorem 4.12]) X' is finite and therefore $X' \leqslant FC(G) = 1$. This means that X is abelian, so $X \leqslant A$. Hence $x \in A$. ■

(4.2). THEOREM. – *Let G be a monolithic just non-CC-group with $A = Fitt(G) \neq 1$ and $FC(G) = 1$. Then*

(i) *Either A is an elementary abelian p-group, for a prime p, or A is a torsion-free divisible abelian group.*

(ii) *A is the unique minimal normal subgroup of G.*

(iii) $A = C_G(A)$.

(iv) *If $H = G/A$ and $S = Soc_a(H)$, then there exists a $J \leqslant S$ such that S/J is locally cyclic and $J_G = 1$.*

(v) *If A is an elementary abelian p-group, then S is a p'-group.*

(vi) *If $Fitt(H) \neq 1$, then G splits conjugately on A.*

PROOF. – Since $FC(G) = 1$, A is abelian and either A is elementary or A is torsion-free ([10]). By corollary (2.4), A is the monolith of G and so, if A is torsion-free, then A becomes divisible. Thus, we have (i) and (ii). Lemma (4.1) shows (iii) and theorem (3.3) gives (iv) and (v). Finally, (vi) is given by the basic result of [18]. ■

(4.3). COROLLARY. – *Let G be a monolithic just non-CC-group with $A = Fitt(G) \neq 1$ and $FC(G) = 1$. Suppose that $H = G/A$ has finite 0-rank. Then*

(i) *A is an elementary abelian p-group, p a prime. Thus G has finite 0-rank.*

(ii) *A is the unique minimal normal subgroup of G and $A = C_G(A)$.*

(iii) *$S = Soc_a(H)$ is a p'-group and there exists a $J \leqslant S$ such that S/J is locally cyclic and $J_G = 1$.*

(iv) *If $Fitt(H) \neq 1$, then G splits conjugately on A.*

REFERENCES

[1] R. BAER, *Irreducible groups of automorphisms of abelian groups*, Pacific J. Math., **14** (1994), 385-406.

[2] K. DOERK - T. HAWKES, *Finite soluble groups*, de Gruyter, Berlin, 1992.

[3] S. FRANCIOSI - F. DE GIOVANNI, *Soluble groups with many Chernikov quotients*, Atti Ac. Naz. Lincei (8), **79** (1985), 12-24.

[4] S. FRANCIOSI - F. DE GIOVANNI - L. A. KURDACHENKO, *Groups whose proper quotients are FC-groups*, J. Algebra, **186** (1996), 544-577.

[5] S. FRANCIOSI - F. DE GIOVANNI - M. J. TOMKINSON, *Groups with Chernikov conjugacy classes*, J. Austral. Math. Soc. Ser A, **50** (1991), 1-14.

[6] J. R. J. GROVES, *Soluble groups with every proper quotient polycyclic*, Illinois J. Math., **22** (1978), 90-95.

[7] P. HALL, *On the finiteness of certain soluble groups*, Proc. London Math. Soc., **9** (1959), 595-622.

[8] B. HARTLEY - D. MCDOUGALL, *Injective modules and soluble groups satisfying the minimal condition for normal subgroups*, Bull. Austral. Math. Soc., **4** (1971), 113-135.

[9] L. A. KURDACHENKO, *Some classes of groups with the weak minimal and maximal conditions*, Ukrainian Math. j., **42** (1990), 937-942.

[10] L. A. KURDACHENKO - J. OTAL, *Groups whose proper factors have Chernikov conjugacy classes*, to appear.

[11] T. LANDOLFI, *On generalized central series of groups*, Ricerche Mat., **44** (1995), 337-347.

[12] D. MCCARTHY, *Infinite groups whose proper quotients are finite*, Comm. Pure Applied Math., **21** (1968), 545-562.

[13] D. MCCARTHY, *Infinite groups whose proper quotients are finite*, Comm. Pure Applied Math., **23** (1970), 767-789.

[14] M. F. NEWMAN, *On a class of metabelian groups*, Proc. London. Math. Soc., **10** (1960), 354-364.

[15] M. F. NEWMAN, *On a class of nilpotent groups*, Proc. London. Math. Soc., **10** (1960), 365-375.

[16] YA. D. POLOVITSKY, *Groups with extremal classes of conjugate elements*, Siberian Math. J., **5** (1964), 891-895.

[17] D. J. S. ROBINSON, *Finiteness conditions and generalized soluble groups*, Ergebnisse der mathematik bands 62-63, Springer-Verlag, Berlin, 1972.

[18] D. J. S. ROBINSON, *The vanishing of certain homology and cohomology groups*, J. Pure and Applied Algebra, **2** (1976), 145-167.

[19] D. J. S. ROBINSON - J. S. WILSON, *Soluble groups with many polycyclic quotients*, Proc. London Math. Soc. (3), **48** (1984), 193-229.

[20] D. J. S. ROBINSON - Z. ZHANG, *Groups whose proper quotients have finite derived group*, J. Algebra, **118** (1988), 346-368.

[21] J. S. WILSON, *Some properties of groups inherited by normal subgroups of finite index*, Math. Z., **114** (1970), 19-21.

[22] J. S. WILSON, *Groups with every proper quotient finite*, Proc. Cambridge Phil. Soc., **69** (1971), 373-391.

[23] D. I. ZAITSEV, *On the existence of direct complements in groups with operators*, Investigations in group theory, Math. Inst. Kiev (1976), 26-44.

[24] D. I. ZAITSEV, *Products of abelian groups*, Algebra and Logic, **19** (1980), 94-106.

Leonid A. Kurdachenko: Department of Algebra, University of Dnepropetrovsk
Prov. Naukovyi, 13. Dnepropetrovsk 50 (UKRAINE 320625)
E-mail address: mmf@ff.dsa.dp.ua

Javier Otal: Departamento de Matemáticas, Facultad de Ciencias
Universidad de Zaragoza. 50009 Zaragoza (SPAIN).
E-mail address: otal@posta.unizar.es

Pervenuta in Redazione
il 7 giugno 1999