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On Lower Semicontinuity in the Calculus of Variations.

GIOVANNI LEONI (*)

Sunto. – *Vengono studiate proprietà di semicontinuitá per integrali multipli*

$$u \in W^{k,1}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(x, u(x), \ldots, \nabla^k u(x))\, dx$$

quando f soddisfa a condizioni di semicontinuità nelle variabili $(x, u, \ldots, \nabla^{k-1} u(x))$ e può non essere soggetta a ipotesi di coercitività, e le successioni ammissibili in $W^{k,1}(\Omega; \mathbb{R}^d)$ convergono fortemente in $W^{k-1,1}(\Omega; \mathbb{R}^d)$.

1. – Introduction.

In this paper we address lower semicontinuity properties for multiple functionals of the form

$$u \in W^{k,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(x, u(x), \ldots, \nabla^k u(x))\, dx\,, \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, with $N \geqslant 1$, and $k, d \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$. Our treatment is mainly expository in intent, based on the references [21], [46], [47], [48].

It is well known that k-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of the functional (1.1) with respect to weak convergence (resp. weak * convergence if $p = \infty$) in $W^{k,p}(\Omega; \mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand f. Indeed this was shown by Morrey [69] when $k = 1$ and later extended by Meyers [68] to the case $k > 1$. We recall that a function $f : E_k^d \to \mathbb{R}$ is said to be *k-qua-*

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siconvex if

$$f(\xi) \leqslant \int_Q f(\xi + \nabla^k w(y))\, dy$$

for all $\xi \in E_k^d$ and all $w \in C_0^\infty(Q; \mathbb{R}^d)$, where $E_1^d := \mathbb{R}^{d \times N}$, while, for $k > 1$, E_k^d stands for the space of symmetric k-linear tensors from $\mathbb{R}^N$ into $\mathbb{R}^d$.

Here we will concentrate mainly on the case $p = 1$, which is more delicate than the case $p > 1$ since, due to lack of reflexivity of the space $W^{k,1}(\Omega; \mathbb{R}^d)$, one can only conclude that an energy bounded sequence $\{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d)$ with

$$\sup_n \|u_n\|_{W^{k,1}} < \infty \tag{1.2}$$

admits a subsequence (not relabelled) such that

$$u_n \to u \quad \text{in } W^{k-1,1}(\Omega; \mathbb{R}^d), \tag{1.3}$$

where $u \in W^{k-1,1}(\Omega; \mathbb{R}^d)$ and $\nabla^{k-1} u$ is a vector-valued function of bounded variation.

In this paper we seek to establish lower semicontinuity in the space $W^{k,1}(\Omega; \mathbb{R}^d)$ under the natural notion of convergence (1.3), and without assuming in general the strong condition (1.2). The main tool in the proofs of the results presented below is the *blow-up method* introduced by Fonseca and Müller [52], [53], which reduces the problem of lower semicontinuity for the functional (1.1) to showing that the inequality

$$\liminf_{h \to \infty} \int_Q f(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k w_h(y), \ldots, \nabla^k w_h(y))\, dy \geqslant f(x_0, u(x_0), \ldots, \nabla^k u(x_0)) \tag{1.4}$$

holds for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$, where Q is the unit cube of $\mathbb{R}^N$, $\varepsilon_h \searrow 0$,

$$T_k(x) := \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,$$

and $\{w_h\} \subset W^{k,1}(Q; \mathbb{R}^d)$ converges strongly in $W^{k-1,1}(Q; \mathbb{R}^d)$ to the function

$$w_0(y) := \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)\, y^\alpha.$$

To prove (1.4) the usual strategy is to localize lower order terms, that is to show that, by truncating the sequence $\{w_h\}$ and rendering $\{w_h(y)\}$,

$\{\nabla w_h(y)\}, \ldots, \{\nabla^{k-1} w_h(y)\}$ uniformly bounded,

$$\liminf_{h\to\infty} \int_Q f(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k w_h(y), \ldots, \nabla^k w_h(y))\, dy \geq \liminf_{h\to\infty} \int_Q f(x_0, u(x_0), \ldots, \nabla^{k-1} u(x_0), \nabla^k w_h(y))\, dy, \tag{1.5}$$

and then to further modify the sequence $\{w_h\}$ in order to match the Dirichlet boundary condition in the definition of k-quasiconvexity. The difficulty in this problem is in the truncation, being the matching of boundary conditions an easy procedure.

A standing open problem is to find necessary and sufficient conditions for (1.5) and, therefore, (1.4) to hold. In this paper we present some *simple* sufficient assumptions, which are easy to verify in the applications. We will start with first order gradients, that is with the case $k = 1$, and then move to the more delicate situation of higher order derivatives. For first order derivatives we also need to distinguish between the scalar case $d = 1$ and the vectorial case $d > 1$.

1.1. *The scalar case: non-coercive integrands.*

In the scalar case $d = 1$ and when $k = 1$ the inequality (1.5) takes the simple form

$$\liminf_{h\to\infty} \int_Q f(x_0 + \varepsilon_h y, u(x_0) + \varepsilon_h w_h(y), \nabla w_h(y))\, dy \geq \liminf_{h\to\infty} \int_Q f(x_0, u(x_0), \nabla w_h(y))\, dy. \tag{1.6}$$

Since for real valued functions $u \in W^{1,p}(\Omega; \mathbb{R})$ one may use simple truncations $\tau_L : \mathbb{R} \to \mathbb{R}$ of the form

$$\tau_L(u) := \begin{cases} L & \text{if } u \geq L, \\ u & \text{if } -L < u < L, \\ -L & \text{if } u \leq -L, \end{cases} \tag{1.7}$$

it is clear that (1.6) holds if we assume that $f(\cdot, \cdot, \xi)$ is lower semicontinuous, uniformly with respect to ξ. Indeed we have the following

THEOREM 1 ([46], Theorem 1.1). – *Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is a Borel integrand, $f(x, u, \cdot)$ is convex in $\mathbb{R}^N$, and for all $(x_0, u_0) \in \Omega \times \mathbb{R}$ and*

$\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$f(x_0, u_0, \xi) - f(x, u, \xi) \leq \varepsilon(1 + f(x, u, \xi)) \tag{1.8}$$

for all $(x, u) \in \Omega \times \mathbb{R}$ *with* $|x - x_0| + |u - u_0| \leq \delta$ *and for all* $\xi \in \mathbb{R}^N$. *Let* $u \in BV(\Omega; \mathbb{R})$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{1,1}(\Omega; \mathbb{R})$ *converging to* u *in* $L^1(\Omega; \mathbb{R})$. *Then*

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx .$$

Here ∇u is the Radon-Nikodym derivative of the distributional derivative Du of u, with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$. Theorem 1 improves a classical result of Serrin (cf. [72], Theorem 11(ii)), where the target function u was assumed to be continuous and the condition corresponding to (1.8) is significantly stronger. Conditions of the type (1.8) appeared already in the papers of Fonseca and Müller [52], [53], Dal Maso and Sbordone [33], Fusco and Hutchinson [57]. All these results deal with the vectorial case and require some type of coercivity conditions. See also the papers of Dal Maso ([32], Theorem 3.2) and of Trombetti [76] for related results in the scalar case. As an immediate corollary of Theorem 1 we have the following

COROLLARY 1 ([46], Corollary 1.2). – *Let* $g : \mathbb{R}^N \to [0, \infty)$ *be a convex function, and let* $h : \Omega \times \mathbb{R} \to [0, \infty)$ *be a lower semicontinuous function. If* $u \in BV(\Omega; \mathbb{R})$ *and* $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R})$ *converges to* u *in* $L^1(\Omega; \mathbb{R})$, *then*

$$\int_\Omega h(x, u)\, g(\nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega h(x, u_n)\, g(\nabla u_n)\, dx .$$

This result seems to be new in this generality.

The lower semicontinuity of f in the u variable is not necessary, but in order to drop it, stronger assumptions on the dependence on x seem to be needed.

THEOREM 2 ([46], Theorem 1.5). – *Assume that* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ *is a Borel integrand,* $f(x, u, \cdot)$ *is convex in* $\mathbb{R}^N$, *and for all* $x_0 \in \Omega$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|f(x_0, u, \xi) - f(x, u, \xi)| \leq \varepsilon(1 + f(x, u, \xi)) \tag{1.9}$$

for all $x \in \Omega$ *with* $|x - x_0| \leq \delta$ *and for all* $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$. *Suppose also that* $f(x_0, \cdot, 0)$ *is lower semicontinuous and*

$$\limsup_{|\xi| \to 0} \frac{(f(x_0, u, 0) - f(x_0, u, \xi))^+}{|\xi|} \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}) .$$

Let $u \in BV(\Omega; \mathbb{R})$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{1,1}(\Omega; \mathbb{R})$ *con-*

verging to u in $L^1(\Omega; \mathbb{R})$. Then

$$\int_\Omega f(x, u(x), \nabla u(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx\,.$$

Theorem 2 extends a classical result of De Giorgi, Buttazzo and Dal Maso [37] to integrands $f = f(x, u, \xi)$ which depend on x. See also the papers of Ambrosio [6], Dal Maso ([32], Theorem 3.5) and of De Cicco [34] for related results.

As for the lower semicontinuity with respect to x, note that for functionals of the form

$$F(u) := \int_\Omega f(x, \nabla u(x))\, dx\,,$$

and *without a coercivity assumption of the type*

$$\liminf_{|\xi|\to\infty} \frac{f(x, \xi)}{|\xi|} = \infty\,,$$

condition (1.8) is rather sharp. Indeed, when $N = 1$ and Ω is bounded, Fusco [55] proved that the functional

$$\int_\Omega h(x)\, |u'(x)|\, dx\,, \qquad u \in W^{1,1}(\Omega; \mathbb{R})\,,$$

where $h(x)$ is a bounded, nonnegative measurable function, is lower semicontinuous in $L^1(\Omega; \mathbb{R})$ if and only if $h(x)$ has a lower semicontinuous representative. Moreover, in [32] Dal Maso, following a counterexample of Aronszajn, constructed a continuous function $\omega : \Omega \to \mathbb{R}$, where $\Omega = (0, 1) \times (0, 1)$ and $x = (x_1, x_2)$, and a sequence of functions $\{u_n\}$ converging to $u(x) = x_2$ in $L^\infty(\Omega; \mathbb{R})$, such that

$$\int_\Omega |(\sin\omega(x), \cos\omega(x))\cdot\nabla u(x)|\, dx > \liminf_{n\to\infty} \int_\Omega |(\sin\omega(x), \cos\omega(x))\cdot\nabla u_n(x)|\, dx\,.$$

If $f : \Omega \times \mathbb{R}^N \to [0, \infty)$ is a Borel integrand, $f(x, \cdot)$ is convex in $\mathbb{R}^N$, and there exists $C > 0$ such that

$$f(x, \xi) \leq C(1 + |\xi|)$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^N$ then we conjecture that a necessary condition for $F(u)$ to be lower semicontinuous with respect to $L^1(\Omega; \mathbb{R})$ convergence is some form of regularity of $f(\cdot, \xi)$.

To be also sufficient it may have to be uniform in ξ for non coercive functionals, as Dal Maso's example seems to indicate. Together with the coercivity assumption

$$\liminf_{|\xi| \to \infty} f(x, \xi) = \infty$$

lower semicontinuity of f in the x variable becomes a sufficient condition for lower semicontinuity of the functional $F(u)$, see Theorem 5 below.

1.2. *The vectorial case: non-coercive integrands.*

We now turn our attention to the vectorial case, and consider nonnegative integrands

$$f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{dN} \to [0, \infty) \qquad \text{where } d > 1 .$$

The situation is considerably more complicated, even when $f(x, u, \cdot)$ is assumed to be convex rather than quasiconvex, which is the natural assumption when $d > 1$. This is due to the fact that in the truncation corresponding to (1.7), namely

$$\tau_L(u) := \begin{cases} u & \text{if } |u| < L , \\ \dfrac{u}{|u|} L & \text{if } |u| \geqslant L , \end{cases} \tag{1.10}$$

the fact that

$$\nabla(\tau_L \circ u)(x) = 0 \qquad \mathcal{L}^N \text{ a.e. in } \{x \in \Omega : u(x) \neq (\tau_L \circ u)(x)\}$$

is no longer valid. In [42] Eisen constructed an integrand of the form

$$f = f(u, \xi) = h(u)\, g(\xi) , \tag{1.11}$$

where h is a nonnegative continuous function and g is nonnegative and convex, for which the corresponding functional ceases to be lower semicontinuos with respect to convergence in L^1 (for later purposes it is important to notice that the function *does not* satisfy the property that $g \to \infty$ as $|\xi| \to \infty$). Thus we cannot hope to fully extend either Theorem 1 or Theorem 2 to the vectorial case. However, we can prove the following:

THEOREM 3 ([46], Theorem 1.7). – *Let f be a nonnegative Borel integrand. Suppose that for all $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ and $\varepsilon > 0$ there exist $\delta > 0$ and a modulus of continuity ϱ, with $\varrho(s) \leqslant C(1+s)$ for $s > 0$ and for some $C > 0$, such that*

$$f(x_0, u_0, \xi) - f(x, u, \xi) \leqslant \varepsilon(1 + f(x, u, \xi)) + \varrho(|u - u_0|) \tag{1.12}$$

for all $x \in \Omega$ *with* $|x - x_0| \leq \delta$, *and for all* $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{dN}$. *Assume also that either*

(a) $f(x_0, u_0, \cdot)$ *is convex in* $\mathbb{R}^{dN}$ *or*

(b) $f(x_0, u_0, \cdot)$ *is quasiconvex in* $\mathbb{R}^{dN}$ *and*

$$0 \leq f(x_0, u_0, \xi) \leq C(|\xi|^q + 1) \quad \text{for all } \xi \in \mathbb{R}^{dN}, \tag{1.13}$$

where $C > 0$ *and the exponent* $q \geq 1$ *may depend on* (x_0, u_0). *In addition, if* $q > 1$ *then assume that*

$$f(x_0, u_0, \xi) \geq \frac{1}{C}|\xi|^q - C \quad \text{for all } \xi \in \mathbb{R}^{dN}. \tag{1.14}$$

Let $u \in BV(\Omega; \mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{1,1}(\Omega; \mathbb{R}^d)$ *which converges to* u *in* $L^1(\Omega; \mathbb{R}^d)$. *Then*

$$\int_\Omega f(x, u, \nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx .$$

Here we intend by *modulus of continuity* a nonnegative, increasing, continuous function ϱ such that $\varrho(0) = 0$. Even in the scalar case, Theorem 3 improves Theorem 11(i) of Serrin in [72], since condition (1.12) is significantly weaker than the corresponding one in [72]. An important class of integrands which satisfy condition (1.12) is given by

$$f = f(x, \xi) = h(x)\, g(\xi),$$

where h is a nonnegative lower semicontinuous function and g is a nonnegative function which satisfies either condition (a) or (b).

Note that without (1.14) L^1 lower semicontinuity may fail even for the simplest case when $f = f(\xi)$. This has been shown by Malý [63] for

$$f = f(\xi) = |\det \xi|, \qquad d = N,$$

who constructed a sequence in $W^{1,N}$ which converges to $u(x) = x$ weakly in $W^{1,p}$, where $p < N - 1$, and for which lower semicontinuity fail (see also Fonseca and Malý [50]).

Theorem 3 covers the case in which the integrand f is essentially of the type $f = f(x, \xi)$. To cover the general case $f = f(x, u, \xi)$, rather than using (1.10) it is more convenient to adopt a truncation of the type

$$\tau_{L,M}(u) := \begin{cases} u & \text{if } |u| < L, \\ 0 & \text{if } |u| \geq L + M. \end{cases} \tag{1.15}$$

A weak form of coercivity will then be needed to control the derivatives of the sequence $\{u_n\}$ in sets of the form $\{x \in \Omega : L_n \leq |u_n| \leq L_n + M_n\}$.

THEOREM 1.4 ([46], Theorem 1.8). – *Theorem* 3 *still holds if we replace condition* (1.12) *with the following: for all* $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ *either* $f(x_0, u_0, \xi) \equiv 0$ *for all* $\xi \in \mathbb{R}^{dN}$, *or for every* $\varepsilon > 0$ *there exist* $C_1, C_2, \delta > 0$ *such that*

$$f(x_0, u_0, \xi) - f(x, u, \xi) \leq \varepsilon(1 + f(x, u, \xi)), \tag{1.16}$$

$$f(x, u, \xi) \geq C_1 |x| - C_2, \tag{1.17}$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$ *with* $|x - x_0| + |u - u_0| \leq \delta$ *and for all* $\xi \in \mathbb{R}^{dN}$.

Theorem 4 was proven by Fonseca and Müller [52], under somewhat stronger hypotheses, and in the case where assumption (b) of Theorem 3 holds with $q = 1$. The convex case can be thought of as a natural extension of Theorem 11(ii) in [72] of Serrin to the vectorial case. Note that Theorem 4 can be applied to integrands of the type

$$f = f(u, \xi) = h(u)\, g(\xi),$$

where h is a nonnegative lower semicontinuous function and g is nonnegative, convex, and

$$g(\xi) \to \infty \qquad \text{as} \qquad |\xi| \to \infty . \tag{1.18}$$

Thus, under the additional condition (1.18), one can recover lower semicontinuity for the class of integrands treated in Eisen's counterexample. Note that we do not assume any control from below on the function $h(u)$, other than $h(u) \geq 0$.

Most of the proofs are carried out firstly for f which grow at most linearly in the gradient variable ξ. While this approach is standard in the convex setting, due to the well known results which allow to approximate from below convex functions by an increasing sequence of convex functions which grow at most linearly, it was only very recently that Kristensen brought this idea to the vectorial setting, exploiting his approximation result for quasiconvex functions (see [60]; also [65]).

1.3. *Coercive integrands.*

In all the previous Theorems we have seen that when coercivity, or weak forms of it, fails then one needs to assume regularity conditions on $f(\cdot, \cdot, \xi)$ which are *uniform with respect to the gradient variable* ξ. *Theorem* 5 *below shows that the uniformity can be dropped at least for convex and polyconvex integrands, if one strengthen the coercivity condition on* f.

THEOREM 5 ([47], Theorem 1.1). – *Let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, \infty]$ *be a lower semicontinuous function, with* $f(x, u, \cdot)$ *convex in* $\mathbb{R}^{d \times N}$. *Suppose that for all* $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ *either* $f(x_0, u_0, \xi) \equiv 0$ *for all* $\xi \in \mathbb{R}^{d \times N}$, *or there exist*

C, $\delta_0 > 0$, and a continuous function $g : B(x_0, \delta_0) \times B(u_0, \delta_0) \to \mathbb{R}^{d \times N}$ such that

$$f(x, u, g(x, u)) \in L^\infty(B(x_0, \delta_0) \times B(u_0, \delta_0); \mathbb{R}), \tag{1.19}$$

$$f(x, u, \xi) \geq C|\xi| - \frac{1}{C} \tag{1.20}$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$ with $|x - x_0| + |u - u_0| \leq \delta_0$ and for all $\xi \in \mathbb{R}^{d \times N}$. Let $u \in BV(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}(\Omega; \mathbb{R}^d)$ converging to u in $L^1(\Omega; \mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx\,.$$

COROLLARY 2 ([47], Corollary 1.2). – *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, \infty]$ be a lower semicontinuous function, with $f(x, u, \cdot)$ convex in $\mathbb{R}^{d \times N}$. Suppose that*

$$f(x, u, \xi) \to \infty \quad \text{as} \quad |\xi| \to \infty$$

and that $f(x, u, 0) \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}^d; \mathbb{R})$. Let $u \in BV(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}(\Omega; \mathbb{R}^d)$ converging to u in $L^1(\Omega; \mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx\,.$$

Corollary 1.9 extends Theorem 12(i) [72] of Serrin to the vectorial case (see also Theorem 3.2 of Ambrosio [6]). The tecniques used in the proof are rather different to those of the authors just mentioned which only work in the scalar case (see also [42], [70]). It is interesting to observe that without a condition of the type (1.19) Theorem 5 is false in general. This has been recently proved by Černý and Malý in [27].

The method used in Theorem 5 can also be applied to polyconvex integrands. For each matrix $\xi \in \mathbb{R}^{d \times N}$ let $\mathcal{M}(\xi) \in \mathbb{R}^\tau$ be the vector whose components are all the minors of ξ, where

$$\tau = \tau(d, N) := \sum_{k=1}^{\min\{d, N\}} \binom{d}{k} \binom{N}{k}.$$

THEOREM 6 ([47], Theorem 1.4). – *Let $h : \Omega \times \mathbb{R}^d \times \mathbb{R}^\tau \to [0, \infty]$ be a lower semicontinuous function, with $h(x, u, \cdot)$ convex in $\mathbb{R}^\tau$. Suppose that for all $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ either $h(x_0, u_0, v) \equiv 0$ for all $v \in \mathbb{R}^\tau$, or there exist C,*

$\delta_0 > 0$, *and a continuous function* $g : B(x_0, \delta_0) \times B(u_0, \delta_0) \to \mathbb{R}^\tau$ *such that*

$$h(x, u, g(x, u)) \in L^\infty(B(x_0, \delta_0) \times B(u_0, \delta_0); \mathbb{R}), \tag{1.21}$$

$$h(x, u, v) \geq C|v| - \frac{1}{C} \tag{1.22}$$

for all $(x, u) \in \Omega \times \mathbb{R}^d$ *with* $|x - x_0| + |u - u_0| \leq \delta_0$ *and for all* $v \in \mathbb{R}^\tau$. *Let* $u \in BV(\Omega; \mathbb{R}^d)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{1,p}(\Omega; \mathbb{R}^d)$ *which converges to* u *in* $L^1(\Omega; \mathbb{R}^d)$, *where* $p = \min\{d, N\}$. *Then*

$$\int_\Omega h(x, u, \mathcal{M}(\nabla u))\, dx \leq \liminf_{n \to \infty} \int_\Omega h(x, u_n, \mathcal{M}(\nabla u_n))\, dx .$$

Theorem 6 is closely related to recent results of Dal Maso and Sbordone [33], and of Fusco and Hutchinson [57], where condition (1.21) is replaced by a condition of the type (1.16).

COROLLARY 3 ([47], Corollary 1.4). – *Let* $\varphi : \Omega \times \mathbb{R}^N \times \mathbb{R} \to [0, \infty)$ *be a continuous function, with* $\varphi(x, u, \cdot)$ *convex in* $\mathbb{R}$. *Suppose that for all* $(x, u) \in \Omega \times \mathbb{R}^N$

$$\varphi(x, u, s) \to \infty \qquad \text{as } |s| \to \infty . \tag{1.23}$$

Let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, *and let* $\{u_n\}$ *be a sequence of functions in* $W^{1,N}(\Omega; \mathbb{R}^N)$ *bounded in* $W^{1,N-1}(\Omega; \mathbb{R}^N)$ *and converging to* u *in* $L^1(\Omega; \mathbb{R}^N)$. *Then*

$$\int_\Omega \varphi(x, u, \det \nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega \varphi(x, u_n, \det \nabla u_n)\, dx .$$

To the author's knowledge Corollary 3 is new in this generality. Lower semicontinuity for polyconvex and quasiconvex integrands of this type has been studied by several authors in the past years, see in particular the papers [1], [13], [15], [22], [26], [31], [33], [57], [50], [51], [58], [63], [64] for the polyconvex case and [17], [49], [65], [66] for the quasiconvex case. It is rather interesting to observe that when φ depends only on the gradient variable s, rather than on the full set of variables (x, u, s), then condition (1.23) can be dropped. This was shown by Celada and Dal Maso in [26]. No analogous results without a coercivity condition of the type (1.22) are known for the case when $d \neq N$.

1.4. *Higher order derivatives.*

We now turn our attention to functionals of the form

$$u \in W^{k,p}(\Omega; \mathbb{R}^d) \mapsto \int_\Omega f(x, u(x), \ldots, \nabla^k u(x))\, dx , \tag{1.24}$$

where $k \geqslant 2$. While in the previous subsections we have chosen to address essentially only the case $p = 1$, we consider here also the case $p > 1$. Indeed for higher order derivatives much less is known compared to the case $k = 1$. Meyers [68] proved that k-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$u \mapsto \int_\Omega f(x, u(x), \ldots, \nabla^k u(x))\, dx\,,$$

with respect to weak convergence (resp. weak * convergence if $p = \infty$) in $W^{k,p}(\Omega; \mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand f, thus extending to the case $k > 1$ the notion of quasi-convexity introduced by Morrey when $k = 1$. Meyers' theorem uses results of Agmon, Douglis and Nirenberg [3] concerning Poisson kernels for elliptic equations. Fusco [56] later gave a simpler proof using De Giorgi's Slicing Lemma. He also extended the result to Carathéodory integrands when $p = 1$, while the case $p > 1$ has been recently established by Guidorzi and Poggiolini [59] under the Lipschitz condition

$$|f(x, \boldsymbol{v}, \xi) - f(x, \boldsymbol{v}, \xi_1)| \leqslant C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})\,|\xi - \xi_1|$$

(note that this condition is automatically satisfied for $k = 1$ and $k = 2$, see [65] and [59]), and by Braides, Fonseca and the author in [21], who obtained the following general relaxation result in $W^{k,p}(\Omega; \mathbb{R}^d)$ with respect to weak convergence.

THEOREM 7 ([21], Theorem 1.3). – *Let* $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}$, *and suppose that* $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0, \infty)$ *is a Carathéodory function satisfying*

$$0 \leqslant f(x, \boldsymbol{u}, v) \leqslant C(1 + |\boldsymbol{u}|^p + |v|^p), \qquad 1 \leqslant p < \infty\,,$$

for $\mathcal{L}^N$ *a.e.* $x \in \Omega$ *and all* $(\boldsymbol{u}, v) \in E^d_{[k-1]} \times E^d_k$, *where* $C > 0$, *and*

$$f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times E^d_{[k-1]} \times E^n_k; [0, \infty)) \qquad \textit{if } p = \infty\,.$$

Then for every $u \in W^{k,p}(\Omega; \mathbb{R}^d)$ *we have*

$$\inf\left\{\liminf_{n\to\infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n)\, dx : \{u_n\} \subset W^{k,p}(\Omega; \mathbb{R}^d),\right.$$

$$\left. u_n \rightharpoonup u \textit{ in } W^{k,p}(\Omega; \mathbb{R}^d) \ (\overset{*}{\rightharpoonup} \textit{ if } p = \infty)\right\} = \int_\Omega \mathcal{Q}^k f(x, u, \ldots, \nabla^k u)\, dx\,,$$

where, for $\mathcal{L}^N$ *a.e.* $x \in \Omega$ *and all* $(\boldsymbol{u}, v) \in E^d_{[k-1]} \times E^d_k$,

$$\mathcal{M}^k f(x, \boldsymbol{u}, v) := \inf \left\{ \int_Q f(x, \boldsymbol{u}, v + \nabla^k w(y))\, dy : w \in C_c^\infty(Q; \mathbb{R}^d) \right\}.$$

Here

$$\nabla^l u := \left(\frac{\partial^l u}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} \right)_{\alpha_1 + \dots + \alpha_N = l}, \qquad l \geqslant 1 .$$

When $k = 1$ we recover classical relaxation results (see e.g. the work of Acerbi and Fusco [2], Dacorogna [30], Marcellini and Sbordone [67], and the references contained therein).

To the best of our knowledge, when $k > 1$, Theorem 7 gives the first integral representation formula for the relaxed energy when the integrand is non convex and depends on the full set of variables, that is $f = f(x, u, \dots, \nabla^k u)$. This is due to the fact that classical truncation methods for $k = 1$ cannot be extended in a simple way to truncate higher order derivatives. The results of Fonseca and Müller (see the proof of Lemma 2.15 in [54]), where the truncation is only on the highest order derivative $\nabla^k u$, allows us to overcome this difficulty. Note however that this technique relies heavily on p-equi-integrability, and thus cannot work in the case $p = 1$ if one replaces weak convergence in $W^{k,1}(\Omega; \mathbb{R}^d)$ with the natural convergence, i.e. strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^d)$.

In this context a relaxation result has been given by Amar and De Cicco [4], but only when $f = f(\nabla^k u)$, so that truncation is not needed. The general case where f depends also on lower order derivatives has been addressed by Fonseca, Malý, Paroni and the author [48]. A first striking difference with the first order case $k = 1$ is that in the scalar case $d = 1$, that is when $u(x)$ is an $\mathbb{R}$-valued function, the analogous of Theorem 1 and of Corollary 1 are false when $k > 2$. Indeed we can show the following:

THEOREM 8 ([48], Theorem 4). – *Let* $\Omega := (0, 1)^N$, $N \geqslant 3$, *and let* h *be a smooth cut-off function on* $\mathbb{R}$ *with* $0 \leqslant h \leqslant 1$, $h(u) = 1$ *for* $u \leqslant 1$, $h(u) = 0$ *for* $u \geqslant 2$. *There exists a sequence of functions* $\{u_n\}$ *in* $W^{2,1}(\Omega; \mathbb{R})$ *converging to zero in* $W^{1,1}(\Omega; \mathbb{R})$ *such that* $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ *is uniformly bounded and*

$$\liminf_{n \to \infty} \int_\Omega h(u_n)(1 - \Delta u_n)^+ \, dx < \int_\Omega h(0)\, dx .$$

A positive result is given in the case where f depends essentially only on x and on the highest order derivatives, that is $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require to truncate the initial sequence $\{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d)$.

THEOREM 9 ([48], Theorem 1). – *Let* $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0, \infty)$ *be a Borel integrand. Suppose that for all* $(x_0, \boldsymbol{v}_0) \in \Omega \times E^d_{[k-1]}$ *and* $\varepsilon > 0$ *there exist* $\delta_0 > 0$ *and a modulus of continuity* ϱ, *with* $\varrho(s) \leq C_0(1+s)$ *for* $s > 0$ *and for some* $C_0 > 0$, *such that*

$$f(x_0, \boldsymbol{v}_0, \xi) - f(x, \boldsymbol{v}, \xi) \leq \varepsilon(1 + f(x, \boldsymbol{v}, \xi)) + \varrho(|\boldsymbol{v} - \boldsymbol{v}_0|) \tag{1.25}$$

for all $x \in \Omega$ *with* $|x - x_0| \leq \delta_0$, *and for all* $(\boldsymbol{v}, \xi) \in E^d_{[k-1]} \times E^d_k$. *Assume also that one of the following three conditions is satisfied*:

(a) $f(x_0, \boldsymbol{v}_0, \cdot)$ *is k-quasiconvex in* E^d_k *and*

$$\frac{1}{C_1}|\xi| - C_1 \leq f(x_0, \boldsymbol{v}_0, \xi) \leq C_1(1 + |\xi|) \quad \textit{for all } \xi \in E^d_k, \tag{1.26}$$

where $C_1 > 0$;

(b) $f(x_0, \boldsymbol{v}_0, \cdot)$ is 1-*quasiconvex in* E^d_k *and*

$$0 \leq f(x_0, \boldsymbol{v}_0, \xi) \leq C_1(1 + |\xi|) \quad \textit{for all } \xi \in E^d_k, \tag{1.27}$$

where $C_1 > 0$;

(c) $f(x_0, \boldsymbol{v}_0, \cdot)$ *is convex in* E^d_k.

Let $u \in BV^k(\Omega; \mathbb{R}^d)$ *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,1}(\Omega; \mathbb{R}^d)$ *converging to* u *in* $W^{k-1,1}(\Omega; \mathbb{R}^d)$. *Then*

$$\int_\Omega f(x, u, \ldots, \nabla^k u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n)\, dx\,.$$

Here $\nabla^k u$ is the Radon-Nikodym derivative of the distributional derivative $D^k u$ of $\nabla^{k-1} u$, with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, and for any integer $k \geq 2$ we define

$$BV^k(\Omega; \mathbb{R}^d) := \{u \in W^{k-1,1}(\Omega; \mathbb{R}^d) : \nabla^{k-1} u \in BV(\Omega; E^d_{k-1})\}\,.$$

An important class of integrands which satisfy (1.25) of Theorem 9 is given by

$$f = f(x, \xi) := h(x)\, g(\xi)\,,$$

where $h(x)$ is a nonnegative lower semicontinuous function and g is a nonnegative function which satisfies either (a) or (b) or (c). The case where $h(x) \equiv 1$ and g satisfies condition (a) was proved by Amar and De Cicco [4]. Theorem 9 extends Theorem 3 to higher order derivatives. Even in the simple case $f = f(\xi)$ it is not known if Theorem 9(a) still holds without the coercivity condition

$$f(\xi) \geq \frac{1}{C_1}|\xi| - C_1.$$

When the integrand f depends on the full set of variables in an essential way, the situation becomes significantly more complicated since one needs to truncate gradients and higher order derivatives in order to localize lower order terms. However, setting $v := (u, \ldots, \nabla^{k-1} u)$ and applying Theorems 4 and 5 to v and to the integrand $\tilde{f}$ such that $\tilde{f}(v, \nabla v) = f(u, \ldots, \nabla^k u)$, we find easily the two results below, Theorems 10 and 11.

THEOREM 10 ([48], Theorem 2). – *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0, \infty)$ be a Borel integrand, with $f(x, \boldsymbol{v}, \cdot)$ 1-quasiconvex in E^d_k. Suppose that for all $(x_0, \boldsymbol{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0, \boldsymbol{v}_0, \xi) \equiv 0$ for all $\xi \in E^d_k$, or for every $\varepsilon > 0$ there exist $C, \delta_0 > 0$ such that*

$$f(x_0, \boldsymbol{v}_0, \xi) - f(x, \boldsymbol{v}, \xi) \leqslant \varepsilon(1 + f(x, \boldsymbol{v}, \xi)), \tag{1.28}$$

$$C|\xi| - \frac{1}{C} \leqslant f(x_0, \boldsymbol{v}_0, \xi) \leqslant C(1 + |\xi|) \tag{1.29}$$

for all $(x, \boldsymbol{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\boldsymbol{v} - \boldsymbol{v}_0| \leqslant \delta_0$ and for all $\xi \in E^d_k$. Let $u \in BV^k(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \ldots, \nabla^k u)\, dx \leqslant \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n)\, dx .$$

A standing open problem is to decide whether Theorem 10 continues to hold under the weaker assumption that $f(x, \boldsymbol{v}, \cdot)$ is k-quasiconvex, which is the natural assumption in this context.

As in Theorem 9, conditions (1.28) and (1.29) can be considerably weakened if we assume that $f(x, \boldsymbol{v}, \cdot)$ is convex rather than 1-quasiconvex. Indeed we have the following result:

THEOREM 11 ([48], Theorem 3). – *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to\to [0, \infty]$ be a lower semicontinuous function, with $f(x, \boldsymbol{v}, \cdot)$ convex in E^d_k. Suppose that for all $(x_0, \boldsymbol{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0, \boldsymbol{v}_0, \xi) \equiv 0$ for all $\xi \in E^d_k$, or there exist C_1, $\delta_0 > 0$, and a continuous function $g : B(x_0, \delta_0) \times B(\boldsymbol{v}_0, \delta_0) \to E^d_k$ such that*

$$f(x, \boldsymbol{v}, g(x, \boldsymbol{v})) \in L^\infty(B(x_0, \delta_0) \times B(\boldsymbol{v}_0, \delta_0); \mathbb{R}), \tag{1.30}$$

$$f(x, \boldsymbol{v}, \xi) \geqslant C_1 |\xi| - \frac{1}{C_1} \tag{1.31}$$

for all $(x, \boldsymbol{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\boldsymbol{v} - \boldsymbol{v}_0| \leqslant \delta_0$ and for all $\xi \in E^d_k$. Let $u \in BV^k(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \ldots, \nabla^k u)\, dx \leqslant \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n)\, dx .$$

The situation is more complicated if the integrand f is k-polyconvex, since the method used to prove Theorems 10 and 11 cannot be applied in this case. For each $\xi \in E_k^d$ let $\mathcal{M}(\xi) \in \mathbb{R}^\tau$ be the vector whose components are all the minors of ξ.

THEOREM 12 ([48], Theorem 5). – *Let $h : \Omega \times E^d_{[k-1]} \times \mathbb{R}^\tau \to [0, \infty]$ be a lower semicontinuous function, with $h(x, \boldsymbol{v}, \cdot)$ convex in $\mathbb{R}^\tau$. Suppose that for all $(x_0, \boldsymbol{v}_0) \in \Omega \times E^d_{[k-1]}$ either $h(x_0, \boldsymbol{v}_0, v) \equiv 0$ for all $v \in \mathbb{R}^\tau$, or there exist C, $\delta_0 > 0$, and a continuous function $g : B(x_0, \delta_0) \times B(\boldsymbol{v}_0, \delta_0) \to \mathbb{R}^\tau$ such that*

$$h(x, \boldsymbol{v}, g(x, \boldsymbol{v})) \in L^\infty(B(x_0, \delta_0) \times B(\boldsymbol{v}_0, \delta_0); \mathbb{R}), \tag{1.32}$$

$$h(x, \boldsymbol{v}, v) \geq C|v| - \frac{1}{C} \tag{1.33}$$

for all $(x, \boldsymbol{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\boldsymbol{v} - \boldsymbol{v}_0| \leq \delta_0$ and for all $v \in \mathbb{R}^\tau$. Let $u \in BV^k(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,p}(\Omega; \mathbb{R}^d)$ which converges to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, where p is the minimum between N and the dimension of the vectorial space E^d_{k-1}. Then

$$\int_\Omega h(x, u, \ldots, \nabla^{k-1} u, \mathcal{M}(\nabla^k u))\, dx \leq \liminf_{n \to \infty} \int_\Omega h(x, u_n, \ldots, \nabla^{k-1} u_n, \mathcal{M}(\nabla^k u_n))\, dx\,.$$

Theorem 12 is closely related to a result of Ball, Currie and Olver [14], where it was assumed that

$$h(x, \boldsymbol{v}, v) \geq \gamma(|v|) - \frac{1}{C},$$

where

$$\frac{\gamma(s)}{s} \to \infty \qquad \text{as } s \to \infty\,.$$

2. – Open problems.

1. In [72] Serrin proved the following

THEOREM 13 ([72], Theorem 12). – *Assume that $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N; [0, \infty))$, $f(x, u, \cdot)$ is convex in $\mathbb{R}^N$ and f satisfies any one of the following conditions:*

(i) *$f(x, u, \xi) \to \infty$ as $|\xi| \to \infty$ for each $(x, u) \in \Omega \times \mathbb{R}$.*
(ii) *$f(x, u, \cdot)$ is strictly convex in $\mathbb{R}^N$ for each $(x, u) \in \Omega \times \mathbb{R}$.*
(iii) *The derivatives f_x, f_ξ and $f_{\xi x}$ exist and are continuous.*

Then $F(u, \Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R})$ with respect to local convergence in L^1.

Theorem 13 was extended to the vectorial case by Morrey in his book on Calculus of Variations ([69], Thms. 4.1.1, 4.1.2). However, several years later Eisen [42] found a gap in Morrey's proof, thus placing in doubt the validity of Theorem 13 when $d > 1$, and constructed counterexamples for Theorem 13(iii) when $d > 1$ (see also [74] for more details and an extensive bibliography). Corollary 2 above shows that Theorem 13(i) continues to hold when $d > 1$, while, to our knowledge the validity of Theorem 13(ii) when $d > 1$ remains open.

2. Consider a continuous integrand $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{dN} \to [0, \infty)$, $d > 1$, such that $f(x, u, \cdot)$ is quasiconvex in $\mathbb{R}^{dN}$ and

$$\frac{1}{C}|\xi| - C \leq f(x, u, \xi) \leq C(|\xi| + 1) \quad \text{for all } (x, u, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{dN}.$$

Is it possible to find a sequence of nonnegative functions $f_j \nearrow f$, which satisfy the same properties of f, but are also continuous in (x, u) *uniformly with respect to ξ*? If so then the functional corresponding to the integrand f would be lower semicontinuous with respect to L^1 convergence. Note that this approximation result is true for convex functions and for quasiconvex functions with superlinear growth (see the paper of Marcellini [65]).

3. Let $f : E^d_k \to [0, \infty)$ be k-quasiconvex and assume that

$$0 \leq f(\xi) \leq C(|\xi| + 1) \quad \text{for all } \xi \in E^d_k.$$

Let $u \in BV^k(\Omega; \mathbb{R}^d)$ and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$. Does the following inequality

$$\int_\Omega f(\nabla^k u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(\nabla^k u_n)\, dx$$

hold? It certainly holds if one assume that $f(\xi) \geq C_1 |\xi|$ for $|\xi|$ large, but for $k = 1$ this coercivity condition can be avoided using De Giorgi's Slicing Lemma.

4. Assume that

$$f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0, \infty)$$

is continuous and such that $f(x, \boldsymbol{v}, \cdot)$ is k-quasiconvex in E^d_k and

$$\frac{1}{C}|\xi| - C \leq f(x, \boldsymbol{v}, \xi) \leq C(|\xi| + 1) \quad \text{for all } (x, \boldsymbol{v}, \xi) \in \Omega \times E^d_{[k-1]} \times E^d_k.$$

Under some condition of the type (1.16), is the corresponding functional lower semicontinuous with respect to strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^d)$?

REFERENCES

[1] E. ACERBI - G. DAL MASO, *New lower semicontinuity results for polyconvex integrals case*, Cal. Var., **2** (1994), 329-372.

[2] E. ACERBI - N. FUSCO, *Semicontinuity problems in the calculus of variations*, Arch. Rat. Mech. Anal., **86** (1984), 125-145.

[3] S. AGMON - A. DOUGLIS - L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., **12** (1959), 623-727.

[4] M. AMAR - V. DE CICCO, *Relaxation of quasi-convex integrals of arbitrary order*, Proc. Roy. Soc. Edin., **124** (1994), 927-946.

[5] G. ALBERTI - C. MANTEGAZZA, *A note on the theory of SBV functions*, Boll. Un. Mat. Ital. B, **11** (1997), 375-382.

[6] L. AMBROSIO, *New lower semicontinuity results for integral functionals*, Rend. Accad. Naz. Sci. XL, **11** (1987), 1-42.

[7] L. AMBROSIO, *A compactness theorem for a special class of functions of bounded variation*, Boll. Un. Mat. Ital., **3B 7** (1989), 857-881.

[8] L. AMBROSIO, *Existence theory for a new class*, Arch. Rat. Mech. Anal., **111** (1990), 291-322.

[9] L. AMBROSIO, *On the lower semicontinuity of quasi-convex integrals in SBV*, Nonlinear Anal., **23** (1994), 405-425.

[10] L. AMBROSIO - G. DAL MASO, *On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasi-convex integrals*, J. Funct. Anal., **109** (1992), 76-97.

[11] L. AMBROSIO - N. FUSCO - D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, Mathematical Monographs, Oxford University Press, 2000.

[12] L. AMBROSIO - S. MORTOLA - V. M. TORTORELLI, *Functional with linear growth defined on vector-valued BV functions*, J. Math. Pures et Appl., **70** (1991), 269-332.

[13] J. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal., **63** (1977), 337-403.

[14] J. BALL - J. CURRIE - P. OLVER, *Null lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal., **41** (1981), 315-328.

[15] J. M. BALL - F. MURAT, *Remarks on Chacon's biting lemma*, Proc. AMS, **107** (1989), 655-663.

[16] L. BOCCARDO - D. GIACHETTI - J. I. DIAZ - F. MURAT, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms*, J. Diff. Eq., **106** (1993), 215-237.

[17] G. BOUCHITTÉ, I. FONSECA - J. MALÝ, *Relaxation of multiple integrals below the growth exponent*, Proc. Royal Soc. Edin., **128A** (1998), 463-479.

[18] G. BOUCHITTÉ - I. FONSECA - L. MASCARENHAS, *A global method for relaxation*, Arch. Rat. Mech. Anal., **145** (1998), 51-98.

[19] A. BRAIDES, *Approximation of free-discontinuity problems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1998.

[20] A. BRAIDES - A. DEFRANCESCHI, *Homogenization of multiple integrals*, Oxford Lecture Series in Mathematics and its Applications, 12, The Clarendon Press, Oxford University Press, New York, 1998.

[21] A. BRAIDES - I. FONSECA - G. LEONI, *A-quasiconvexity: relaxation and homogenization*, to appear in COCV.

[22] M. Carbone L. - R. De Arcangelis, *Further results on Γ-convergence and lower semicontinuity of integral functionals depending on vector-valued functions*, Richerche Mat., **39** (1990), 99-129.
[23] M. Carriero - A. Leaci - F. Tomarelli, *Special bounded hessian and elastic-plastic plate*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5), **16** (1992), 223-258.
[24] M. Carriero - A. Leaci - F. Tomarelli, *Strong minimizers of Blake & Zisserman functional*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **25** (1997), 257-285.
[25] M. Carriero - A. Leaci - F. Tomarelli, *A second order model in image segmentation: Blake & Zisserman functional*, Progr. Nonlinear Differential Equations Appl., 25, Birkhäuser, **25** (1996), 57-72.
[26] P. Celada - G. Dal Maso, *Further remarks on the lower semicontinuity of polyconvex integrals*, Ann. Inst. Henri Poincaré, Ann. Non Lin., **11** (1994), 661-691.
[27] R. Černý - J. Malý, *Counterexample to lower semicontinuity in Calculus of Variations*, to appear in Math. Z.
[28] R. Choksi - I. Fonseca, *Bulk and interfacial energy densities for structured deformations of continua*, Arch. Rat. Mech. Anal., **138** (1997), 37-103.
[29] B. Dacorogna, *Quasiconvexity and relaxation of nonconvex problems in the calculus of variations*, J. Funct. Anal., **46** (1982), 102-118.
[30] B. Dacorogna, *Direct methods in the calculus of variations*, Springer-Verlag, New York, 1989.
[31] B. Dacorogna - P. Marcellini, *Semicontinuité pour des intégrandes polyconvexes sans continuité des determinants*, C. R. Acad. Sci. Paris Sér. I Math., **311, 6** (1990), 393-396.
[32] G. Dal Maso, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math., **30** (1980), 387-416.
[33] G. Dal Maso - C. Sbordone, *Weak lower semicontinuity of polyconvex integrals: a borderline case*, Math. Z., **218** (1995), 603-609.
[34] V. De Cicco, *A lower semicontinuity result for functionals defined on $BV(\Omega)$*, Ricerche di Mat., **39** (1990), 293-325.
[35] V. De Cicco, *Lower semicontinuity for certain integral functionals on $BV(\Omega)$*, Boll. U.M.I., **5-B** (1991), 291-313.
[36] E. De Giorgi - L. Ambrosio, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), **82** (1988), 199-210.
[37] E. De Giorgi - G. Buttazzo - G. Dal Maso, *On the lower semicontinuity of certain integral functions*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur., Rend., **74** (1983), 274-282.
[38] E. De Giorgi - M. Carriero - A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rat. Mech. Anal., **108** (1989), 195-218.
[39] G. Del Piero - D. R. Owen, *Structured deformations of continua*, Arch. Rat. Mech. Anal., **124** (1993), 99-155.
[40] F. Demengel, *Fonctions á hessien borné*, Ann. Inst. Fourier (Grenoble), **34** (1984), 155-190.
[41] F. Demengel, *Compactness theorems for spaces of functions with bounded derivatives and applications to limit analysis problems in plasticity*, Arch. Rat. Mech. Anal., **105** (1989), 123-161.
[42] G. Eisen, *A counterexample for some lower semicontinuity results*, Math. Z., **162** (1978), 241-243.

[43] I. EKELAND I. - R. TEMAM, *Convex analysis and variational problems*, North-Holland Publishing Company (1976).

[44] L. C. EVANS - R. F. GARIEPY, *Lecture Notes on Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, 1992.

[45] I. FONSECA, *The lower quasiconvex envelope of the stored energy function for an elastic crystal*, J. Math. Pures et Appl., **67** (1988), 175-195.

[46] I. FONSECA - G. LEONI, *On lower semicontinuity and relaxation*, to appear in the Proc. Royal Soc. Edin.

[47] I. FONSECA - G. LEONI, *Some remarks on lower semicontinuity and relaxation*, to appear in Indiana Univ. Math. J.

[48] I. FONSECA - G. LEONI, J. MALÝ - R. PARONI, *A note on Meyers' Theorem in $W^{k,1}$*, to appear.

[49] I. FONSECA - J. MALÝ, *Relaxation of multiple integrals below the growth exponent*, Ann. Inst. H. Poincaré, Analyse non Linéaire, **14** (1997), 308-338.

[50] I. FONSECA - J. MALÝ, *Weak convergence of minors*, to appear.

[51] I. FONSECA - P. MARCELLINI, *Relaxation of multiple integrals in subcritical Sobolev spaces*, J. Geom. Anal., **7** (1997), 57-81.

[52] I. FONSECA - S. MÜLLER, *Quasi-convex integrands and lower semicontinuity in L^1*, SIAM J. Math. Anal., **23** (1992), 1081-1098.

[53] I. FONSECA - S. MÜLLER, *Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$*, Arch. Rat. Mech. Anal., **123** (1993), 1-49.

[54] I. FONSECA I. - S. MÜLLER, *A-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal., **30** (1999), 1355-1390.

[55] N. FUSCO, *Dualità e semicontinuità per integrali del tipo dell'area*, Rend. Accad. Sci. Fis. Mat., IV. Ser., **46** (1979), 81-90.

[56] N. FUSCO, *Quasiconvessità e semicontinuità per integrali multipli di ordine superiore*, Ricerche Mat., **29** (1980), 307-323.

[57] N. FUSCO - J. E. HUTCHINSON, *A direct proof for lower semicontinuity of polyconvex functionals*, Manuscripta Math., **85** (1995), 35-50.

[58] W. GANGBO, *On the weak lower semicontinuity of energies with polyconvex integrands*, J. Math. Pures et Appl., **73** (1994), 455-469.

[59] M. GUIDORZI - L. POGGIOLINI, *Lower semicontinuity for quasiconvex integrals of higher order*, NoDEA, **6** (1999), 227-246.

[60] J. KRISTENSEN, *Lower semicontinuity in spaces of weakly differentiable functions*, Math. Ann., **313** (1999), 653-710.

[61] C. LARSEN, *Quasiconvexification in W^p and optimal jump microstructure in BV relaxation*, SIAM J. Math. Anal., **29** (1998), 823-848.

[62] F. C. LIU, *A Luzin type property of Sobolev functions*, Indiana Univ. Math. J., **26** (1977), 645-651.

[63] J. MALÝ, *Weak lower semicontinuity of polyconvex integrals*, Proc. Royal Soc. Edin., **123A** (1993), 681-691.

[64] J. MALÝ, *Lower semicontinuity of quasiconvex integrals*, Manuscripta Math., **85** (1994), 419-428.

[65] P. MARCELLINI, *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals quasiconvex integrals*, Manuscripta Math., **51** (1985), 1-28.

[66] P. MARCELLINI, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré, Analyse non Linéaire, **3** (1986), 391-409.

[67] P. MARCELLINI - C. SBORDONE, *Semicontinuity problems in the calculus of variations*, Nonlinear Analysis, **4** (1980), 241-257.
[68] N. MEYERS, *Quasi-convexity - lower semi-continuity of multiple variational integrals of any order*, Trans. Amer. Math. Soc., **119** (1965), 125-149.
[69] C. B. MORREY, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math., **2** (1952), 25-53.
[70] C. B. MORREY, *Multiple integrals in the Calculus of Variations*, Springer, Berlin, 1966.
[71] D. R. OWEN - R. PARONI, *Second-order structured deformations*, accepted by Arch. Rat. Mech. Anal.
[72] J. SERRIN, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc., **161** (1961), 139-167.
[73] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.
[74] M. STRUWE, *Variational methods*, Springer, Berlin, 2000.
[75] T. TEMAM, *Problémes mathématiques en plasticité*, Gauthier-Villars, Paris, 1983.
[76] C. TROMBETTI, *On lower semicontinuity and relaxation properties of certain classes of variational integrals*, Rend. Accad. Naz. Sci. XL., **21** (1997), 25-51.
[77] W. P. ZIEMER, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Springer-Verlag, New York, 1989.

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