Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **4-B** (2001), n.1, p. 239–267.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_1_239_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 239-267

On Decompositions in Generalised Lorentz-Zygmund Spaces (*).

J. S. NEVES

Sunto. – *Il lavoro presenta diverse caratterizzazioni degli spazi Lorentz-Zygmund generalizzati (GLZ) $L_{p,q;\boldsymbol{\alpha}}(R)$, con $p, q \in (0, +\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$ e (R, μ) spazio misurato con misura $\mu(R)$ finita. Dato uno spazio misurato (R, μ) e $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, otteniamo representazioni equivalenti per la (quasi-) norma dello spazio GLZ $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. Inoltre, se (R, μ) è uno spazio misurato con misura finita e $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, viene presentata in termini di decomposizioni una norma equivalente per lo spazio $L_{1,1;\boldsymbol{\alpha}}(R)$. Si dimostra che le norme equivalenti considerate per $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, con (R, μ) uno spazio a misura finita, e la norma di decomposizione in $L_{1,1;\boldsymbol{\alpha}}(R)$ possono essere utilizzate per ottenere semplici dimostrazioni di alcuni risultati di estrapolazione concernenti questi spazi.*

1. – Introduction.

In [7], Edmunds and Krbec obtained some decompositions for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, usually denoted by $E_\alpha(\Omega)$, with Young function Φ_1 given by $\Phi_1(t) = \exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite n-dimensional Lebesgue measure $|\Omega|_n$. Without loss of generality, it was assumed that $|\Omega|_n = 1$. They showed that considering a suitable decomposition of $(0, 1)$ into a union of disjoint intervals $\{(t_k, t_{k-1})\}_{k \in \mathbb{N}}$ it is enough to control only the blow up of the norms $\|f^*\|_{L_k(t_k, t_{k-1})}$, where f^* is the non-increasing rearrangement of f, by the same power $k^{-1/\alpha}$ to have $L_{\Phi_1}(\Omega)$. The proof was based on the fact that $L_{\Phi_1}(\Omega)$ coincides with the Zygmund space $L^\infty(\log L)^{-1/\alpha}(\Omega)$ (see [2, Theorem D] or [3, Lemma IV.6.2]). In Section 3, we extend this result to the generalised Lorentz-Zygmund (GLZ) spaces $L_{p,q;\boldsymbol{\alpha}}(R)$, with $p, q \in (0, +\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, and (R, μ) a finite measure space, cf. Theorem 3.2. The method of the proof is different from, and in our opinion easier than, that used in [7].

In [19], Triebel gave an equivalent norm for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with finite volume; see also [6]. With this equivalent norm, he proved that the embeddings $id: B^{n/p}_{p,p}(\Omega) \to E_\alpha(\Omega)$ and $id: H^{n/p}_p(\Omega) \to E_\alpha(\Omega)$, with $1 < p < +\infty$, $0 < \alpha < p'$ and Ω a bounded C^∞-domain in $\mathbb{R}^n$, are compact and obtained estimates for the appro-

(*) 1991 Mathematics subject Classification: 46 E 30.

ximation and entropy numbers of those embeddings. Let us just mention that $B^{n/p}_{p,p}(\Omega)$ and $H^{n/p}_p(\Omega)$ are classical Besov spaces and fractional Sobolev spaces, respectively. We refer to [19] for more details. Equivalent norms for the double exponential Orlicz space $L_{\Phi_2}(\Omega)$, usually denoted by $EE_\alpha(\Omega)$, with Young function Φ_2 given by $\Phi_2(t) = \exp\exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, were obtained by Edmunds, Gurka and Opic in [6]. The proof was also based on the fact that $L_{\Phi_2}(\Omega)$ coincides with the GLZ space $L_{\infty,\infty;0,-1/\alpha}(\Omega)$, see [4, Lemma 3.9]. Following the same technique as in [6], we obtain in Section 4 equivalent representations for the (quasi-) norms of the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with (R,μ) a measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, *i.e.* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \ldots, \alpha_{m-1} \leqslant 0$ and $\alpha_m < 0$, cf. Theorem 4.1 and its Corollaries. In particular, when (R,μ) has finite measure we obtain equivalent norms for the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, extending in this way the results in [19] and [6]. Still in Section 4, we give an equivalent norm for the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with (R,μ) a non-atomic finite measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, *i.e.* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \ldots, \alpha_{m-1} \geqslant 0$ and $\alpha_m > 0$, in terms of decompositions. This result extends a result obtained by Edmunds and Triebel, cf. [8, Theorem 2, p. 72], for the spaces $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$ and Ω a measurable subset of $\mathbb{R}^n$ with finite volume. We refer to [9, Theorem 3.4] for a different proof of this result.

In Section 5, we show how the equivalent norms obtained in Section 4 for $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, and the decomposition norm in $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, can be employed to get simple proofs of some extrapolation results involving these spaces. Let us remark that we do not follow a general setting in terms of abstract extrapolation methods considered by Jawerth and Milman, cf. [11] (see also [14]). We mention that the starting point of the extrapolation theory was the Theorem of Yano [20] which can be described as follows. Suppose that T is a bounded linear operator on $L_p(0,1)$ for $p > 1$ with $\|T\|_{L_p \to L_p} = \mathcal{O}((p-1)^{-\alpha})$ as $p \downarrow 1$, for some $\alpha > 0$; then these estimates can be extrapolated to $L^1(\log L)^\alpha(0,1) \to L_1(0,1)$; see [22, Theorem XII.4.11 (ii), p. 119] for a more general formulation. We refer to [17, Theorem IV.5.3, p. 92] where T was supposed to be sublinear. We also refer to [9, Theorem 4.2] where T was supposed to be subadditive. In [16, p. 23] and [8, p. 74] the case was considered when T is the Hardy-Littlewood maximal operator. It should be emphasised that the decomposition approach, used in [8] and [9], skips completely the machinery of weak type inequalities and the Marcinkiewicz interpolation Theorem, since it follows at once from the expression of the norm in $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$. There is also a dual statement for operators acting from $L_p(R_0)$ into $L_p(R_1)$, with (R_0, μ_0) and (R_1, μ_1) finite measure spaces, for p close to $+\infty$, such that $\|T\|_{L_p \to L_p} = \mathcal{O}(p^{1/\alpha})$ as $p \to +\infty$, for some $\alpha > 0$; then there exist positive constants λ, K such that $\int_{R_1} \exp(\lambda |Tf|^\alpha)\, d\mu_1 \leqslant K$ for each f with

$|f| \leqslant 1$; see [22, Theorem XII.4.11 (i), p. 119]. There is also a version of this result for sublinear operators. We refer to Section 5 for more details.

2. – Notation and preliminaries.

As usual, $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Let (R, Σ, μ), usually denoted by (R, μ), be a totally σ-finite measure space and referred in the sequel only as a measure space. A set $E \in \Sigma$ is called an atom of (R, Σ, μ) if $\mu(E) > 0$ and $F \subset E$, $F \in \Sigma$ implies either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. If there are no atoms, then (R, Σ, μ) is called non-atomic. A measure space (R, μ) is called resonant if it is one of the following two types: (i) non-atomic; (ii) completely atomic, with all atoms having equal measure. We refer to [3, pp. 45-51] for more details and for a different, but equivalent, definition. When $R = \mathbb{R}^n$ we shall always take μ to be Lebesgue measure μ_n, and shall write $|\Omega|_n = \mu_n(\Omega)$ for any measurable subset Ω of $\mathbb{R}^n$. The family of all extended scalar-valued (real or complex) μ-measurable functions on R will be denoted by $\mathcal{M}(R, \mu)$; $\mathcal{M}_0(R, \mu)$ will stand for the subset of $\mathcal{M}(R, \mu)$ consisting of all those functions which are finite μ-a.e. and $\mathcal{M}^+(R, \mu)$ ($\mathcal{M}_0^+(R, \mu)$) will represent the subset of $\mathcal{M}(R, \mu)$ ($\mathcal{M}_0(R, \mu)$) made up of all those functions which are non-negative μ-a.e.

DEFINITION 2.1. – *Let $f \in \mathcal{M}_0(R, \mu)$. The distribution function μ_f of f is defined by*

$$\mu_f(\lambda) = \mu\{x \in R: |f(x)| > \lambda\}, \qquad \text{for all } \lambda \geqslant 0, \tag{1}$$

and the non-increasing rearrangement of f is the function f^* defined on $[0, +\infty)$ by

$$f^*(t) = \inf\{\lambda \geqslant 0: \mu_f(\lambda) \leqslant t\}, \qquad \text{for all } t \geqslant 0. \tag{2}$$

The non-increasing rearrangement of the characteristic function $f = \chi_E$, where E is a μ-measurable subset of R with finite measure $\mu(E)$, is $f^* = \chi_{[0, \mu(E))}$.

If (R, μ) is a finite measure space, then the distribution function μ_f is bounded by $\mu(R)$ and so $f^*(t) = 0$ for all $t \geqslant \mu(R)$. In this case we may regard f^* as a function defined on the interval $[0, \mu(R))$; for more details we refer to [3].

DEFINITION 2.2. – *Two functions $f \in \mathcal{M}_0(R, \mu)$ and $g \in \mathcal{M}_0(S, \nu)$ are said to be equimeasurable if they have the same distribution function, i.e.*, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geqslant 0$.

Let $p \in (0, +\infty]$. We denote by $L_p(R)$ the Lebesgue space endowed with the (quasi-) norm $\|.\|_{p;R}$. An alternative description of $\|.\|_{p;R}$ is given by the next result, cf. Proposition II.1.8 in [3] or Theorem 1.8.5 in [21].

PROPOSITION 2.1. – *Let $f \in L_p(R)$. If $0 < p < +\infty$, then*

$$\|f\|_{p;R}^p = \int_R |f|^p \, d\mu = \int_0^{+\infty} (f^*(t))^p \, dt = \|f^*\|_{p;(0,+\infty)}^p .$$

Furthermore, in the case $p = +\infty$,

$$\|f\|_{\infty;R} = \operatorname{ess\,sup}_{x \in R} |f(x)| = f^*(0).$$

Now let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by ϑ_α^m and ω_α^m the real functions defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t), \qquad \text{for all } t \in (0, +\infty), \tag{3}$$

and

$$\omega_\alpha^m(t) = \prod_{i=1}^m l_{i-1}^{\alpha_i}(t), \qquad \text{for all } t \in [1, +\infty), \tag{4}$$

where $l_0, l_1, \ldots, l_m$ are non-negative functions defined on $(0, +\infty)$ by

$$l_0(t) = t, \quad l_1(t) = 1 + |\log t|, \quad l_i(t) = 1 + \log l_{i-1}(t), \quad i \in \{2, \ldots, m\}. \tag{5}$$

DEFINITION 2.3. (cf. [5]) – *Let $p, q \in (0, +\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. The generalised Lorentz-Zygmund (GLZ) space $L_{p,q;\boldsymbol{\alpha}}(R)$ is defined to be the set of all functions $f \in \mathcal{M}_0(R, \mu)$ such that*

$$\|f\|_{p,q;\boldsymbol{\alpha};R} := \|t^{1/p - 1/q} \vartheta_\alpha^m(t) \, f^*(t)\|_{q,(0,+\infty)} \tag{6}$$

is finite. Here $\|.\|_{q,(0,+\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0, +\infty)$.

We remark that in [5], the space $L_{p,q;\boldsymbol{\alpha}}(R)$ and the quasi-norm $\|.\|_{p,q;\boldsymbol{\alpha};R}$ defined above are denoted by $L_{p,q;\alpha_1,\ldots,\alpha_m}(R)$ and $\|.\|_{p,q;\alpha_1,\ldots,\alpha_m;R}$, respectively. We use the notation in [5] only when we are considering particular cases.

Let us observe that when we consider $\boldsymbol{\alpha} = (0, \ldots, 0)$ in the previous Definition, we get the Lorentz space $L_{p,q}(R)$ endowed with the (quasi-) norm $\|.\|_{p,q;R}$, which is just the Lebesgue space $L_p(R)$ endowed with the (quasi-) norm $\|.\|_{p;R}$ when $p = q$; if $p = q$, $m = 1$ and $(R, \mu) = (\Omega, \mu_n)$, we get the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-) norm $\|.\|_{p;\alpha_1;\Omega}$.

Let us introduce some more notation, that will be needed in Section 4. Let $m \in \mathbb{N}$ with $m \geq 2$. We define the numbers $exp_0, \ldots, exp_m$ by

$$exp_0 = 1, \qquad exp_i = e^{exp_{i-1}}, \qquad i \in \{1, \ldots, m\}.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by γ_α^m the non-negative function defined by

$$\gamma_\alpha^m(t) = \prod_{i=1}^{m} \ell_{i-1}^{\alpha_i}(t), \qquad \text{for all } t \in [exp_{m-2}, +\infty), \tag{7}$$

where $\ell_0, \ldots, \ell_m$ are the non-negative functions defined by

$$\ell_0(t) = t, \quad t \geq 1; \qquad \ell_i(t) = \log \ell_{i-1}(t), \quad t \geq exp_{i-1}, \quad i \in \{1, \ldots, m\}.$$

We are going to need in Section 3 the following Lemma, which is very easy to prove.

LEMMA 2.1 *(i)*. – *Let* $m, k \in \mathbb{N}$. *Then*

$$l_m(e^{-k+1}) = l_{m-1}(k).$$

(ii) Let $m \in \mathbb{N}_0$ *and* $k \in \mathbb{N}$. *Then*

$$l_m(k) \leq l_m(k+1) \leq e\, l_m(k).$$

(iii) Let $\alpha \in \mathbb{R}$ *and* $m, k \in \mathbb{N}$. *Then for each* $t \in (e^{-k}, e^{-k+1})$, *we have the inequalities*

$$\min\{1, e^{\alpha}\}\, l_{m-1}^{\alpha}(k) \leq l_m^{\alpha}(t) \leq \max\{1, e^{\alpha}\}\, l_{m-1}^{\alpha}(k).$$

(iv) Let $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$ *and* $k \geq 2$. *Then the inequalities*

$$\min\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k) \leq l_m^{\alpha}(t) \leq \max\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k)$$

hold for each $t \in (e^{-k+1}, e^{-k+2})$.

The following Lemma, with an obvious proof, will be used later on.

LEMMA 2.2. – *Let* $k \in \mathbb{N}$ *and* $q_0 > exp_{k-1}$. *Then*

(i) $\ell_k(q) \leq l_k(q)$, *for each* $q \in [exp_{k-1}, +\infty)$;

(ii) $l_k(q) \leq e^k \ell_k(q)$, *for each* $q \in [exp_k, +\infty)$;

(iii) $l_k(q) \leq \left(\dfrac{k}{\ell_k(q_0)} + 1\right) \ell_k(q)$, *for each* $q \in [q_0, +\infty)$.

By a Young function Φ we mean a continuous non-negative, strictly increasing and convex function on $[0, +\infty)$ satisfying

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = \lim_{t\to +\infty} \frac{t}{\Phi(t)} = 0 .$$

Given a Young function Φ and any measurable subset Ω of $\mathbb{R}^n$, $L_\Phi(\Omega)$ will denote the corresponding Orlicz space, *i.e.* the collection of functions $f \in \mathcal{M}_0(\Omega, \mu_n)$ for which there is a $\lambda > 0$ such that $\int_\Omega \Phi(|f(x)|/\lambda)\, dx < +\infty$, equipped with the Luxemburg norm $\|.\|_{\Phi,\Omega}$ given by

$$\|f\|_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}.$$

We refer to [1, Chapter VIII] and [12, Chapter III] for more details.

Let Φ_1 and Φ_2 be Young functions. Recall that Φ_2 *dominates* Φ_1 *globally* if there is a positive constant κ such that

$$\Phi_1(t) \leqslant \Phi_2(\kappa t) \tag{8}$$

for all $t \geqslant 0$. Similarly, Φ_2 *dominates* Φ_1 *near infinity* if there are positive constants κ and t_0 such that (8) holds for all $t \in [t_0, +\infty)$. Two Young functions are said to be *equivalent globally* (*near infinity*) if each dominates the other globally (near infinity). We have from [1, Theorem 8.12, pp. 234-235] the following result: If Φ_1 and Φ_2 are equivalent globally (or near infinity and $|\Omega|_n < +\infty$), then $L_{\Phi_1}(\Omega) = L_{\Phi_2}(\Omega)$ and the corresponding norms are equivalent.

LEMMA 2.3. (cf. [6]) – *Let Ω be a measurable subset of $\mathbb{R}^n$ with finite volume and let $\alpha > 0$. Then*

(*i*) *the space $L^\infty(\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;-1/\alpha}(\Omega)$ coincides with the Orlicz space $L_{\Phi_1}(\Omega)$, where $\Phi_1(t) = \exp t^\alpha$ for all $t \geqslant t_0$ with some $t_0 \in (0, +\infty)$, and the corresponding (quasi-) norms are equivalent*;

(*ii*) *the space $L^\infty(\log\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;0,-1/\alpha}(\Omega)$ coincides with the Orlicz space $L_{\Phi_2}(\Omega)$, where $\Phi_2(t) = \exp\exp t^\alpha$ for all $t \geqslant t_0$ with some $t_0 \in (0, +\infty)$, and the corresponding (quasi-) norms are equivalent.*

We will denote the Orlicz spaces $L_{\Phi_1}(\Omega)$ and $L_{\Phi_2}(\Omega)$, considered in Lemma 2.3, by $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, respectively. In view of the same Lemma, we may endow these spaces with the quasi-norms

$$\|.\|_{E_\alpha(\Omega)} := \|.\|_{\infty,\infty;-1/\alpha;\Omega} \quad \text{and} \quad \|.\|_{EE_\alpha(\Omega)} := \|.\|_{\infty,\infty;0,-1/\alpha;\Omega}.$$

For more details we refer to [6].

Let $m \in \mathbb{N}$. We denote by $\mathcal{R}^m_+$ and $\mathcal{R}^m_-$ the following subsets of $\mathbb{R}^m$:

$$\mathcal{R}^m_+ = \{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m \colon \alpha_1, \dots, \alpha_{m-1} \geqslant 0 \text{ and } \alpha_m > 0\}$$

$$\mathcal{R}^m_- = \{(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m \colon \alpha_1, \dots, \alpha_{m-1} \leqslant 0 \text{ and } \alpha_m < 0\}.$$

Given a Banach space X let us denote by X^* its dual space.

Let $j_0 \in \mathbb{N}$ and let $\{A_j\}_{j \geqslant j_0}$ be a sequence of Banach spaces. We denote by $l_1(A_j)$ the space of all sequences $a = \{a_j\}_{j \geqslant j_0}$ with $a_j \in A_j$, $j \geqslant j_0$, such that

$$\|a\|_{l_1(A_j)} = \sum_{j=j_0}^{+\infty} \|a_j\|_{A_j} < +\infty .$$

By $l_\infty(A_j)$ we denote the space of all sequences $a = \{a_j\}_{j \geqslant j_0}$ with $a_j \in A_j, j \geqslant j_0$, for which $\|a\|_{l_\infty(A_j)} = \sup\limits_{j \geqslant j_0} \|a_j\|_{A_j}$ is finite. The space $c_0(A_j)$ is the subspace of $l_\infty(A_j)$ consisting of all sequences $a = \{a_j\}_{j \geqslant j_0}$ such that

$$\lim_{j \to +\infty} \|a_j\|_{A_j} = 0 .$$

By Lemma 1.11.1 in [18, pp. 68-69], generalised in an obvious way,

$$[c_0(A_j)]^* = l_1(A_j^*), \tag{9}$$

with the usual interpretation (not only isomorphic but also isometric). More precisely, given $g = \{g_j\}_{j \geqslant j_0} \in l_1(A_j^*)$, the functional $\tilde{g}$ defined by

$$\tilde{g}(f) = \sum_{j=j_0}^{+\infty} g_j(f_j), \qquad \text{for all } f = \{f_j\}_{j \geqslant j_0} \in c_0(A_j), \tag{10}$$

is an element of $[c_0(A_j)]^*$ and is such that

$$\|\tilde{g}\|_{[c_0(A_j)]^*} = \sum_{j=j_0}^{+\infty} \|g_j\|_{A_j^*} = \|g\|_{l_1(A_j^*)}. \tag{11}$$

Conversely, let us consider $\tilde{g} \in [c_0(A_j)]^*$. Then $\tilde{g}$ can be identified with an element $g = \{g_j\}_{j \geqslant j_0} \in l_1((A_j^*)$ by (10) and such that (11) holds; see [18] for more details.

For general facts about Banach function spaces with Banach function norm (or simply a function norm) ϱ on a measure space (R, μ) we refer to [3, Chap. 1, Chap. 2]. Nevertheless, let us recall a few concepts and results. A function norm ϱ over a measure space (R, μ) is said to be *rearrangement-invariant* if $\varrho(f) = \varrho(g)$ for every pair of equimeasurable functions f and g in $\mathcal{M}_0^+(R, \mu)$.

Let (R, μ) be a measure space and let ϱ be a function norm. The associate function norm ϱ' of ϱ is defined on $\mathcal{M}^+(R,\mu)$ by

$$\varrho'(g) = \sup\left\{\int_R fg\,d\mu : f \in \mathcal{M}^+(R,\mu),\ \varrho(f) \leq 1\right\}, \tag{12}$$

for each $g \in \mathcal{M}^+(R,\mu)$. The collection $X = X(\varrho)$ of all functions f in $\mathcal{M}(R,\mu)$ for which $\varrho(|f|)$ is finite is called a *Banach function space*. The norm of a function f in X is given by

$$\|f\|_X = \varrho(|f|). \tag{13}$$

The Banach function space $X = X(\varrho)$ generated by a rearrangement-invariant function norm ϱ is called a *rearrangement-invariant space*. The Banach function space $X(\varrho')$ determined by ϱ', where ϱ' is the associate norm of ϱ, is called the *associate space* of $X(\varrho)$ and is denoted by X'. It follows from (12) and (13) that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} := \sup\left\{\int_R |fg|\,d\mu : f \in X,\ \|f\|_X \leq 1\right\}.$$

Let X be a Banach function space over the measure space (R,μ). The closure in X of the set of simple functions is denoted by X_b.

PROPOSITION 2.2. (cf. [3], Proposition I.3.10) – *The subspace X_b is the closure in X of the set of bounded functions supported in sets of finite measure.*

Let us recall the Lorentz-Luxemburg Theorem, cf. Theorem I.2.7 in [3].

THEOREM 2.1. – *Every Banach function space X coincides with its second associate space $X'' := (X')'$. In other words, a function f belongs to X if, and only if, it belongs to X'', and in that case $\|f\|_X = \|f\|_{X''}$.*

REMARK 2.1. – If X and Y are two Banach function spaces such that $Y = X'$, up to equivalence of norms, then it follows, by the Lorentz-Luxemburg Theorem, cf. Theorem 2.1, and by the definition of Y', that $Y' = X$, up to equivalence of norms. In other words, X and Y are *mutually associate*.

Now we recall the Luxemburg representation theorem, cf. [3, Theorem II.4.10].

THEOREM 2.2. – *Let ϱ be a rearrangement-invariant function norm over a resonant measure space (R,μ). Then there is a (not necessarily unique) rearrangement-invariant function norm $\overline{\varrho}$ over $(\mathbb{R}^+,\mu_1)$ such that $\varrho(f) = \overline{\varrho}(f^*)$, for all f in $\mathcal{M}_0^+(R,\mu)$.*

Furthermore, if σ is any rearrangement-invariant function norm over $(\mathbb{R}^+, \mu_1)$ which represents ϱ, in the sense that $\varrho(f) = \sigma(f^*)$, for all f in $\mathcal{M}_0^+(R, \mu)$, then the associate norm ϱ' of ϱ is represented in the same way by the associate norm σ' of σ, that is, $\varrho'(g) = \sigma'(g^*)$, for all g in $\mathcal{M}_0^+(R, \mu)$.

Let X be a rearrangement-invariant Banach function space over a resonant measure space (R, μ). For each finite value of t belonging to the range of μ, let E be a μ-measurable subset of R with $\mu(E) = t$ and let

$$\varphi_X(t) = \|\chi_E\|_X. \tag{14}$$

The function φ_X so defined is called the *fundamental function* of X. Observe that the particular choice of the set E with $\mu(E) = t$ is immaterial since if F is any other subset of R with $\mu(F) = t$, then χ_E and χ_F are equimeasurable and so $\|\chi_E\|_X = \|\chi_F\|_X$, because of the rearrangement invariance of X. Therefore, φ_X is well defined by (14).

THEOREM 2.3. (cf. [3], Theorem II.5.5) – *Let (R, μ) be a non-atomic measure space and let X be an arbitrary rearrangement-invariant space over (R, μ). Then*

$$\lim_{t \to 0^+} \varphi_X(t) = 0 \quad \text{if, and only if,} \quad (X_b)^* = X'.$$

For two non-negative expressions (*i.e.* functions or functionals) $\mathcal{A}$, $\mathcal{B}$ we use the symbol $\mathcal{A} \lesssim \mathcal{B}$ to mean that $\mathcal{A} \leqslant c\mathcal{B}$, for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$.

We adopt the convention that $(a/+\infty) = 0$ and $(a/0) = +\infty$ for all $a > 0$. If $p \in [1, +\infty]$, the conjugate number p' is given by $(1/p) + (1/p') = 1$.

3. – Decompositions.

As was said in the Introduction, the following results extend the decompositions considered in [7] for the exponential Orlicz spaces $E_\alpha(\Omega)$.

Let us assume, in this Section, that (R, μ) is a finite measure space. Without loss of generality we suppose that $\mu(R) = 1$; see Remark 3.1. In the sequel, we shall consider the decomposition of $(0, 1)$ into $\{(e^{-k}, e^{-k+1})\}_{k \geqslant 1}$.

THEOREM 3.1. – *Let $p, q \in (0, +\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^n$. Then for each $f \in L_{p,q;\boldsymbol{\alpha}}(R)$ we have* (*i*) *if* $0 < q < +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};\mathrm{R}} \approx \left[\sum_{k=1}^{+\infty} (e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k}))^q\right]^{1/q} \tag{15}$$

$$\approx \left[\sum_{k=2}^{+\infty} (e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k+1}))^q\right]^{1/q}; \tag{16}$$

(*ii*) *if* $q = +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \approx \sup_{k \geqslant 1} \{e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k})\} \tag{17}$$

$$\approx \sup_{k \geqslant 2} \{e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k+1})\}. \tag{18}$$

PROOF. – (i) Let $0 < q < +\infty$ and suppose $f \in L_{p,q;\boldsymbol{\alpha}}(R)$. Then by Lemma 2.1 it follows that

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \geqslant c_1 \sum_{k=2}^{+\infty} (e^{-k(1/p-1/q)}\,\vartheta_{\alpha}^{m}(e^{-k+1})\, f^*(e^{-k+1}))^q e^{-k}$$

$$\geqslant c_2 \sum_{k=1}^{+\infty} (e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k}))^q.$$

Conversely, for $f \in L_{p,q;\boldsymbol{\alpha}}(R)$, we have again by Lemma 2.1

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \leqslant c_3 \sum_{k=2}^{+\infty} (e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k+1}))^q \leqslant c_4 \sum_{k=1}^{+\infty} (e^{-k/p}\,\omega_{\alpha}^{m}(k)\, f^*(e^{-k}))^q,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. ■

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.1 we conclude that

$$\|f\|_{E_\alpha(\Omega)} \approx \sup_{k \geqslant 1} \frac{f^*(e^{-k})}{k^{1/\alpha}} \approx \sup_{k \geqslant 2} \frac{f^*(e^{-k+1})}{k^{1/\alpha}}, \qquad \text{for each} \quad f \in E_\alpha(\Omega),$$

and

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k \geqslant 1} \frac{f^*(e^{-k})}{(1+\log k)^{1/\alpha}} \approx \sup_{k \geqslant 2} \frac{f^*(e^{-k+1})}{\log^{1/\alpha} k}, \qquad \text{for each} \quad f \in EE_\alpha(\Omega).$$

The next Lemma, with an easy proof, will be used to prove the last result of this Section.

LEMMA 3.1. – *Let* $f \in \mathcal{M}_0(R, \mu)$, $J_k = (e^{-k}, e^{-k+1})$, $k \geqslant 1$. *Then*

(*i*) *for each* $k \in \mathbb{N}$ *we have*

$$c_1 f^*(e^{-k+1}) \leqslant \|f^*\|_{k, J_k} \leqslant c_2 f^*(e^{-k}), \tag{19}$$

where c_1 *and* c_2 *are positive constants independent of* f *and* k;

(*ii*) *for each* $k \geqslant 2$ *we have*

$$c_1\, f^*(e^{-k+2}) \leqslant \|f^*\|_{k, J_{k-1}} \leqslant c_2\, f^*(e^{-k+1}), \tag{20}$$

where c_1 *and* c_2 *are positive constants independent of* f *and* k.

THEOREM 3.2. – *Let* $p, q \in (0, +\infty]$, $m \in \mathbb{N}$ *and* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^n$. *Let* $J_k = (e^{-k}, e^{-k+1})$, $k \geqslant 1$, *and* $I_k = J_{k-1}$, $k \geqslant 2$. *Then for each* $f \in L_{p, q; \boldsymbol{\alpha}}(R)$ *we have*

(*i*) *if* $0 < q < +\infty$,

$$\|f\|_{p, q; \boldsymbol{\alpha}; R} \approx \left[\sum_{k=1}^{+\infty} (e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, J_k})^q \right]^{1/q} \tag{21}$$

$$\approx \left[\sum_{k=2}^{+\infty} (e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, I_k})^q \right]^{1/q}; \tag{22}$$

(*ii*) *if* $q = +\infty$,

$$\|f\|_{p, q; \boldsymbol{\alpha}; R} \approx \sup_{k \geqslant 1} \{ e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, J_k} \} \tag{23}$$

$$\approx \sup_{k \geqslant 2} \{ e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, I_k} \}. \tag{24}$$

PROOF. – (i) Suppose $0 < q < +\infty$ and let $f \in L_{p, q; \boldsymbol{\alpha}}(R)$. Then by (15) and by (19), we have

$$\|f\|_{p, q; \boldsymbol{\alpha}; R}^q \geqslant c_1 \sum_{k=1}^{+\infty} (e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, J_k})^q.$$

By (16) and by (20), we also have

$$\|f\|_{p, q; \boldsymbol{\alpha}; R}^q \geqslant c_2 \sum_{k=2}^{+\infty} (e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, I_k})^q.$$

Conversely, for $f \in L_{p, q; \boldsymbol{\alpha}}(R)$, by (16) and by (19), we have

$$\|f\|_{p, q; \boldsymbol{\alpha}; R}^q \leqslant c_3 \sum_{k=1}^{+\infty} (e^{-k/p}\, \omega_{\alpha}^{m}(k) \|f^*\|_{k, J_k})^q.$$

By (16), by Lemma 2.1 and by (20), we have

$$\|f\|^q_{p,\,q;\,\boldsymbol{\alpha};\,R} \leqslant c_4 \sum_{k=3}^{+\infty} (e^{-k/p}\,\omega^m_{\alpha}(k)\, f^*(e^{-k+2}))^q \leqslant c_5 \sum_{k=2}^{+\infty} (e^{-k/p}\,\omega^m_{\alpha}(k)\|f^*\|_{k,\,I_k})^q,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. ■

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.2 we conclude that for each $f \in E_\alpha(\Omega)$

$$\|f\|_{E_\alpha(\Omega)} \approx \sup_{k \geqslant 1} \frac{\|f^*\|_{k,\,J_k}}{k^{1/\alpha}} \approx \sup_{k \geqslant 2} \frac{\|f^*\|_{k,\,I_k}}{k^{1/\alpha}}. \tag{25}$$

The first estimate in (25) is given in [7] by Corollary 2.3. The counterpart for the spaces $EE_\alpha(\Omega)$ is given by

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k \geqslant 1} \frac{\|f^*\|_{k,\,J_k}}{(1+\log k)^{1/\alpha}} \approx \sup_{k \geqslant 2} \frac{\|f^*\|_{k,\,I_k}}{\log^{1/\alpha} k}, \qquad \text{for all } f \in EE_\alpha(\Omega).$$

REMARK 3.1. – If (R, μ) is a finite measure space with measure $\mu(R)$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$, we have $\vartheta^m_\alpha(s) \approx \vartheta^m_\alpha(s\mu(R))$, for all $s \in (0, 1)$. This follows from the estimates $e^{-j} l_i(s) \leqslant l_i(s\mu(R)) \leqslant e^j l_i(s)$, for all $s \in (0, 1)$ and $i = 1, \ldots, m$ where j is a positive integer such that $e^{j-1} \leqslant l_1(\mu(R)) \leqslant e^j$.

With the previous considerations, it is easy to see that the estimates in Theorem 3.1 and Theorem 3.2 still hold, up to constants, if we replace $f^*(e^{-k})$ by $f^*(e^{-k}\mu(R))$, for each $k \in \mathbb{N}$, and $J_k = (e^{-k}, e^{-k+1})$ by $J_k = (e^{-k}\mu(R), e^{-k+1}\mu(R))$, for each $k \in \mathbb{N}$, respectively.

4. – Equivalent (quasi-) norms for some generalised Lorentz-Zygmund spaces.

In this Section, we are going to consider in the first part the GLZ spaces $L_{\infty,\,\infty;\,\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, and in the second part the GLZ spaces $L_{1,\,1;\,\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$.

4.1. *The GLZ spaces $L_{\infty,\,\infty;\,\boldsymbol{\alpha}}(R)$.*

First we are going to recall a Lemma.

LEMMA 4.1. (cf. [10], Lemma 5.1) – *Let $m \in \mathbb{N}$ and $\nu > 0$. Then there is a constant $c \in (0, +\infty)$ such that for all $s \in (0, 1)$,*

$$\sup_{q \in [1,\,+\infty)} l^{-\nu}_{m-1}(q)\, s^{1/q} \leqslant c\, l^{-\nu}_m(s).$$

With the help of the previous result, it is not difficult to prove the next Lemma.

LEMMA 4.2. – *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0, +\infty)$. Then there is a positive constant c such that $\omega^m_\alpha(q) s^{1/q} \leqslant c \vartheta^m_\alpha(s)$, for all $s \in (0, t_0)$ and all $q \in [1, +\infty)$.*

The following result generalises Theorem 3.1 in [6].

THEOREM 4.1. – *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0, +\infty)$.*

(*i*) *Let $p \in (0, +\infty]$. Then for each $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{p,\infty;\boldsymbol{\alpha};R} \approx \sup_{q \in [1,+\infty)} \omega^m_\alpha(q) \|f^*\|_{(q/(q/p+1)),\infty;(0,t_0)} + \sup_{t_0 \leqslant t < +\infty} \{t^{1/p} \vartheta^m_\alpha(t) f^*(t)\}. \tag{26}$$

(*ii*) *Then for each $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx f^*(t_0) + \sup_{q \in [1,+\infty)} \omega^m_\alpha(q) \|f^*\|_{q;(0,t_0)}. \tag{27}$$

PROOF. – We follow the proof of Theorem 3.1 in [6], where the case $p = +\infty$, $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$ and $\mu(R) < +\infty$ with $t_0 = \mu(R)$ was considered.

(i) Let $t_0 \in (0, +\infty)$ and $\mathcal{A} := \mathcal{B} + \mathcal{C}$ where

$$\mathcal{B} := \sup_{q \in [1,+\infty)} \omega^m_\alpha(q) \|f^*\|_{(q/(q/p+1)),\infty;(0,t_0)} \quad \text{and} \quad \mathcal{C} := \sup_{t_0 \leqslant t < +\infty} \{t^{1/p} \vartheta^m_\alpha(t) f^*(t)\}.$$

Suppose $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$. By Lemma 4.2 there is a constant $c_1 > 0$ such that for all $q \in [1, +\infty)$,

$$\omega^m_\alpha(q) \|f^*\|_{(q/(q/p+1)),\infty;(0,t_0)} \leqslant c_1 \sup_{0 < s < t_0} \{\vartheta^m_\alpha(s)\, s^{1/p} f^*(s)\}.$$

Passing to the supremum over all $q \in [1, +\infty)$, we get the inequality

$$\mathcal{B} \leqslant c_1 \|f\|_{p,\infty;\boldsymbol{\alpha};R}.$$

Hence

$$\mathcal{A} \leqslant 2 \max\{1, c_1\} \|f\|_{p,\infty;\boldsymbol{\alpha};R}. \tag{28}$$

Conversely, suppose the right hand-side of (26) is finite. Fix $s \in (0, t_0)$ and set $q = 1 + |\log s|$. Then $\mathcal{B} \geqslant \omega^m_\alpha(q) s^{1/q+1/p} f^*(s) \geqslant e^{-1} \vartheta^m_\alpha(s)\, s^{1/p} f^*(s)$. Taking the supremum over all $s \in (0, t_0)$, we obtain the inequality

$$\mathcal{B} \geqslant e^{-1} \sup_{0 < t < t_0} \{t^{1/p} \vartheta^m_\alpha(t)\, f^*(t)\}.$$

So $\mathcal{A} \geqslant e^{-1} \|f\|_{p,\infty;\boldsymbol{\alpha};R}$, which together with (28) gives the estimate (26).

(ii) Let $t_0 \in (0, +\infty)$. First we prove the following estimate

$$\mathcal{A} + \mathcal{B} \approx f^*(t_0) + \sup_{q \in [1, +\infty)} \omega_\alpha^m(q) \|f^*\|_{q;(0, t_0)}, \tag{29}$$

where

$$\mathcal{A} := \sup_{q \in [1, +\infty)} \omega_\alpha^m(q) \|f^*\|_{q, \infty;(0, t_0)} \quad \text{and} \quad \mathcal{B} := \sup_{t_0 \leqslant t < +\infty} \{\vartheta_\alpha^m(t)\, f^*(t)\}.$$

Suppose the right hand-side of (29) is finite. First we verify that

$$\|f^*\|_{q, \infty;(0, t_0)} \leqslant \|f^*\|_{q;(0, t_0)}, \quad \text{for each } q \in [1, +\infty). \tag{30}$$

Let $t \in (0, t_0)$. Using the fact that f^* is decreasing, we have

$$t^{1/q} f^*(t) = \left\{ \int_0^t [s^{1/q} f^*(t)]^q \frac{ds}{s} \right\}^{1/q} \leqslant \left\{ \int_0^t [s^{1/q} f^*(s)]^q \frac{ds}{s} \right\}^{1/q}$$
$$\leqslant \|f^*\|_{q;(0, t_0)}.$$

Hence taking the supremum over all $t \in (0, t_0)$, we obtain (30). Using inequality (30) and since $\mathcal{B} \leqslant f^*(t_0)$, we immediately obtain

$$\mathcal{A} + \mathcal{B} \leqslant f^*(t_0) + \sup_{q \in [1, +\infty)} \omega_\alpha^m(q) \|f^*\|_{q;(0, t_0)}.$$

Now we prove the converse inequality. Suppose that $\mathcal{A} + \mathcal{B} < +\infty$. If $1 \leqslant q < q_1$ then

$$\|f^*\|_{q;(0, t_0)} \leqslant \|f^*\|_{q_1, \infty;(0, t_0)}\, t_0^{1/q - 1/q_1} \left(1 - \frac{q}{q_1}\right)^{-1/q}. \tag{31}$$

Let $q \in [1, +\infty)$. Since $l_j(q) \leqslant l_j(2q) \leqslant e l_j(q)$, for all $j \in \mathbb{N}_0$ we have by (31), with $q_1 = 2q$, the following inequalities

$$\omega_\alpha^m(q) \|f^*\|_{q;(0, t_0)} \leqslant c_1 \omega_\alpha^m(2q) \|f^*\|_{2q, \infty;(0, t_0)} \leqslant c_1 \sup_{r \in [2, +\infty)} \omega_\alpha^m(r) \|f^*\|_{r, \infty;(0, t_0)}.$$

Therefore, passing to the supremum over all $q \in [1, +\infty)$, we get the inequality

$$\sup_{q \in [1, +\infty)} \omega_\alpha^m(q) \|f^*\|_{q;(0, t_0)} \leqslant c_1 \mathcal{A}. \tag{32}$$

Now it easily follows from (32) that

$$f^*(t_0) + \sup_{q \in [1, +\infty)} \omega_\alpha^m(q) \|f^*\|_{q;(0, t_0)} \leqslant \max\{c_1, \vartheta_{-\alpha}^m(t_0)\}(\mathcal{A} + \mathcal{B})$$

and (29) is proved. The estimate (27) follows from (26), with $p = +\infty$, and from (29). ■

When (R, μ) is a finite measure space the previous estimates are much nicer.

COROLLARY 4.1. – *Suppose (R, μ) is a measure space such that $\mu(R) < +\infty$. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$.*

(*i*) *Let $p \in (0, +\infty]$. Then for each $f \in L_{p, \infty; \boldsymbol{\alpha}}(R)$,*

$$\|f\|_{p, \infty; \boldsymbol{\alpha}; R} \approx \sup_{q \in [1, +\infty)} \omega^m_\alpha(q) \|f\|_{(q/(q/p+1)), \infty; R}. \tag{33}$$

(*ii*) *Then for each $f \in L_{\infty, \infty; \boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty, \infty; \boldsymbol{\alpha}; R} \approx \sup_{q \in [1, +\infty)} \omega^m_\alpha(q) \|f\|_{q; R}. \tag{34}$$

PROOF. – The results follow from the theorem with $t_0 = \mu(R)$ and from the fact that $f^*(t) = 0$, $t \geq \mu(R)$. For the part (ii) we use also Proposition 2.1. ■

From (ii) of Corollary 4.1 we recover the results of Theorem 3.1 in [6] for the spaces $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n < +\infty$.

COROLLARY 4.2. – *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0, +\infty)$. If $j_0 \in \mathbb{N}$ and $q_0 \geq 1$ then for all $f \in L_{\infty, \infty; \boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty, \infty; \boldsymbol{\alpha}; R} \approx f^*(t_0) + \sup_{j \in \mathbb{N}, j \geq j_0} \omega^m_\alpha(j) \|f^*\|_{j; (0, t_0)} \tag{35}$$

$$\approx f^*(t_0) + \sup_{q \in [q_0, +\infty)} \omega^m_\alpha(q) \|f^*\|_{q; (0, t_0)}. \tag{36}$$

PROOF. – We follow the proof of Corollary 3.2 in [6], where the case $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$ and $\mu(R) < +\infty$ with $t_0 = \mu(R)$ was proved. For $f \in L_{\infty, \infty; \boldsymbol{\alpha}}(R)$, $j_0 \in \mathbb{N}$ and $q_0 \geq 1$ we denote

$$S_1(f) = f^*(t_0) + \sup_{q \in [1, +\infty)} \omega^m_\alpha(q) \|f^*\|_{q; (0, t_0)},$$

$$S_2(f) = f^*(t_0) + \sup_{q \in [1, +\infty)} \omega^m_\alpha(q)\, t_0^{-1/q} \|f^*\|_{q; (0, t_0)},$$

$$S_3(f) = f^*(t_0) + \sup_{q \in [q_0, +\infty)} \omega^m_\alpha(q) \|f^*\|_{q; (0, t_0)},$$

$$\sigma_1(f) = f^*(t_0) + \sup_{j \in \mathbb{N}, j \geq j_0} \omega^m_\alpha(j) \|f^*\|_{j; (0, t_0)},$$

$$\sigma_2(f) = f^*(t_0) + \sup_{j \in \mathbb{N},\, j \geq j_0} \omega_\alpha^m(j) t_0^{-1/j} \|f^*\|_{j;(0,\, t_0)},$$

$$\sigma_3(f) = f^*(t_0) + \sup_{j \in \mathbb{N},\, j \geq [q_0]+1} \omega_\alpha^m(j) \|f^*\|_{j;(0,\, t_0)},$$

where $[q_0]$ denotes the integer part of q_0.

(i) Let $t_0 \in (0, +\infty)$, $j_0 \in \mathbb{N}$ and $f \in L_{\infty,\, \infty;\, \boldsymbol{\alpha}}(R)$. First we prove that

$$\|f\|_{\infty,\, \infty;\, \boldsymbol{\alpha};\, R} \approx f^*(t_0) + \sup_{j \in \mathbb{N},\, j \geq j_0} \omega_\alpha^m(j) \|f^*\|_{j;(0,\, t_0)}.$$

If $q \in [1, +\infty)$, we put $j = \max\{j_0, [q]+1\}$ and choose $n \in \mathbb{N}$ such that $e^{n-1} \geq j_0$. Then

$$j \leq j_0([q]+1) < j_0 q \leq e^{n-1}(q+1) \leq e^{n-1} 2q \leq e^n q$$

and hence

$$l_{k-1}(j) \leq e^n l_{k-1}(q),\ k = 2, \ldots, m\,.$$

Therefore

(37) $$e^{n(\alpha_1 + \ldots + \alpha_m)} \omega_\alpha^m(q) \leq \omega_\alpha^m(j)\,.$$

Since $j \geq [q]+1 > q$, we get by Hölder's inequality together with (37) the inequality

$$\omega_\alpha^m(q) t_0^{-1/q} \|f^*\|_{q;(0,\, t_0)} \leq c\omega_\alpha^m(j)\, t_0^{-1/j} \|f^*\|_{j;(0,\, t_0)},$$

where $c = e^{-n(\alpha_1 + \ldots + \alpha_m)} > 1$, and hence

(38) $$S_2(f) \leq c\sigma_2(f)\,.$$

It is easy to see that $S_1(f) \approx S_2(f)$, $\sigma_1(f) \approx \sigma_2(f)$, and since $\sigma_1(f) \leq S_1(f)$ we have, together with (38), the estimates

(39) $$\sigma_1(f) \leq S_1(f) \approx S_2(f) \leq c\sigma_2(f) \approx \sigma_1(f)\,.$$

So (35) it follows from (27) and (39).

(ii) Let $t_0 \in (0, +\infty)$, $q_0 \geq 1$ and $f \in L_{\infty,\, \infty;\, \boldsymbol{\alpha}}(R)$. From (27) it follows that

(40) $$S_3(f) \leq S_1(f) \approx \|f\|_{\infty,\, \infty;\, \boldsymbol{\alpha};\, R}.$$

Since $\sigma_3(f) = \sigma_1(f)$ if $j_0 = [q_0]+1$, we have by (35)

(41) $$\|f\|_{\infty,\, \infty;\, \boldsymbol{\alpha};\, R} \approx \sigma_3(f) \leq S_3(f)\,.$$

Therefore, by (40) and (41) we get (36) and the proof is finished. ■

When (R, μ) is a measure space of finite measure we obtain simple equivalent norms.

COROLLARY 4.3. – *Suppose* (R,μ) *is a measure space such that* $\mu(R)<+\infty$. *Let* $m\in\mathbb{N}$ *and* $\boldsymbol{\alpha}\in\mathcal{R}^m_-$. *If* $j_0\in\mathbb{N}$ *and* $q_0\geqslant 1$ *then for all* $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx\sup_{j\in\mathbb{N},\,j\geqslant j_0}\omega^m_\alpha(j)\|f\|_{j;R} \tag{42}$$

$$\approx\sup_{q\in[q_0,+\infty)}\omega^m_\alpha(q)\|f\|_{q;R}. \tag{43}$$

PROOF. – The results follow from Corollary 4.2 with $t_0=\mu(R)$ and Proposition 2.1. ■

If we consider $m=1$, $\alpha_1<0$ and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n<+\infty$ in the above Corollary we recover part (i) of Corollary 3.2 in [6].

COROLLARY 4.4. – *Let* $m\in\mathbb{N}$, $m\geqslant 2$ *and* $\boldsymbol{\alpha}\in\mathcal{R}^m_-$. *Let* $t_0\in(0,+\infty)$. *If* $j_0\in\mathbb{N}$, $j_0\geqslant[exp_{m-2}]+1$ *and* $q_0>exp_{m-2}$ *then for all* $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx f^*(t_0)+\sup_{j\in\mathbb{N},\,j\geqslant j_0}\gamma^m_\alpha(j)\|f^*\|_{j;(0,t_0)} \tag{44}$$

$$\approx f^*(t_0)+\sup_{q\in[q_0,+\infty)}\gamma^m_\alpha(q)\|f^*\|_{q;(0,t_0)}. \tag{45}$$

PROOF. – (i) Let $j_0\in\mathbb{N}$, $j_0\geqslant[exp_{m-2}]+1$. Since $j_0>exp_{m-2}$, it follows from (i) and (iii) of Lemma 2.2 that, for each $k\in\{1,\dots,m-1\}$, $\ell_k(j)\approx l_k(j)$, for all $j\geqslant j_0$. Therefore, the estimate (44) follows from (35).

(ii) Let $q_0>exp_{m-2}$. Then for $k=1,\dots,m-1$, the estimate $\ell_k(q)\approx l_k(q)$, for all $q\geqslant q_0$, follows from (i) and (iii) of Lemma 2.2. Therefore, the estimate (45) follows from (36). ■

COROLLARY 4.5. – *Suppose* (R,μ) *is a measure space such that* $\mu(R)<+\infty$. *Let* $m\in\mathbb{N}$, $m\geqslant 2$ *and* $\boldsymbol{\alpha}\in\mathcal{R}^m_-$. *If* $j_0\in\mathbb{N}$, $j_0\geqslant[exp_{m-2}]+1$ *and* $q_0>exp_{m-2}$ *then for all* $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx\sup_{j\in\mathbb{N},\,j\geqslant j_0}\gamma^m_\alpha(j)\|f\|_{j;R} \tag{46}$$

$$\approx\sup_{q\in[q_0,+\infty)}\gamma^m_\alpha(q)\|f\|_{q;R}. \tag{47}$$

PROOF. – The results follow from Corollary 4.4 with $t_0=\mu(R)$ and Proposition 2.1. ■

If we consider $m=2$, $\alpha_1=0$, $\alpha_2<0$, and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n<+\infty$ in the above Corollary we recover part (ii) of Corollary 3.2 in [6].

4.2. *The GLZ spaces* $L_{1,1;\boldsymbol{\alpha}}(R)$.

Let us assume, in this Subsection, that (R,μ) is a finite measure space. Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}^m_+$. Let us consider the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$ and $L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ endowed with $\|.\|_{1,1;\boldsymbol{\alpha};R}$ and $\|.\|_{\infty,\infty;-\boldsymbol{\alpha};R}$, respectively.

Again, without loss of generality we suppose that $\mu(R)=1$, because if (R,μ) is a finite measure space with measure $\mu(R)$, after a change of variables, we have by Remark 3.1

$$\|f\|_{1,1;\boldsymbol{\alpha};R}\approx\int_0^1 \vartheta^m_\alpha(s)\, f_1^*(s)\,ds\,,$$

for each $f\in L_{1,1;\boldsymbol{\alpha}}(R)$, and

$$\|f\|_{\infty,\infty;-\boldsymbol{\alpha};R}\approx\sup_{0<s<1}\vartheta^m_{-\alpha}(s)\, f_1^*(s)\,ds\,,$$

for each $f\in L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$, where $f_1^*(s)=f^*(s\mu(R))$, for each $s\in(0,1)$, which is the non-increasing rearrangement with respect to the measure $\mu_1=\mu/\mu(R)$.

The triangle inequality for $\|.\|_{1,1;\boldsymbol{\alpha};R}$ follows immediately by the property,

$$\int_0^t\varphi(s)(f+g)^*(s)\,ds\leq\int_0^t\varphi(s)\,f^*(s)\,ds+\int_0^t\varphi(s)\,g^*(s)\,ds\,,\qquad 0<t<1\,,$$

whenever φ is a non-negative decreasing function on $(0,1)$, cf. [13, p. 38] or [2, p. 23].

Let us introduce the functional $\|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)}=\sup\limits_{0<t<1}\vartheta^m_{-\alpha}(t)\,f^{**}(t)$. Then by Lemma 3.2 in [5], we have

$$\|f\|_{\infty,\infty;-\boldsymbol{\alpha};R}\leq\|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)}\lesssim\|f\|_{\infty,\infty;-\boldsymbol{\alpha};R},$$

for all $f\in L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$. The triangle inequality for $\|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)}$ it follows from the sub-additivity of $f\mapsto f^{**}$, cf. Theorem II.3.4 in [3].

LEMMA 4.3. – *Let* $m\in\mathbb{N}$ *and* $\boldsymbol{\alpha}\in\mathcal{R}^m_+$. *If* (R,μ) *is a resonant measure space, then*

$$X=(L_{1,1;\boldsymbol{\alpha}}(R),\ \|.\|_{1,1;\boldsymbol{\alpha};R})$$

and

$$Y=(L_{\infty,\infty;-\boldsymbol{\alpha}}(R),\ \|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)})$$

are rearrangement-invariant Banach function spaces and they are mutually associate (*up to equivalence of norms*).

PROOF. – There is no difficulty in verifying that X and Y are Banach function spaces and the rearrangement invariance is obvious, since two equimeasurable functions have the same non-increasing rearrangement.

Now we are going to prove that X and Y are mutually associate. We follow the proof of Theorem IV.6.5 in [3] and the proof of Lemma 3.4 in [5].

Suppose $g \in Y$. Then for any $f \in X$ with $\|f\|_X \leqslant 1$, we have by the Hardy-Littlewood inequality, cf. Theorem II.2.2 in [3],

$$\int_R |fg|\,d\mu \leqslant \int_0^1 f^*(t)\,g^*(t)\,dt \leqslant \sup_{0<t<1}\{g^{**}(t)\,\vartheta^m_{-\alpha}(t)\}\,\|f\|_X = \|g\|_Y\|f\|_X.$$

Hence taking the supremum over all $f \in X$ with $\|f\|_X \leqslant 1$, we get

$$\|g\|_{X'} = \sup\left\{\int_R |fg|\,d\mu : f \in X,\ \|f\|_X \leqslant 1\right\} \leqslant \|g\|_Y. \tag{48}$$

To establish an inequality reverse to (48), it is sufficient by the Luxemburg representation Theorem, cf. Theorem 2.2, to do so for the measure space $(\mathbb{R}^+, \mu_1)$ and functions g in $\mathbb{R}^+$ for which $g = g^*$. Suppose g belongs to the associate space X' of X, and also under the previous conditions, then by Hölder's inequality, cf. Corollary II.4.5 in [3], for $0 < t < 1$,

$$tg^{**}(t) = \int_0^1 \chi_{[0,t]}(s)\,g^*(s)\,ds \leqslant \|\chi_{[0,t]}\|_X\|g\|_{X'}.$$

Since

$$\|\chi_{[0,t]}\|_X = \int_0^1 \chi_{[0,t]}(s)\,\vartheta^m_\alpha(s)\,ds = \int_0^t \vartheta^m_\alpha(s)\,ds \approx t\vartheta^m_\alpha(t),$$

we get

$$\|g\|_Y \lesssim \|g\|_{X'}. \tag{49}$$

The estimates (48) and (49) together show that Y is equivalent to the associate of X. Hence, it follows immediately from Remark 2.1 that the spaces X and Y are mutually associate. ■

PROPOSITION 4.1. – *Suppose* (R, μ) *is a non-atomic measure space. Let* $m \in \mathbb{N}$ *and* $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. *Then, up to equivalence of norms,*

$$(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R))^* = L_{1,1;-\boldsymbol{\alpha}}(R), \tag{50}$$

where $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ *is the completion of* $L_\infty(R)$ *in* $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

PROOF. – We apply Theorem 2.3 to the space $X = L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. It is easy to see that $\lim_{t\to 0^+} \varphi_X(t) = 0$, where φ_X is the fundamental function of X. Therefore, by Theorem 2.3, $(X_b)^* = X'$. But by Lemma 4.3, X' coincides with $L_{1,1;-\boldsymbol{\alpha}}(R)$, up to equivalence of norms, and, by Proposition 2.2, X_b coincides with the space $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$. ■

Let $j_0, m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. We denote by $c_0^s(L_j(R))$ the subspace of $c_0(L_j(R))$ which consists of all elements $\{F_j\}_{j \geq j_0}$ of $c_0(L_j(R))$ with $F_j = \omega_\alpha^m(j)\, f$, for all $j \geq j_0$, where $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. In what follows, and according to Corollary 4.3, we consider the space $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ endowed with the norm

$$\|.\|^d_{\infty,\infty;\boldsymbol{\alpha};R} = \sup_{j\in\mathbb{N},\, j\geq j_0} \omega_\alpha^m(j)\|.\|_{j;R}.$$

PROPOSITION 4.2. – *Let* $j_0, m \in \mathbb{N}$ *and* $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. *Then*

$$L^0_{\infty,\infty;\boldsymbol{\alpha}}(R) = \left\{ f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)\colon \lim_{j\to+\infty} \omega_\alpha^m(j)\|f\|_{j;R} = 0 \right\}$$

and $(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R), \|.\|^d)$ *is isometric to* $(c_0^s(L_j(R)), \|.\|_{l_\infty(L_j(R))})$.

PROOF. – If $f \in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$, the results follow easily.

Conversely, suppose $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ with $\lim_{j\to+\infty} \omega_\alpha^m(j)\|f\|_{j;R} = 0$. Let $\varepsilon > 0$. Then there is $j_1 \in \mathbb{N}$, with $j_1 \geq j_0$, such that for all $j \geq j_1$ we have the inequality

$$\omega_\alpha^m(j)\|f\|_{j;R} < \frac{\varepsilon}{2}. \tag{51}$$

Since $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, f is finite $\mu - a.e.$ For each $n \in \mathbb{N}$ let us consider the set $R_n = \{x \in R\colon |f(x)| > n\}$. Now we introduce a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L_\infty(R)$ by $f_n(x) = f(x)$ if $x \in R \setminus R_n$, and $f_n(x) = 0$ otherwise. Then, for each $n \in \mathbb{N}$, we have by (51)

$$\|f - f_n\|^d_{\infty,\infty;\boldsymbol{\alpha};R} \leq \max_{j\in\mathbb{N},\, j_0\leq j\leq j_1} \omega_\alpha^m(j)\|f\|_{j;R_n} + \frac{\varepsilon}{2} = \omega_\alpha^m(k)\|f\|_{k;R_n} + \frac{\varepsilon}{2}. \tag{52}$$

Now

$$\|\omega_\alpha^m(k)\, f\|_{k;\, R_n}^k = \|(\omega_\alpha^m(k)\, f)^k \chi_{R_n}\|_{1;\, R}.$$

Let us consider, for each $n \in \mathbb{N}$, a function defined $\mu - a.e.$ on R by

$$g_n = (\omega_\alpha^m(k)\, |f|)^k \chi_{R_n}.$$

We note that for all $n \in \mathbb{N}$, $|g_n| \leq h$, $\mu - a.e.$ on R, where $h = (\omega_\alpha^m(k)\, |f|)^k$, $\mu - a.e.$ on R, is a function in $L_1(R)$. Since $\lim_{n \to +\infty} \chi_{R_n} = 0$ $\mu - a.e.$ it follows from the Lebesgue dominated convergence Theorem that $\lim_{n \to +\infty} \|\omega_\alpha^m(k)\, f\|_{k;\, R_n}^k = 0$. Hence, there is $n_0 \in \mathbb{N}$ such that

$$\omega_\alpha^m(k) \|f\|_{k;\, R_n} < \frac{\varepsilon}{2}, \qquad \text{for each } n \geq n_0. \tag{53}$$

Therefore, from (52) and (53), we get $\lim_{n \to +\infty} \|f - f_n\|_{\infty,\, \infty;\, \boldsymbol{\alpha};\, R}^d = 0$, which shows that $f \in L_{\infty,\, \infty;\, \boldsymbol{\alpha}}^0(R)$.

Now we can define a linear mapping H from $L_{\infty,\, \infty;\, \boldsymbol{\alpha}}^0(R)$ onto $c_0^s(L_j(R))$ by

$$H(f) = \{\omega_\alpha^m(j)\, f\}_{j \geq j_0}, \qquad \text{for all } f \in L_{\infty,\, \infty;\, \boldsymbol{\alpha}}^0(R).$$

We also have $\|H(f)\|_{c_0^s(L_j(R))} = \|f\|_{\infty,\, \infty;\, \boldsymbol{\alpha};\, R}^d$, for all $f \in L_{\infty,\, \infty;\, \boldsymbol{\alpha}}^0(R)$, and the proof is finished. ■

The next result gives an equivalent norm for the GLZ spaces $L_{1,\, 1;\, \boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_+^m$, in terms of decompositions.

THEOREM 4.2. – *Suppose (R, μ) is a non-atomic measure space. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_+^m$. Let $j_0 \in \mathbb{N}$ with $j_0 \geq 2$. Then $L_{1,\, 1;\, \boldsymbol{\alpha}}(R)$ is the set of all measurable functions $g: R \to \mathbb{C}$ which can be represented as*

$$g = \sum_{j = j_0}^{+\infty} g_j, \tag{54}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \geq j_0$, such that

$$\sum_{j = j_0}^{+\infty} \omega_\alpha^m(j) \|g_j\|_{j';\, R} < +\infty. \tag{55}$$

The infimum of the expression (55) *taken over all admissible representations* (54) *is an equivalent norm on $L_{1,\, 1;\, \boldsymbol{\alpha}}(R)$ and it will be denoted by $|g|_{1,\, 1;\, \boldsymbol{\alpha};\, R}$.*

PROOF. – Let $j_0 \in \mathbb{N}$. Let us consider a measurable function $h\colon R \to \mathbb{C}$ that can be represented as

$$h = \sum_{j=j_0}^{+\infty} g_j, \tag{56}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \geqslant j_0$, such that

$$\sum_{j=j_0}^{+\infty} \omega_{\alpha}^{m}(j)\|g_j\|_{j';R} < +\infty$$

and let us define

$$\Phi_h(f) = \int_R hf\,d\mu, \qquad \text{for all } f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R). \tag{57}$$

Then $\Phi_h \in (L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*$ and

$$\|\Phi_h\,|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \leqslant \inf \sum_{j=j_0}^{+\infty} \omega_{\alpha}^{m}(j)\|g_j\|_{j';R}, \tag{58}$$

where the infimum is taken over all admissible representations (56). In fact, for all $f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$, we have by Theorem 1.27 in [15, p. 22] and by Hölder's inequality, the following

$$|\Phi_h(f)| \leqslant \sum_{j=j_0}^{+\infty} \|g_j\|_{j';R}\|f\|_{j;R} \leqslant \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R} \sum_{j=j_0}^{+\infty} \omega_{\alpha}^{m}(j)\|g_j\|_{j';R}.$$

Thus, Φ_h is a bounded linear functional on $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ (the linearity of Φ_h is obvious) such that

$$\|\Phi_h\,|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \leqslant \sum_{j=j_0}^{+\infty} \omega_{\alpha}^{m}(j)\|g_j\|_{j';R}$$

and we get (58).

Now we follow the reasoning in the proof of Theorem 2.6.2/2 in [8, pp. 72-74]. Let $G \in (L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*$. Since $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ is isometric to $c_0^s(L_j(R))$, cf. Proposition 4.2, $G \circ H^{-1} \in (c_0^s(L_j(R)))^*$, where H is the isometry considered in the referred proposition. By Hahn-Banach theorem, there exists a bounded linear functional $\widetilde{G \circ H^{-1}}$ on $c_0(L_j(R))$, which is an extension of $G \circ H^{-1}$ to $c_0(L_j(R))$ and has the same norm

$$\|\widetilde{G \circ H^{-1}}\,|(c_0(L_j(R)))^*\| = \|G \circ H^{-1}\,|(c_0^s(L_j(R)))^*\|.$$

But by (9), $\widetilde{G \circ H^{-1}}$ can be identified with an element $\{\widetilde{G}_j\}_{j \geqslant j_0} \in l_1((L_j(R))^*)$

such that

$$(59)\qquad \left\|G\circ H^{-1}\,|\,(c_0^s(L_j(R)))^*\right\| = \left\|\widetilde{G\circ H^{-1}}\,|\,(c_0(L_j(R)))^*\right\| = \sum_{j=j_0}^{+\infty}\|\tilde{G}_j\,|\,(L_j(R))^*\|\,.$$

Since each $\tilde{G}_j$ can be identified with a $\tilde{g}_j\in L_{j'}(R)$ by

$$\tilde{G}_j(f)=\int_R \tilde{g}_j f d\mu,\qquad \text{for all } f\in L_j(R),$$

with $\|\tilde{G}_j\,|\,(L_j(R))^*\| = \|\tilde{g}_j\|_{j',R}$, it follows from (59) that

$$(60)\qquad \|G\,|\,(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| = \left\|G\circ H^{-1}\,|\,(c_0^s(L_j(R)))^*\right\| = \sum_{j=j_0}^{+\infty}\|\tilde{g}_j\|_{j',R}.$$

Using Theorem 1.38 in [15, p. 29] we get

$$G(f)=\sum_{j=j_0}^{+\infty}\tilde{G}_j(\omega^m_{-\alpha}(j)\,f)=\int_R hf\,d\mu\,,\qquad \text{for all } f\in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$h=\sum_{j=j_0}^{+\infty} g_j\qquad\text{and}\qquad g_j=\tilde{g}_j\,\omega^m_{-\alpha}(j),\qquad j\geq j_0,$$

because, for each $f\in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$,

$$\sum_{j=j_0}^{+\infty}\int_R |f\omega^m_{-\alpha}(j)\,\tilde{g}_j\,|\,d\mu \leq \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R}\sum_{j=j_0}^{+\infty}\|\tilde{g}_j\|_{j';R} < +\infty\,.$$

From (60), we get

$$(61)\qquad \|G\,|\,(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \geq \inf\sum_{j=j_0}^{+\infty}\omega^m_{\alpha}(j)\|g_j\|_{j',R},$$

where the infimum is taken over all admissible representations of h that satisfy (55). But since $G=\Phi_h$, we have from (58) and (61) that

$$\|G\,|\,(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| = \inf\sum_{j=j_0}^{+\infty}\omega^m_{\alpha}(j)\|g_j\|_{j',R},$$

where the infimum is taken over all admissible representations of h that satisfy (55).

Now given a function h represented as (54) and satisfying (55), we infer by (50) that there is a $g\in L_{1,1;\boldsymbol{\alpha}}(R)$ such that

$$\Phi_h(f)=\int_R fg\,d\mu,\qquad \text{for all } f\in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Phi_h\,|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|\,.$$

Then it follows, by Theorem 1.39 in [15, p. 30], that $g = h\ \mu - a.e.$, because it is easy to see that $g, h \in L_1(R)$, and

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx |g|_{1,1;\boldsymbol{\alpha};R}.$$

Conversely, let $g \in L_{1,1;\boldsymbol{\alpha}}(R)$. By (50), g defines a linear functional Λ_g on $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ such that

$$\Lambda_g(f) = \int_R fg\,d\mu\,, \qquad \text{for all } f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)\,,$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Lambda_g\,|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|\,.$$

Since there is a function h that can be represented as (54) and satisfying (55) for which $\Lambda_g = \Phi_h$, it follows as above that $g = h\ \mu - a.e.$ and

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx |g|_{1,1;\boldsymbol{\alpha};R}.$$

In order to verify that $|\,.\,|_{1,1;\boldsymbol{\alpha};R}$ is a norm on $L_{1,1;\boldsymbol{\alpha}}(R)$, we just prove the triangle inequality, because the other conditions are not difficult to prove. Let $f, g \in L_{1,1;\boldsymbol{\alpha}}(R)$. Let us consider representations of f and g as (54),

$$f = \sum_{j=j_0}^{+\infty} f_j \qquad \text{and} \qquad g = \sum_{j=j_0}^{+\infty} g_j,$$

and satisfying (55), respectively. Then $f+g$ can be represented as

$$f + g = \sum_{j=j_0}^{+\infty} (f_j + g_j) \tag{62}$$

and, by Minkowski's inequality,

$$\sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|f_j + g_j\|_{j';R} \leqslant \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|f_j\|_{j';R} + \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j';R} < +\infty\,. \tag{63}$$

Now, it follows from (62) and (63) that

$$\begin{aligned} |f+g|_{1,1;\boldsymbol{\alpha};R} &= \inf_{f+g=\sum_{j=j_0}^{+\infty} z_j} \omega^m_\alpha(j)\|z_j\|_{j';R} \\ &\leqslant \inf_{\substack{f=\sum_{j=j_0}^{+\infty} f_j \\ g=\sum_{j=j_0}^{+\infty} g_j}} \omega^m_\alpha(j)\|f_j + g_j\|_{j';R} \\ &\leqslant |f|_{1,1;\boldsymbol{\alpha};R} + |g|_{1,1;\boldsymbol{\alpha};R}, \end{aligned}$$

and the triangle inequality is verified. ■

5. – Applications.

As was referred in the Introduction, there is a version of the extrapolation result in [22, Theorem XII.4.11 (i), p. 119] for sublinear operators. Therefore we start this section by defining sublinear operator and by recalling that extrapolation result; see [17, Theorem V.3.3, p. 124] or [9, Theorem 4.1] for instance.

DEFINITION 5.1. – *Let (R_0, μ_0) and (R_1, μ_1) be measure spaces. Let T be an operator whose domain is some linear subspace of $\mathcal{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathcal{M}(R_1, \mu_1)$. Then T is said to be sublinear if the relations*

$$|T(f+g)| \leqslant |Tf| + |Tg| \qquad \text{and} \qquad |T(\lambda f)| = |\lambda|\,|Tf|$$

hold μ_1-a.e. on R_1 for all f and g in the domain of T and for all scalars λ.

THEOREM 5.1. – *Suppose Ω is a measurable subset of $\mathbb{R}^n$ with finite volume. Let $\alpha > 0$ and $q_0 \in [1, +\infty)$. If A is a bounded sublinear operator in $L_q(\Omega)$, $q_0 \leqslant q < +\infty$, such that*

$$\|Af\|_q \leqslant c q^{1/\alpha} \|f\|_q, \qquad q \geqslant q_0 \geqslant 1\,,$$

then

$$\|Af\|_{E_\alpha(\Omega)} \leqslant c\|f\|_\infty, \qquad \text{for all } f \in L_\infty(\Omega)\,.$$

Now, by the results of Section 4, the following Theorem is an obvious generalisation of the previous one.

THEOREM 5.2. – *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Suppose (R_0, μ_0) and (R_1, μ_1) are finite measure spaces.*

(*i*) *Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either*

$$\|Af\|_{q;\, R_1} \leqslant c\omega^m_{-\alpha}(q)\|f\|_{q;\, R_0}, \qquad \text{for all } f \in L_q(R_0)\,,$$

for each $q \in [q_0, +\infty)$ with $q_0 \geqslant 1$, or

$$\|Af\|_{q;\, R_1} \leqslant c\gamma^m_{-\alpha}(q)\|f\|_{q;\, R_0}, \qquad \text{for all } f \in L_q(R_0)\,,$$

for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geqslant 2$. Then

$$A \colon L_\infty(R_0) \to L_{\infty,\,\infty;\,\boldsymbol{\alpha}}(R_1)\,,$$

and

$$\|Af\|_{\infty,\,\infty;\,\boldsymbol{\alpha};\, R_1} \leqslant c\|f\|_{\infty;\, R_0}, \qquad \text{for all } f \in L_\infty(R_0)\,.$$

(*ii*) *Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either*

$$\|Af\|_{q;\,R_1} \leq c\omega_{\alpha}^{m}(q)\|f\|_{q;\,R_0}, \qquad \text{for all } f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 \geq 1$, or

$$\|Af\|_{q;\,R_1} \leq c\gamma_{\alpha}^{m}(q)\|f\|_{q;\,R_0}, \qquad \text{for all } f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geq 2$. Then

$$A: L_{\infty,\,\infty;\,\boldsymbol{\alpha}}(R_0) \rightarrow L_{\infty}(R_1),$$

and

$$\|Af\|_{\infty;\,R_1} \leq c\|f\|_{\infty,\,\infty;\,\boldsymbol{\alpha};\,R_0}, \qquad \text{for all } f \in L_{\infty,\,\infty;\,\boldsymbol{\alpha}}(R_0).$$

PROOF. – The proof is a consequence of Corollaries 4.3, 4.5 and [12, Theorem 2.11.4, p. 84]. ■

If we take $m = 1$, $\boldsymbol{\alpha} = -1/\alpha$, with $\alpha > 0$ in part (i) of the previous Theorem, we recover Theorem 5.1.

Now we present an extrapolation result involving the GLZ spaces $L_{1,\,1;\,\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_{+}^{m}$, the proof of which is similar to that of Theorem 4.2 in [9].

THEOREM 5.3. – *Let (R_0, μ_0) and (R_1, μ_1) be non-atomic finite measure spaces. Let $m \in \mathbb{N}$, $j_0 \geq 2$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}_{+}^{m}$. Suppose A is an operator whose domain is $\mathfrak{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathfrak{M}(R_1, \mu_1)$ such that:*

(*i*) *for every possible representation of $f \in \mathfrak{M}_0(R_0, \mu_0)$ by $f = \sum_{j=j_0}^{+\infty} f_j$ (convergent μ_0-a.e. on R_0), with $\{f_j\}_j \subset \mathfrak{M}_0(R_0, \mu_0)$, we have $\sum_{j=j_0}^{+\infty} Af_j$ convergent μ_1-a.e. on R_1 and the inequality*

$$|Af| \leq \Big|\sum_{j=j_0}^{+\infty} Af_j\Big| \qquad \mu_1\text{-a.e. on } R_1; \tag{64}$$

(*ii*) *for all $p \in (1, +\infty)$ and all $f \in L_p(R_0)$,*

$$\|Af\|_{p;\,R_1} \leq c\omega_{\boldsymbol{\beta}}^{m}\left(\frac{1}{p-1}\right)\|f\|_{p;\,R_0}, \tag{65}$$

where c is independent of f, p and $\boldsymbol{\beta}$.

Then

$$\|Af\|_{1,\,1;\,\boldsymbol{\alpha};\,R_1} \leq c'\,\|f\|_{1,\,1;\,\boldsymbol{\alpha}+\boldsymbol{\beta};\,R_0}, \tag{66}$$

for all $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$, *for some constant* c' *independent of* f, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$.

PROOF. – Let $j_0 \geq 2$. Fix $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$ and $f = \sum_{j=j_0}^{+\infty} f_j$, with

$$\sum_{j=j_0}^{+\infty} \omega^m_{\alpha+\beta}(j) \| f_j \|_{j', R_0} < +\infty . \tag{67}$$

We remark that $\sum_{j=j_0}^{+\infty} Af_j$ converges $\mu_1 - a.e.$ on R_1, because by Hölder's inequality and (65) we get

$$\sum_{j=j_0}^{+\infty} \int_{R_1} |Af_j| \, d\mu_1 \leq c \sum_{j=j_0}^{+\infty} \omega^m_{\alpha+\beta}(j) \| f_j \|_{j', R_0},$$

and the rest it follows from (67) and from Theorem 1.38 in [15, p. 29].

Now, by (64), (65) and Theorem 4.2, we have

$$\begin{aligned} \|Af\|_{1,1;\boldsymbol{\alpha};R_1} &\leq \left\| \sum_{j=j_0}^{+\infty} Af_j \right\|_{1,1;\boldsymbol{\alpha};R_1} \leq c_1 \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j) \|Af_j\|_{j', R_1} \\ &\leq c_2 \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\, \omega^m_\beta\left(\frac{1}{j'-1} \right) \|f_j\|_{j', R_0} \\ &\leq c_2 \sum_{j=j_0}^{+\infty} \omega^m_{\alpha+\beta}(j) \|f_j\|_{j', R_0}. \end{aligned}$$

Taking the infimum over all the decompositions of f we get (66). ∎

REMARK 5.1. – In the Theorem above we only need the condition (65) be satisfied for all p such that $1 < p \leq p_0$, for some $p_0 \in (1, +\infty)$, because in that case we can consider j_0 large enough. We could also replace (65) by the condition

$$\|Af\|_{p; R_1} \leq c\omega^m_\beta\left(\frac{p}{p-1} \right) \|f\|_{p; R_0},$$

for all $p \in (1, +\infty)$ (or $p \in (1, p_0]$) and for all $f \in L_p(R_0)$, where c is a positive constant independent of f, p and $\boldsymbol{\beta}$.

Since the Hardy-Littlewood maximal operator satisfies part (i) of the previous Theorem trivially and condition (65) with $m = 1$ and $\boldsymbol{\beta} = 1$, we recover the result already known for the maximal operator, *i.e.*

$$M : L^1(\log L)^{\alpha+1}(\Omega) \to L^1(\log L)^{\alpha}(\Omega),$$

and

$$\|Mf|L^1(\log L)^a(\Omega)\| \leqslant c_2 \|f|L^1(\log L)^{a+1}(\Omega)\|,$$

for all $f \in L^1(\log L)^{a+1}(\Omega)$, where $a > 0$; see the literature mentioned in the Introduction.

Acknowledgements. It is a pleasure to thank Prof. D. E. Edmunds for his helpful suggestions during the preparation of this paper. The author is also grateful to Calouste Gulbenkian Foundation and to Mathematics Department of the University of Coimbra for the financial support.

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Pervenuta in Redazione
il 20 settembre 1999