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Operator Semigroups in Banach Space Theory.

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Sunto. – *In questo lavoro, motivati dalla teoria di Fredholm in spazi di Banach e dalla cosiddetta teoria degli ideali di operatori nel senso di Pietsch, viene definito un nuovo concetto di semigruppo di operatori. Questa nuova definizione include quella di molte classi di operatori già studiate in letteratura, come la classe degli operatori di semi-Fredholm, quella degli operatori tauberiani ed altre ancora. Inoltre permette un nuovo ed unificante approccio ad una serie di problemi in teoria degli operatori su spazi di Banach. Ad un ideale di operatori $\mathcal{A}$ vengono associati, in modo naturale, due semigruppi di operatori $\mathcal{A}_+$ e $\mathcal{A}_-$. In particolare, se $\mathcal{W}$ è la classe degli operatori debolmente compatti, il semigruppo di operatori associato $\mathcal{W}_+$ è la classe degli operatori tauberiani. Questo lavoro contiene, oltre che una panoramica sulle proprietá, gli esempi e le applicazioni di tali semigruppi, diversi nuovi risultati. Vengono inoltre posti in evidenza una serie di nuovi problemi aperti che meritano di essere studiati..*

1. – Introduction.

The theory of Fredholm operators stems from the study of the existence and uniqueness of solutions of differential and integral equations from an abstract point of view. In this theory we find two fundamentally different classes of operators: semigroups, like the classes of Fredholm and upper and lower semi-Fredholm operators; and ideals, like the classes of compact, strictly singular, strictly cosingular and inessential operators. The aim of Fredholm theory is to determine the algebraic and topological properties of these semigroups and its stability under perturbation by operators of certain kinds; in particular, by operators in the mentioned ideals.

The ideals of Fredholm theory were one of the sources of inspiration for Pietsch's concept of *operator ideal* [82]. Moreover, many classes of operators which are operator ideals, like the weakly compact, the completely continuous, or the unconditionally converging operators, have been applied to study the isomorphic properties of Banach spaces from a homological point of view. For

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example, Grothendieck [62] isolated the Banach spaces with the *Dunford-Pettis property* as those spaces X such that weakly compact operators from X into any Banach space are completely continuous. His aim was to determine a class of Banach spaces that shares some of the pleasant properties of the spaces $C(K)$ and $L_1(\mu)$. During the sixties and the seventies, the theory of operator ideals became an important branch of functional analysis, with a wealth of results, examples and applications. A systematic account of this theory can be found in [82].

The aim of this paper is to introduce a concept of *operator semigroup* which includes the semigroups of Fredholm theory and some other classes of operators, like the tauberian operators, that have been considered in the literature. The definition of operator semigroup is inspired in that of operator ideal, but these concepts are opposite: For every Banach space X, the identity I_X belongs to all the operator semigroups and the null operator 0_X belongs to all the operator ideals.

Using operator semigroups we can study the isomorphic properties of Banach spaces from a new point of view, similar to that used in the study of the Radon-Nikodym property by means of semiembeddings, G_δ-embeddings and similar classes of operators [21, 34]. In fact, the operator semigroups, like the tauberian operators [79], preserve some isomorphic properties of Banach spaces.

We show that every operator ideal has associated several semigroups. In this way, Fredholm and semi-Fredholm operators are associated with the ideal of compact operators, and tauberian and cotauberian operators are associated with the ideal of weakly compact operators. Some operator semigroups associated with certain classical operator ideals had been previously introduced in a different way [69, 79, 80, 55, 56, 58, 44, 20]. Moreover, the concept of perturbation class allows us to associate operator ideals to operator semigroups.

We develop the basic properties of the operator semigroups, present plenty of examples and indicate some open questions. We give a survey of the results in the literature that may be included in the theory of operator semigroups. Moreover, we describe some methods that can be used to construct or to characterize operator semigroups.

Let us now review the contents of the different sections. After this introduction we give some definitions and some basic results which will be needed later on.

In section 2 we introduce the concept of *operator semigroup* and some related concepts. Moreover, we give some fundamental results for which we include the proofs because, in its full generality, they have not appeared before in the literature.

We show that an operator ideal $\mathcal{A}$ has associated two semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ in a natural way. For some operator ideals defined in terms of sequences, these semigroups have been studied before. We introduce several properties of semigroups like right and left stability, injectivity and surjectivity, openess, and left and right three-space properties, most of them suggested by the corresponding properties of the operator ideals. We show examples of semigroups with and without these properties and examples of semigroups that are not associated with any operator ideal.

Moreover, every operator ideal $\mathcal{A}$ has associated two other semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ defined in terms of left and right invertibility modulo the operator ideal $\mathcal{A}$. The classical case, corresponding to the ideal of all compact operators, was studied by Atkinson [14]. We prove that these semigroups are open and that they allow us to give a nice description of the radical $\mathcal{A}^{rad}$ of the operator ideal $\mathcal{A}$.

We introduce the *perturbation class* $P\mathcal{S}$ associated with a semigroup $\mathcal{S}$, defined as the class of all operators K such that $T+K \in \mathcal{S}$ for every $T \in \mathcal{S}$.

We also associate operator ideals to some semigroups that satisfy a perturbative characterization, and using them we solve a problem of Herman [67].

In section 3 we give a survey of the results about concrete semigroups and some related classes that have appeared in the literature. For the proofs, we refer most of the times to the original source.

First, we consider the semigroups of classical Fredholm theory. These semigroups have been intensely studied and there are good monographies that describe its properties [23, 36, 63]. So we only recall the main properties of the perturbation classes of the upper and lower semi-Fredholm operators. Moreover, we describe the semigroups associated with the operator ideal of inessential operators. This operator ideal $\mathfrak{I}$, introduced in [70], coincides with the perturbation class of Fredholm operators, but its associated semigroups $\mathfrak{I}_+$ and $\mathfrak{I}_-$ are quite different from the semi-Fredholm operators.

We give a detailed description of the semigroups associated with the weakly compact operators $\mathcal{W}$: the tauberian operators $\mathcal{W}_+$ and the cotauberian operators $\mathcal{W}_-$. Apart from the semigroups of Fredholm theory, these are the semigroups that have been studied more intensely. We describe some shortcomings of the class $\mathcal{W}_+$; in particular the imperfect duality between $\mathcal{W}_+$ and $\mathcal{W}_-$ can be seen as an asymmetry for the class of weakly compact operators. For operators acting on $L_1(\mu)$ spaces, these shortcomings do not appear.

We present some of the applications that tauberian operators have found in Banach space theory: preservation of isomorphic properties [79], refinements of James' characterization of reflexive spaces [80], equivalence between the Radon-Nykodim property and the Krein-Milman property [85], and factorization of operators [31].

We describe some semigroups that can be characterized in terms of sequences. They are related with some results similar to Lohman's lifting [73]. Among them, the semigroup $\mathcal{R}_+$ associated with the weakly precompact operators admits some nice characterizations. We also show how these semigroups allow us to characterize some classes of Banach spaces.

Moreover, we show that the semiembeddings and the G_δ-embeddings, although do not form semigroups, have analogous properties and have close relations with some of the semigroups previously considered. These classes were introduced in relation with the study of the Radon-Nikodym property and the inclusion of subspaces isomorphic to $L_1[0, 1]$.

In section 4 we describe some additional methods that have been employed to define or to characterize semigroups. Here again, we do not include proofs of the results, but refer to the literature.

We consider some semigroups that can be described as ultrapowers of other semigroups. In general they have better behaviour than the original semigroups. Some results about local reflexivity for dual spaces [45] allow us to obtain characterizations of the semigroups. Additional properties are showed for operators acting on $L_1(\mu)$ spaces.

Also, we describe some semigroups associated with incomparability of Banach spaces; namely, the total incomparability, introduced by Rosenthal [83] and the total coincomparability, studied in [10, 54]. These semigroups satisfy a kind of three-space property and admit a perturbative characterization. Moreover, they allow us to characterize the notions of incomparability.

Finally, we show how to define semigroups in terms of certain operational quantities associated with a space ideal. These semigroups are open and, consequently, they do not coincide in general with the corresponding semigroups defined in terms of operator ideals in Section 2.1.

1.1. *Preliminaries.*

Along the paper, X, Y and Z are Banach spaces, and we denote by X^* and X^{**} the *dual space* and the *second dual* of X, respectively. As usual, we identify X with a subspace of X^{**}, and we say that the space X is *reflexive* if $X = X^{**}$.

We denote by $\mathcal{L}(X, Y)$ the set of all (continuous linear) operators from X into Y. For $T \in \mathcal{L}(X, Y)$, we denote by $N(T)$ and $R(T)$ the *kernel* and the *range* of T, respectively. Moreover, $T^* \in \mathcal{L}(Y^*, X^*)$ denotes the *conjugate* operator of T, and $T^{**} \in \mathcal{L}(X^{**}, Y^{**})$ the *second conjugate* of T.

A *subspace* is always a closed linear subspace. Given a subspace M of X, we denote by J_M the inclusion of M into X, and by Q_M the quotient map from X onto X/M.

A series $\sum_{n=1}^{\infty} x_n$ is a Banach space X is said to be *weakly unconditionally Cauchy* if $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for every $f \in X^*$. The series $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if every subseries $\sum_{k=1}^{\infty} x_{n_k}$ is convergent.

We denote by $\mathcal{L}$ the class of all operators between Banach spaces; i.e. the union of all $\mathcal{L}(X, Y)$, and by $\mathcal{F}$ the subclass of all *finite dimensional range* operators. Given a subclass $\mathcal{A} \subset \mathcal{L}$, the *components of* $\mathcal{A}$ are the subsets

$$\mathcal{A}(X, Y) := \mathcal{A} \cap \mathcal{L}(X, Y).$$

In the case $X = Y$, we simply write $\mathcal{A}(X)$ for $\mathcal{A}(X, X)$.

DEFINITION 1.1 [82]. – *A subclass* $\mathcal{A} \subset \mathcal{L}$ *is said to be an* operator ideal *if it satisfies the following conditions*:

(α_1) $\mathcal{F} \subset \mathcal{A}$.

(α_2) $\mathcal{A}(X, Y)$ *is a subspace of* $\mathcal{L}(X, Y)$.

(α_3) *If* $T \in \mathcal{L}(W, X)$, $S \in \mathcal{A}(X, Y)$ *and* $R \in \mathcal{L}(Y, Z)$, *then* $RST \in \mathcal{A}(W, Z)$.

Observe that in the definition of operator ideal we may replace condition (α_2) by

$$(\alpha_2')\ \ S \in \mathcal{A}(W, Y), T \in \mathcal{A}(X, Z) \Leftrightarrow S \oplus T \in \mathcal{A}(W \oplus X, Y \oplus Z),$$

where the operator $S \oplus T$ is defined by $S \oplus T(w, x) := (Sx, Tw)$.

The following classes of operators are well-known operator ideals that admit sequential characterizations.

DEFINITION 1.2. – *An operator* $T \in \mathcal{L}(X, Y)$ *is said to be* (weakly) compact *if it takes bounded sets into relatively* (*weakly*) *compact sets; equivalently, if for every bounded sequence* (x_n) *in* X, (Tx_n) *admits a* (*weakly*) *convergent subsequence.*

It is said to be weakly precompact *if for every bounded sequence* (x_n) *in* X, (Tx_n) *admits a weakly Cauchy subsequence.*

It is said to be (weakly) completely continuous *if it takes weakly Cauchy sequences into* (*weakly*) *convergent sequences.*

It is said to be unconditionally converging *if it takes weakly unconditionally Cauchy series into unconditionally convergent series.*

We denote by $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{CC}$, $\mathcal{WCC}$ *and* $\mathcal{U}$ *the classes of all compact, weakly compact, weakly precompact, completely continuous, weakly completely continuous and unconditionally converging operators, respectively.*

Now we define the operator ideals that appeared in Fredholm theory.

DEFINITION 1.3. – *An operator* $T \in \mathcal{L}(X, Y)$ *is said to be* strictly singular *if no restriction* TJ_M *of* T *to an infinite dimensional subspace* M *of* X *is an isomorphism.*

The operator T is said to be strictly cosingular *if there is no infinite codimensional subspace N of Y such that $Q_N T$ is surjective.*

We denote by $\mathcal{SS}$ and $\mathcal{SC}$ the classes of all strictly singular and strictly cosingular operators, respectively.

The following result provides basic relations between $\mathcal{K}$, $\mathcal{SS}$ and $\mathcal{SC}$. The converse implications are not true and the classes $\mathcal{SS}$ and $\mathcal{SC}$ are not comparable [4].

PROPOSITION 1.4 [82]. – *Let $T \in \mathcal{L}(X, Y)$.*

(a) $T \in \mathcal{K} \Rightarrow T \in \mathcal{SS} \cap \mathcal{SC}$.

(b) $T^* \in \mathcal{SC} \Rightarrow T \in \mathcal{SS}$ *and* $T^* \in \mathcal{SS} \Rightarrow T \in \mathcal{SC}$.

We denote by F the class of all finite dimensional Banach spaces.

DEFINITION 1.5 [82]. – *A class A of Banach spaces is said to be a* space ideal *if it satisfies the following conditions*:

(β_1) $\mathsf{F} \subset \mathsf{A}$.

(β_2) $X, Y \in \mathsf{A} \Leftrightarrow X \oplus Y \in \mathsf{A}$.

(β_3) X *isomorphic to* $Y \in \mathsf{A} \Rightarrow X \in \mathsf{A}$.

An operator ideal $\mathcal{A}$ has associated a space ideal $Sp(\mathcal{A})$, given by

$$Sp(\mathcal{A}) := \{X : I_X \in \mathcal{A}\}.$$

The space ideals associated with the operator ideals $\mathcal{W}$, $\mathcal{R}$, $\mathcal{CC}$, $\mathcal{WCC}$ and $\mathcal{U}$ introduced in Definition 1.2 are the reflexive spaces, the spaces containing no copies of l_1, the spaces with the Schur property, the weakly sequentially complete spaces and the spaces containing no copies of c_0, respectively. For $\mathcal{A}$ equal to $\mathcal{K}$, $\mathcal{SS}$ or $\mathcal{SC}$, we have $Sp(\mathcal{A}) = \mathsf{F}$ [82].

For every operator ideal $\mathcal{A}$, the *dual operator ideal* $\mathcal{A}^d$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^d = \{T \in \mathcal{L} : T^* \in \mathcal{A}\}.$$

Observe that $\mathcal{F}$, $\mathcal{K}$ and $\mathcal{W}$ are self-dual. Moreover, $Sp(\mathcal{A}^d) = \{X : X^* \in Sp(\mathcal{A})\}$.

DEFINITION 1.6 [82]. – *Let $\mathcal{A}$ be an operator ideal.*

$\mathcal{A}$ is closed *if its components $\mathcal{A}(X, Y)$ are closed subsets of $\mathcal{L}(X, Y)$.*

$\mathcal{A}$ is injective *if given a subspace Y of Z, $T \in \mathcal{L}(X, Y)$, $J_Y T \in \mathcal{A} \Rightarrow T \in \mathcal{A}$.*

$\mathcal{A}$ is surjective *if given a subspace Y of Z, $T \in \mathcal{L}(Z/Y, X)$, $TQ_Y \in \mathcal{A} \Rightarrow T \in \mathcal{A}$.*

The operator ideals $\mathcal{K}$, $\mathcal{W}$ and $\mathcal{R}$ are closed, injective and surjective; $\mathcal{CC}$, $\mathcal{WCC}$, $\mathcal{U}$ and $\mathcal{SS}$ are closed and injective, but not surjective; $\mathcal{SC}$ is closed and surjective, but not injective. Moreover, $\mathcal{A}$ injective (surjective) implies $\mathcal{A}^d$ surjective (injective).

Now we recall the definition of the semigroups that appeared in Fredholm theory.

DEFINITION 1.7. – *An operator $T \in \mathcal{L}(X, Y)$ is said to be* upper semi-Fredholm *if $R(T)$ is closed and $N(T)$ is finite dimensional.*

It is said to be lower semi-Fredholm *if $R(T)$ is finite co-dimensional, hence closed.*

It is said to be Fredholm *if it is upper semi-Fredholm and lower semi-Fredholm.*

We denote by Φ_+, Φ_- and Φ the classes of all upper semi-Fredholm, lower semi-Fredholm and Fredholm operators, respectively.

It easily follows from the basic relations of duality that an operator T is upper (lower) semi-Fredholm if and only if its conjugate T^* is lower (upper) semi-Fredholm.

For every operator $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y)$, we define the *index* $\operatorname{ind}(T)$ by

$$\operatorname{ind}(T) := \dim N(T) - \dim Y/R(T) \in \boldsymbol{Z} \cup \{\pm \infty\}.$$

The next result shows that the index is continuous.

THEOREM 1.8 [91, Theorem IV.13.8]. – *For every operator $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y)$, there exists a number $\delta_T > 0$ so that*

$$A \in \mathcal{L}(X, Y),\ \|A\| < \delta_T \Rightarrow T + A \in \Phi_+ \cup \Phi_- \ \text{ and } \operatorname{ind}(T+A) = \operatorname{ind}(T).$$

The study of the following classes of operators was initiated by Atkinson [14].

DEFINITION 1.9. – *An operator $T \in \mathcal{L}(X, Y)$ is said to be* left-Atkinson *if there exists $A \in \mathcal{L}(Y, X)$ such that $I_X - AT \in \mathcal{K}(X)$.*

It is said to be right-Atkinson *if there exists $B \in \mathcal{L}(Y, X)$ such that $I_Y - TB \in \mathcal{K}(Y)$.*

We denote by Φ_l and Φ_r the classes of all left-Atkinson and right-Atkinson operators, respectively.

In the definition of Atkinson operators we can replace $\mathcal{K}$ by $\mathcal{F}$. Moreover, Atkinson operators are semi-Fredholm as the following result shows.

PROPOSITION 1.10 [91, IV.13 Problems]. – *Let* $T \in \mathcal{L}(X, Y)$.

(a) $T \in \Phi_l$ *if and only if* $T \in \Phi_+$ *and* $R(T)$ *is complemented.*

(b) $T \in \Phi_r$ *if and only if* $T \in \Phi_-$ *and* $N(T)$ *is complemented.*

(c) $\Phi(X, Y) = \Phi_l(X, Y) \cap \Phi_r(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y)$.

The following class of operators appeared in the study of the perturbation of Fredholm operators.

DEFINITION 1.11 [81] (see also [82, 4.3.5, 26.7.2]). – *An operator* $T \in \mathcal{L}(X, Y)$ *is said to be* inessential *if* $I_X - ST \in \Phi(X)$, *for every* $S \in \mathcal{L}(Y, X)$.

We denote by $\mathfrak{I}(X, Y)$ *the set of all inessential operators from* X *into* Y.

In the case $X = Y$, the class $\mathfrak{I}$ was introduced by Kleinecke [70] as the radical of the approximable operators. This class admits some weaker characterizations.

THEOREM 1.12 [81, 2, 3]. – *For* $T \in \mathcal{L}(X, Y)$ *the following assertions are equivalent*:

(a) T *is inessential.*

(b) *For every* $S \in \mathcal{L}(Y, X)$ *the kernel* $N(I_X - ST)$ *is finite dimensional.*

(c) *For every* $S \in \mathcal{L}(Y, X)$ *the cokernel* $X/\overline{R(I_X - ST)}$ *is finite dimensional.*

(d) *For every* $S \in \mathcal{L}(Y, X)$ *the kernel* $N(I_Y - TS)$ *is finite dimensional.*

(e) *For every* $S \in \mathcal{L}(Y, X)$ *the cokernel* $Y/\overline{R(I_Y - TS)}$ *is finite dimensional.*

Moreover, $\mathfrak{I}$ is the closed operator ideal that properly contains the other ideals $\mathcal{K}$, $\mathcal{SS}$ and $\mathcal{SC}$ that occur in Fredholm theory [82, 26.7.3].

Apart from the semi-Fredholm and the Atkinson operators, the tauberian and cotauberian operators are the semigroups that have been more studied. Introduced in [69] and [87], they were considered in [96] as the counterparts of upper and lower semi-Fredholm operators in a generalized Fredholm theory in which the reflexive spaces play the role of the finite dimensional spaces in the classical case.

DEFINITION 1.13 [69, 87]. – *An operator* $T \in \mathcal{L}(X, Y)$ *is said to be* tauberian *if its second conjugate* T^{**} *satisfies* $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$.

It is said to be cotauberian *if* T^* *is tauberian.*

For every $T \in \mathcal{L}(X, Y)$, we consider the operator $T^{co}: X^{**}/X \to Y^{**}/Y$, given by

$$T^{co}(\alpha + X) = T^{**}\alpha + Y, \qquad \alpha \in X^{**}.$$

Since we can identify $(T^{co})^*$ and $(T^*)^{co}$, we obtain the following result.

PROPOSITION 1.14. – *Let* $T \in \mathcal{L}(X, Y)$.

(a) *T is tauberian if and only if T^{co} is injective.*

(b) *T is cotauberian if and only if T^{co} has dense range.*

Fredholm theory is better understood for operators acting on spaces that contain many complemented subspaces, like those introduced in the following definition.

DEFINITION 1.15. – *A Banach space X is said to be* subprojective *if every infinite dimensional subspace of X contains an infinite dimensional, complemented subspace of X.*

The space X is said to be superprojective *if every infinite codimensional subspace of X is contained in an infinite codimensional, complemented subspace of X.*

The spaces l_p $(1 < p < \infty)$ are subprojective and superprojective. Moreover, $L_p[0, 1]$ is subprojective for $2 < p < \infty$ and superprojective for $1 < p < 2$ [95]. The spaces $L_1[0, 1]$ and $C[0, 1]$ are neither subprojective nor superprojective. For further examples see [3].

2. – Fundamental results.

In this section we introduce a general notion of *operator semigroup*, and give some basic properties and examples. Every operator ideal $\mathcal{A}$ has associated four semigroups $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{A}_l$ and $\mathcal{A}_r$ in an analogous way as the classes Φ_+, Φ_-, Φ_l and Φ_r of classical Fredholm theory are associated with the operator ideal $\mathcal{K}$ of all compact operators.

The semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ are defined in an algebraic manner. Some properties of $\mathcal{A}$ can be characterized in terms of $\mathcal{A}_+$ and $\mathcal{A}_-$, and this fact allows us to define special classes of semigroups.

The semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ are defined in terms of left and right invertibility modulo $\mathcal{A}$. The radical $\mathcal{A}^{rad}$ of $\mathcal{A}$ admits a nice description in terms of $\mathcal{A}_l$ and $\mathcal{A}_r$.

We also define the perturbation class $P\mathcal{S}$ of a semigroup $\mathcal{S}$ and describe its basic properties. In particular, $P\mathcal{S}(X)$ is a two-sided ideal in $\mathcal{L}(X)$.

2.1. *From operator ideals to operator semigroups.*

The following Definition should be compared with that of operator ideal (Definition 1.1). We denote $\mathcal{G} := \{T \in \mathcal{L} : T \text{ bijective}\}$.

DEFINITION 2.1. – *A subclass* $\mathcal{S} \subset \mathcal{L}$ *is said to be an* operator semigroup (*a semigroup, for short*) *if it satisfies the following conditions*:

(σ_1) $\mathcal{G} \subset \mathcal{S}$.

(σ_2) $S \in \mathcal{S}(W, Y)$ *and* $T \in \mathcal{S}(X, Z)$ *if and only if* $S \oplus T \in \mathcal{S}(W \oplus X, Y \oplus Z)$.

(σ_3) *If* $T \in \mathcal{S}(X, Y)$ *and* $S \in \mathcal{S}(Y, Z)$, *then* $ST \in \mathcal{S}(X, Z)$.

REMARKS 2.2. – (a) Although the definitions of operator ideal and operator semigroup are similar, these are opposite concepts. Indeed, for a Banach space X, the zero operator 0_X belongs to every ideal, and the identity operator I_X belongs to every semigroup.

(b) As a consequence of (σ_1) and (σ_3), a semigroup is stable under multiplication by isomorphisms; in particular, by non-zero scalars.

(c) The class $\mathcal{G}$ is the smallest semigroup. Moreover, the classes Φ, Φ_+, Φ_-, Φ_l and Φ_r of Fredholm theory are also semigroups.

DEFINITION 2.3. – *Let* $\mathcal{A}$ *be an operator ideal. We define the class* $\mathcal{A}_+$ *by*

$$\mathcal{A}_+ := \{T \in \mathcal{L} : S \in \mathcal{L},\, TS \in \mathcal{A} \Rightarrow S \in \mathcal{A}\};$$

i.e., $T \in \mathcal{L}(X, Y)$ *belongs to* $\mathcal{A}_+$ *if and only if for every Banach space* Z *and every operator* $S \in \mathcal{L}(Z, X)$, *if* $TS \in \mathcal{A}$ *then* $S \in \mathcal{A}$.

Analogously, we define the class $\mathcal{A}_-$ *by*

$$\mathcal{A}_- := \{T \in \mathcal{L} : S \in \mathcal{L},\, ST \in \mathcal{A} \Rightarrow S \in \mathcal{A}\}.$$

PROPOSITION 2.4. – *The classes* $\mathcal{A}_+$ *and* $\mathcal{A}_-$ *are operator semigroups and* $\Phi \subset \mathcal{A}_+ \cap \mathcal{A}_-$.

PROOF. – We shall give the proof only for $\mathcal{A}_+$, since the proof for $\mathcal{A}_-$ is analogous.

(σ_1): We show that $\Phi \subset \mathcal{A}_+$. Assume $T \in \Phi$, $S \in \mathcal{L}$ and $TS \in \mathcal{A}$. Then there exists $U \in \mathcal{L}$ such that $K := I_X - UT \in \mathcal{F} \subset \mathcal{A}$. Hence $S = UTS + KS \in \mathcal{A}$.

(σ_2): Assume $S, T \in \mathcal{A}_+$, $R \in \mathcal{L}$ and $(S \oplus T)R \in \mathcal{A}$. Writing $R = (R_1, R_2)$, by (α_2) we have $SR_1, TR_2 \in \mathcal{A}$. Since $S, T \in \mathcal{A}_+$ we have $R_1, R_2 \in \mathcal{A}$; hence $R \in \mathcal{A}$.

Conversely, if $(S \oplus T) \in \mathcal{A}_+$, $R \in \mathcal{L}$ and $SR \in \mathcal{A}$, we have $(S \oplus T)(R, 0) \in \mathcal{A}$.

Then $(R, 0) \in \mathcal{A}$, hence $R \in \mathcal{A}$ and we conclude $S \in \mathcal{A}_+$. Analogously, we have $T \in \mathcal{A}_+$.

(σ_3): Assume $S, T \in \mathcal{A}_+$, $A \in \mathcal{L}$ and $STA \in \mathcal{A}$. Since $S \in \mathcal{A}_+$ we obtain $TA \in \mathcal{A}$; hence $A \in \mathcal{A}$, because $T \in \mathcal{A}_+$. ■

REMARKS 2.5. – (*a*) Let $J_X: X \to X^{**}$ denote the natural injection. An operator ideal $\mathcal{A}$ is said to be *regular* if $T \in \mathcal{L}(X, Y)$, $J_Y T \in \mathcal{A} \Rightarrow T \in \mathcal{A}$ [82]. Note that $\mathcal{A}$ is regular if and only if $J_X \in \mathcal{A}_+$ for every X.

(*b*) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are semigroups, then $\mathcal{S}_1 \cap \mathcal{S}_2$ is clearly a semigroup.

EXAMPLE 2.6. – Here we describe some semigroups associated with the operator ideals introduced in Definition 1.2 and with the finite dimensional operators $\mathcal{F}$.

(*a*) The semigroups of classical Fredholm theory are associated with the compact operators $\mathcal{K}$:

$$\mathcal{K}_+ = \Phi_+, \quad \mathcal{K}_- = \Phi_- \quad \text{and} \quad \mathcal{K}_+ \cap \mathcal{K}_- = \Phi \ [71].$$

(*b*) The semigroups $\mathcal{W}_+$ and $\mathcal{W}_-$ associated with the weakly compact operators $\mathcal{W}$ coincide with the tauberian and the cotauberian operators, respectively [55, 56].

(*c*) For $\mathcal{A}$ the completely continuous $\mathcal{CC}$, the weakly completely continuous $\mathcal{WCC}$, the unconditionally converging $\mathcal{U}$, or the weakly precompact operators $\mathcal{R}$, the semigroups $\mathcal{A}_+$ and $\{T \in \mathcal{L}: T^* \in \mathcal{A}_+\}$ were studied in [55, 56, 58, 44] (see section 3.4).

(d) It is not difficult to see that for the finite dimensional operators $\mathcal{F}$ we obtain

$$\mathcal{F}_+ = \{T \in \mathcal{L}: \dim N(T) < \infty\} \quad \text{and} \quad \mathcal{F}_- = \{T \in \mathcal{L}: \dim Y/\overline{R(T)} < \infty\}.$$

EXAMPLE 2.7. – There are semigroups that contain Φ, but cannot be obtained as $\mathcal{A}_+$ or $\mathcal{A}_-$ for any operator ideal $\mathcal{A}$. An example is the class of those operators $T \in \mathcal{L}$ such that the second conjugate T^{**} has finite dimensional kernel.

Indeed, assume that $\mathcal{A}_+ = \{T \in \mathcal{L}: \dim N(T^{**}) < \infty\}$. If $S \in \mathcal{L}(l_2, Y)$ has infinite dimensional range, composing if necessary with a suitable operator on l_2 we may assume that S is also injective. Clearly, $T \in \mathcal{A}_+$, $K \in \mathcal{A} \Rightarrow T - K \in \mathcal{A}_+$. In our case $S \in \mathcal{A}_+$; hence $S \notin \mathcal{A}$, because the zero operator in $\mathcal{L}(l_2, Y)$ does not belong to $\mathcal{A}_+$. Thus $\mathcal{A}(l_2, Y) = \mathcal{F}(l_2, Y)$ for every Banach space Y.

Moreover, the operator $T \in \mathcal{L}(l_1, l_1)$ given by $T(x_n) = (x_n/n)$ satisfies $\dim N(T) < \infty$ and $\dim N(T^{**}) = \infty$. Since $\mathcal{F}_+ = \{T \in \mathcal{L}: \dim N(T) < \infty\}$, there exist a Banach space X and an operator $S \notin \mathcal{A}(X, l_1)$, so that $TS \in \mathcal{A}$ and

$R(TS)$ is infinite dimensional. Now, it is not difficult to obtain an operator $U \in \mathcal{L}(l_2, X)$ so that $TSU \in \mathcal{A}$ and $R(TSA)$ is infinite dimensional. This is a contradiction.

An interesting example of operator semigroup is the class of all *strongly tauberian* operators, studied by Rosenthal [84], which consists of those $T \in \mathcal{L}(X, Y)$ such that $T^{co} \in \mathcal{L}(X^{**}/X, Y^{**}/Y)$ is an isomorphism (into). It is not difficult to see that strongly tauberian operators are tauberian, but the converse implication fails.

QUESTION 1. – We do not know if there exists an operator ideal $\mathcal{A}$ such that $\mathcal{A}_+$ coincides with the strongly tauberian operators.

The following result shows that $\mathcal{A}_+$ and $\mathcal{A}_-$ satisfy certain cancellation properties. Its proof is a direct application of the definitions.

PROPOSITION 2.8. – *Let $\mathcal{A}$ be an operator ideal. If $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$, then*

(a) $ST \in \mathcal{A}_+ \Rightarrow T \in \mathcal{A}_+$.

(b) $ST \in \mathcal{A}_- \Rightarrow S \in \mathcal{A}_-$.

DEFINITION 2.9. – *We say that a semigroup $\mathcal{S}$ is* left stable *if $ST \in \mathcal{S}$ implies $T \in \mathcal{S}$. We say that $\mathcal{S}$ is* right stable *if $ST \in \mathcal{S}$ implies $S \in \mathcal{S}$.*

Note that Φ_+ is left but not right stable, Φ_- is right but not left stable, and Φ is neither left nor right stable.

Now we associate a class of Banach spaces to a semigroup in a similar way that an operator ideal $\mathcal{A}$ has associated its space ideal $Sp(\mathcal{A})$.

DEFINITION 2.10. – *Given an operator semigroup $\mathcal{S}$ we define*

$$Sp(\mathcal{S}) := \{X : 0_X \in \mathcal{A}\}.$$

It is immediate to check that $Sp(\mathcal{S})$ is a space ideal when $\Phi_0 \subset \mathcal{S}$. This fact and the following result show that the definition of $Sp(\mathcal{S})$ is natural.

PROPOSITION 2.11. – *For every operator ideal $\mathcal{A}$ we have $Sp(\mathcal{A}) = Sp(\mathcal{A}_+) = Sp(\mathcal{A}_-)$.*

PROOF. – If $X \in Sp(\mathcal{A})$, then any operator defined on X or taking values in X belongs to $\mathcal{A}$; hence $X \in Sp(\mathcal{A}_+) \cap Sp(\mathcal{A}_-)$. On the other hand, if $0_X \in \mathcal{A}_+(X)$, since $0_X I_X \in \mathcal{A}$, we get $I_X \in \mathcal{A}$; and analogously when $0_X \in \mathcal{A}_-(X)$. ■

Recall that in the preliminaries we included the definitions of injective and surjective operator ideal.

PROPOSITION 2.12. – *Let $\mathcal{A}$ be an operator ideal.*

(a) *$\mathcal{A}$ is injective if and only if $\Phi_+ \subset \mathcal{A}_+$.*

(b) *$\mathcal{A}$ is surjective if and only if $\Phi_- \subset \mathcal{A}_-$.*

PROOF. – (a) Clearly $\mathcal{A}$ is injective if and only if the isomorphic embeddings belong to $\mathcal{A}_+$. So, assume that $\mathcal{A}$ is injective and $T \in \Phi_+(X, Y)$. We can write $X = M \oplus N(T)$, where M is a finite codimensional (closed) subspace such that the restriction $T|_M$ is an isomorphic embedding. Thus, if P is a projection from X onto M and $TK \in \mathcal{A}$ then $PK \in \mathcal{A}$ and $(I - P)K$ is finite dimensional; hence $K \in \mathcal{A}$ and we conclude that $T \in \mathcal{A}_+$.

The proof of (b) is analogous. ∎

DEFINITION 2.13. – *We say that a semigroup $\mathcal{S}$ is* injective *if $\Phi_+ \subset \mathcal{S}$. We say that $\mathcal{S}$ is* surjective *if $\Phi_- \subset \mathcal{S}$.*

PROPOSITION 2.14. – *Let $\mathcal{S}$ be an operator semigroup.*

(a) *If $\mathcal{S}$ is injective and left stable, then $T \in \mathcal{S}(X, Y) \Rightarrow N(T) \in Sp(\mathcal{S})$.*

(b) *If $\mathcal{S}$ is surjective and right stable, then $T \in \mathcal{S}(X, Y) \Rightarrow Y/\overline{R(T)} \in Sp(\mathcal{S})$.*

PROOF. – (a) If $T \in \mathcal{S}(X, Y)$ and $J : N(T) \to X$ is the inclusion, we have $J \in \mathcal{S}$ because $\mathcal{S}$ is injective. Then $TJ = 0 \in \mathcal{S}$. Since $\mathcal{S}$ is left stable we get $0_{N(T)} \in \mathcal{S}$; i.e., $N(T) \in Sp(\mathcal{S})$.

(b) Analogous. ∎

COROLLARY 2.15. – *Let $\mathcal{A}$ be an operator ideal and let $T \in \mathcal{L}(X, Y)$.*

(a) *$\mathcal{A}$ injective, $T \in \mathcal{A}_+ \Rightarrow N(T) \in Sp(\mathcal{A})$.*

(b) *$\mathcal{A}$ surjective, $T \in \mathcal{A}_- \Rightarrow Y/\overline{R(T)} \in Sp(\mathcal{A})$.*

We say that a class of Banach spaces A satisfies the *three-space property* if given a subspace M of a Banach space X we have $M, X/M \in \mathsf{A} \Rightarrow X \in \mathsf{A}$.

Many classes of Banach spaces, like the reflexive spaces and the weakly sequentially continuous spaces have the three-space property. On the other hand, the class of spaces isomorphic to a Hilbert space fails this property. We refer to [25] for a complete survey on the three space property in Banach spaces.

Sometimes, we can characterize the operators T with closed range $R(T)$ which belong to a semigroup $\mathcal{S}$ in terms of their kernel $N(T)$ or their cokernel $Y/R(T)$. In such a case $Sp(\mathcal{S})$ has the three-space property.

DEFINITION 2.16. – *We say that an operator semigroup* $\mathcal{S}$ *satisfy the* left three-space property *if every* $T \in \mathcal{L}(X, Y)$ *with* $R(T)$ *closed and* $N(T) \in Sp(\mathcal{S})$ *belongs to* $\mathcal{S}$.

We say that $\mathcal{S}$ *satisfy the* right three-space property *if an operator* $T \in \mathcal{L}(X, Y)$ *with* $R(T)$ *closed and* $Y/R(T) \in Sp(\mathcal{S})$ *belongs to* $\mathcal{S}$.

REMARKS 2.17. – (*a*) It is easy to see that if $\mathcal{S}$ satisfies the left (right) three-space property and is left (right) stable, then $Sp(\mathcal{S})$ has the three-space property.

(*b*) If $\mathcal{A}$ is one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{CC}$, $\mathcal{WCC}$ or $\mathcal{U}$ of Definition 1.2, then $\mathcal{A}_+$ satisfies the left three-space property and $\mathcal{A}_-^d$ satisfies the right three-space property. From these facts, it easily follows that $Sp(\mathcal{A})$ has the three-space property.

QUESTION 2. – To determine the operator ideals $\mathcal{A}$ such that $\mathcal{A}_+$ satisfies the left three-space property or $\mathcal{A}_-$ satisfies the right three-space property. In particular, what happens with $\mathcal{R}_-$ and $\mathcal{R}_+^d$?

Note that if $\mathcal{A}_+$ satisfies the left three-space property then $\mathcal{A}$ is injective; if $\mathcal{A}_-$ satisfies the right three-space property then $\mathcal{A}$ is surjective and in both cases $Sp(\mathcal{A})$ has the three-space property.

DEFINITION 2.18. – *We say that a semigroup* $\mathcal{S}$ *is* open *if the components* $\mathcal{S}(X, Y)$ *are open in* $\mathcal{L}(X, Y)$.

The semigroups $\mathcal{K}_+$ and $\mathcal{K}_-$ of semi-Fredholm operators are open, as well as some semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ considered in section 4.1. However, the following example shows that the semigroups $\mathcal{W}_+$ and $\mathcal{W}_-$ of tauberian and cotauberian operators are not open.

EXAMPLE 2.19 [87]. – Let X be a non-reflexive Banach space, and consider the operators T and T_n defined in $l_2(X)$ by

$$T(x_k) := (x_k/k); \quad \text{and } T_n(x_k) := (x_1, x_2/2, \ldots x_n/n, 0, 0, \ldots).$$

It is not difficult to see that T is tauberian and cotauberian (see Definition 1.13). However, T_n is neither tauberian nor cotauberian and $\|T - T_n\| < 1/n$, for every integer n. Thus $\mathcal{W}_+(l_2(X))$ and $\mathcal{W}_-(l_2(X))$ are not open in $\mathcal{L}(l_2(X))$.

The following result shows that the operators in $\mathcal{A}$ are admissible perturbations for the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$. Its proof is immediate.

PROPOSITION 2.20. – *Let* $\mathcal{A}$ *an operator ideal and let* $K \in \mathcal{A}(X, Y)$.

(a) $T \in \mathcal{A}_+(X, Y) \Rightarrow T + K \in \mathcal{A}_+(X, Y)$.

(b) $T \in \mathcal{A}_-(X, Y) \Rightarrow T + K \in \mathcal{A}_-(X, Y)$.

Some semigroups admit a perturbative characterization; for example, those associated with the operator ideals presented in Definition 1.2.

PROPOSITION 2.21. – *Let* $\mathcal{A}$ *be one of the operator ideals* $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{CC}$, $\mathcal{WCC}$ *or* $\mathcal{U}$ *of Definition* 1.2. *Then for every* $T \in \mathcal{L}(X, Y)$, *we have*

(a) $T \in \mathcal{A}_+$ *if and only if* $N(T+K) \in Sp(\mathcal{A})$ *for every* $K \in \mathcal{K}(X, Y)$.

(b) $T \in \mathcal{A}_-^d$ *if and only if* $T^* \in \mathcal{A}_+$.

(c) $T \in \mathcal{A}_-^d$ *if and only if* $Y/\overline{R(T+K)} \in Sp(\mathcal{A}^d)$ *for every* $K \in \mathcal{K}(X, Y)$.

PROOF. – (*a*) The case of the compact operators $\mathcal{K}$ is classic [71]. For the unconditionally converging operators $\mathcal{U}$ see [44], and for the remaining cases see [58].

(*b*) Assume that $T^* \in \mathcal{A}_+$ and $BT \in \mathcal{A}^d$. We have $(BT)^* = T^*B^* \in \mathcal{A}$. Then $B^* \in \mathcal{A}$, hence $B \in \mathcal{A}^d$, and we conclude that $T \in \mathcal{A}_-^d$.

Conversely, assume that $T \in \mathcal{A}_-^d(X, Y)$. It was proved in [44] and [58] that $T^* \in \mathcal{A}_+$ if and only if $Y/\overline{R(T+K)} \in Sp(\mathcal{A}^d)$ for every $K \in \mathcal{K}(X, Y)$. Clearly, $T + K \in \mathcal{A}_-^d$ for every compact operator K. So it is enough to prove that $Y/\overline{R(T)} \in Sp(\mathcal{A}^d)$.

If Q denotes the quotient map from Y onto $Y/\overline{R(T)}$, we have $QT = 0 \in \mathcal{A}^d$. Then $Q \in \mathcal{A}^d$ and, since the operator ideals $\mathcal{A}^d$ are surjective, we conclude that $Y/\overline{R(T)} \in Sp(\mathcal{A}^d)$.

(*c*) It follows from (*b*) and the perturbative characterization, mentioned in the proof of (*b*), of the operators T such that $T^* \in \mathcal{A}_+$. ■

DEFINITION 2.22. – *Let* $\mathcal{A}$ *be an operator ideal. We say that* $\mathcal{A}_+$ admits a perturbative characterization *if for every pair* X, Y,

$$\mathcal{A}_+(X, Y) = \{T \in \mathcal{L}(X, Y) : N(T+K) \in Sp(\mathcal{A}) \text{ for every } K \in \mathcal{K}(X, Y)\}.$$

We say that $\mathcal{A}_-$ admits a perturbative characterization *if for every pair* X, Y,

$$\mathcal{A}_-(X, Y) = \{T \in \mathcal{L}(X, Y) : Y/\overline{R(T+K)} \in Sp(\mathcal{A}) \text{ for every } K \in \mathcal{K}(X, Y)\}.$$

REMARKS 2.23. – Let $\mathcal{A}$ be an operator ideal. If $\mathcal{A}_+$ ($\mathcal{A}_-$) admits a perturbative characterization, then $\mathcal{A}$ is injective (surjective).

QUESTION 3. – To determine the operator ideals $\mathcal{A}$ such that $\mathcal{A}_+$ or $\mathcal{A}_-$ admits a perturbative characterization. In particular, for the weakly precompact operators $\mathcal{R}$, does a similar characterization exists for $\mathcal{R}_+^d$ and $\mathcal{R}_-$?

Note that $\mathcal{R}$ and $\mathcal{R}^d$ are injective and surjective.

QUESTION 4. – Given an operator ideal $\mathcal{A}$, let $\mathcal{A}_k$ and $\mathcal{A}_c$ be the classes defined by

$$\mathcal{A}_k(X, Y) := \{T \in \mathcal{L}(X, Y) : N(T+K) \in Sp(\mathcal{A}), \quad \text{for every } K \in \mathcal{K}(X, Y)\},$$

$$\mathcal{A}_c(X, Y) := \{T \in \mathcal{L}(X, Y) : Y/\overline{R(T+K)} \in Sp(\mathcal{A}), \quad \text{for every } K \in \mathcal{K}(X, Y)\}.$$

Are $\mathcal{A}_k$ and $\mathcal{A}_c$ operator semigroups?

NOTES AND REMARKS 2.24. – In [71] the authors study the upper semi-Fredholm operators and some related classes by means of an abstract notion of semigroup in a Banach algebra. This treatment and the theory of operator ideals [82] are the main sources of inspiration for our definition of operator semigroup.

2.2. *Invertibility modulo an operator ideal.*

Here we consider semigroups that are defined in terms of left or right invertibility, modulo the elements of an operator ideal. The classical examples of semigroups of this kind are the left Atkinson and the right Atkinson, presented in Definition 1.9, which are invertible modulo the compact operators.

DEFINITION 2.25. – *Let $\mathcal{A}$ be an operator ideal and let $T \in \mathcal{L}(X, Y)$. We define the classes $\mathcal{A}_l$ and $\mathcal{A}_r$ as follows*:

$T \in \mathcal{A}_l$ if there exists $A \in \mathcal{L}(Y, X)$ such that $I_X - AT \in \mathcal{A}(X)$.

$T \in \mathcal{A}_r$ if there exists $B \in \mathcal{L}(Y, X)$ such that $I_Y - TB \in \mathcal{A}(Y)$.

REMARKS 2.26. – Clearly $\mathcal{A}_l(X, Y) \neq \emptyset$ if and only if $\mathcal{A}_r(Y, X) \neq \emptyset$.

PROPOSITION 2.27. – *For every operator ideal $\mathcal{A}$, the classes $\mathcal{A}_l$ and $\mathcal{A}_r$ are operator semigroups. Moreover, $\Phi_l \subset \mathcal{A}_l \subset \mathcal{A}_+$ and $\Phi_r \subset \mathcal{A}_r \subset \mathcal{A}_-$.*

PROOF. – The fact that $\mathcal{A}_l$ and $\mathcal{A}_r$ are semigroups can be proved in a similar way as Proposition 2.4. Moreover, we have $\Phi_l = \mathcal{F}_l$ and $\Phi_r = \mathcal{F}_r$ [23]. Hence $\Phi_l \subset \mathcal{A}_l$ and $\Phi_r \subset \mathcal{A}_r$.

In order to show that $\mathcal{A}_l \subset \mathcal{A}_+$, assume that $T \in \mathcal{A}_l(X, Y)$ and take $A \in \mathcal{L}(Y, X)$ such that $I_X - AT \in \mathcal{A}(X)$. If $B \in \mathcal{L}(Z, X)$ and $TB \in \mathcal{A}$, then $B(I_X - AT) = B - ATB \in \mathcal{A}$; hence $B \in \mathcal{A}$.

The proof of $\mathcal{A}_r \subset \mathcal{A}_-$ is analogous. ■

PROPOSITION 2.28. – *For every operator ideal $\mathcal{A}$, the semigroup $\mathcal{A}_l$ is left stable and the semigroup $\mathcal{A}_r$ is right stable; i.e., given $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$, we have*

(a) $ST \in \mathcal{A}_l \Rightarrow T \in \mathcal{A}_l$,

(b) $ST \in \mathcal{A}_r \Rightarrow S \in \mathcal{A}_r$.

PROOF. – It immediately follows from the definitions. ■

REMARKS 2.29. – Observe that both classes $\mathcal{A}_+ \cap \mathcal{A}_-$ and $\mathcal{A}_l \cap \mathcal{A}_r$ are semigroups. They coincide for $\mathcal{A} = \mathcal{K}$, since $\mathcal{K}_+ \cap \mathcal{K}_- = \mathcal{K}_l \cap \mathcal{K}_r = \Phi$, but are different in general.

Let us see that $\mathcal{W}_+ \cap \mathcal{W}_- \neq \mathcal{W}_l \cap \mathcal{W}_r$:

We saw in Proposition 2.6 that an operator $T \in \mathcal{L}(X, Y)$ is tauberian (cotauberian) if and only if the associated operator $T^{co} \in \mathcal{L}(X^{**}/X, Y^{**}/Y)$ is injective (has dense range). In particular, $T \in \mathcal{W}_+ \cap \mathcal{W}_-$ if T^{co} is bijective.

On the other hand, it was proved in [59] that, if $X = Y = l_2(J)$, where J is James' quasireflexive space, then $X^{**}/X = l_2$ and $T \in \mathcal{L}(X)$ belongs to $\mathcal{W}_l \cap \mathcal{W}_r$ if and only if T^{co} is regular with respect to the usual lattice structure in l_2.

Now, it is not difficult to find examples of operators $T \in \mathcal{L}(X, X)$ such that T^{co} is bijective but not regular. Hence $T \in \mathcal{W}_+ \cap \mathcal{W}_- \setminus \mathcal{W}_l \cap \mathcal{W}_r$. We refer to [59] for the details.

Recall that the *radical* $\mathcal{A}^{rad}$ of an operator ideal $\mathcal{A}$ is defined [82, 4.3.1] as follows:

$$\mathcal{A}^{rad}(X, Y) := \left\{ T \in \mathcal{L}(X, Y) : \begin{array}{l} \text{for every } S \in \mathcal{L}(Y, X), \text{ there exists } U \in \mathcal{L}(X), \\ \text{so that } I_X - U(I_X - ST) \in \mathcal{A} \end{array} \right\}.$$

$\mathcal{A}^{rad}$ is a closed operator ideal that contains $\mathcal{A}$ and satisfies $Sp(\mathcal{A}^{rad}) = Sp(\mathcal{A})$ [82, Section 4.3]. Observe that the definition of $\mathcal{A}^{rad}$ and some characterizations given in [82] can be written in a more compact form, in terms of $\mathcal{A}_l$ and $\mathcal{A}_r$.

PROPOSITION 2.30. – *Let $\mathcal{A}$ be an operator ideal. For $T \in \mathcal{L}(X, Y)$, the following assertions are equivalent*:

(a) $T \in \mathcal{A}^{rad}(X, Y)$.

(b) *For every* $S \in \mathcal{L}(Y, X)$, $I_X - ST \in \mathcal{A}_l$.

(c) *For every* $S \in \mathcal{L}(Y, X)$, $I_X - ST \in \mathcal{A}_r$.

(d) *For every* $S \in \mathcal{L}(Y, X)$, $I_Y - TS \in \mathcal{A}_r$.

(e) *For every* $S \in \mathcal{L}(Y, X)$, $I_Y - TS \in \mathcal{A}_l$.

COROLLARY 2.31. – *We have $\mathcal{K}^{rad} = \mathfrak{I}$, the inessential operators.*

PROOF. – It is enough to look to the definition of $\mathfrak{I}$, and observe that $\Phi = \Phi_l \cap \Phi_r$. ∎

Now we show that the radical $\mathcal{A}^{rad}$ consists of admissible perturbations for the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$.

PROPOSITION 2.32 [8]. – *Let $\mathcal{A}$ an operator ideal and let $K \in \mathcal{A}^{rad}(X, Y)$.*

(a) $T \in \mathcal{A}_l(X, Y) \Rightarrow T + K \in \mathcal{A}_l(X, Y)$.

(b) $T \in \mathcal{A}_r(X, Y) \Rightarrow T + K \in \mathcal{A}_r(X, Y)$.

PROPOSITION 2.33 [8]. – *For every operator ideal $\mathcal{A}$ we have $\mathcal{A}_l = \mathcal{A}_l^{rad}$ and $\mathcal{A}_r = \mathcal{A}_r^{rad}$.*

Moreover, the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ are open.

NOTES AND REMARKS 2.34. – The semigroups Φ_l and Φ_r of operators which are invertible modulo the compact operators were introduced by Atkinson [14]. Moreover, Yang [97] studied the operators that are left or right invertible modulo the weakly compact operators.

2.3. *Perturbation class of an operator semigroup*

We define the perturbation class for a semigroup in a similar way as Lebow and Schechter [71] did for some subsets of Banach spaces.

DEFINITION 2.35. – *Let $\mathcal{S}$ be an operator semigroup and let X, Y be Banach spaces such that $\mathcal{S}(X, Y) \neq \emptyset$. The component $P\mathcal{S}(X, Y)$ of the* perturbation class *$P\mathcal{S}$ of $\mathcal{S}$ is defined by*

$$P\mathcal{S}(X, Y) := \{K \in \mathcal{L}(X, Y) : T + K \in \mathcal{S}(X, Y) \text{ for every } T \in \mathcal{S}(X, Y)\}.$$

REMARKS 2.36. – (*a*) For every operator semigroup $\mathcal{S}$, the components $P\mathcal{S}(X, Y)$ are (not necessarily closed) linear subspaces of $\mathcal{L}(X, Y)$.

(*b*) The perturbation class of the operator semigroup $\mathcal{G}$ of all bijective operators is the class $\{\mathbf{0}\}$ of all null operators.

PROPOSITION 2.37. – *Let $\mathcal{S}$ be an operator semigroup.*

(a) *The components $P\mathcal{S}(X)$ on single Banach spaces of the perturbation class $P\mathcal{S}$ are two-sided ideals in $\mathcal{L}(X)$.*

(b) *If $\mathcal{S}(X, Y)$ is an open subset of $\mathcal{L}(X, Y)$, then $P\mathcal{S}(X, Y)$ is closed.*

PROOF. – (*a*) Clearly $P\mathcal{S}(X)$ is a subspace of $\mathcal{L}(X)$. Moreover, if $K \in P\mathcal{S}(X)$, $A \in \mathcal{L}(X)$ is invertible and $T \in \mathcal{S}(X)$, then $T + AK = A(A^{-1}T + K) \in \mathcal{S}$. Consequently $AK \in P\mathcal{S}(X)$.

Now, since every $A \in \mathcal{L}(X)$ is the sum of two invertible elements $A = tI_X + (A - tI_X)$, we obtain that $P\mathcal{S}$ is a left ideal. Analogously we may prove that it is a right ideal.

(*b*) Assume $K_n \in P\mathcal{S}(X)$ converges to $K \in \mathcal{L}(X)$. Given $T \in \mathcal{S}(X)$ there exists $c > 0$ so that $T + A \in \mathcal{S}(X)$ for every $A \in \mathcal{L}(X)$ with $\|A\| < c$. Now, writing $T + K = T + K_n + (K - K_n)$, it is clear that $T + K \in \mathcal{S}(X)$. Hence $A \in P\mathcal{S}(X)$. ∎

Proposition 2.37 shows that the following Remarks can be applied to the components $P\mathcal{S}(X)$ of the perturbation class of a semigroup $\mathcal{S}$.

REMARKS 2.38. – It was observed in [82, Proposition 1.1.3] that if we have a two-sided ideal $\mathcal{A}(X) \neq \{0\}$ for every Banach space X, then these sets are the components of a unique operator ideal $\mathcal{A}$ if and only if the following *condition of compatibility* holds:

$$A \in \mathcal{L}(X, Y), B \in \mathcal{L}(Y, X), K \in \mathcal{A}(X) \Rightarrow AKB \in \mathcal{A}(Y). \tag{1}$$

In this case, the components $\mathcal{A}(X, Y)$ are determined by

$$T \in \mathcal{L}(X, Y) \text{ belongs to } \mathcal{A} \text{ if and only if } \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in \mathcal{A}(X \oplus Y). \tag{2}$$

So the question arises:

QUESTION 5. – Assume that $P\mathcal{S}(X) \neq \{0\}$ for every Banach space X. Do the components $P\mathcal{S}(X)$ satisfy the condition of compatibility 1?

This is an open problem even for the perturbation classes of the semi-Fredholm operators Φ_+ and Φ_-.

For many semigroups $\mathcal{S}$, $P\mathcal{S}(X) \neq \emptyset$ for every non-zero Banach space X.

PROPOSITION 2.39 [8]. – *Let $\mathcal{S}$ be an operator semigroup such that $\Phi \subset \mathcal{S}$. Then $\mathcal{F}(X, Y) \subset P\mathcal{S}(X, Y)$, whenever $\mathcal{S}(X, Y) \neq \emptyset$.*

REMARKS 2.40. – The converse to Proposition 2.39 fails: The semigroup Φ_0 of Fredholm operators with index equal to 0 is a counterexample. See [8] for details.

Lebow and Schechter [71] proved that the answer Question 5 is positive for the semigroups of Atkinson operators. We derive this fact from our previous results.

PROPOSITION 2.41 [71, Theorem 2.7]. – *We have* $P\Phi(X) = P\Phi_l(X) = P\Phi_r(X) = \mathfrak{I}(X)$, *for every Banach space* X.

PROOF. – It follows from Corollary 2.31 that $\mathfrak{I}(X) = \mathcal{K}^{rad}(X)$. Since $\Phi_l = \mathcal{K}_l$, $\Phi_r = \mathcal{K}_r$ and $\Phi = \mathcal{K}_l \cap \mathcal{K}_r$, it follows from Proposition 2.32 that $\mathfrak{I}(X)$ is contained in the three perturbation classes.

On the other hand, Proposition 2.37 tells us that $P\Phi(X)$ is a two-sided ideal. Therefore, if $T \in P\Phi(X)$ and $S \in \mathcal{L}(X)$, then $I_X - ST \in \Phi$; hence $T \in \mathfrak{I}$.

For $P\Phi_l(X)$ and $P\Phi_r(X)$ the proof is analogous. ■

REMARKS 2.42. – Let $\mathcal{A}$ be an operator ideal. It follows from Propositions 2.20 and 2.32 that

$$\mathcal{A} \subset P\mathcal{A}_+ \cap P\mathcal{A}_- \quad \text{and} \quad \mathcal{A}^{rad} \subset P\mathcal{A}_l \cap P\mathcal{A}_r.$$

QUESTION 6. – Is it possible to characterize the perturbation classes for $\mathcal{A}_+$ and $\mathcal{A}_-$?

For which operator ideals $\mathcal{A}$ we have $P(\mathcal{A}_+) = \mathcal{A}$ or $P(\mathcal{A}_-) = \mathcal{A}$?

QUESTION 7. – Is it true that $\mathcal{A}^{rad} = P\mathcal{A}_l = P\mathcal{A}_r$ for every operator ideal $\mathcal{A}$?

We have seen that the answer is positive in the case $\mathcal{A} = \mathcal{K}$.

It is a well-known open question whether $\mathcal{A}^{rad}$ is the biggest operator ideal whose space ideal coincides with $Sp(\mathcal{A})$ [82]. The answer is not known even in the case $\mathcal{A} = \mathcal{K}$.

NOTES AND REMARKS 2.43. – The concept of perturbation class was introduced by Lebow and Schechter [71]. They also proved that the perturbation class for the Fredholm operators Φ and for the Atkinson operators Φ_l and Φ_r is the operator ideal of inessential operators.

2.4. *Back to operator ideals from operator semigroups.*

Now we associate some operator ideals to the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$, whenever they admit a perturbative characterization.

DEFINITION 2.44. – *Let* $\mathcal{A}$ *be an operator ideal. We define the classes* $\mathcal{A}\mathcal{S}$, $\mathcal{A}\mathcal{C}$ *by*

$$\mathcal{A}\mathcal{S}(X, Y) := \{K \in \mathcal{L}(X, Y) : A \in \mathcal{L}(Z, X),\ KA \in \mathcal{A}_+ \Rightarrow A \in \mathcal{A}\},$$

$$\mathcal{A}\mathcal{C}(X, Y) := \{K \in \mathcal{L}(X, Y) : B \in \mathcal{L}(Y, Z),\ BK \in \mathcal{A}_- \Rightarrow B \in \mathcal{A}\}.$$

Recall that $\mathcal{A}_+$ *admits a perturbative characterization if* $T \in \mathcal{L}(X, Y)$ *belongs to* $\mathcal{A}_+$ *whenever* $N(T+K)$ *belongs to* $Sp(\mathcal{A})$ *for every compact operator* $K \in \mathcal{K}(X, Y)$; *and* $\mathcal{A}_-$ *admits a perturbative characterization if* $T \in \mathcal{L}(X, Y)$

belongs to $\mathcal{A}_-$ *whenever* $Y/\overline{R(T+K)}$ *belongs to* $Sp(\mathcal{A})$ *for every compact operator* $K \in \mathcal{K}(X, Y)$. *See Proposition 2.21 for examples of semigroups of this kind.*

PROPOSITION 2.45. – *Let* $\mathcal{A}$ *be an operator ideal.*

(a) *If* $\mathcal{A}_+$ *admits a perturbative characterization then* $\mathcal{A}\mathcal{S}$ *is an operator ideal and* $\mathcal{A}(X, Y) \subset \mathcal{A}\mathcal{S}(X, Y) \subset P\mathcal{A}_+(X, Y)$, *whenever* $\mathcal{A}_+(X, Y) \neq \emptyset$.

(b) *If* $\mathcal{A}_-$ *admits a perturbative characterization then* $\mathcal{A}\mathcal{C}$ *is an operator ideal and* $\mathcal{A} \subset \mathcal{A}\mathcal{C} \subset P\mathcal{A}_-$, *whenever* $\mathcal{A}_-(X, Y) \neq \emptyset$.

PROOF. – (a) First we show that $\mathcal{A} \subset \mathcal{A}\mathcal{S}$. We take $K \in \mathcal{A}(X, Y)$ and $A \in \mathcal{L}(Z, X)$, and assume that $KA \in \mathcal{A}_+$. Then $I_Z \in \mathcal{A}$; hence $A \in \mathcal{A}$, and we conclude $K \in \mathcal{A}\mathcal{S}(X, Y)$. In particular, $\mathcal{A}\mathcal{S}$ satisfies the property (α_1) in the definition of operator ideal.

In order to prove (α_2), assume $K, L \in \mathcal{A}\mathcal{S}(X, Y)$ and $A \in \mathcal{L}(Z, X)$. If $A \notin \mathcal{A}$, then KA and LA are not in $\mathcal{A}_+$. Since $\mathcal{A}_+$ admits a perturbative characterization, we can find a subspace M of Z such that $M \notin \mathcal{A}$ and KAJ_M is compact. Moreover $LAJ_M \notin \mathcal{A}_+$; otherwise, by the definition of $\mathcal{A}\mathcal{S}$ we would have $AJ_M \in \mathcal{A}$; then AJ_M belongs both to $\mathcal{A}_+$ and $\mathcal{A}$; hence $M \in Sp(\mathcal{A})$, a contradiction. Thus $(K+L)\,AJ_M \notin \mathcal{A}_+$; hence $(K+L)A \notin \mathcal{A}_+$, and we have proved that $K+L \in \mathcal{A}\mathcal{S}$.

Suppose $K \in \mathcal{A}\mathcal{S}(X, Y)$ and $A \in \mathcal{L}(Z, X)$. If $S \in \mathcal{L}(W, Z)$ and $KAS \in \mathcal{A}_+$, then doing as in the previous paragraph we can show that $W \in Sp(\mathcal{A})$; hence $S \in \mathcal{A}$ and we conclude that $KA \in \mathcal{A}\mathcal{S}$. Analogously, given $K \in \mathcal{A}\mathcal{S}(X, Y)$ and $A \in \mathcal{L}(Y, Z)$, if $S \in \mathcal{L}(W, X)$ and $AKS \in \mathcal{A}_+$, we get that $W \in Sp(\mathcal{A})$, and we conclude $AK \in \mathcal{A}\mathcal{S}$. In this way we have proved that $\mathcal{A}\mathcal{S}$ is an operator ideal.

It remains to show that $\mathcal{A}\mathcal{S} \subset P\mathcal{A}_+$. In order to do that, we take $T \in \mathcal{A}_+(X, Y)$ and $K \in \mathcal{L}(X, Y)$, and assume that $T+K \notin \mathcal{A}_+$. Since $\mathcal{A}_+$ admits a perturbative characterization, we can find a subspace M of X such that $M \notin \mathcal{A}$ and $(T+K)\,J_M$ is compact. Since $TJ_M \in \mathcal{A}_+$ we have $KJ_M \in \mathcal{A}_+$ (they differ in a compact operator). However, $J_M \notin \mathcal{A}$; hence $K \notin \mathcal{A}\mathcal{S}$.

Moreover, if $K \in \mathcal{A}\mathcal{S}(X, Y)$, $T \in \mathcal{A}_+(X, Y)$ and $T+K \notin \mathcal{A}_+$, then we can find $A \in \mathcal{L}(Z, X)$ such that $(T+K)A \in \mathcal{A}$ but $A \notin \mathcal{A}$.

(*b*) It is similar. ■

COROLLARY 2.46. – *Let* $\mathcal{A}$ *be an operator ideal.*

(*a*) *If* $\mathcal{A}_+$ *admits a perturbative characterization then* $Sp(\mathcal{A}\mathcal{S}) = Sp(\mathcal{A})$.

(*b*) *If* $\mathcal{A}_-$ *admits a perturbative characterization then* $Sp(\mathcal{A}\mathcal{C}) = Sp(\mathcal{A})$.

PROOF. – (*a*) Observe that, by the compatibility condition presented in Remark 2.38, we have $\mathcal{A} \subset \mathcal{A}\mathcal{S}$; hence $Sp(\mathcal{A}) \subset Sp(\mathcal{A}\mathcal{S})$. Moreover, if $X \in Sp(\mathcal{A}\mathcal{S})$, then $I_X \in \mathcal{A}_+$ implies $I_X \in \mathcal{A}$; i.e., $X \in Sp(\mathcal{A})$.

The proof of (*b*) is analogous. ■

REMARKS 2.47. – It is not difficult to see that for the compact operators $\mathcal{K}$, we have $\mathcal{K}\mathcal{S} = \mathcal{S}\mathcal{S}$ and $\mathcal{K}\mathcal{C} = \mathcal{S}\mathcal{C}$.

QUESTION 8. – To characterize the operator ideals $\mathcal{A}$ for which $\mathcal{A}\mathcal{S}$ or $\mathcal{A}\mathcal{C}$ are operator ideals.

NOTES AND REMARKS 2.48. – For the class $\mathcal{W}$ of weakly compact operators, the operator ideal $\mathcal{W}\mathcal{S}$ is the natural candidate to be the smallest ideal containig the almost weakly compact operators introduced in [67].

3. – Examples in the literature.

In this section we describe the properties and applications of some semigroups that have been previously considered in the literature, because they could be taken as models for further development of the theory of operator semigroups.

A detailed study of the semigroups Φ_+, Φ_-, Φ_l, Φ_r and Φ may be found in several monographies [23, 36, 63, 91]. Moreover, they are «trivial» from the point of view of Banach space theory. Therefore, we only give a brief description of their perturbation classes and of the semigroups $\mathcal{J}_+$ and $\mathcal{J}_-$ associated with the inessential operators $\mathcal{J}$.

We give some properties and applications of tauberian operators and refer to [37] for a survey about this topic. We also describe the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ for the operator ideals $\mathcal{A}$ presented in Definition 1.2. These semigroups, studied in [20, 44, 55, 56, 58], have similar properties to that of the tauberian and the cotauberian operators.

We also show that although the semiembeddings and the G_δ-embeddings do not form semigroups, they are contained in semigroups that share their nice properties.

3.1. *Semigroups in classical Fredholm theory.*

Recall that $\mathcal{S}\mathcal{S}$ and $\mathcal{S}\mathcal{C}$ denote the operator ideals of all strictly singular and strictly cosingular operators, respectively. We have $\mathcal{S}\mathcal{S} \subset P\Phi_+$ and $\mathcal{S}\mathcal{C} \subset P\Phi_-$. However, the following question remains open.

QUESTION 9. – It is not known whether $\mathcal{S}\mathcal{S} = P\Phi_+$ and $\mathcal{S}\mathcal{C} = P\Phi_-$.

The answer to this question is positive in the case in which one of the spaces has many projections; more precisely, if one of the spaces is subprojective or superprojective (see Section 2 for definitions and examples).

THEOREM 3.1. – *Let X and Z be Banach spaces.*

(a) *If X is subprojective, then $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$ for every space Y.*

(b) *If Z is superprojective, then $\mathcal{SC}(Y, Z) = P\Phi_-(Y, Z)$ for every space Y.*

Gowers and Maurey [60, 61] constructed spaces admitting only trivial projections, for which we have a positive answer to Question 9, as we shall see in Theorem 3.3.

DEFINITION 3.2. – *A Banach space X is said to be* indecomposable *if there is no infinite dimensional subspaces M and N of X so that $M \cap N = \{0\}$ and $M + N = X$.*

A result of Weis [93] characterizes the Banach spaces such that any operator either is semi-Fredholm or belongs to the corresponding perturbation class, in terms of the decomposability of their subspaces and quotients.

THEOREM 3.3 [93]. – *Let X and Z be a Banach space.*

(a) *We have $L(X, Y) = \mathcal{SS}(X, Y) \cup \Phi_+(X, Y)$ for every Banach space Y if and only if all the subspaces of X are indecomposable.*

(b) *We have $L(Y, Z) = \mathcal{SC}(Y, Z) \cup \Phi_-(Y, Z)$ for every Banach space Y if and only if all the quotients of Z are indecomposable.*

We observe that, at the time Weis proved this result, the existence of indecomposable Banach spaces, proved in [61], was open problem.

EXAMPLE 3.4 [61]. – There exists a complex, infinite dimensional, reflexive Banach space X_{GM} such that all of its subspaces are indecomposable. Therefore,

$$L(X_{GM}, Y) = \mathcal{SS}(X_{GM}, Y) \cup \Phi_+(X_{GM}, Y) \text{ for every Banach space Y, and}$$

$$L(X_{GM}) = \{zI + K : z \in \boldsymbol{C}, K \in \mathcal{SS}(X_{GM})\}.$$

Moreover, the dual space X_{GM}^* satisfies $L(Y, X_{GM}^*) = \mathcal{SC}(Y, X_{GM}^*) \cup \Phi_-(Y, X_{GM}^*)$ for every Banach space Y, and $L(X_{GM}^*) = \{zI + K : z \in \boldsymbol{C}, K \in \mathcal{SC}(X_{GM}^*)\}$.

We showed in Proposition 2.41 that the perturbation class of the semigroup Φ, in the case $X = Y$, coincides with the class $\mathfrak{I}$ of inessential operators. Since $\mathfrak{I}$ is an operator ideal, the whole class is determined by the components $\mathfrak{I}(X)$ acting on a single space, as we observed in Remarks 2.38. In the following result we show that in the complex case $\mathfrak{I}(X)$ admits spectral characterizations.

Let X be a complex Banach space. Recall that $T \in \mathcal{L}(X)$ is said to be a *Riesz operator* if for every non-zero complex number z we have $zI_X - T \in \Phi(X)$.

We denote by $\Omega(X)$ the set of all $T \in \mathcal{L}(X)$ such that for no infinite dimensional, invariant subspace M of T, the restriction $T_M: M \to M$ is bijective.

THEOREM 3.5 [1]. – *For a complex Banach space X, the class $\mathfrak{I}(X)$ is the largest ideal contained in $\Omega(X)$; equivalently, it is the largest ideal in $\mathcal{L}(X)$ consisting of Riesz operators.*

On the other hand, we have the following related problem.

QUESTION 10. – Is it true that $\mathfrak{I}$ is the biggest operator ideal whose space ideal is F, the finite dimensional spaces?

A well-known result of classical Fredholm theory tells us that $\mathfrak{I}_l = \mathcal{K}_l$, the left-Atkinson operators, and $\mathfrak{I}_r = \mathcal{K}_r$, the right Atkinson operators. However, the semigroups $\mathfrak{I}_+$ and $\mathfrak{I}_-$ are not comparable with Φ_+ and Φ_-.

PROPOSITION 3.6 [8]. – *Let M and N be subspaces of l_∞ such that $l_\infty/M \cong N \cong l_2$. Then $Q_N \in \mathfrak{I}_+$ and $J_M \in \mathfrak{I}_-$.*

The operator ideal $\mathfrak{I}$ is neither injective nor surjective. So it follows from Propositions 2.12 and 3.6 that $\mathfrak{I}_+$ and $\mathfrak{I}_-$ are not comparable with the semi-Fredholm operators:

$$\Phi_+ \not\subset \mathfrak{I}_+ \not\subset \Phi_+ \quad \text{and} \quad \Phi_- \not\subset \mathfrak{I}_- \not\subset \Phi_-.$$

QUESTION 11. – Is it possible to give a good description of $\mathfrak{I}_+$ and $\mathfrak{I}_-$? Are these semigroups open?

In the next result we show that the class of inessential operators has a certain symmetry.

PROPOSITION 3.7 [39, Proposition 1]. – *Given a pair X, Y of Banach spaces, we have*

$$\mathcal{L}(X, Y) = \mathfrak{I}(X, Y) \quad \text{if and only if} \quad \mathcal{L}(Y, X) = \mathfrak{I}(Y, X).$$

Let us see that the $\mathfrak{I}$ is a much bigger class than the compact operators. Here H^∞ denotes the space of all bounded analytic functions on the disc.

THEOREM 3.8 [3, 39]. – *We have $\mathcal{L}(X, Y) = \mathfrak{I}(X, Y)$ in the following cases*:

(a) *X is reflexive and Y has the Dunford-Pettis property*;

(b) *X has the reciprocal Dunford-Pettis property and Y has the Schur property*;

(c) *X contains no copies of* l_∞ *and* $Y = l_\infty$, H^∞ *or* $C(K)$ *with* K *σ-stonian*;

(d) *X contains no copies of* c_0 *and* $Y = C(K)$;

(e) *X contains no complemented copies of* c_0 *and* $Y = C[0, 1]$;

(f) *X contains no complemented copies of* l_1 *and* $Y = L_1(\mu)$;

(g) *X contains no complemented copies of* l_p *and* $Y = L_p[0, 1]$, *or* l_p $1 < p < \infty$.

Observe that the available characterizations of inessential operators, like those in Definition 1.11 and Theorem 1.12, are not intrinsic: They depend on the properties of the products of the given operator by operators in a large set.

If $T \in \mathfrak{I}(X, Y)$, then there is no infinite dimensional subspace of M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is complemented in Y. It was conjectured by Tarafdar [90] (see also [5]) that this property characterizes the inessential operators. However, it was proved in [6] that the conjecture was not correct. Therefore, the following question remains.

QUESTION 12. – Is it possible to give an intrinsic characterization of inessential operators?

NOTES AND REMARKS 3.9. – In [94] Weis studied the perturbation classes for not necessarily continuous, closed semi-Fredholm operators, but his results cannot be applied to our situation, because the perturbation classes of the closed semi-Fredholm operators could be smaller than that of the corresponding classes of continuous operators.

The investigation of the semigroups $\mathfrak{I}_+$ and $\mathfrak{I}_-$ associated with the inessential operators may be interesting, in view of the wealth of cases in which $\mathcal{L}(X, Y) = \mathfrak{I}(X, Y)$.

3.2. *Tauberian operators.*

Besides of the semigroups of Fredholm theory, the classes of tauberian and cotauberian operators are the operator semigroups that have received more attention in the literature. Moreover, since these semigroups are nontrivial, they constitute good models for the theory of operator semigroups.

Recall that an operator $T \in \mathcal{L}(X, Y)$ is *tauberian* if $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$; equivalently, if $T \in \mathcal{W}_+$. Moreover, T is *cotauberian* if T^* is tauberian; equivalently, if $T \in \mathcal{W}_-$.

The semigroup $\mathcal{W}_+$ has some shortcomings. We showed in Example 2.19 that, given a a non-reflexive Banach space X, the component $\mathcal{W}_+(l_2(X))$ is not open. Moreover, $\mathcal{W}_+$ has an asymmetric behaviour under duality.

EXAMPLE 3.10 [9]. – *There exists a tauberian operator* $T \in \mathcal{W}_+$ *such that* $T^{**} \notin \mathcal{W}_+$.

Here we use a construction due to Bellenot [18]. Let X_n denote the subspace of l_1 generated by the first n elements of the unit basis. We consider the space

$$J(X_n) = \{(x_n) : x_n \in X_n,\ \|x_n\|_1 \to 0 \text{ and } \|(x_n)\|_J < \infty\},$$

where $\|\cdot\|_1$ is the norm in l_1 and $\|\cdot\|_J$ is given by

$$\|(x_n)\|_J = sup\left\{\|x_{n_k}\|_1^2 + \sum_{i=1}^{k-1} \|x_{n_{i+1}} - x_{n_i}\|_1^2 : n_1 < n_2 < \ldots n_k\right\}^{1/2}.$$

$(J(X_n), \|\cdot\|_J)$ is a Banach space and $J(X_n)^{**}/J(X_n) \cong l_1$ [18]. Moreover, the operator

$$T : (x_n) \in J(X_n) \to (x_n/n) \in J(X_n)$$

satisfies $T \in \mathcal{W}_+$, but $T^{**} \notin \mathcal{W}_+$. We refer to [9] for details.

REMARKS 3.11. – The last result shows an «asymmetry» for an apparently perfectly symmetric class, the weakly compact operators $\mathcal{W}$. For every q$T \in \mathcal{L}(X, Y)$ we have $T \in \mathcal{W}$ if and only if $T^* \in \mathcal{W}$, and the associated semigroups $\mathcal{W}_+$, $\mathcal{W}_-$ admit «symmetric» characterizations:

$$T \in \mathcal{W}_+ \text{ if and only if } TA \in \mathcal{W} \Rightarrow A \in \mathcal{W},$$

$$T \in \mathcal{W}_- \text{ if and only if } BT \in \mathcal{W} \Rightarrow B \in \mathcal{W}.$$

However, the duality relations between $\mathcal{W}_+$ and $\mathcal{W}_-$ are not symmetric. We have

$$T \in \mathcal{W}_- \Leftrightarrow T^* \in \mathcal{W}_+ \text{ and } T^* \in \mathcal{W}_- \Rightarrow T \in \mathcal{W}_+,$$

but $T \in \mathcal{W}_+ \not\Rightarrow T^* \in \mathcal{W}_-$.

It is not difficult to show that an operator $T \in \mathcal{L}(X, Y)$ is tauberian if and only if $N(T^{**}) = N(T)$ and $T(B_X)$ is closed, where B_X is the closed unit ball of X. Moreover, Neidinger and Rosenthal obtained the following refinement.

THEOREM 3.12 [80, Theorem 2.3]. – *For a non-zero operator* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent*:

(a) T *is tauberian.*

(b) $T(B_E)$ *is closed, for all subspaces* E *of* X.

(c) $T(K)$ *is (weakly) closed, for all weakly closed bounded subsets of* X.

(d) $T(K)$ *is closed, for all closed convex bounded subsets of* X.

Theorem 3.12 should be compared with the following well-known results.

PROPOSITION 3.13. – *Let* $T \in \mathcal{L}(X, Y)$ *be a non-zero operator.*

(a) *T is an into isomorphism if and only if $T(K)$ is closed, for all closed $K \subset X$.*

(b) *$T \in \Phi_+$ if and only if $T(K)$ is closed, for all closed bounded subsets $K \subset X$.*

QUESTION 13. – Let $\mathcal{A}$ be an injective operator ideal; hence $\Phi_+ \subset \mathcal{A}_+$, by Proposition 2.12.

Is it possible to find a suitable class of closed bounded sets so that $T \in \mathcal{A}_+$ if and only if $T(K)$ is closed for every set K in the class?

Holub obtained several characterizations of tauberian operators in terms of their action over basic sequences. We include below some of them.

THEOREM 3.14 [68]. – *For* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent*:

(a) *T is tauberian.*

(b) *If (x_n) is a normalized basic sequence in X for which (Tx_n) is (weakly) convergent, then (x_n) is weakly null.*

(c) *If (x_n) is a bounded basic sequence in X for which* $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$, *then (x_n) is (both shrinking and) boundedly complete.*

Now, following [41], we show that the shortcomings of tauberian operators do not appear for operators acting on $L_1(\mu)$, where μ is a finite measure. The class of all tauberian operators from $L_1(\mu)$ into Y is open and the second conjugate of a tauberian operator $T: L_1(\mu) \to Y$ is tauberian. We also present several characterizations of tauberian operators $T: L_1(\mu) \to Y$ and we show that the corresponding perturbation class consists of weakly precompact operators.

THEOREM 3.15 [41]. – *For* $T \in \mathcal{L}(L_1(\mu), Y)$, *the following statements are equivalent*:

(a) *T is tauberian;*

(b) *T^{**} is tauberian;*

(c) $N(T) = N(T^{**})$;

(d) $\liminf_n \|Tf_n\| > 0$ *for every normalized disjoint sequence (f_n) in* $L_1(\mu)$;

(e) *there exists $r > 0$ such that* $\liminf_n \|Tf_n\| > r$ *for every normalized disjoint sequence (f_n) in $L_1(\mu)$.*

It easily follows from Theorem 3.15 (e) that $\mathcal{W}_+(L_1(\mu), Y)$ is open, and that it is non-empty if and only if Y contains a subspace isomorphic to $L_1(\mu)$.

Recall that the *dyadic tree* in $[0, 1]$ is given by the following multiples of the characteristic functions of the dyadic intervals

$$\chi_{n,i} := 2^n \chi_{((i-1)/2^n,\, i/2^n)}, \qquad \text{where } n = 0, 1, 2, \dots \text{ and } i = 1, \dots, 2^n.$$

It is well-known [32] that an operator $T : L_1[0, 1] \to Y$ is determined by the values of $\{T\chi_{n,i} : n = 0, 1, 2, \dots \text{ and } i = 1, \dots, 2^n\}$.

THEOREM 3.16 [41]. – *An operator $T : L_1[0, 1] \to Y$ is tauberian if and only if for every sequence (f_n) in the dyadic tree of $L_1[0, 1]$ equivalent to the unit basis of l_1, there is some k so that $(Tf_n)_{n \geq k}$ is also equivalent to the unit basis of l_1.*

Finally, we observe that the perturbation class of $\mathcal{W}_+(L_1(\mu), Y)$ consists of weakly precompact operators.

PROPOSITION 3.17 [41]. – $P\mathcal{W}_+(L_1(\mu), Y) = \mathcal{R}(L_1(\mu), Y)$ *for every Y.*

It follows as a consequence of the Dunford-Pettis property of $L_1(\mu)$ that $\mathcal{R}(L_1(\mu)) = \mathcal{SS}(L_1(\mu))$, which coincides with $P\Phi(L_1(\mu))$ [93]. These facts and some of the previous results about $\mathcal{W}_+(L_1(\mu), Y)$ suggest the following question.

QUESTION 14. – Is it true that $\Phi_+(L_1(\mu)) = \mathcal{W}_+(L_1(\mu))$?

A special case of this question: Given an infinite dimensional reflexive subspace R of $L_1(\mu)$, is the quotient $L_1(\mu)/R$ isomorphic to a subspace of $L_1(\mu)$?

For more information about these questions, we refer to [43].

NOTES AND REMARKS 3.18. – The tauberian operators have been studied in other contexts different from the one considered here. For example, Cross has studied not necessarily continuous tauberian operators [26, 27, 28, 29] and tauberian linear relations [30]. Bonet and Ramanujan [22] have considered tauberian operators acting between Fréchet locally convex spaces, and Martínez and Pellón [77] have analyzed tauberian operators in the context of non-archimedean analysis.

The class of those operators T such that T^{**} is injective has many of the properties of tauberian operators. It was studied by Neidinger in his Ph.D. Thesis [79].

3.3. *Applications of tauberian operators.*

Tauberian operators were introduced by Kalton and Wilansky [69] as the abstract counterpart for a property of conservative matrices, in order to solve a summability problem. Since then, they have found many applications in Banach space theory: preservation of isomorphic properties [79], refinements of James' characterization of reflexive spaces [80], equivalence between the Radon-Nikodym property and the Krein-Milman property [85], and factorization of operators [31], for example. Here we describe these applications. Additional information may be found in [79] and [37]. We begin with some results of Neidinger concerning the preservation of isomorphic properties.

THEOREM 3.19 [79]. – *Let $\mathcal{G}$ be one of the following properties: Reflexivity, weak sequential completeness, Radon-Nikodym property, containing no copies of l_1 or containing no copies of c_0. If Y has $\mathcal{G}$ and there exists a tauberian operator $T \in \mathcal{L}(X, Y)$, then X has $\mathcal{G}$.*

Most of the results of Theorem 3.19 can be «localized»; i.e., tauberian operators preserve some isomorphic properties of bounded sets. See [79] for details.

In Theorem 3.12 we presented a remarkable characterization of tauberian operators, due to Neidinger and Rosenthal [80]. As an application, they obtained a refinement of James' characterization of reflexive Banach spaces as those spaces X such that every element of X^* attains its norm (in the unit ball of X). Observe that, if X is non-reflexive, then any nonzero $f \in X^*$ is a non-tauberian operator, and f attains its norm if and only if $f(B_X)$ is closed.

THEOREM 3.20 [80]. – *Let X be a non-reflexive Banach space and let $f \in X^*$, $f \neq 0$. Then there exists a subspace Y of X such that the restriction $f|_Y$ does not attain its norm.*

The following result allows us to construct tauberian operators in certain cases. For a simpler proof we refer to [39].

THEOREM 3.21 [85]. – *If there exists an injective tauberian operator $i: X \times X \to X$, then there exists an injective tauberian operator $j: l_2(X) \to X$.*

COROLLARY 3.22 [85]. – *If there exists an injective tauberian operator $i: X \times X \to X$, then X has the Radon-Nikodym property if and only if it has the Krein-Milman property.*

The main source of non-trivial examples of tauberian and cotauberian operators is the celebrated factorization of operators obtained by Davis, Figiel, Johnson and Pełczyński.

THEOREM 3.23 [31]. – *For every $T \in \mathcal{L}(X, Y)$, there exists a Banach space Z and operators $A \in \mathcal{L}(X, Z)$ and $j \in \mathcal{L}(Z, Y)$ so that j is tauberian, A is cotauberian and $T = jA$.*

COROLLARY 3.24. – *Every weakly compact operator factors through a reflexive Banach space.*

Let us say that an operator ideal $\mathcal{A}$ has the *interpolation property* if for every $T \in \mathcal{A}(X, Y)$, the intermediate space Z in the factorization of Theorem 3.23 belongs to $Sp(\mathcal{A})$. Using essentially Theorem 3.19, Heinrich proved the following result.

THEOREM 3.25 [66]. – *The operator ideals of all weakly compact, weakly precompact, decomposing and Banach-Saks operators have the interpolation property.*

Analyzing the duality properties of the factorization in Theorem 3.23, it is possible to extend the class of operator ideals with the interpolation property.

Recall that $T \in \mathcal{L}(X, Y)$ has associated an operator $T^{co} \in \mathcal{L}(X^{**}/X, Y^{**}/Y)$. Given an operator ideal $\mathcal{A}$, it is not difficult to see that $\mathcal{A}^{co} := \{T \in \mathcal{L} : T^{co} \in \mathcal{A}\}$ defines a new operator ideal.

THEOREM 3.26 [38]. – *Let $\mathcal{A}$ be an operator ideal with the interpolation property. Then $\mathcal{A}^d$ and $\mathcal{A}^{co}$ have the interpolation property.*

NOTES AND REMARKS 3.7. – Bombal and Fierro [19] have studied relatively weakly compact sets in vector-valued Orlicz spaces, using the fact that the natural embedding into the corresponding vector-valued $L_1(\mu)$-space is a tauberian operator.

3.4. *Lifting results and «sequential» semigroups.*

The semigroups $\mathcal{A}_+$ associated with the operator ideals $\mathcal{A}$ presented in Definition 1.2 admit a sequential characterization [55, 44], that is similar to Lohman's lifting [73]: Let X be a Banach space and let M be a subspace of X. If M contains no copies of l_1, then every weakly Cauchy sequence in X/M admits a subsequence that can be lifted to a weakly Cauchy sequence in X; equivalently, the quotient map from X onto X/M belongs to $\mathcal{R}_+$.

In [57, 44] several lifting results for sequences were obtained.

THEOREM 3.28 [57, 44]. – *Let* (x_n) *be a sequence in* X. *Let* M *be a subspace of* X *and let* $q : X \to X/M$ *and* $p : X^* \to X^*/M^\perp$ *denote the quotient maps.*

(a) *If* M *is reflexive (respectively finite dimensional, contains no copies of* l_1*),* (x_n) *is bounded and* (qx_n) *is weakly convergent (respectively convergent, weakly Cauchy), then* (x_n) *admits a weakly convergent (respectively convergent, weakly Cauchy) subsequence.*

(b) *If* M *is weakly sequentially complete (respectively Schur),* (x_n) *is weakly Cauchy and* (qx_n) *is weakly convergent (respectively convergent), then* (x_n) *admits a weakly convergent (respectively convergent) subsequence.*

(c) *If* X/M *is Grothendieck (respectively has no quotients isomorphic to* c_0*),* (f_n) *is a weak*-convergent sequence in* X^* *and* (pf_n) *is weakly convergent (respectively weakly Cauchy), then* (f_n) *admits a weakly convergent (respectively weakly Cauchy) subsequence.*

(d) *If* M *contains no copies of* c_0, $\sum_{n=1}^{\infty} x_n$ *is a weakly unconditionally Cauchy and* $\sum_{n=1}^{\infty} qx_n$ *is unconditionally converging, then* $\sum_{n=1}^{\infty} x_n$ *is unconditionally converging.*

REMARKS 3.29. – Let $\mathcal{A}$ be one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{CC}$, $\mathcal{WCC}$ or $\mathcal{U}$. It follows from parts (a), (b) and (d) of Theorem 3.28 that the semigroup $\mathcal{A}_+$ has the left-three space property.

QUESTION 15. – To study the semigroups associated with part (c) in Theorem 3.28.

The semigroup $\mathcal{R}_+$ associated with the weakly precompact operators admits a nice characterization in terms of the kernel of the second conjugate of the operators. Given a Banach space X, we denote by $\mathcal{B}_1(X)$ the set of all *first Baire class* elements of X^{**}; i.e., the elements of X^{**} that can be obtained as weak*-limits of sequences in X.

THEOREM 3.30 [20]. – *Let* X *be a separable Banach space. For every* $T \in \mathcal{L}(X, Y)$*, the following assertions are equivalent:*

(a) $T \in \mathcal{R}_+$.

(b) $(T^{**})^{-1}(Y) \subset \mathcal{B}_1(X)$.

(c) $ker(T^{**}) \subset \mathcal{B}_1(X)$.

In the non-separable, we can obtain an analogous result. We refer to [20] for the definitions of the relevant concepts.

THEOREM 3.31 [20, Theorem 3.3 and Corollary 3.4]. – *For an operator* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent*:

(a) $T \in \mathcal{R}_+$.

(b) *Every element of* $(T^{**})^{-1}(Y)$ *is universally measurable on* K.

(c) *Every* $z \in ker(T^{**})$ *is universally measurable on* (B_{X^*}, w^*).

We saw in Proposition 2.21 that if $\mathcal{A}$ is one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{CC}$ or $\mathcal{WCC}$, then the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-^d$ admit a perturbative characterization. As a consequence we can derive some relations between the semigroups. For example, taking account that the reflexive Banach spaces contain no copies of c_0, we obtain that for every couple X, Y of Banach spaces we have

$$\mathcal{K}_+(X,Y) \subset \mathcal{W}_+(X,Y) \subset \mathcal{U}_+(X,Y) \quad \text{and} \quad \mathcal{K}_-(X,Y) \subset \mathcal{W}_-(X,Y) \subset \mathcal{U}_-(X,Y).$$

From these inclusions we derive characterizations of some classes of Banach spaces.

PROPOSITION 3.32 [44, Propositions 2.13 and 2.23]. – (a) *X is hereditarily c_0 if and only if* $\mathcal{K}_+(X, Y) = \mathcal{U}_+(X, Y)$ *for every space* Y.

(b) *Non-reflexive subspaces of X contain copies of c_0 if and only if* $\mathcal{W}_+(X, Y) = \mathcal{U}_+(X, Y)$ *for every space* Y.

(c) *Reflexive subspaces of X are finite dimensional if and only if* $\mathcal{K}_+(X, Y) = \mathcal{W}_+(X, Y)$ *for every space* Y.

(a′) *Quotients of X containing no complemented copies of l_1 are finite dimensional if and only if* $\mathcal{K}_-(Z, X) = \mathcal{U}_-(Z, X)$ *for every space* Z.

(b′) *Quotients of X containing no complemented copies of l_1 are reflexive if and only if* $\mathcal{W}_-(Z, X) = \mathcal{U}_-(Z, X)$ *for every space* Z.

(c′) *Reflexive quotients of X are finite dimensional if and only if* $\mathcal{K}_-(Z, X) = \mathcal{W}_-(Z, X)$ *for every space* Z.

NOTES AND REMARKS 3.33. – The operators in the semigroup $\mathcal{R}_+$ were applied in [75] to characterize semi-Fredholm operators acting on a Banach space that contains no copies of l_1. They were called semitauberian operators in [20].

3.5. *Semiembeddings and G_δ-embeddings.*

The semiembeddings and the G_δ-embeddings of Banach spaces have been studied in [21, 34, 35]. These concepts are weaker than that of isomorphim (into), but yet the operators of these classes preserve some isomorphic properties of Banach spaces. For example, if a Banach space

X semiembeds in another Banach space with the Radon-Nikodym property, then X has the Radon-Nikodym property.

DEFINITION 3.34. – *An operator $T \in \mathcal{L}(X, Y)$ is said to be a* semiembedding *if T is injective and $T(B_X)$ is closed, where B_X is the (closed) unit ball of X.*

It is said to be a G_δ-embedding if T is injective and for every closed bounded subset A of X, $T(A)$ is a G_δ-set in Y.

The concept of semiembedding is isometric, but there is an isomorphic version. An injective operator $T \in \mathcal{L}(X, Y)$ is a semiembedding under some equivalent norm of X if and only if $T(U)$ is a F_σ-set for all open subsets U of X [21, Proposition 1.6]. These operators are called *F_σ-embeddings*.

We have that every semiembedding of a separable space is a G_δ-embedding [21, Proposition 1.8]. Moreover, the G_δ-embeddings are obviously stable by equivalent renormings of the spaces and they satisfy one of the properties that characterizes the operator semigroups.

PROPOSITION 3.25 [34, Proposition III.6]. – *Let $T_i \in \mathcal{L}(X_i, Y_i)$ be a G_δ-embedding for $i = 1, 2$. Then $T_1 \oplus T_2$ is a G_δ-embedding.*

However, the product of two G_δ-embeddings is not a G_δ-embedding, in general.

The G_δ-embeddings were applied to investigate the presence of copies of L_1, the space of all integrable functions on the unit interval, in Banach spaces.

THEOREM 3.36 [21]. – *Let $T : L_1 \to X$ be a G_δ-embedding. Then there exists a subspace Y of L_1 isomorphic to l_1 so that the restriction $T|_Y$ is an isomorphism.*

If additionally X is isomorphic to a dual space or to a subspace of L_1, then X contains a copy of L_1.

However, Talagrand showed that this is a very subtle problem.

THEOREM 3.37 [89, Theorem 1.4]. – *There exist Banach spaces X and Y that do not contain copies of L_1 but such that L_1 embeds in $X \times Y$ in such a way that the restrictions to L_1 of the projections onto X and Y are semiembeddings.*

The G_δ-embeddings share some properties with the operators in $\mathcal{CC}_+$.

PROPOSITION 3.38 [35, Theorem II.6]. – *Let* $T \in \mathcal{L}(X, Y)$ *be a* G_δ*-embedding. If* $S \in \mathcal{L}(L_1, X)$ *and* TS *is completely continuous, then* S *is completely continuous.*

The G_δ-embeddings preserve in some way the separable dual subspaces.

THEOREM 3.39 [34, Theorem I.2]. – *Let* X *be a separable Banach space. Then every* G_δ*-embedding of* X *into any Banach space is an isomorphism if and only if* X *does not contain any infinite dimensional subspace isomorphic to a separable dual.*

COROLLARY 40 [34]. – *If there exists a* G_δ*-embedding* $T \in \mathcal{L}(X, Y)$ *and the space* Y *is hereditarily separable dual, then* X *is hereditarily separable dual.*

Semiembeddings have a close relation with the semigroup $\mathcal{U}_+$ associated with the unconditionally converging operators $\mathcal{U}$.

THEOREM 3.41 [44, Theorem 3.1]. – *A Banach space contains no copies of* l_∞ *if and only if every semiembedding of* X *under any equivalent norm belongs to* $\mathcal{U}_+$.

This result shows that for a large family of Banach spaces X, the semiembeddings of X are examples of operators in $\mathcal{U}_+$. Note that, for $1 \leq p < \infty$, the natural inclusion of $L_\infty[0, 1]$ into $L_p[0, 1]$ is a semiembedding that does not belong to $\mathcal{U}_+$.

Semiembeddings have been applied to characterize scattered compact spaces.

THEOREM 3.42 [74, Theorem 11]. – *Let* K *be a compact space. Then* K *is scattered if and only if every semiembedding of* $C(K)$ *into a Banach space is an isomorphism.*

NOTES AND REMARKS 3.43. – Semiembeddings were introduced by Lotz, Peck and Porta [74] and G_δ-embeddings by Bourgain and Rosenthal [21].

Ghoussoub [33] studied the operators T whose conjugates T^* are G_δ-embeddings. He called them *semi-quotient maps*.

Other similar concepts, like the H_δ-embeddings and the nice G_δ-embeddings have been introduced to study Banach spaces. We refer to [34] for more information.

4. – Methods to define operator semigroups.

In this section we present some methods that allow us to define new semigroups or to characterize the old ones.

We consider some semigroups that can be characterized as the ultrapowers of the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ associated to the ideals $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{U}$. In general they have better behaviour than their classic counter-parts.

We describe other semigroups associated with the total incomparability [83] and the total coincomparability [10, 54] of Banach spaces. These semigroups satisfy a kind of three-space property and admit a perturbative characterization. Moreover, they allow us to characterize the concepts of incomparability.

We also show how to define semigroups in terms of certain operational quantities associated to a space ideal. Since these semigroups are open, they do not coincide in general with the corresponding semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$.

REMARKS 4.1. – There are procedures to define new operator ideals from a given one $\mathcal{A}$. Several of them have already appeared in the paper, like $\mathcal{A}^d$ and $\mathcal{A}^{co}$. Other examples may be found in [82, Chapter 4]. Many of these procedures can be also applied to operator semigroups.

4.1. *Ultrapowers of operators.*

We have already observed that $\mathcal{W}_+$ is not open [87] and that there are tauberian operators T such that T^{**} is not tauberian [9]. However, Tacon [87, 88] introduced a subclass of the tauberian operators, the supertauberian operators, and its dual class, the cosupertauberian operators that have a better behaviour. Let us denote $S_X := \{x \in X : \|x\| = 1\}$.

An operator $T \in \mathcal{L}(X, Y)$ is said to be *supertauberian* if for every $0 < \varepsilon < 1$ there exists a positive integer n for which there do not exist finite sets $\{x_1, \ldots, x_n\}$ in S_X and $\{f_1, \ldots, f_n\}$ in S_{X^*} for which $f_k(x_l) > \varepsilon$ for $1 \leq k \leq l \leq n$, $f_k(x_l) = 0$ for $1 \leq l < k \leq n$, and $\|Tx_k\| < 1/k$ for $k = 1, \ldots, n$.

An operator T is said to be *cosupertauberian* if the conjugate T^* is supertauberian.

Ultrapowers of Banach spaces are useful to study these operators. Given an ultrafilter $\mathcal{U}$ on an infinite set I, we denote by $l_\infty(I, X)$ the Banach space of all bounded families $(x_i)_{i \in I}$ in X endowed with the supremum norm, and by $N_{\mathcal{U}}(X)$ the subspace of the null families $(x_i)_{i \in I}$ following $\mathcal{U}$. The *ultrapower of X following* $\mathcal{U}$ is defined as the quotient

$$X_{\mathcal{U}} := \frac{l_\infty(I, X)}{N_{\mathcal{U}}(X)}.$$

Note that $X_{\mathcal{U}}$ contains an isometric copy of X generated by the constant families. Moreover, every operator $T \in \mathcal{L}(X, Y)$ admits a natural extension $T_{\mathcal{U}} \in \mathcal{L}(X_{\mathcal{U}}, Y_{\mathcal{U}})$. For more details about ultrapowers, we refer to [65].

For the rest of this section, we assume that $\mathcal{U}$ is a countably incomplete ultrafilter on a set I; i.e., there is a countable partition of I whose elements do not belong to $\mathcal{U}$. Given an operator ideal $\mathcal{A}$, it is immediate to show that the class

$$\mathcal{A}^{up} := \{T : T_{\mathcal{U}} \in \mathcal{A} \text{ for all ultrafilters } \mathcal{U}\}$$

is also an operator ideal. For the weakly compact operators $\mathcal{W}$, $\mathcal{W}^{up}$ is the class of all *superweakly compact* operators [64], called *uniformly convexifying* operators in [16].

Recall that an operator ideal $\mathcal{A}$ is *regular* if for every Banach space X, the natural inclusion of X into X^{**} belongs to $\mathcal{A}_+$. In this case we have $(\mathcal{A}^{up})^d = (\mathcal{A}^d)^{up}$ [65]. Note that $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{U}$ are regular. Moreover, $\mathcal{W} = \mathcal{W}^d$.

In [40], the class of the supertauberian operators is identified with the semigroup %%%$\mathcal{W}_+^{up}$, and the class of the cosupertauberian operators is identified with $\mathcal{W}_-^{up}$. These facts suggest that the semigroups $\mathcal{A}_+^{up}$ and $(\mathcal{A}^d)_-^{up}$ may have, in general, better behaviour than $\mathcal{A}_+$ and $\mathcal{A}_-$. In the case $\mathcal{A}$ is one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$, some recent results about finite representability, described below, allow us to confirm it.

Recall that X is *finitely representable* in Y if for every $0 < \varepsilon < 1$ and every finite dimensional subspace E of X, there is a *ε-isometry* $T : E \to Y$; i.e., an operator $L \in \mathcal{L}(E, Y)$ so that $(1-\varepsilon)\|x\| < \|Lx\| < (1+\varepsilon)\|x\|$, for every $x \in E$.

DEFINITION 4.2 [45]. – *Given a subspace E of a dual space X^*, we say that X^* is* finitely representable in E preserving the duality (*f.d.-r. in short*) *if for every couple of finite dimensional subspaces F of X^* and G of X, and for every $0 < \varepsilon < 1$, there is an ε-isometry $L : F \to E$ such that $(Lx)(y) = x(y)$ for all $x \in F$ and all $y \in G$.*

Let X be a Banach space and k, l two positive integers. A linear function $f : \boldsymbol{R}^k \to \boldsymbol{R}^l$, represented by a matrix $(a_{ij})_{i=1j=1}^{lk}$, induces an operator

$$f_X : X \times .^{k}. \times X \to X \times .^{l}. \times X$$

in the natural way $f_X(x_i) := \left(\sum_{j=1}^{k} a_{ij} x_j\right)$. Note that $(f_X)^* = f^*_{X^*}$. We denote by $l_1^k(X)$ and $l_\infty^k(X)$ the space $X \times .^{k}. \times X$, endowed with the norms $\sum_{j=1}^{k} \|x_j\|$ and $\sup_{1 \leq j \leq k} \|x_j\|$, respectively. Given a subset $B \subset X^*$, we denote $B_\circ := \{z \in X : |f(z)| \leq 1 \text{ for all } f \in B\}$.

DEFINITION 4.3 [45]. – *A subspace Z of a dual space X^* is said to have the* polar property *if for every k, l in $\mathbf{N}$ and every linear function $f: \mathbf{R}^k \to \mathbf{R}^l$, we have*

$$\overline{f_X(B_{l1}^k(X))} = (f_{X^*}^* \,|_{l_\infty^l(Z)}^{-1} B_{l_\infty^k(Z)})_\circ .$$

Next we show that the concepts introduced in Definitions 4.2 and 4.3 coincide.

THEOREM 4.4 [45]. – *A subspace Z of a dual space X^* has the polar property if and only if X^* is f.d.-r. in Z.*

The polar property turns out to be a powerful test to check if X^* is f.d.-r. in Z. For instance, an easy application of the Hahn-Banach Theorem shows that every Banach space X has the polar property as a subspace of X^{**}, so we get the main result of the Principle of Local Reflexivity: X^{**} is f.d.-r. in X.

Recall that for every space X, the ultrapower of its conjugate $X_{\mathcal{U}}^*$ is contained in $X_{\mathcal{U}}^*$. Hence $T_{\mathcal{U}}^*$ is an extension of $T_{\mathcal{U}}^*$ for every $T \in \mathcal{L}(X, Y)$. Using Theorem 4.4 and a similar result for a generalization of the polar property, we obtain the following result.

PROPOSITION 4.5 [45]. – *For every operator $T \in \mathcal{L}(X, Y)$, we have*:

(a) *$N(T_{\mathcal{U}}^*)$ is finitely dual representable in $N(T_{\mathcal{U}}^*)$.*

(b) *$N(T_{\mathcal{U}}^{**})$ is finitely representable in $N(T_{\mathcal{U}})$.*

In order to apply the previous results about finite representability, we need the following characterization of the semigroups $\mathcal{W}_+^{up}$, $\mathcal{R}_+^{up}$ and $\mathcal{U}_+^{up}$ in terms of the kernel of the ultrapowers of the operators.

PROPOSITION 4.6 [40, 47]. – *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$. Then the following statements are equivalent*:

(a) $T \in \mathcal{A}_+^{up}$;

(b) $N(T_{\mathcal{U}}) \in Sp(\mathcal{A}^{up})$;

(c) $N(T_{\mathcal{U}}) \in Sp(\mathcal{A})$.

Note that $Sp(\mathcal{W}^{up})$ is the ideal of all superreflexive subspaces, $Sp(\mathcal{U}^{up})$ is the ideal of all Banach spaces which do not uniformly contain copies of l_∞^n for every $n \in \mathbf{N}$, and $Sp(\mathcal{R}^{up})$ is the ideal of all spaces which do not uniformly contain copies of l_1^n for every $n \in \mathbf{N}$.

Propositions 4.6 and 4.5(b) lead to the following symmetry under duality.

PROPOSITION 4.7 [40, 45, 47]. – *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$. Then*

(a) $T \in \mathcal{A}_+^{up} \Leftrightarrow T^* \in (\mathcal{A}^d)_-^{up} \Leftrightarrow T^{**} \in \mathcal{A}_+^{up}$.

(b) $T \in (\mathcal{A}^d)_-^{up} \Leftrightarrow T^* \in \mathcal{A}_+^{up}$.

We can also characterize the semigroups $\mathcal{A}_+^{up}$ and $(\mathcal{A}^d)_-^{up}$ in terms of ultrapowers.

PROPOSITION 4.8 [40, 47]. – *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$. Then we have*:

$$\mathcal{A}_+^{up} = \{T : T_{\mathcal{U}} \in \mathcal{A}_+\} \quad \textit{and} \quad (\mathcal{A}^d)_-^{up} = \{T : T_{\mathcal{U}} \in \mathcal{A}_-^d\}.$$

Moreover, these semigroups have good topological properties and admit a perturbative characterization.

PROPOSITION 4.9 [40, 47]. – *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$. Then $\mathcal{A}_+^{up}$ and $(\mathcal{A}^d)_-^{up}$ are open, and for every $T \in \mathcal{L}(X, Y)$, we have*:

(a) $T \in \mathcal{A}_+^{up} \Leftrightarrow N(T+K) \in Sp(\mathcal{A}^{up})$ *for every* $K \in \mathcal{K}(X, Y)$;

(b) $T \in (\mathcal{A}^d)_-^{up} \Leftrightarrow Y/\overline{R(T+K)} \in Sp((\mathcal{A}^d)^{up})$ *for every* $K \in \mathcal{K}(X, Y)$.

An immediate consequence of this result is that

$$\mathcal{W}_+^{up} \subset \mathcal{R}_+^{up} \subset \mathcal{U}_+^{up} \text{ and } \mathcal{W}_-^{up} \subset (\mathcal{R}^d)_-^{up} \subset (\mathcal{U}^d)_-^{up}.$$

We can also characterize the operators $T \in (\mathcal{A}^d)_-^{up}$ in terms of the kernels $N(T_{\mathcal{U}}^*)$. Note that the same results are true with $N(T_{\mathcal{U}}^*)$ instead of $N(T_{\mathcal{U}}^*)$.

PROPOSITION 4.10 [47]. – *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{W}$, $\mathcal{R}$ or $\mathcal{U}$. Then the following statements are equivalent*:

(a) $T \in (\mathcal{A}^d)_-^{up}$;

(b) $N(T_{\mathcal{U}}^*) \in Sp(\mathcal{A}^{up})$;

(c) $N(T_{\mathcal{U}}^*) \in Sp(\mathcal{A})$.

For cosupertauberian operators we have a better result.

PROPOSITION 4.11 [45]. – *An operator $T \in \mathcal{W}_-^{up}$ if and only if $N(T_{\mathcal{U}}^*) = N(T_{\mathcal{U}}^*)$.*

Recently, Rosenthal has characterized supertauberian operators in terms of wide-(s) finite sequences. Recall [84, Definition 3] that given $\lambda > 0$, a finite sequence $x_1, \ldots, x_n \in X$ is said to be a *λ-wide-(s) sequence* if

(*a*) $\left\| \sum_{i=1}^{k} c_i x_i \right\| \leq 2\lambda \left\| \sum_{i=1}^{n} x_i \right\|$ for all $k < n$ and scalars $c_1, \ldots, c_n$,

(*b*) $\|x_i\| \leq \lambda$ for every i, and

(*c*) $\left| \sum_{i=k}^{n} c_i \right| \leq \lambda \left\| \sum_{i=1}^{n} c_i x_i \right\|$ for all $1 \leq k \leq n$ and scalars $c_1, \ldots, c_n$.

THEOREM 4.12 [84]. – *An operator $T \in \mathcal{L}(X, Y)$ is not supertauberian if and only if for every $\varepsilon > 0$ there are finite $(1 + \varepsilon)$-wide-(s) sequences of arbitrary lenght whose images have norm at most ε.*

The ultrapower $L_1(\mu)_{\mathcal{U}}$ can be decomposed as $L_1(\mu)_{\mathcal{U}} = L_1(\mu_{\mathcal{U}}) \oplus L_1(\nu_{\mathcal{U}})$, where $L_1(\mu_{\mathcal{U}})$ consists of all $\boldsymbol{f} \in L_1(\mu)_{\mathcal{U}}$ which admit an equiintegrable representative, and $L_1(\nu_{\mathcal{U}})$ consists of those $\boldsymbol{f} = (f_i)_{i \in I}$ such that $\lim_{i \to \mathcal{U}} \mu(\{f_i \neq 0\}) = 0$ [48].

For operators $T : L_1(\mu) \to L_1(\mu)$, the decomposition of $L_1(\mu)_{\mathcal{U}}$ induces the following matricial representation of $T_{\mathcal{U}}$:

$$T_{\mathcal{U}} = \begin{pmatrix} T_{\mathcal{U}}^{11} & T_{\mathcal{U}}^{12} \\ 0 & T_{\mathcal{U}}^{22} \end{pmatrix},$$

Using this decomposition, we can characterize the supertauberian operators on $L_1(\mu)$.

THEOREM 4.13 [46]. – *For $T : L_1(\mu) \to Y$, the following statements are equivalent*:

(1) *T is supertauberian*;

(2) *$N(T_{\mathcal{U}}) \subset L_1(\mu_{\mathcal{U}})$*;

(3) *$T_{\mathcal{U}}|_{L_1(\nu_{\mathcal{U}})}$ is an isomorphism*;

(4) *$T_{\mathcal{U}}|_{L_1(\nu_{\mathcal{U}})}$ is injective.*

(5) *$T_{\mathcal{U}}^{22}$ is an isomorphism.*

Finally, we give some examples of operators in the semigroups $\mathcal{A}_+^{up}$ and $(\mathcal{A}^d)_-^{up}$.

EXAMPLES. – (*a*) *Semi-Fredholm operators:* For the operator ideal of compact operators we have that $\mathcal{K} = \mathcal{K}^{up}$. Thus,

$$\Phi_+ = \mathcal{K}_+^{up} \text{ and } \Phi_- = \mathcal{K}_-^{up}.$$

(*b*) *Operators with closed range:* Let $T \in \mathcal{L}(X, Y)$ be an operator with closed range. Then $T \in \mathcal{A}_+^{up}$ if and only if $N(T) \in Sp(\mathcal{A}^{up})$. Moreover, $T \in (\mathcal{A}^d)_-^{up}$ if and only if $Y/R(T) \in Sp(\mathcal{A}^d)$ [47].

(c) *Operators on* $L_1(\mu)$: Since a subspace of $L_1(\mu)$ is either superreflexive or contains a copy of l_1 [17], Theorem 3.15 and Propositions 2.21 and 4.9 yield

$$\mathcal{W}_+(L_1(\mu), Y) = \mathcal{W}_+^{up}(L_1(\mu), Y) = \mathcal{R}_+(L_1(\mu), Y) = \mathcal{R}_+^{up}(L_1(\mu), Y)$$

for every Banach space Y.

(d) Let $\mathcal{N}^*$ be the original Tsirelson space. Since $\mathcal{N}^*$ is reflexive, every operator $T: \mathcal{N}^* \to Y$ is tauberian. However, since l_1 is finite representable in every infinite dimensional subspace of $\mathcal{N}^*$ [24], we have that $\Phi_+(\mathcal{N}^*, Y) = \mathcal{R}_+^{up}(\mathcal{N}^*, Y)$ for every Banach space Y.

(e) The natural inclusion $i : J \to c_0$ of the classical James space into c_0 is tauberian. However, $i \notin \mathcal{U}_+^{up}$ [47].

NOTES AND REMARKS 4.14. – The supertauberian operators were studied by Tacon [87, 88] using nonstandard analysis. Further study using ultrapowers was done in [15, 40, 76].

4.2. *Incomparable Banach spaces.*

Fredholm theory has been fruitfully applied to the study of Banach spaces throughout the concept of incomparability between Banach spaces. We refer to [52] for a description of these applications. Here we follow [7] to show that using these concepts we can define some operator ideals whose associated semigroups have a good behaviour: they admit a perturbative characterization and the operators $T \in \mathcal{L}(X, Y)$ with closed range $R(T)$ in the semigroups can be characterized in terms of the properties of the kernel $N(T)$ or the cokernel $Y/R(T)$ of the operator.

DEFINITION 4.15 [83, 54]. – *We say that two Banach spaces X and Y are* totally incomparable *if there is no infinite dimensional subspace of X isomorphic to a subspace of Y.*

We say that X and Y are totally coincomparable *if there is no infinite dimensional quotient of X isomorphic to a quotient of Y.*

EXAMPLE 4.6. – (a) The spaces l_1 and l_p are totally incomparable but not totally coincomparable for $1 < p < \infty$.

(b) l_∞ and l_q are totally coincomparable but not totally incomparable for $2 < q < \infty$.

(c) l_p and l_q are totally incomparable and totally coincomparable for $1 < p < q < \infty$.

Indeed, every separable Banach space is isomorphic to a quotient of l_1 and to a subspace of l_∞. Moreover, every operator from l_q into l_p is compact [72, Proposition 2.c.3] for $p < q < \infty$, and every infinite dimensional subspace of l_p contains a complemented subspace isomorphic to l_p [72, Proposition 2.a.2]. These results imply (a).

We can prove (*b*) and (*c*) in a similar way.

It is easy to see that if X^* and Y^* are subspace (quotient) incomparable then X and Y are quotient (subspace) incomparable, but the converse implications fail [54].

These concepts admit the following structural characterization.

THEOREM 4.17 ([83, Theorem 2] and [54]). – (a) *X and Y are totally incomparable if and only if for every Banach space Z with subspaces M and N isomorphic to X and Y, the sum $M+N$ is closed.*

(b) *X and Y are totally coincomparable if and only if for every Banach space Z with subspaces M and N such that Z/M and Z/N are isomorphic to X and Y, the sum $M+N$ is closed.*

For a class of Banach spaces A, the incomparability classes A^i and A^c are defined by

$$\mathsf{A}^s := \{X : X \text{ totally incomparable with every } Y \in \mathsf{A}\} \text{ and}$$

$$\mathsf{A}^q := \{X : X \text{ totally coincomparable with every } Y \in \mathsf{A}\}.$$

We can repeat the procedure and define A^{ss}, A^{cc}, etc. It is not difficult to see that $\mathsf{A} \subset \mathsf{A}^{ss}$, $\mathsf{A}^s \cap \mathsf{A}^{ss} = \mathsf{F}$, the finite dimensional spaces, $\mathsf{A}^s = \mathsf{A}^{sss}$, and we have analogous results for A^q.

PROPOSITION 4.18 [10]. – *For every class of Banach spaces A, the incomparability class A^s (respectively, A^q) is an injective (respectively, surjective) space ideal which satisfies the three-space property.*

The incomparability classes of a space ideal $Sp(\mathcal{A})$ can be characterized in terms of the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ for some operator ideals. In the next Proposition we assume $\mathcal{A}$ closed so that $\mathcal{A}$ contains the nuclear operators.

PROPOSITION 4.19 [7]. – *Let $\mathcal{A}$ be a closed operator ideal.*

(a) *If $\mathcal{A}$ is injective and satisfies the left three-space property, then*

$$Sp(\mathcal{A})^s = \{X : \mathcal{A}_+(X, Y) = \Phi_+(X, Y) \text{ for every space } Y\}.$$

(b) *If $\mathcal{A}$ is surjective and satisfies the right three-space property, then*

$$Sp(\mathcal{A})^q = \{X : \mathcal{A}_-(Z, X) = \Phi_-(Z, X) \text{ for every space } Z\}.$$

Using these concepts of incomparability, for every class A of Banach spaces, we define the A-singular operators $\mathsf{A}\mathcal{S}$ and the A-cosingular operators $\mathsf{A}\mathcal{C}$, that are generalizations of the strictly singular and the strictly cosingular operators, in the following way:

$$\mathsf{A}\mathcal{S}(X, Y) := \{T \in \mathcal{L}(X, Y) : TJ_M \text{ isomorphism } \Rightarrow M \in \mathsf{A}\},$$

$$\mathsf{A}\mathcal{C}(X, Y) := \{T \in \mathcal{L}(X, Y) : Q_N T \text{ surjective } \Rightarrow Y/N \in \mathsf{A}\}.$$

PROPOSITION 4.20 [11]. – *Let* A *be a space ideal.*

(a) *If* $\mathsf{A} = \mathsf{A}^{ss}$, *then* $\mathsf{A}\mathcal{S}$ *is an operator ideal and* $Sp(\mathsf{A}\mathcal{S}) = \mathsf{A}$.

(b) *If* $\mathsf{A} = \mathsf{A}^{qq}$, *then* $\mathsf{A}\mathcal{C}$ *is an operator ideal and* $Sp(\mathsf{A}\mathcal{C}) = \mathsf{A}$.

EXAMPLE 4.21. – (a) The class Nc_0 of Banach spaces containing no copies of c_0 satisfies $Nc_0 = Nc_0^{ss}$ and $Nc_0\mathcal{S} = \mathcal{U}$, the unconditionally converging operators.

(b) The class Nl_1 of Banach spaces containing no copies of l_1 satisfies $Nl_1 = Nl_1^{ss}$ and $Nl_1\mathcal{S} = \mathcal{R}$, the weakly precompact operators.

QUESTION 16. – If A does not satisfy $\mathsf{A} = \mathsf{A}^{ss}$ (respectively, $\mathsf{A} = \mathsf{A}^{cc}$), we do not know if $\mathsf{A}\mathcal{S}(X, Y)$ (respectively, $\mathsf{A}\mathcal{C}(X, Y)$) is always a subspace of $\mathcal{L}(X, Y)$.

The problem for A the class of reflexive Banach spaces was raised in [67].

Next we show that, in some cases, the semigroups $\mathsf{A}\mathcal{S}_+$ and $\mathsf{A}\mathcal{C}_-$ admit algebraic and perturbative characterizations.

THEOREM 4.22 [7]. – *Let* A *be a space ideal satisfying* $\mathsf{A} = \mathsf{A}^{ss}$. *Then for* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent*:

(a) $T \in \mathsf{A}\mathcal{S}_+$;

(b) $N(T + K) \in \mathsf{A}$ for every $\in \mathcal{K}(X, Y)$;

(c) $TJ_M \in \Phi_+$ *for every subspace* $M \in \mathsf{A}^s$.

THEOREM 4.23 [7]. – *Let* A *be a space ideal satisfying* $\mathsf{A} = \mathsf{A}^{qq}$. *Then for* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent*:

(a) $T \in \mathsf{A}\mathcal{C}_-$;

(b) $Y/\overline{R(T + K)} \in \mathsf{A}$ *for every* $\in \mathcal{K}(X, Y)$;

(c) $Q_N T \in \Phi_-$ *for every subspace* N *such that* $Y/N \in \mathsf{A}^q$.

From these characterizations we derive that the semigroups have the left (or the right) three-space property.

COROLLARY 4.24 [7]. – *Let* $T \in \mathcal{L}(X, Y)$ *be an operator with* $R(T)$ *closed.*

(a) *If* $\mathsf{A} = \mathsf{A}^{ss}$ *and* $N(T) \in \mathsf{A}$, *then* $T \in \mathsf{A}\mathcal{S}_+$.

(b) *If* $\mathsf{A} = \mathsf{A}^{qq}$ *and* $Y/R(T) \in \mathsf{A}$, *then* $T \in \mathsf{A}\mathcal{C}_-$.

EXAMPLE 4.25. – In the case $\mathsf{A} = Nc_0$ we have $\mathsf{A}\mathcal{S}_+ = \mathcal{U}_+$, where $\mathcal{U}$ is the ideal of the unconditionally converging operators.

For $\mathsf{A} = Nl_1$ we have $\mathsf{A}\mathcal{S}_+ = \mathcal{R}_+$, where $\mathcal{R}$ is the ideal of the weakly precompact operators. These semigroups were considered in Section 3.4.

QUESTION 17. – Do the semigroups $\mathsf{A}\mathcal{S}_-$ and $\mathsf{A}\mathcal{S}_+^d$ admit a perturbative characterization in the case $\mathsf{A} = \mathsf{A}^{ss}$?

Do they have the left (or the right) three-space property?

The same questions can be asked for $\mathsf{A}\mathcal{C}_+$ and $\mathsf{A}\mathcal{C}_-^d$ in the case $\mathsf{A} = \mathsf{A}^{qq}$?

NOTES AND REMARKS 4.26. – For other notions of incomparability we refer to [52].

4.3. *Operational quantities.*

Here we show that it is possible to define operator ideals and operator semigroups in terms of operational quantities.

Semi-Fredholm operators and strictly singular (cosingular) operators have been studied by means of some operational quantities. Let us denote by $n(T) := \|T\|$ the norm of $T \in \mathcal{L}(X, Y)$. Schechter [86] (with a different notation) considered the operational quantities in and sin, defined as follows:

$$in(T) := \inf\{n(TJ_M) : dim M = \infty\},$$

$$sin(T) := \sup\{in(TJ_M) : dim M = \infty\},$$

and for $K, T \in \mathcal{L}(X, Y)$, he proved that $T \in \Phi_+$ if and only if $in(T) > 0$, $T \in \mathcal{SS}$ if and only if $sin(T) = 0$ and

$$sin(K) < in(T) \Rightarrow T + K \text{ is upper semi} - \text{Fredholm}.$$

The last result unifies and improves previous results about the stability of upper semi-Fredholm operators under perturbation by small-norm and strictly singular operators. Note that in the definition of in and sin we need the space X to be infinite dimensional.

Lower semi-Fredholm operators and strictly cosingular operators may be also characterized in terms of operational quantities derived from the norm, using quotients instead of subspaces. We refer to [4, Section 3] for a brief description. Moreover, similar results have been obtained for operational quantities derived from the *injection modulus*, the *surjection modulus* and other operational quantities [78, Section 3].

These operational quantities that characterize the classes of operators of Fredholm theory are defined in terms of the class F of finite dimensional spaces. Using other space ideals instead of F, other operational quantities were introduced in [50, 51], that allowed to obtain operator semigroups.

Given a space ideal A, we say that X is hereditarily in A if every subspace of X belongs to A. We say that X is co-hereditarily in A if every quotient of X belongs to A.

Now, assuming that X is not hereditarily in a space ideal A, for every operator $T \in \mathcal{L}(X, Y)$ we define the operational quantities sj_{A} and isj_{A} by

$$sj_{\mathsf{A}}(T) := \sup\{ j(TJ_M) : M \notin \mathsf{A}\},$$

$$isj_{\mathsf{A}}(T) := \inf\{ sj_{\mathsf{A}}(TJ_M) : M \notin \mathsf{A}\}.$$

Moreover, assuming that Y is not co-hereditarily in A, we define the operational quantities sq_{A} and isq_{A} by

$$sq_{\mathsf{A}}(T) := \sup\{ q(Q_N T) : Y/N \notin \mathsf{A}\},$$

$$isq_{\mathsf{A}}(T) := \inf\{ sq_{\mathsf{A}}(Q_N T) : Y/N \notin \mathsf{A}\}.$$

In the case $\mathsf{A} = \mathsf{F}$, the finite dimensional spaces, these quantities characterize the operators in classical Fredholm theory:

$$T \in \Phi_+ \Leftrightarrow isj_{\mathsf{F}}(T) > 0; \quad T \in \mathcal{SS} \Leftrightarrow sj_{\mathsf{F}}(T) = 0\,;$$

$$T \in \Phi_- \Leftrightarrow isq_{\mathsf{F}}(T) > 0; \quad \text{and } T \in \mathcal{SC} \Leftrightarrow sj_{\mathsf{F}}(T) = 0\,.$$

Moreover, it is not difficult to see that the quantities sj_{A} and sq_{A} characterize the classes $\mathsf{A}\mathcal{S}$ and $\mathsf{A}\mathcal{C}$ introduced in section 4.2 in the following way:

$$T \in \mathsf{A}\mathcal{S} \Leftrightarrow sj_{\mathsf{A}}(T) = 0; \quad T \in \mathsf{A}\mathcal{C} \Leftrightarrow sq_{\mathsf{A}}(T) = 0\,.$$

The following result shows that these quantities are suitable to define semigroups in the form

$$\{T \in \mathcal{L} : isj_{\mathsf{A}}(T) > 0\} \text{ or } \{T \in \mathcal{L} : isq_{\mathsf{A}}(T) > 0\}.$$

THEOREM 4.27 [51, Theorem 2.5]. – *Let* A *be a space ideal, and let* $T \in \mathcal{L}(X, Y)$ *and* $S \in \mathcal{L}(Y, Z)$.

(a) *If* X *and* Y *are not hereditarily in* A, *then* $isj_{\mathsf{A}}(S) \cdot isj_{\mathsf{A}}(T) \leq isj_{\mathsf{A}}(ST)$.

(b) *If* Y *and* Z *are not co-hereditarily in* A, *then* $isq_{\mathsf{A}}(S) \cdot isq_{\mathsf{A}}(T) \leq isq_{\mathsf{A}}(ST)$.

The semigroups obtained in this way are clearly open; hence they do not coincide with the semigroups $\mathsf{A}\mathcal{S}_+$ and $\mathsf{A}\mathcal{C}_-$ of section 4, in general. Indeed, if A is either the class of Banach spaces containing no copies of l_1 or the class of Banach spaces containing no copies of c_0, the semigroup $\mathsf{A}\mathcal{S}_+$ is $\mathcal{R}_+$ or U_+ (Example 4.25), which are not open.

NOTES AND REMARKS 4.28. – It is possible to define ideal variations associated with an operator ideal [12, 92], which in turn allow us to define semigroups. Some of these semigroups, in the case of the weakly compact operators, were considered in [13].

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