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An Electromagnetic Damping Machine: Model, Analysis and Numerics.

A. Buffa - Y. Maday - F. Rapetti (*)

Sunto. – *In questo lavoro viene considerato il modello bidimensionale completo di sistema elettromagnetico in movimento: le equazioni dei campi elettromagnetici sono accoppiate con quelle della meccanica e il sistema così ottenuto risulta essere non lineare nell'accoppiamento. Vengono analizzate la buona posizione del problema e la regolarità della soluzione continua; si propone inoltre uno schema di discretizzazione di tipo esplicito. Si dimostra la buona posizione e la convergenza della formulazione discreta e si propongono alcuni test numerici comprovanti la convergenza dello schema proposto. L'algoritmo fornisce la simulazione completa di un freno magnetico e permette di mettere in evidenza i fenomeni non lineari legati al suo funzionamento.*

Introduction.

In a coupled magneto-mechanical system, the forces due to the magnetic field make the free structure move and the resulting variation in the structure configuration modifies the distribution of the magnetic field and consequently of the induced forces. Therefore, the interaction between magnetic and mechanical phenomena cannot be simulated independently (see also Gaspalou *et al.*, 1995). The modeling of this coupled system requires to take simultaneously into account the electromagnetic and mechanical equations. To carry out such a coupling, it is necessary to compute the global magnetic force acting on the moving part of the system, through the numerical evaluation of the magnetic field. As an example, we study a system composed of two solid parts: the stator, which stands still, and the rotor, which can rotate around its rotation axis.

The algorithm we consider is based on an «explicit» coupling procedure: at each time step, the magnetic force obtained from the field solution is inserted into the mechanical equation to compute the displacement. The latter is imposed to the moving part for the next step of the magnetic field calculation. In presence of a friction coefficient, the procedure naturally ends when the free part has reached its equilibrium position corresponding to a zero magnetic torque. In this model, the time step has to be small enough so that the induced

(*) Comunicazione presentata dal primo autore a Napoli in occasione del XVI Congresso U.M.I.

force does not change too much from one step to the next one. If it is not the case, a procedure to check the convergence of either the force or the displacement is necessary (see Vassent *et al.*, 1991).

The magnetic force is obtained from the generalized Lorentz law and the magnetic field is computed by applying the sliding mesh mortar finite element strategy to the magnetic vector potential formulation of the eddy currents problem, as proposed in Buffa *et al.*, 1999 (see Bernardi *et al.*, 1994 for more details on the mortar element method). To avoid the presence of a convective term in the equations, we work in Lagrangian variables: the problem equations are solved in their own frames, that are one fixed with the stator and the other rotating with the rotor. We remark that in a Lagrangian approach, the mesh nodes always coincide with the same material particles throughout the movement. Due to this feature, even if the stator and the rotor meshes have been generated in such a way that the two sets of nodes lying on the sliding interface coincide at the initial configuration, it could not be the case when the rotor part has moved. The mortar method is a non-conforming non-overlapping domain decomposition technique which allows for independent (and thus, in general, non-matching at the interface) meshes in the stator and rotor domains. The idea of the mortar method is to weakly impose the transmission conditions at the sliding interface by means of Lagrangian multipliers. The key argument is the explicit construction of a particular Lagrangian multipliers space in order to ensure good properties on the discrete problem. In the context of the node finite element method, the space of Lagrangian multipliers is the space of the shape functions' traces at the sliding interface. The possibility of using such a method allows us to work with a whole mesh composed of a fixed part and a rotating one, without imposing constraints between the mesh element size at the interface and the rotation angle associated with each time step. In the simulation of a coupled magneto-mechanical, this is important since at each time step the rotation angle is not constant but depends on the magnetic torque.

1. – The continuous model problem.

In this section, we present the mathematical model we consider to study the proposed coupled magneto-mechanical moving system. Eddy currents problems are mathematically described by Maxwell equations where the displacement currents are neglected with respect to the conducting ones. In the two dimensional transverse magnetic formulation of such problems, the magnetic vector potential $\boldsymbol{A} = (0, 0, u(\boldsymbol{x}, t))$ satisfies, in the (x, y) section Ω of a cylinder of $\mathbb{R}^3$, the scalar equation

$$\sigma \partial_t u - \mathrm{div}\,(\nu \,\mathrm{grad}\, u) = j_z \tag{1}$$

where j_z is the non-zero third component of the source currents density, σ the electric conductivity and $\mu = \nu^{-1}$ the magnetic permeability. The presence of a

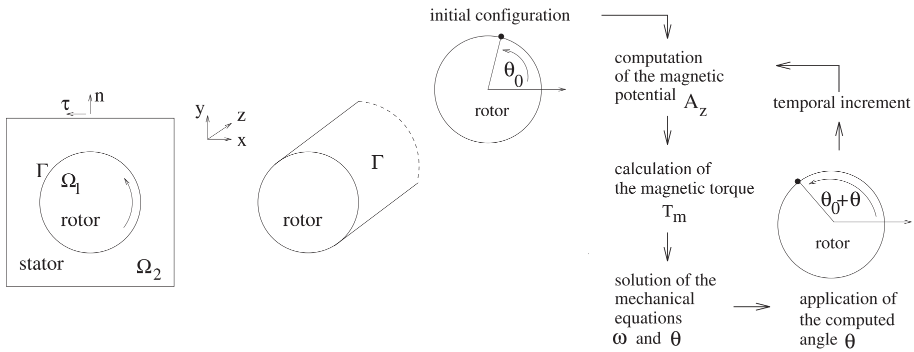

Fig. 1. – The computational domain $\Omega = \Omega_1 \cup \Omega_2$. The iterative «explicit» procedure.

magnetic field in the considered system generates an induced electromagnetic force which acts as a torque on the moving part $\Omega_1 \subset \Omega$ (see Fig. 1). The motion equation of this part around its center is described by

$$J \frac{d\omega}{dt} + k\omega = T_m, \qquad \omega = \frac{d\theta}{dt} \tag{2}$$

where $\omega = \omega(t)$ is the rotation speed, $\theta = \theta(t)$ the rotation angle, $T_m = T_m(t)$ the acting global magnetic torque, J the inertial momentum per unit lenght of Ω_1 and k the friction coefficient.

If we denote, for any time $t > 0$, by $r_t\colon \Omega_1 \to \Omega_1$ the rotation operator which rotates the domain Ω_1 with an angle equal to $\theta = \theta(t)$ and r_{-t} the inverse operator, the coupled system reads

$$\begin{aligned}
&\sigma(\boldsymbol{x}) \frac{\partial u_i}{\partial t}(\boldsymbol{x}, t) - \operatorname{div}(\nu \operatorname{grad} u_i)(\boldsymbol{x}, t) = j_z(\boldsymbol{x}, t) && \Omega_i \times]0, T[\quad (i = 1, 2)\\
&u_1(r_{-t}\boldsymbol{x},\, t) = u_2(\boldsymbol{x},\, t) && \Gamma \times]0,\, T[\\
&\nu(r_{-t}\boldsymbol{x}) \frac{\partial u_1}{\partial \boldsymbol{n}}(r_{-t}\boldsymbol{x},\, t) = \nu(\boldsymbol{x}) \frac{\partial u_2}{\partial \boldsymbol{n}}(\boldsymbol{x},\, t) && \Gamma \times]0,\, T[\\
&u_2(\boldsymbol{x}, t) = u_0(\boldsymbol{x}) \quad (\partial\Omega)_D \times]0, T[\,; \qquad \frac{\partial u_2}{\partial \boldsymbol{n}_{\partial\Omega}} = 0 && (\partial\Omega)_N \times]0,\, T[\\
&u(\boldsymbol{x},\, 0) = 0 && \Omega_1 \cup \Omega_2 \times \{0\}
\end{aligned} \tag{3}$$

$$\begin{aligned}
&(0, 0, T_m) = \int_{\Omega_1} \boldsymbol{r} \wedge \left[\left(-\sigma\left(0, 0, \frac{\partial u_1}{\partial t}\right) + (0, 0, j_z) \right) \wedge \mathbf{curl}\,(0, 0, u_1) \right] d\Omega\\
&J \frac{d\omega}{dt} + k\omega = T_m, \quad \omega = \frac{d\theta}{dt} \quad]0,\, T[\\
&\omega(0) = 0\,, \quad \theta(0) = \theta_0
\end{aligned} \tag{4}$$

where $\boldsymbol{n}$ is at every $\boldsymbol{x} \in \Gamma$ the unit vector normal outward to Ω_2, $\boldsymbol{n}_{\partial\Omega}$ the unit vector normal outward to Ω and u_i $(i = 1, 2)$ are the restrictions to Ω_i of the third component of the magnetic vector potential. We remark that for the studied example, σ and j_z are supposed equal to zero in $\Omega_2 = \Omega \backslash \Omega_1$. We set $\mathcal{H}^s(\Omega) = H^s(\Omega_1 \cup \Omega_2)$ for any $s \geq 1$.

The following theorem holds:

THEOREM 1.1. – *Let $J_z \in H^1(0, T, L^2(\Omega_1))$. The system of equations* (3) *admits at least one solution $(u, T_m, \omega) \in L^2(0, T, \mathcal{H}^2(\Omega)) \cap L^\infty(0, T, \mathcal{H}^1(\Omega)) \times L^2(0, T) \times C^0(0, T)$. For J large enough the solution is also unique.*

PROOF. – Let $T \in \mathbb{R}$ and $\mathcal{G}: C^0(0, T) \to C^0(0, T)$ be the open feedback operator, that associates to every angular speed $\overline{\omega} \in C^0(0, T)$, the angular speed ω calculated by means of (4) when the torque T_m i s computed by means of the solution u of (3) associated with the speed $\overline{\omega}$.

Using the results obtained in Buffa *et al.*, 1999 we know that the system (3) admits a unique solution $u \in L^2(0, T, \mathcal{U}^t) \cap C^0(0, T, L^2(\Omega))$. Now, thanks to the regularity results proved in Bouillault *et al.*, 2000 in the «open feedback» system, for any $\overline{\omega} \in C^0(0, T)$, the integral in the equation in (4) is meaningfull and moreover the resulting torque $T_m = T_m(t)$ belongs to $L^2(0, T)$.

Using finally the second and third equations in (4) we find the angular speed ω which turns out to belong to $H^1(0, T)$. Moreover the following stability holds:

$$\|\omega\|_{H^1(0, T)} \leq C \|\overline{\omega}\|_{C^0(0, T)}.$$

Since the embedding $H^1(0, T) \hookrightarrow C^0(0, T)$ is compact, the operator $\mathcal{G}$ is also compact. By applying the Schauder fixed-point theorem, we deduce that $\mathcal{G}$ has at least one fixed point which corresponds to a solution of the system (3).

It is not difficult to see that, when J is large enough, the operator $\mathcal{G}$ is also contractive and, by applying the Banach fixed-point theorem, we have that such a fixed point is also unique. ■

2. – Discretization of the coupled system.

As in Buffa *et al.*, 1999, we discretize the following functional space, that is defined at each time $t \geq 0$:

$$\mathcal{U}^t = \{u := (u_1, u_2) \in H^1(\Omega_1) \times H^1_{0, \partial\Omega_D}(\Omega_2) \quad \text{such that } u_1(r_{-t}\boldsymbol{x}) = u_2(\boldsymbol{x}) \text{ a.e. } \boldsymbol{x} \in \Gamma\}. \tag{5}$$

Let $X_{i, h}$ be finite element spaces of degree one over Ω_i $(i = 1, 2)$ and M_h

be the space of traces over Γ of elements of, say, $X_{2,h}$. The choice of M_h as the space of traces over Γ of elements of $X_{1,h}$ provides a different but similar method.

To solve problem (3-4) at each time step, we use an «explicit» coupling procedure (see Fig. 2): given a computed solution consisting in u_h^n, T_{mh}^n at time $t^n = n\delta t$ and θ_h^{n+1} at time t^{n+1}, we compute the magnetic field u_h^{n+1} belonging to the discrete version of $\mathcal{U}_0^t$ at time t^{n+1} given by

$$\mathcal{U}_h^0(n+1) = \left\{ v_h = (v_{1,h}, v_{2,h}) \in X_{1,h} \times X_{2,h} \text{ such that } \int_\Gamma (v_{1,h}(r_{-t^{n+1},h}\boldsymbol{x}) - v_{2,h}(\boldsymbol{x}))\, \varphi_h(\boldsymbol{x})\, d\Gamma = 0 \quad \forall \varphi_h \in M_h \right\} \tag{6}$$

where $r_{-t^{n+1},h}$ is the rotation operator relative to θ_h^{n+1}, by solving

$$\forall n = 0, \dots, N-1, \text{ find } u_h^{n+1} \in \mathcal{U}_h^0(n+1) \text{ such that } \forall v_h^{n+1} \in \mathcal{U}_h^0(n+1):$$

$$\int_{\mathcal{C}} \sigma \frac{u_h^{n+1} - u_h^n}{\delta t} v_h^{n+1}\, d\Omega + \sum_{i=1}^{2} \int_{\Omega_i} \nu \nabla u_h^{n+1} \cdot \nabla v_h^{n+1}\, d\Omega = \int_{\Omega_2} j_z^{n+1} v_h^{n+1}\, d\Omega\,, \tag{7}$$

where $\mathcal{C}$ denotes the conducting region ($\sigma \neq 0$).

Once we have u_h^{n+1} from system (7), we can compute the associated *discrete torque* by:

$$(0, 0, T_{mh}^{n+1}) =$$

$$\int_{\Omega_1} \left(\boldsymbol{r} \wedge \left[\left(-\sigma \left(0, 0, \frac{u_h^{n+1} - u_h^n}{\delta t} \right) + (0, 0, j_z^{n+1}) \right) \right] \wedge \mathbf{curl}\,(0, 0, u_h^{n+1}) \right) d\Omega\,.$$

Concerning the discretization in (2) of the angular speed, we choose the same time step, δt, as for the time discretization of the magnetic equation and propose an implicit first order Euler scheme that reads

$$\text{Find} \quad w_h^{n+1}, n = 0, \dots, N-1 \quad \text{such that} \quad J \frac{w_h^{n+1} - w_h^n}{\delta t} + k w_h^{n+1} = T_{mh}^{n+1}.$$

For what concerns the angle θ_h^{n+2}, we use an explicit first order Euler scheme of the form

$$\theta_h^{n+2} = \theta_h^{n+1} + \delta t w_h^{n+1}$$

and we can go back to problem (7) in the new space $\mathcal{U}_h^0(n+2)$. In the following Theorem, based on convergence results proven in Bouillault *et al.*, 2000, and in Buffa *et al.*, 1999, we state that this explicit algorithm has an optimal convergence rate:

THEOREM. – *Under suitable regularity assumptions on the solution* (u, ω, T_m) *of* ((3)-(4)), *the following error estimate holds*:

$$\|u^n - u_h^n\|_{0,\Omega}^2 + \sum_{i=1}^{n} \delta t \|u^i - u_h^i\|_\star^2 + |w^n - w_h^n|^2 + |\theta_h^{n+1} - \theta^{n+1}|^2 + \sum_{i=1}^{n} \delta t |T_m^n - T_{mh}^n|^2 \leqslant c(h^2 + \delta t^2)$$

where the constant c *depends neither on* h *nor on* δt (*here* $\|u\|_\star^2 = \|u_1\|_{1,\Omega_1}^2 + \|u_2\|_{1,\Omega_2}^2$).

In order to write the discrete version of (7) in a matrix form, we need to construct a basis of the approximation space $\mathcal{U}_h^0$. As it is standard with finite element methods, the elements of the chosen basis are built from the node element basis functions. At the nodes lying on Γ, the basis elements are linked through the matching condition stated in (6). At the algebraic level, this involves a rectangular matrix $\widetilde{Q}$ that allows for coupling at the sliding interface the information coming from the stator and rotor domains at time t (see Rapetti *et al.*, 1999, for more details). The matrix form of the fully discrete problem (7) has the following layout

$$\widetilde{Q}_{n+1}^T \left(K + \frac{M}{\Delta t} \right) \widetilde{Q}_{n+1} \boldsymbol{U}_{n+1} = \widetilde{Q}_{n+1}^T \frac{M}{\Delta t} \widetilde{Q}_n \boldsymbol{U}_n + \widetilde{Q}_{n+1}^T \widetilde{Q}_n \boldsymbol{J}_n \tag{8}$$

where $\boldsymbol{J}_n = (\boldsymbol{0}, \boldsymbol{J}_{1,n}^\Gamma, \boldsymbol{J}_{1,n}^{\mathrm{Int}})$ and $\boldsymbol{U}_{n+1} = (\boldsymbol{U}_{2,n+1}^{\mathrm{Int}}, \boldsymbol{U}_{1,n+1}^\Gamma, \boldsymbol{U}_{1,n+1}^{\mathrm{Int}})$ are the real degrees of freedom. Here we have defined $\boldsymbol{U}_{i,n+1}^{\mathrm{Int}}$ as the vector at time t_{n+1} of the unknown magnetic potential values at the mesh nodes internal to domain Ω_i; similarly, $\boldsymbol{U}_{i,n+1}^\Gamma$ is the vector at time t_{n+1} of the unknown magnetic potential values at the mesh nodes lying on Γ and belonging to Ω_i. We denote by ^{T} the transpose operator, K and M, respectively, the classical stiffness and mass matrices and by $\widetilde{Q}_{n+1}$, the coupling matrix at time t_{n+1}. We remark that the coupling matrix has to be rebuilt at each time step (i.e. at each new rotor position) whereas the matrices K, M do not depend on time. The final system (2.10) has a symmetric and positive matrix and can be solved iteratively by a Conjugate Gradient procedure.

3. – Accuracy of the method.

The accuracy of the sliding-mesh mortar element method, when applied to compute the induced currents in a stator-rotor system as the one considered here, has been already analyzed in Buffa *et al.*, 1999. In this subsection we are going to analyze the accuracy of the same method when applied to the coupled problem. The concerned quantities are the magnetic torque T_m, the rotation angle θ and the angular speed ω. For these three quantities, we will make a comparison between the «exact value» (v) and the one (v_h) numerically computed with different time steps on different meshes, with the fixed value $\sigma = 10^7$ S/m. The term «exact value» actually refers to a numerical value computed on the finest mesh with the smallest time step.

Temporal error: it is given by

$$\|v - v_h\|_{L^\infty([0,\,T])} = \sup\{\,|v(t) - v_h(t)|\,, \qquad t \in [0,\,T]\}\,. \tag{9}$$

The considered time steps are $\delta t = 2^s\,\delta t_1$ with $s = 0, 1, 2, 3, 4$ and $\delta t_1 = 2.\,510^{-4}$ s and all simulations are led on an unstructured mesh. In Figure 2 are reported the time errors on the position of the rotor and on the magnetic torque with respect to the time step. Both figures show that the error depends linearly on the time step. Results confirm the theoretical linear dependence of the torque value on the time step.

Spatial error: analyzed for the magnetic torque only, it shows the influence of the mesh triangles size h on the torque values. All simulations have been done with $\delta t = \delta t_1$ and using the same stator mesh. The considered rotor meshes have triangles of size $h = 2^r h_1$ with $h_1 = 5\cdot 10^{-3}$ m and $r = -2, -1, 0, 1, 2$. The analysis that has been done in Section 3 on the coupled problem is related to the use of curved finite elements. Since we have used more simple «flat» fi-

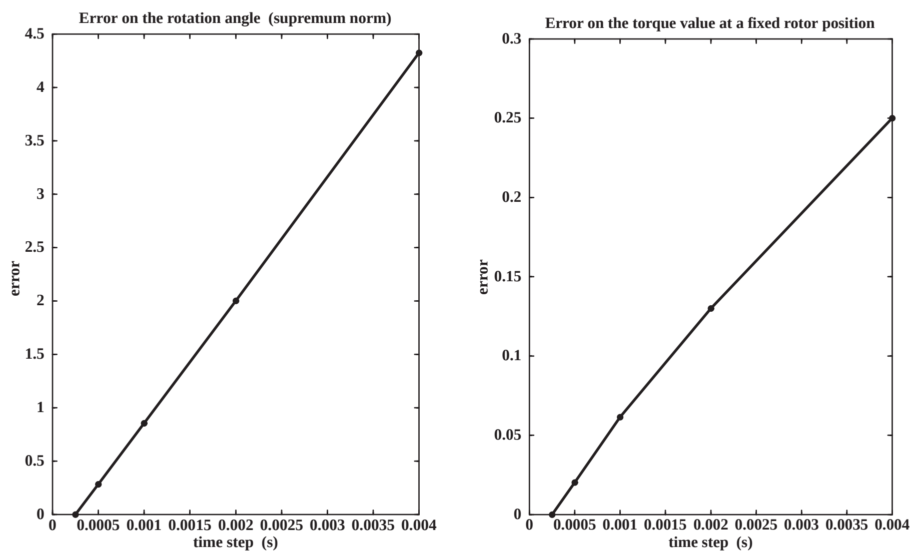

Fig. 2. – Temporal error on the rotor position θ (left) and on the magnetic torque T_m (right).

nite elements we are facing additional errors. The first one is related to the resolution of the magnetic problem. The corresponding error analysis is performed in Buffa *et al.*, 1999. The second one is related to the computation of the torque itself when the domain of integration is replaced by the polygon composed of all mesh triangles included in Ω_1. The difference of the surface integral is naturally of order $O(h^2)$. The third source of error is due to the geometrical nature of the parameter distribution that is also not well represented by the triangulation: the domain when $\sigma > 0$ may be larger or smaller than the exact one. This is also a surface contribution that involves again an error of $O(h^2)$. These arguments are in agreement with the torque behavior displayed in Figures 3. In Figure 3 (left) are presented the torque values, computed on three rotor meshes (corresponding to h_1, $2h_1$, $4h_1$), as a function of the rotation angle. Looking to Figure 3 (left), we can see that, for a given rotor position (about 30° for example), the distance (i.e. the error in the L^∞-norm) between the computed torque values on the meshes with elements of diameters $4h_1$ and $2h_1$ is roughly twice that between the computed torque values on the meshes with elements of diameters $2h_1$ and h_1. This asymptotic first order accuracy in space of the proposed method can be observed also in Figure 3 (right) where the relative error on the magnetic torque for the rotor position corresponding to $\theta = 30°$ is displayed, in logarithm scale, with respect to the mesh element si-

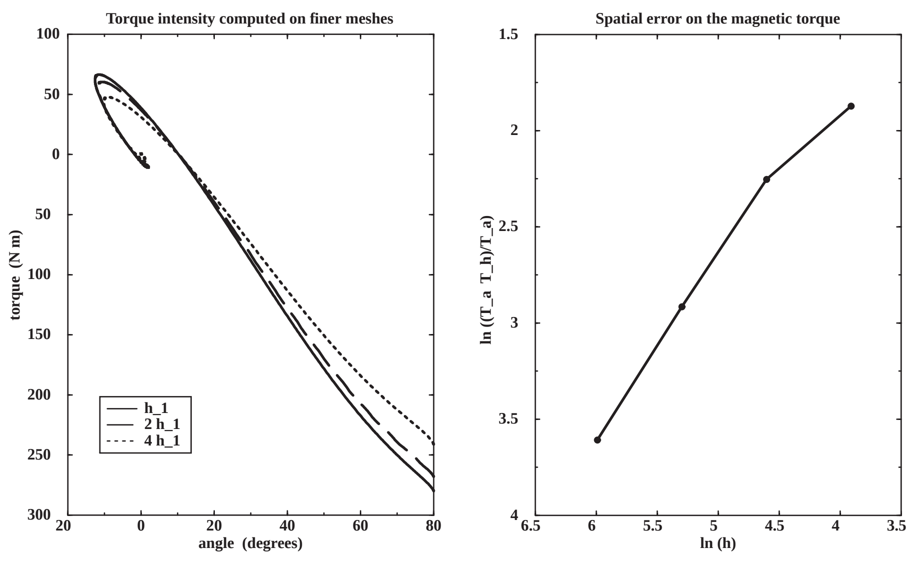

Fig. 3. – Influence of the spatial discretization step on the magnetic torque value T_m (left). Spatial error on the magnetic torque value T_m for $\theta = 30°$. The analytic value T_a is equal to the one computed on the finest rotor mesh with element size equal to $1.25 \cdot 10^{-3}$ m (right).

ze h. The «analytical» value T_a is given by $T_{h_1/4}$, i.e. the one computed on the finest rotor mesh.

CONCLUSIONS. – The proposed method, through a weak coupling at the sliding interface Γ that allows for non-matching grids at the interface, is well suited for treating such a magneto-mechanical problem. It has several advantages (flexibility, symmetry, robustness, accuracy, ...) with respect to other approaches and it provides an optimal approximation of the solution, as proved in Bouillault *et al.*, 2000.

REFERENCES

[1] C. BERNARDI - Y. MADAY - A. PATERA, *A new nonconforming approach to domain decomposition: The mortar elements method*, in *Nonlinear partial differential equations and their applications*, H. Brezis and J. Lions, eds., Pitman, 1994, pp. 13–51.

[2] F. BOUILLAULT - A. BUFFA - Y. MADAY - F. RAPETTI, *Simulation of a magneto-mechanical damping machine: analysis, discretisation, numerical results*, submitted to Comput. Meth. Appl. Mech. Engrg.

[3] A. BUFFA - Y. MADAY - F. RAPETTI, *A sliding mesh-mortar method for a two-dimensional eddy currents model of electric engines*, Laboratoire d' Analyse Numerique, Université Paris VI, technical report R99002 (1999) (submitted to M^2AN).

[4] B. GASPALOU - F. COLAMARTINO - C. MARCHAND - Z. REN, *Simulation of an electromagnetic actuator by a coupled magneto-mechanical modelling*, COMPEL, **14** (1995), 203-206.

[5] F. RAPETTI - L. SANTANDREA - F. BOUILLAULT - A. RAZEK, *Calculation of eddy currents in moving structures using a finite element method on non-matching grids*, COMPEL, **19** (2000) 1.

[6] E. VASSENT - G. MEUNIER - A. FOGGIA - G. REYNE, *Simulation of induction machine operation using a step by step finite element method coupled with circuits and mechanical equations*, IEEE Transactions on Magnetics, **27** (1991), 5232-5234.

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