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On Group Automorphisms
Fixing Subnormal Subgroups Setwise (*)

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Dedicated to Mario Curzio on his 70th birthday

Sunto. – In questo lavoro si studiano i gruppi $\text{Aut}_{sn}(G)$, $\text{Aut}_d(G)$, $\text{Aut}_x(G)$ degli automorfismi di un gruppo $G$ che fissano — come insiemi — tutti i sottogruppi di $G$ che risultano essere rispettivamente subnormali, subnormali di difetto al più $d$, oppure che sono compresi tra un sottogruppo caratteristico ed il suo derivato. Si danno condizioni sufficienti affinché tali gruppi siano parasolubili di para-altezza al più 2 o 3. Si generalizzano così risultati da [4], [7], [8], [10].

1. – Introduction and statement of main results.

The group $\text{Aut}_{sn}(G)$ of all automorphisms of a group $G$ fixing every subnormal subgroup of $G$ setwise featured recently in a few papers. It has been shown that there are restrictions on its structure, when $G$ is either finite or soluble. In particular, D. J. S. Robinson [10] has shown that $\text{Aut}_{sn}(G)/\text{Inn}(G) \cap \text{Aut}_{sn}(G)$ is always soluble with derived length at most 4, if $G$ is finite. Concerning the soluble case, S. Franciosi and F. de Giovanni [8] proved that if $G$ is any soluble group then $\text{Aut}_{sn}(G)$ is metabelian and that it is either abelian or finite, if $G$ is polycyclic. Moreover, M. Dalle Molle [4] has shown that if $G$ is soluble then $\text{Aut}_{sn}(G)$ normalizes each subgroup of its derived subgroup (whence it is locally supersoluble), provided that $G$ is a Chernikov group or the Fitting subgroup $\text{Fit}(w(G))$ of the Wielandt subgroup of $G$ either has finite exponent or is non-periodic (recall that $w(G)$ is the subgroup of the elements of $G$ normalizing all subnormal subgroups). We improve these results to:

**Theorem 1.** – Let $G$ be a soluble group. Then the metabelian group $\text{Aut}_{sn}(G)$ acts by means of power automorphisms on its derived subgroup, provided either $G$ is nilpotent-by-(finitely generated) or $\text{Fit}(G)$ is non-periodic.

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We show that similar restrictions actually hold for some groups of automorphisms larger than the above one.

For each \( d \geq 2 \), let \( \text{Aut}_d(G) \) be the group of all automorphisms of \( G \) fixing subnormal subgroups with defect at most \( d \) setwise. Similarly, let \( \text{Aut}_\gamma(G) \) be the group of all automorphisms of \( G \) setwise fixing all subgroups which lie between a characteristic subgroup of \( G \) and its derived subgroup. Clearly, \( \text{Aut}_d(G) \) and \( \text{Aut}_\gamma(G) \) are normal subgroups of \( \text{Aut}(G) \) and

\[
\text{Aut}_d(G) \supseteq \text{Aut}_2(G) \supseteq \text{Aut}_d(G) \supseteq \text{Aut}_{d+1}(G) \supseteq \bigcap_d \text{Aut}_d(G) = \text{Aut}_{sn}(G).
\]

Note that \( \text{Aut}_\gamma(G) \) induces power automorphisms on the factors of the derived series of \( G \). Thus \( \text{Aut}_\gamma(G) \) is locally supersoluble, if \( G \) is soluble and \( G' \) has finite exponent (see [5]). However, more can be said as in the next theorem below.

**Theorem 2.** – Let \( G \) be a soluble group. Then the derived subgroup of \( \text{Aut}_\gamma(G) \) is nilpotent of class at most 2. Moreover \( \text{Aut}_\gamma(G) \) induces power automorphisms on the factors of a series of length at most 2 in the torsion-subgroup of its derived subgroup.

Furthermore, \( \text{Aut}_3(G) \) is metabelian and normalizes each periodic subgroup of its derived subgroup.

We will see that the bound 3 for the derived length of \( \text{Aut}_2(G) \) is best possible (see Proposition 2). Moreover, the consideration of the torsion-subgroup in the above statement cannot be avoided (see Proposition 3). However, sufficient conditions for the groups \( \text{Aut}_d(G) \) to be locally supersoluble will be given in Theorem 1’.

Apart from its intrinsic interest, the group \( \text{Aut}_\gamma(G) \) will be also a tool to prove the following result, concerning the non-soluble case.

**Theorem 3.** – If the group \( G \) is finite, then \( \text{Aut}_2(G) \) is abelian-by-(completely reducible)-by-(soluble with derived length at most 3).

Finally, we consider the group \( \text{Aut}_\gamma(G) \) of all automorphisms of a nilpotent group \( G \) which induce power automorphisms on the factors of the lower central series of \( G \). By Proposition 5 we generalize results in [7] on the group \( \text{Aut}_n(G) \) of all automorphisms fixing each normal subgroup of a nilpotent group \( G \) setwise. Clearly, \( \text{Aut}_n(G) \leq \text{Aut}_\gamma(G) \), but the inclusion may be strict.

For notation and terminology we refer mainly to [9]. By a dihedral group over an abelian group \( A \) we will mean a group isomorphic to the split extension of \( A \) by the inversion map, i.e. the automorphism
For each subgroup $H$ of $G$ we denote by $\mathcal{H}$ the group of inner automorphisms of $G$ induced by $H$.

2. – Proofs and related results.

Recall that power automorphisms of a group $G$ form an abelian normal subgroup $\text{PAut}(G)$ of $\text{Aut}(G)$. Moreover $\text{PAut}(G) \lhd Z(\text{Aut}(G))$, if $G$ is abelian.

Furthermore, if $G$ is abelian and has finite exponent then its power automorphisms are even universal, i.e. of type $x \mapsto x^n$ where $n$ is independent of $x$. The same holds, with $n = \pm 1$, if $G$ is a non-periodic nilpotent group (see [2]).

Denote by $u_d(G)$ the subgroup of all elements of $G$ determining an inner automorphism in $\text{Aut}_d(G)$ (as in [1]) and by $u_x(G)$ the normalizer in $G$ of all subgroups lying between a characteristic subgroup and its derived subgroup.

From now on letter $d$ either denotes a natural number greater than 1 or stands for $x$, where we set $x = 46$.

Clearly $[G, \text{Aut}_d(G)] \leq u_d(G)$, that is $\text{Aut}_d(G)$ acts trivially on $G/u_d(G)$. The next statement concerns solubility in $\text{Aut}_d(G)$.

**Proposition 1.** – Let $G$ be a group.

(i) If $\Gamma$ is an $\text{Inn}(G)$-subgroup of $\text{Aut}_d(G)$ such that $[G, \Gamma]$ is hyperabelian, then $\Gamma$ and $[G, \Gamma]$ are metabelian and $[G, \Gamma]$ is a Dedekind group.

(ii) If $\Delta$ is an $\text{Inn}(G)$-subgroup of $\text{Aut}_d(G)$ (resp. $\text{Aut}(G)$-subgroup of $\text{Aut}_d(G)$), such that $[G, \Delta]$ is hypoabelian, then $\Delta'$ and $[G, \Delta]$ are nilpotent of class at most 2.

(iii) $\text{Aut}_d(G)$ is soluble (resp. soluble-by-finite) if and only if $u_d(G)$ is soluble (resp. soluble-by-finite).

**Proof.** – To prove the first part of the statement set $K := [G, \Gamma]$. Then $K = [\bar{G}, \Gamma] \leq \Gamma$ induces a group of power automorphisms on each abelian section $S/T$ of $G$, where $S$, $T$ are subnormal subgroups of $G$ with defect at most 2. Therefore $K'$ stabilizes $S/T$. Hence each soluble normal subgroup $N$ of $K'$ stabilizes its own derived series and therefore it is nilpotent of class at most 2, as $\gamma_3(N) = [N, N'] \leq N'' \leq \gamma_4(N)$. Since $K'$ is clearly hyperabelian, it follows that $\gamma_3(K') = 1$ and $K$ is a soluble normal subgroup of $G$. Then $K$ stabilizes its own derived series and therefore it is nilpotent of class at most 2. Hence each subgroup of $K$ is subnormal in $G$ with defect at most 3 and $\Gamma$ induces power automorphisms on $K$. Thus $K$ is a Dedekind group and $\Gamma'$ is abelian, since it stabilizes the series $G \triangleright K \triangleright 1$. The second part of the statement can be proved in a similar way.

If $\text{Aut}_d(G)$ is soluble (resp. soluble-by-finite), then it is clear that $u_d(G)$ is soluble (resp. soluble-by-finite) since $\frac{u_d(G)}{G} \leq \bar{G} \cap \text{Aut}_d(G)$. Conversely, if the
soluble radical $R$ of $K := [G, \text{Aut}_d(G)] \leq u_d(G)$ has finite index and is soluble, then $I := C_{\text{Aut}_d(G)}(K/R)$ has finite index in $\text{Aut}_d(G)$ and $I'$ stabilizes the characteristic finite series $G \supseteq K \supseteq R \supseteq R' \supseteq R'' \supseteq \ldots \supseteq 1$. Hence $I'$ is nilpotent by the Hall-Kaloujnine theorem, and $\text{Aut}_d(G)$ is soluble-by-finite. ■

**COROLLARY** If $\text{Aut}_d(G)$ is soluble, then its derived subgroup is nilpotent of class at most 2 or even abelian if $d \geq 3$.

The difference between the structure of $\text{Aut}_2(G)$ and $\text{Aut}_3(G)$ is a genuine one, as we are going to show that actually the derived subgroup of a soluble $\text{Aut}_2(G)$ need not be even a Dedekind group.

**PROPOSITION 2.** Let $p$ be any prime and let $U$ be either the extra-special group of order $p^3$ and exponent $p$ if $p > 2$ or the quaternion group with order 8 if $p = 2$. Then $U$ has an automorphism $\sigma$ with order $p^2 - 1$ such that the group $G := U \times \langle \sigma \rangle$ has the following properties:

(i) $\text{Aut}_2(G)$ has derived length exactly 3 and its derived subgroup has nilpotency class exactly 2, if $p \neq 2$;

(ii) $[G, \text{Aut}_2(G)]$ has nilpotency class exactly 2;

(iii) $[G, \text{Aut}_n(G)]$ is the quaternion group with order 8, if $p = 2$.

**PROOF.** Recall that $\text{Out}(U)$ is isomorphic to the general linear group $\text{GL}(2, p)$ (see [6]) and pick $\sigma \in \text{Aut}(U)$ such that $\langle \sigma \rangle \text{Inn}(U)$ corresponds to a Singer cycle of $\text{GL}(2, p)$ in the above isomorphism.

Then each element of $G \setminus U$ induces on $U/U'$ a fixed-point-free automorphism, since it acts as a non-trivial element of $\langle \sigma \rangle$. It follows that $[U, x] = U$ for each $x \in G \setminus U$. Thus each subnormal subgroup of $G$ either contains $U$ or is contained in $U$ and $\text{Fit}(G) = U$. Moreover $\langle \sigma \rangle$ is irreducible as a group of linear transformations of $U/U'$. Hence $U'$ is the only non-trivial normal subgroup of $G$ properly contained in $U$ and the subnormal subgroups of $G$ with defect exactly 2 are the maximal subgroups of $U$.

If $p > 2$, then $Z(G) = 1$ and $u_2(G) = U/\sigma^{(p^2 - 1)/2}$, since $\sigma^{(p^2 - 1)/2}$ acts on $U/U'$ as the inversion map. Thus $\text{Aut}_2(G)' \supseteq u_2(G)' = U$. The remaining part of statement follows now in all cases from the relation

$$U \geq [G, \text{Aut}_2(G)] \geq [G, u_2(G)] = [G, u_2(G)] = U. ■$$

For the proofs of Theorems 1 and 2 we need the following technical lemmas. We omit the easy proof of the second one.

**LEMMA 1.** Let $G \geq N \geq 1$ be a series in a group $G$ and let $\sigma, \gamma \in \text{Aut}(G)$. Assume that $\sigma$ stabilizes the series, $N$ and $[G, \sigma]$ are $\gamma$-invariant, $\gamma^{-1}$
acts as a universal power \( n_1 \) on \( G/N \) and \( \gamma \) acts as a universal power \( n_2 \) on \([G, \sigma]\). If either \( n_1 = 1 \) or \([G, \sigma] \leq Z(G)\), then \( \sigma'' = \sigma^{n_1 n_2} \).

**Proof.** – For each \( g \in G \), there exists \( a \in N \) such that \( g^{\gamma^{-1}} = ag^{n_1} \). Thus

\[
[g, \sigma^{\gamma}] = [g^{\gamma^{-1}}, \sigma^{\gamma}] = [ag^{n_1}, \sigma^{\gamma}] = [g^{n_1}, \sigma^{\gamma}] = [g, \sigma^{n_1} \gamma] = [g, \sigma^{n_1}]^{n_2} = [g, \sigma^{n_1 n_2}].
\]

**Lemma 2.** – Let \( \Gamma \leq \text{Aut}(G) \) stabilize a series \( G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_m = 1 \).

If \( m = 2 \), for each \( \gamma \in \Gamma \) the subgroup \([G, \gamma]\) has exponent \( s \) if and only if \( \gamma \) has order \( s \). If \( G_1 \) has finite exponent \( e \), then \( \Gamma \) has finite exponent dividing \( e^{m-1} \), for each \( m \geq 1 \).

**Proof of Theorem 2.** – Denote \( \Gamma := \text{Aut}_\gamma(G) \). By Proposition 1, the subgroups \( \Gamma' \) and \( K:= [G, \Gamma] \) are nilpotent with class at most 2. Set \( \Sigma:= C_\Gamma(K) \).

Then \( \Gamma \) acts by means of power automorphisms on \([G, \Sigma] \leq Z(K)\) and therefore \( \Gamma \) induces power automorphisms on the torsion-subgroup of \( \Sigma \), by Lemmas 1 and 2. Now regard \( \Gamma/\Sigma \) as a group of automorphisms of \( K \). Clearly, \( \Gamma/\Sigma \) induces power automorphisms on each factor of the lower central series of \( K \) and \( \Gamma' \Sigma/\Sigma \) stabilizes this series. Thus if \( \sigma \in \Gamma' \Sigma/\Sigma \), then \( C_K(\sigma) \triangleright K' \triangleright [K, \sigma] \).

Furthermore if \( \sigma \) has order \( s \), then \( K/C_K(\sigma) \) and \([K, \sigma] \) both have finite exponent, since \([k^s, \sigma] = [k, \sigma]^s = [k, \sigma^s] \), for each \( k \in K \). Hence applying Lemma 1 to the series \( K \triangleright C_K(\sigma) \triangleright 1 \) we get that \( \Gamma/\Sigma \) induces power automorphisms on the torsion-subgroup of \( \Gamma' \Sigma/\Sigma \), and the statement follows. If \( \Gamma = \text{Aut}_\gamma(G) \), then it induces power automorphisms on \( K \) and so \( \Gamma' \leq \Sigma \).

**Proposition 3.** – There exists a metabelian group \( G \) of rank 2 and trivial centre such that \( \Gamma := \text{Aut}_\gamma(G) \) is not locally supersoluble and does not normalize any non-trivial subgroup of \( \Gamma' / \text{tor}(\Gamma') \).

**Proof.** – Let \( G = \text{Dr}_n G_n \), where \( G_n \) is the subgroup of order \( p_n q_n \) of the holomorph of the cyclic group of order \( p_n \) and \( q_1, p_1, \ldots, q_n, p_n, \ldots \) is an increasing sequence of distinct primes such that \( q_n \) divides \( p_n - 1 \), for each \( n \).

Recall that such a sequence exists by Dirichlet Theorem. Then \( \Gamma = \text{Aut}_{\gamma_n}(G) = \text{Cr}_n \text{Aut}_{\gamma_n}(G_n) = \text{Cr}_n \text{Aut}(G_n) \). Clearly \( \text{Aut}(G_n) = A_n \rtimes B_n \), with \( A_n \) and \( B_n \) cyclic subgroups. Then \( \Gamma = \Gamma' \rtimes B_n \), where \( \Gamma' = \text{Cr}_n A_n \) and \( B = \text{Cr}_n B_n \).

An argument similar to that in Example 1 of [4] shows that \( \Gamma \) is not locally supersoluble. Moreover, if \( \gamma = (\gamma_n)_{n \in N} \in \Gamma' \) we have \( (\gamma)^{\gamma} = (\gamma)^{\gamma} = \text{Cr}_n A_n \), where \( X := \{ n \in N \mid \gamma_n \neq 1 \} \). It follows that \( \Gamma \) does not normalize any non-trivial subgroup of \( \Gamma' / \text{tor}(\Gamma') \).

We show now that for soluble groups \( G \) of finite exponent the picture is rather clear since \( \text{Aut}_d(G) \) is in the same condition as \( G \) is.
LEMMA 3. – If $a \in \text{Aut}(G)$ acts as a power automorphism on $G/G^\prime$, then $a$ acts as a power automorphism on each factor of the lower central series of $G$. In particular, if $a$ has form $x \mapsto x^n$ on $G/G^\prime$, then $a$ has form $x \mapsto x^{n_i}$ on $\gamma_i(G)/\gamma_{i+1}(G)$, for each $i$.

PROOF. – By a well known argument by D. J. S. Robinson, each element of $\gamma_i(G)/\gamma_{i+1}(G)$ can be regarded as a (finite) sum $t = \sum t_j$, where $t_j = a_{j_1} \otimes \ldots \otimes a_{j_i}$, with $a_{j_k} \in G/G^\prime$. Since power automorphisms of abelian groups are locally universal, there is an integer $n$ such that $a_{j_k}^n = na_{j_k}$ (in additive notation). Thus $t_j^a = n_t^j$ and $t^a = n^i t$.

PROPOSITION 4. – Let $G$ be a soluble group such that $\text{Fit}(u_d(G))$ has finite exponent $e$. Then $\text{Aut}_d(G)$ has finite exponent at most $e^3$ (resp. $e^2$) and the exponent of its derived subgroup divides $e^2$ (resp. $e$, if $d \geq 3$).

PROOF. – As above, denote $\Gamma := \text{Aut}_d(G)$, $K := [G, \Gamma]$ and $\Sigma_1 := C_1(K/K^\prime)$. By Proposition 1, the subgroup $K$ is nilpotent of class at most 2, so $K \leq \text{Fit}(u_d(G))$. Moreover $\Sigma_1$ stabilizes the series $G \triangleright K \triangleright K^\prime \triangleright 1$, by Lemma 3. It follows that $\Gamma^\prime \leq \Sigma_1$ has finite exponent dividing $e^2$, by Lemma 2. Furthermore $\Gamma/\Sigma_1$ has order at most $e$ as a group of power automorphisms of $K/K^\prime$.

If $d \geq 3$, then $K$ is a Dedekind group and $\Gamma$ induces power automorphisms on $K$. The statement follows as above since the group of power automorphisms of a hamiltonian group with exponent $e$ has exponent at most $e$.

Note that the consideration of the dihedral group $G$ over a quasicyclic $p$-group shows that there is no hope for $\text{Aut}_{sn}(G)^\prime$ to have finite exponent, if only $\pi(G)$ is finite.

Next lemma proves Theorem 1 in the case $\text{Fit}(G)$ is non-periodic, while the remaining part of the statement will follow from Theorem 1′.

LEMMA 4. – If $G$ is a soluble group with a nilpotent non-periodic normal (resp. characteristic if $d = \chi$) subgroup $N$ with class at most $d - 1$ if $d \geq 3$ or 2 otherwise, then $\text{Aut}_d(G)$ has a central subgroup $\Delta$ such that $\text{Aut}_d(G)/\Delta$ is either abelian or dihedral. Moreover, if $d \geq 3$ the above holds with $\Delta = 1$.

PROOF. – Set $\Gamma := \text{Aut}_d(G)$ and $K := [G, \Gamma]$. Let us first prove that $NK$ has class at most $d - 1$ if $d \geq 3$ or 2 otherwise. By Proposition 1, $\gamma_3(K) = 1$. Moreover $K$ stabilizes the lower central series of $N$. Thus, applying the Three Subgroup Lemma to $K$, $\gamma_i(K)$, $N$, we get $[\gamma_i(K), N] \leq \gamma_{i+1}(N)$ for each $i$. Then,
by an inductive argument, we have
\[ \gamma_{i+1}(NK) = [\gamma_i(NK), NK] = [\gamma_i(N)\gamma_i(K), NK] = \]
\[ [\gamma_i(N), N][\gamma_i(N), K][\gamma_i(K), N][\gamma_i(K), K] = \gamma_{i+1}(N)\gamma_{i+1}(K) \]
and \( NK \) is nilpotent with either the same class as \( N \) if \( d \geq 3 \), or 2 otherwise.

To prove now the statement, note that since \( C_r(K) \) is abelian, if \( [K, \Gamma] = 1 \), there is nothing to prove. Suppose then \( [K, \Gamma] \neq 1 \), from now on. It follows that if \( d \geq 3 \), then \( H := NK \) is necessarily abelian, since it is a non-periodic nilpotent group with a non-trivial power automorphism (see [2]). Hence \( H \) is nilpotent of class at most 2 in all cases.

Since \( H \) is non-periodic, then \( H/H' \) is non-periodic and therefore \( \Gamma \) acts on \( H/H' \) as either the identity or the inversion map. In both cases \( \Gamma \) acts trivially on \( H' \), by Lemma 3. Set \( \Sigma := C_r(H/H'), \Delta_1 := C_r(H), \Delta_2 := C_r(G/H') \) and \( \Delta := \Delta_1 \cap \Delta_2 \). Then \( \Gamma/\Sigma \) has order at most 2 as a group of power automorphisms of \( H/H' \). Clearly \( \Delta = 1 \) if \( H \) is abelian. Applying Lemma 1 to the series \( G \geq H \geq H' \), we get \( [\Delta, \Gamma] = 1 \). Moreover \( \Sigma/\Delta_1 \) and \( \Sigma/\Delta_2 \) are abelian, if regarded as groups stabilizing the series \( H \geq H' \geq 1 \) and \( G \geq H \geq H' \) respectively. Lemma 1 applied to the previous series yields that each \( \gamma \in \Gamma \setminus \Sigma \) acts on \( \Sigma/\Delta_1 \) and \( \Sigma/\Delta_2 \) as the inversion map. Thus \( \gamma \) acts as the inversion map on \( \Sigma/\Delta \).

Finally for each \( \gamma \in \Gamma \setminus \Sigma, g \in G \), we have: \( [g, \gamma^2] = [g, \gamma][g, \gamma]^{-1} \equiv [g, \gamma][g, \gamma]^{-1} \equiv 1 \mod H' \). Moreover, if \( h \in H \), then \( hh^\gamma \in H' \leq Z(H) \) and so \([h, \gamma^2] = h^{-1}h^{-1}h^\gamma = h^{-1}h^{-1}h^\gamma(h^\gamma)^{-1} = h^{-1}h^\gamma(h^\gamma)^{-1} = 1 \). Therefore \( \gamma^2 \in \Delta \) and the statement follows by considering the series \( \Gamma \geq \Sigma \geq \Delta \geq 1 \).

Note that when \( G \) is polycyclic, \( \text{Aut}_d(G) \) is either finite or described by Lemma 4.

Recall that a group \( G \) is parasoluble of paraheight at most \( n \) if it has an abelian normal series of length \( n \) on whose factors it induces power automorphisms (see [11]).

**Theorem 1’.** – If \( G \) is a soluble group, then \( \text{Aut}_d(G) \) is parasoluble with paraheight at most 2 if \( d \geq 3 \) and 3 otherwise, provided one of the following holds:

(i) either \( \text{Fit}(u_d(G)) \) has finite exponent or \( \text{Fit}(u_3(G)) \) is non-periodic;

(ii) \( G/N \) is finitely generated, where \( N \) is a normal nilpotent subgroup of class at most \( d - 1 \) (and \( d \neq \chi \));

(iii) \( G/N \) is finitely generated, where \( N \) is a characteristic abelian subgroup of \( G \) (and \( d = \chi \)).
PROOF. – Denote \( \Gamma := \text{Aut}_d(G) \) and \( K := [G, \Gamma] \). Note that since \( \text{Fit}(u_3(G)) \) is a Dedekind group, if either \( \text{Fit}(u_3(G)) \) or \( K \) is non-periodic, by Lemma 4 there is nothing left to prove. Let us show that in all remaining cases, \( \Gamma' \) is periodic and the statement follows from Theorem 2. If \( \text{Fit}(u_3(G)) \) has finite exponent, then \( \Gamma \) is periodic, by Proposition 4.

Let now \( K \) be periodic and (ii) or (iii) hold. Since \( G/N \) is finitely generated, there is a subgroup \( X = \langle x_1, \ldots, x_n \rangle \) such that \( G = NX \). On the other hand \( \Gamma' \) acts trivially on \( N \), thus for each \( \sigma \in \Gamma' \) the group \( [G, \sigma] = [X, \sigma] \leq K \) is a periodic nilpotent group generated by conjugates of the \([x_i, \sigma]'s and the \([x_i^{-1}, \sigma]'s. Therefore \([G, \sigma] \) has finite exponent. Thus if \( \Sigma := C_G(K) \), then \( \Gamma' \cap \Sigma \) is periodic, by Lemma 2. Then, in case \( d \geq 3 \) the proof is already achieved, since \( \Gamma' \leq \Sigma \). Otherwise regard \( \Gamma'/\Sigma \) as a group of automorphisms of \( K \) and observe that \( \Gamma' \) stabilizes the series \( K \triangleright K' \triangleright 1 \). Then, by arguing as above, \( \Gamma' \Sigma/\Sigma \) is periodic, and so \( \Gamma' \) is periodic, as we claimed. \( \blacksquare \)

REMARK. – Note that Theorem 1’ and Theorem 2 in case \( d \geq 3 \) can be proved also for any group \( G \) and the soluble radical of \( \text{Aut}_d(G) \) instead of the whole \( \text{Aut}_d(G) \).

We show now a result which implies Theorem 3.

THEOREM 3’. – Let \( G \) be a soluble-by-finite group with soluble radical \( S \). Then the 3rd term \( \Gamma^{(3)} \) of the derived series of \( \Gamma := \text{Aut}_2(G) \) has got an abelian normal subgroup \( A \) such that \( \Gamma^{(3)}/A \) is isomorphic to the completely reducible radical of \( G/S \). Moreover, \( \Gamma^{(5)} \) is finite.

Furthermore, \( \text{Aut}_2(G)/\text{Inn}(G) \cap \text{Aut}_2(G) \) is soluble with derived length at most 4, or even 3 if \( H^1(G/S, Z(S)) = 0 \).

PROOF. – Let \( G \) be a finite semisimple group with completely reducible radical \( R \). Then \( R = S_1 \times \ldots \times S_k \), where the \( S_i \)'s are non-abelian simple groups. Via the Three Subgroup Lemma applied to \( G, R \) and \( C_{\text{Aut}(G)}(R) \) from equality \( C_G(R) = 1 \) we deduce \( C_{\text{Aut}(G)}(R) = 1 \). Thus \( \text{Aut}_2(G) \) can be regarded as a subgroup of \( \text{Aut}(R) \) fixing all \( S_i \)'s setwise. Therefore \( \text{Aut}_2(G) \) is isomorphic to a subgroup of \( \text{Aut}(S_1) \times \ldots \times \text{Aut}(S_k) \). Because of Schreier Conjecture, each \( \text{Out}(S_i) \) is soluble with derived length at most 3. Therefore \( \text{Aut}_2(G)^{(3)} = \overline{R} \) and, by a theorem by Wielandt (see [9, Th. 13.3.2, p. 383]), \( S_i \leq w(G) \) for each \( i \). Thus \( \overline{R} \leq \text{Aut}_{\text{sn}}(G) \leq \text{Aut}_2(G) \) and so \( \text{Aut}_2(G)^{(3)} = \overline{R} \).

From now on let \( G \) be any group as in the statement, \( S \) its soluble radical, \( \Gamma := \text{Aut}_2(G) \) and \( \Delta := C_G(S) \). Clearly all characteristic subgroups of \( S \) are characteristic in \( G \), since \( S \) in turn is characteristic. Then \( \Gamma/\Delta \) can be regarded as a subgroup of \( \text{Aut}_2(S) \) and we have \( \Gamma^{(3)} \leq \Delta \), by Theorem 2. On the other hand \( G/S \) is semisimple and the argument of the previous paragraph applies. Therefore, denoting by \( \Sigma \) the subgroup of \( \Gamma \) stabilizing the series \( G \triangleright S \triangleright 1 \) we
have that $\Sigma$ is abelian and that the 3rd term of the derived series of $\Gamma/\Sigma$ is isomorphic to the completely reducible radical $R/S$ of $G/S$.

Moreover, by the same argument used in [10], we have that $\Gamma^{(4)} \leq C_n(S)$, and $\Gamma^{(3)} \leq G$ if $H^1(G/S, Z(S)) = 0$. Then, since $C_n(S)$ is central-by-finite, a well known theorem by Schur implies that $\Gamma^{(5)}$ is finite. ■

Recall that in [3] it is shown that if $G := D_4(3) \times (SL_3(7) \rtimes \langle \alpha \rangle)$, where $\alpha$ is an automorphism of $SL_3(7)$ with order 2 acting as the inversion map on the centre of $SL_3(7)$, then the 3rd term of the derived series of $\text{Aut}_{on}(G)$ is a non-trivial abelian-by-(completely reducible) group.

Finally we consider the group $\Gamma := \text{Aut}_{\gamma}(G)$, where $G$ is a nilpotent group with class $c$. Note that by Lemma 2, $\text{Aut}_{\gamma}(G)$ is just the group of all automorphisms of $G$ which induce power automorphisms on $G/G'$. Clearly $\Gamma'$ is nilpotent with class at most $c - 1$. Moreover $\Gamma$ is parasoluble with paraheight at most $\frac{c(c - 1)}{2} + 1$, provided either $G'$ has finite exponent or $G/G'$ is not a periodic group with infinite exponent, by [5]. Moreover, if $G$ is periodic then $\text{Aut}_{\gamma}(G) = \text{Cr}_p \text{Aut}_{\gamma}(G_p)$, where $G_p$ is the $p$-component of $G$.

**Proposition 5.** Let $G$ be a nilpotent group of class $c$ and $\Gamma := \text{Aut}_{\gamma}(G)$.

If $G/G'$ has finite exponent $p^n$, then $\Gamma = \Delta_0 \rtimes \langle \gamma \rangle$, where $\gamma$ has order dividing $p - 1$ and $\Delta_0 := C_\Gamma(G/G'G^p)$ has exponent dividing $p^{cn - 1}$ and is nilpotent of class at most $cn - 1$.

If $G/G'$ is non-periodic, then $\Delta := C_\Gamma(G/G')$ is a nilpotent subgroup of class at most $c - 1$ and index at most 2. Moreover $\pi(\Delta) \subseteq \pi(G')$.

**Proof.** Assume $G/G'$ has finite exponent $p^n$ and set $\Delta_i := C_\Gamma(G/\gamma_{i+1}(G))$ for $i \geq 2$. Then the group $\Delta_i/\Delta_{i+1}$ can be seen as a subgroup of $\text{Aut}(G/\gamma_{i+1}(G))$ stabilizing the series $G \supseteq \gamma_i(G) \supseteq \gamma_{i+1}(G)$ and so by Lemma 2, its exponent divides $p^n$. Thus $\Delta_2 = C_\Gamma(G/G')$ has exponent dividing $p^{(c-1)n}$, since $\Delta_{c+1} = 1$. On the other hand, as a group of power automorphisms of $G/G'$, the group $\Gamma/\Delta_2$ is abelian with order dividing $p^{n-1}(p - 1)$. If $\Gamma_p/\Delta_2$ denotes its $p$-component, then $\Gamma/\Gamma_p$ is cyclic with order dividing $p - 1$ (where $\Gamma_2 = \Gamma$) and $\Gamma_p$ stabilizes all refinements of the lower central series of $G$ with elementary abelian factors. The shortest of them has length at most $cn$, hence $\Gamma_p$ is nilpotent of class at most $cn - 1$ and has exponent at most $p^{cn - 1}$. Clearly $\Gamma_p = \Delta_0$.

Assume now that $G/G'$ is non-periodic. Since $\Gamma$ induces power automorphisms on $G/G'$, we have that $|\Gamma/\Delta| \leq 2$. Moreover $\Delta$ stabilizes the lower central series of $G$, and so it is nilpotent of class at most $c - 1$. The remaining part of the statement follows by Lemma 2. ■
Remark. – We have seen that in the above cases the picture of $\text{Aut}_\gamma(G)$ is not that different from that of the group $\text{Aut}_n(G)$ of automorphisms fixing all normal subgroups setwise (see [7]). However it is easily seen that if $G = \text{Dr}_{n \in \mathbb{N}} U(3, \mathbb{Z}_p^n)$, then the group $\text{Aut}_\gamma(G)$ is not nilpotent, while $\text{Aut}_n(G)$ is. Here $U(3, \mathbb{Z}_p^n)$ is the group of unitriangular $3 \times 3$ matrices with entries in the ring of integers modulo $p^n$.

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