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Good and Very Good Magnifiers.

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Sunto. – *Un elemento a di un semigruppo S è un elemento accrescitivo sinistro se la traslazione λ_a di S, associata all'elemento a, è surgettiva e non è iniettiva* (*E. S. Ljapin*, [13], § 5). *Così, per ogni elemento accrescitivo sinistro a, esiste un sottoinsieme proprio M di S tale che la restrizione a M di λ_a è biunivoca. Se M è un sottosemigruppo* (*risp. un ideale destro*) *di S, l'elemento accrescitivo sinistro a viene detto buono* (*risp. molto buono*) (*F. Migliorini* [15], [16], [17]). *Utilizzando il monoide biciclico, i semigruppi con elementi accrescitivi sinistri e identità sinistre sono stati ben caratterizzati da E. S. Ljapin* [13] *e da R. Desq* [3], [4]. *In questo articolo, mediante i risultati dimostrati in* [7], *si caratterizzano i semigruppi i cui elementi accrescitivi sinistri sono tutti molto buoni. Come applicazione, si costruiscono semigruppi nei quali ogni elemento accrescitivo sinistro è buono ma non molto buono.*

1. – Introduction.

An element a of a semigroup S is a *left* (resp. *right*) *magnifier* if the inner left translation λ_a (resp. the inner right translation ϱ_a) of S associated with a is surjective and is not injective. The notion of magnifying element in a semigroup has been introduced by E. S. Ljapin [13], § 5.

Denote by $\mathfrak{LM}(S)$ the set of all left magnifiers of S and by $\mathfrak{RM}(S)$ the set of all right magnifiers of S.

An element a of a semigroup S is called *right* (resp. *left*) *invertible* if $aS = S$ (resp. $Sa = S$). Magnifying elements are strongly connected with invertible elements.

Denote by $\mathfrak{RI}(S)$ (resp. $\mathfrak{LI}(S)$) the set of all right (resp. left) invertible elements of S. It is known that if S is a semigroup without left identity then $\mathfrak{LM}(S) = \mathfrak{RI}(S)$ ([5], Lemma 1).

An important example of semigroup with magnifiers is the bicyclic monoid $\mathcal{B} = \mathcal{B}(p, q) = \langle p, q \mid pq = 1 \rangle$ for which $\mathfrak{LM}(\mathcal{B}) = \langle p \rangle$ and $\mathfrak{RM}(\mathcal{B}) = \langle q \rangle$. An interesting generalization of the bicyclic monoid, the *polycyclic monoids,* has been given in [18]. These monoids also contain magnifiers.

An other example is the semigroup $\Sigma = \Sigma(a, p, q)$, the disjoint union of the free monogenic semigroup generated by a with the bicyclic monoid $\mathcal{B} =$

$\mathcal{B}(p, q)$. The semigroup operation on Σ is given by:

$$a^m(q^k p^l) = \begin{cases} q^{k-m} p^l & \text{if } m \leqslant k \\ a^{m-k+l} & \text{otherwise} \end{cases}$$

$$(q^k p^l)\, a^m = q^k p^{l+m},$$

for every $m \in \mathbb{N}^*$ and every k, l in $\mathbb{N}$, where $q^0 = p^0 = 1$, the identity of $\mathcal{B}$. Then $\mathfrak{LM}(\Sigma) = \langle a \rangle$ and $\mathfrak{RM}(\Sigma) = \langle q \rangle$ ([11], Theorem 10. 2 and [17], Theorem II. 2. 3).

Other semigroups with magnifiers are the idempotent free right simple (left simple) semigroups, for which all the elements are left (right) magnifiers.

Remarkable classes of semigroups without magnifiers are : finite semigroups, periodic semigroups, cancellative semigroups, commutative semigroups, groups, compact semigroups.

If a is a left magnifier in a semigoup S then there exists a proper subset M of S such that, for every $s \in S$, the set $M \cap \lambda_a^{-1}(s)$ is a singleton (whence the restriction to M of the map λ_a is bijective). When this happens M is said to be a *minimal subset for the left magnifier a.* If such a set M is a subsemigroup (resp. right ideal) of S the element a is said to be a *good* (resp. *very good*) *magnifier* and M is called a *minimal subsemigroup* (resp. *right ideal*) *associated with a.*

The semigroups admitting good left magnifiers have been characterised in [7], where it has been proved that every such semigroup is an extension of a semigroup M, with left identities and left magnifiers, by an endomorphism of M satisfying some conditions. However, it is rather difficult to construct intricate semigroups with good magnifiers because, in that case, M is also intricate, therefore it is not easy to find its endomorphisms.

It is known that if a semigroup contains a very good left magnifier then all its left magnifiers are of this kind ([7], Proposition 3.7; [11], Theorem 10.15; [19]) but no example of semigroup having good left magnifiers such that none of them be very good has been given yet. The main motivation of this paper is to give methods for obtaining such semigroups.

In § 2, some remarkable subsemigroups of semigroups with good magnifiers are studied. The main result of this paper (Theorem 3.1) concerns the characterization of semigroups for which all the left magnifiers are very good. We establish that in order to obtain semigroups with good but not very good magnifiers we must consider, from the start, semigroups M with left identities and left magnifiers for which the set of idempotents is not a subsemigroup of M (whence semigroups which are not orthodox). The aim of § 4 is to present some classes of semigroups with very good magnifiers which have analogous properties to the semigroups $\mathcal{B}$ and Σ. In § 5, using Theorem 3.1 and infinite matrices, semigroups for which all their left magnifiers are good but not very good are constructed (5. 3, 5. 4, 5. 5).

2. – Remarkable subsemigroups in semigroups with good magnifiers.

Let S be a semigroup and a be a good left magnifier of S. Consider M a minimal subsemigroup associated with a. Hence M is a proper subsemigroup of S such that $\lambda_{a|_M}: M \to S$, the restriction to M of λ_a, is a bijection, that is, for every $s \in S$, there exists a unique $m \in M$ for which $s = am$. As it has been shown in [16] (Theorem 1) and [17] (Theorem I.1), M contains three unique elements e, u, v such that $a = ae$, $e = av$ and $a^2 = au$. Hence the following conditions are satisfied:

$$
\text{(1)} \qquad \begin{cases} uv = e & \text{and } vu \neq e \\ em = m\,, & \text{for every } m \in M \\ ue = u & \text{and } ve = v\,. \end{cases}
$$

Also, for every $m \in M$ there exists a unique element $\psi(m) \in M$ such that $ma = a\psi(m)$. In this way we get a map $\psi : M \to M$ satisfying the properties given in the next.

Lemma. 2.1 ([7], 2.7). – *Let m and m' be elements of M. Then*:

$$
\begin{array}{lll}
(\alpha) & \psi(mm') & = \psi(m)\,\psi(m') \\
(\beta) & \psi(m) & = u\psi(vm)\,e \\
(\gamma) & \psi(m)\,v & = vme \\
(\delta) & u\psi(u\psi(m)) & = u\psi(m)\,u \\
(\varepsilon) & u & = u\psi(e)\,. \qquad \blacksquare
\end{array}
$$

We also have:

Lemma 2.2. – *Let ψ be an endomorphism of M. Then ψ fulfils conditions (β)-(ε) if and only if it fulfils (β'), (γ), (δ') and (ε), where*:

$$
\begin{array}{llll}
(\beta') & \psi(e) = u\psi(v) & \textit{and} & \psi(M) \subset Me \\
(\delta') & u\psi(u) = u^2 & \textit{and} & u\psi(\psi(m)) = \psi(m)\,u\,, \quad \textit{for every } m \in M\,.
\end{array}
$$

Proof. – Obviously (β) and (β') are equivalent and (δ') implies (δ). It remains to prove that the conditions (α)-(ε) imply (δ'). This results as follows:

$u^2 \overset{(\varepsilon)}{=} u\psi(e)\,u \overset{(\delta)}{=} u\psi(u\psi(e)) \overset{(\varepsilon)}{=} u\psi(u)$ and $u\psi(\psi(m)) \overset{(\alpha)}{=} u\psi(\psi(e)\,\psi(m)) \overset{(\beta)}{=}$ $\overset{(\beta)}{=} u\psi(u\psi(v)\,\psi(m)) \overset{(\alpha)}{=} u\psi(u\psi(vm)) \overset{(\delta)}{=} u\psi(vm)\,u \overset{(\beta)}{=} \psi(m)\,u.$ ■

In [7] we have established that the semigroup S is completely determined by M, u, v and ψ.

Conversely, let M be a semigroup which contains three elements e, u, v such that conditions (1) hold. Consider $\psi : M \to M$ a map satisfying conditions (α)-(ε) and $\tau : M \backslash vM \to A$ a bijection, where A is a set which is disjoint from M.

Denote $\tau(e) = a$ and $S = A \cup M$. Define $\phi : M \to S$ by

$$\phi(m) = \begin{cases} um & \text{if } m \in vM \\ \tau(m) & \text{if } m \in M \backslash vM . \end{cases}$$

Using ϕ we endow S with an operation «$\cdot$» as follows:

$$\phi(m) \cdot \phi(m') = \phi(u\psi(m)\,m'), \qquad \text{for every } m, m' \text{ in } M. \tag{2}$$

Then $(S, \cdot)$ is a semigroup (denoted $S = \mathfrak{S}(M, u, v, \psi)$) for which a is a good left magnifier and M is a minimal subsemigroup associated with a.

In fact, in [7] we have proved:

Theorem 2.3 ([7], Theorem 5.1). – *Every semigroup with good left magnifiers is a semigroup of type* $\mathfrak{S}(M, u, v, \psi)$. ■

In the rest of this section we suppose that M is a semigroup containing three elements e, u, v such that conditions (1) hold and ψ is an endomorphism of M which fulfils (β)-(ε). We also consider that $S = \mathfrak{S}(M, u, v, \psi)$ and $a = \phi(e)$.

Lemma 2.4. – *Let* $P = P(M, u, v, \psi) = \{m \in M \mid u\psi(m) = mu\}$. *Then*:

(i) *P is a subsemigroup of M*;

(ii) $\psi(M) \subset P$;

(iii) $u^k \in P$, *for every* $k \in \mathbb{N}$.

Proof. – (i) If m, m' are in P then $u\psi(mm') = u\psi(m)\,\psi(m') = mu\psi(m') = mm'u$, whence $mm' \in P$.

(ii) Let $m' = \psi(m) \in \psi(M)$. We have $u\psi(m') = u\psi(\psi(m)) \overset{(\delta')}{=} \psi(m)\,u = m'u$, so $m' \in P$.

(iii) As $u\psi(e) \overset{(\varepsilon)}{=} u \overset{(1)}{=} eu$, it follows that $e \in P$. On the other hand, $u\psi(u) \overset{(\delta')}{=} u^2$, thus $u \in P$. Therefore, by (i), we get $u^k \in P$, for every $k \in \mathbb{N}$. ■

LEMMA 2.5. – *Suppose that a is a very good left magnifier for the semigroup S. Let N be a right ideal of S, which is minimal for a, and e' be the unique element of N such that $a = ae'$. Then $N = e'S$ and there exists $m \in M$, a right inverse for u with respect to e, such that $e' = am$.*

PROOF. – Let u', v' be the unique elements of N for which $a^2 = au'$ and $e' = av'$. As $e'a \in N$ and $a(e'a) = a^2 = au'$, we get $e'a = u'$. Also according to 2.2 and 2.6 of [7], e' is a left identity and u' is a left magnifier for N. Hence $N = u'N = e'aN = e'S$. Suppose $e' = am$, where m belongs to M. Then $a = ae' = aam = aum$, thus $um = e$, that is m is a right inverse for u, with respect to e. ■

If ϱ is a binary relation on S, $s \in S$ and $m \in M$ denote

$$\begin{aligned} s\varrho &= \{(sx, sy) \mid (x, y) \in \varrho\} \\ \operatorname{Ker}(\lambda_s) &= \{(x, y) \in S \times S \mid sx = sy\} \\ \ker(\lambda_m) &= \operatorname{Ker}(\lambda_m) \cap (M \times M) = \{(m_1, m_2) \in M \times M \mid mm_1 = mm_2\}. \end{aligned}$$

The next lemma allows us to reduce the calculation of kernels for inner left translations of S to the one of kernels for inner left translations of M.

LEMMA 2.6. – *If $m \in M$ then* $\operatorname{Ker}(\lambda_{am}) = a \ker(\lambda_{u\psi(m)})$.

PROOF. – This is immediate because $(am_1, am_2) \in \operatorname{Ker}(\lambda_{am})$ if and only if $au\psi(m)\, m_1 = au\psi(m)\, m_2$.

In the following we denote $\psi^0 = Id_M$ and, for every $k \in \mathbb{N}^*$, $\psi^k = \underbrace{\psi \circ \psi \circ \ldots \circ \psi}_{k \text{ times}}$.

COROLLARY 2.7.

(i) $\mathfrak{RI}(S) = a\{m \in M \mid u\psi(m) \in \mathfrak{RI}(M)\}$;

(ii) $\mathfrak{LM}(S) = a\{m \in M \mid u\psi(m) \in \mathfrak{LM}(M)\}$;

(iii) *am is a left identity for S if and only if $u\psi(m) = e$;*

(iv) *If $mm' = e$, where m and m' are in M, then am is a good left magnifier for S and $m'M$ is a minimal subsemigroup of S associated with am.* ■

In the next, we also consider others subsemigroups of S, namely $Q = Q(M, u, v, \psi) = uP(M, u, v, \psi)$, $Q' = Q'(M, u, v, \psi) = aQ(M, u, v, \psi)$ and $P' = P'(M, u, v, \psi) = aP(M, u, v, \psi)$. Here are some of their properties:

LEMMA 2.8.

(i) $u^k \in Q$, *for every* $k \in \mathbb{N}^*$;

(ii) $\psi(M) \subset Q$;

(iii) $Q = QP = PQ = uQ = u\psi(P)\,P$ *and* $Q' = P'P'$;

(iv) Q' *and* P' *are left ideals of* S;

(v) $Sa \subset Q'$.

PROOF. – (i) $u\psi(\{e, u, u^2, \ldots\}) = \{u, u^2, \ldots\}$, whence, from Lemma 2.4. (ii), $u^k \in \mathbb{N}$, for every $k \in \mathbb{N}^*$.

(ii) $\psi(M) \overset{(\beta)}{=} u\psi(vM) \subset uP = Q$.

(iii) $QP = uPP = uP = Q$ and since $Q \subset PQ = PuP = u\psi(P)\,P \subset uQP = uQ \subset Q$, it results $Q = PQ = uQ = u\psi(P)\,P$. Also $Q' = aQ = au\psi(P)\,P = aa\psi(P)\,P = aPaP = P'P'$.

(iv) $SP' = aMaP = aa\psi(M)\,P = au\psi(M)\,P \subset P'$, whence P' is a left ideal of S. The fact that Q' is a left ideal of S yields from (iii).

(v) $Sa = aMa = aa\psi(M) = au\psi(M) \subset aQ = Q'$. ■

3. – The main result.

The purpose of this section is to establish a characterization of semigroups for which all the left magnifiers are very good.

If a semigroup contains a very good left magnifier then all its left magnifiers are very good (Proposition 3.7 of [7], Theorem 10.15 of [11], [19]). It follows that it suffices to find the conditions satisfied by M, e, u, v, ψ (when (1) and $(\alpha) - (\varepsilon)$ hold) in order that the element $a = \phi(e)$ be a very good left magnifier in $S = \mathfrak{S}(M, u, v, \psi)$. These conditions will be obtained using the semigroups $P(M, u, v, \psi)$ and $Q(M, u, v, \psi)$, which have been studied in the previous section.

Denote by $R(M, e, u) = \{v' \in M \mid uv' = e\}$ the set of all right inverses of u with respect to e in M.

THEOREM 3.1. – *Let M be a semigroup, e, u, v be elements of M satisfying* (1) *and let ψ be an endomorphism of M such that conditions* (β)-(ε) *hold. Then all the left magnifiers of the semigroup* $\mathfrak{S}(M, u, v, \psi)$ *are very good if and only if* $P(M, u, v, \psi) \cap R(M, e, u) \neq \emptyset$.

PROOF. – Suppose that there exists $m \in M$ such that $um = e$ and $u\psi(m) = mu$. Denote $e' = am$ and $N = e'S$. Then $ae' = a$, $aN = S$ and $N = amaM = au\psi(m)\,M = amM$. As u is not left invertible it follows that $mM \neq M$, whence N is a proper right ideal in S. We prove that $\lambda_{a|_N}\colon N \to S$ is injective.

Consider m_1, m_2 in M and $n_1 = amm_1$, $n_2 = amm_2$ in N such that $an_1 = an_2$. As $an_i = a^2mm_i = aumm_i = am_i (i \in \{1, 2\})$ we deduce that $an_1 = an_2$ implies $m_1 = m_2$, so $n_1 = n_2$. It follows that a is a very good left magnifier of S, whence, according to Proposition 3.7 of [7], in S all the left magnifiers are very good.

For proving the converse, suppose that in $S = \mathfrak{S}(M, u, v, \psi)$ every left magnifier, particularly a, is very good. Therefore S contains a right ideal N which is a minimal subset for a. Then, by Lemma 2.5 there exists $m \in R(M, e, u)$ such that $a = ae'$ and $N = e'S$, where $e' = am$.

We have that $\mathrm{Ker}(\lambda_a) \subset \mathrm{Ker}(\lambda_{e'})$. Indeed, $(s_1, s_2) \in \mathrm{Ker}(\lambda_a)$ if and only if $as_1 = as_2$, that is $ae's_1 = ae's_2$. On the other hand $n_1 = e's_1$ and $n_2 = e's_2$ are in N and $\lambda_{a|_N}$ is injective. Thus $e's_1 = e's_2$, whence $(s_1, s_2) \in \mathrm{Ker}(\lambda_{e'})$. The inclusion $\mathrm{Ker}(\lambda'_e) \subset \mathrm{Ker}(\lambda_a)$ results immediately from $a = ae'$. Hence $\mathrm{Ker}(\lambda_a) = \mathrm{Ker}(\lambda_{e'})$.

Now using Lemma 2.6 we get that $a \ker(\lambda_u) = a \ker(\lambda_{u\psi(m)})$ so $\ker(\lambda_u) = \ker(\lambda_{u\psi(m)})$.

As $(e, vu) \in \ker(\lambda_u)$ it follows that $u\psi(m) = u\psi(m)e = u\psi(m)vu = uvmeu = mu$, whence $m \in P(M, u, v, \psi) \cap R(M, e, u)$. ∎

Using Lemma 2.8 we get:

REMARK 3.2. – *The following three statements are equivalent*:

(i) $P(M, u, v, \psi) \cap R(M, e, u) \neq \emptyset$;

(ii) $e \in Q(M, u, v, \psi)$;

(iii) $a \in Q'(M, u, v, \psi)$.

LEMMA 3.3. – *Let M be a semigroup which contains the elements e, u, v satisfying* (1) *and ψ be an endomorphism of M such that the conditions* (β)-(ε) *hold.*

Then only the following three cases can occur:

(I) $\psi(e) = vu$;

(II) $\psi(e) = e$;

(III) $\psi(e) \notin \mathcal{B}$, *where $\mathcal{B}$ is the subsemigroup of M generated by u and v.*

PROOF. – It results from the fact that the equation $ux = x$ has only two solutions in $\mathcal{B}$, namely $x = vu$ and $x = e$. ∎

We next deal with each case separately.

I. *Case* $\psi(e) = vu$. Then $v \in P(M, u, v, \psi) \cap R(M, e, u)$ and, using the previous theorem, it follows that every left magnifier of the semigroup

$S = \mathfrak{S}(M, u, v, \psi)$ is very good. Moreover in this case M is a right ideal of S.

It worth mentioning that in Theorem 3.1 of [7] a characterization of this kind of semigroups have been established: they are extensions of a semigroup M, which contains left identities and left magnifiers, by a right translation of M fulfilling conditions that are analogous to (β)-(ε).

For instance, the semigroups $\mathcal{B}$ and Σ, presented in Section 1, are such semigroups.

REMARK 3.4. – *The following three statements are equivalent*:

(i) $\psi(e) = vu$;

(ii) $\psi(e) \in vM$;

(iii) $\psi(M) \subset vM$.

II. *Case* $\psi(e) = e$. Then $\psi(v) \in P(M, u, v, \psi) \cap R(M, e, u)$. As $ea = a\psi(e) = a \notin M$, it follows that M is not a right ideal in S. Also, e is a left identity for $S = \mathfrak{S}(M, u, , v, \psi)$ whence all the left magnifiers of S are very good.

An example of such an endomorphism ψ has been given in [7]. For that ψ, the corresponding semigroup $S = \mathfrak{S}(M, u, v, \psi)$ has an identity, whence all the magnifiers of S are very good.

III. *Case* $\psi(e) \notin \mathcal{B}$. Then M is not a right ideal of S.

In this case, the sets $\psi^k(\mathcal{B})$, $k \in \mathbb{N}$, are disjoint. Indeed, notice first that, by (γ), $\psi_{|Me}$, the restriction to Me of ψ, is injective. As $\bigcup_{k \in \mathbb{N}} \psi^k(\mathcal{B}) \subset Me$, it results that it suffices to show that $\mathcal{B} \cap \psi^k(\mathcal{B}) = \emptyset$, for every $k \in \mathbb{N}^*$.

Let $b = v^m u^n \in \mathcal{B}$, where m, n are in $\mathbb{N}$. From (β'), we deduce that $u^m \psi(b) = \psi(u^n)$. As $\psi(u^k) \notin \mathcal{B} \backslash v\mathcal{B}$, for every $k \in \mathbb{N}^*$, and $\psi(M) = \psi(u^m)\psi(M)$, it follows that $\psi(b) \notin \mathcal{B}$. Hence $\mathcal{B} \cap \psi(\mathcal{B}) = \emptyset$. Also, $u\psi^{k+1}(b)\, v = \psi^k(v)$, therefore, by recurrence, $\mathcal{B} \cap \psi^k(\mathcal{B}) = \emptyset$, for every $k \in \mathbb{N}^*$.

In Section 5 we will give examples of endomorphisms ψ of this kind.

Finally, we prove that for obtaining semigroups with good but not very good left magnifiers we must take as M semigroups which are not orthodox (for the definition and properties of orthodox semigroups, see 6.2 [9]).

PROPOSITION 3.5. – *Let S be a semigroup which contains a good left magnifier admitting a minimal subsemigroup M. If* Idemp(M) *is a subsemigroup of M then every left magnifier of S is very good.*

PROOF. – Suppose $S = \mathfrak{S}(M, u, v, \psi)$. Then vu and $\psi(vu)$ belong to Idemp(M), so $vu\psi(vu) \in$ Idemp(M). Now, from (β), we get $vu\psi(vu) = v\psi(u)$.

We also have $v\psi(u)\, v\psi(u) \overset{(\gamma)}{=} v^2 u\psi(u) \overset{(\delta')}{=} v^2 u^2$. It follows that $v\psi(u) = v^2 u^2$, whence $\psi(u) = vu^2$. Then $\psi(e) = \psi(u)\,\psi(v) = vu^2 \psi(v) = vu\psi(e) = vu$ and thus $\psi(M) = \psi(e)\,\psi(M) \subset vM$. Hence M is a right ideal of S, so every left magnifier of S is a very good one. ■

Using the previous proposition, we deduce that if M is an orthodox semigroup which contains three elements e, u, v satisfying (1), then, for every endomorphism ψ of M which fulfils (β)-(ε), the left magnifiers of the semigroup $\mathfrak{S}(M, u, v, \psi)$ are all very good.

4. – Analogs of semigroups $\mathcal{B}$ and Σ.

Let G be a commutative totally ordered group. For every $x \in G$ let us put:

$$x^+ = \begin{cases} x & \text{if } x \geqslant 0 \\ 0 & \text{otherwise}. \end{cases}$$

Notice that $x^+ - (-x)^+ = x$ and $(x + y^+)^+ = x^+ + (y - (-x)^+)^+$. The set $H = G \times G$ endowed with an operation defined by:

$$(a, b)(c, d) = (a + (c-b)^+,\, d + (b-c)^+)$$

is a semigroup.

In the following we will study some subsemigroups of H having very good magnifiers and which coincide with $\mathcal{B}$ or Σ when we choose as a totally ordered group the group $(\mathbb{Z}, +)$.

4.1. *The left principal ideals of H are:*

$$\pi_a = [a;\, +\infty[\, \times G = (a, b)\, H, \text{ where } a \text{ and } b \text{ are in } G.$$

The right principal ideals of H are:

$$\varrho_b = G \times [b;\, +\infty[\, = H(a, b), \text{ where } a \text{ and } b \text{ are in } G.$$

The lattice of left ideals and that of right ideals of H are totally ordered and H has no two-sided proper ideal.

For every $(a, b) \in H$, $P(a, b) = \pi_a \cap \varrho_b$ is a subsemigroup of H.

Denote $\tilde{\pi}_a =]a;\, +\infty[\, \times G$ and $\tilde{\varrho}_b = G \times]b;\, +\infty[$.

4.2. $\mathrm{Idemp}(H) = \{(a, a) \mid a \in G\}$.

4.3. *The centralizers of the elements of H are*:

$$\begin{aligned} \mathcal{C}(a, a) &= \{(x, x) \mid x \in G \text{ and } x \leqslant a\} \cup P(a, a) \\ \mathcal{C}(a, a+b) &= \{(x+b, x) \mid x \in G \text{ and } x \leqslant a\} \cup P(a+b, a) \\ \mathcal{C}(a+b, a) &= \{(x, x+b) \mid x \in G \text{ and } x \leqslant a\} \cup P(a, a+b) \end{aligned}$$

where a and b belong to G and $b > 0$.

4.4. *The kernels of inner left translations of H* are:

$$\ker(\lambda_{(a,b)}) = \ker(\lambda_{(0,b)}) = \{\{(c, d)\} \mid c, d \in G, c > b\} \cup \{A_d \mid d \in G\}$$

where $A_d = \{(c, c+d) \mid c \in G, c \leqslant b\}$.

Hence the restriction of $\lambda_{(a,b)}$ to π_b is injective.

4.5. Notice that if we designate by H° the opposite semigroup of H then the map $\tau : H^\circ \to H$, $\tau(a, b) = (b, a)$, for every a, b in G, is a semigroup isomorphism. Therefore in H there exists a left-right symmetry.

4.6. Let $e \in G$. Then $\{(a, b) \in H \mid (e, e)(a, b) = (a, b)\} = \pi_e$. Thus (a, a) is the identity of $P(a, a)$. If $a < b$ then:

- (b, b) is a right identity for $P(a, b)$ and $P(a, b)$ does not contain left identities;
- $P(b, a)$ admits (b, b) as left identity and $P(b, a)$ has no right identity.

4.7. If a, b, c are elements of G then the semigroups $P(a, b)$ and $P(a+c, b+c)$ are isomorphic. For instance, the map $\psi : P(a, b) \to P(a+c, b+c)$ defined by $\psi(x, y) = (x+c, y+c)$, for every $(x, y) \in P(a, b)$, is an isomorphism between these two semigroups.

4.8. We establish below some properties of the subsemigroup $P(0, 0)$ of H.

(i) If x and y belong to $P(0, 0)$ then $(x, y)\, P(0, 0) = (x, 0)$.

(ii) $\mathfrak{RI}(P(0, 0)) = \{(0, a) \mid a \in G, a \geqslant 0\}$
$\mathfrak{LM}(P(0, 0)) = \{(0, a) \mid a \in G, a > 0\}$

(iii) If $a \in G$ and $a > 0$ then $(0, a) P(a, 0) = P(0, 0)$. Hence, by 4.4, we deduce that $P(a, 0)$ is a minimal right ideal of $P(0, 0)$ associated with the left magnifier $(0, a)$.

It follows that every left magnifier of $P(0, 0)$ is very good.

(iv) We prove now that $P(a, 0)$ is the unique minimal subsemigroup of $P(0, 0)$ associated with the left magnifier $(0, a)$.

To see that, consider that M is a subsemigroup of $P(0, 0)$ such that $(0, a)\, M = P(0, 0)$ and $\lambda_{(0,a)|_M}$ is a bijection. From 4.4, $\{(a, 0)\} \cup \widetilde{P}(a, 0) \subset M$, where $\widetilde{P}(a, b) = \widetilde{\pi}_a \cap \varrho_b$. Consider $(a, y) \in M$, with $y \geqslant 0$ and $0 \leqslant x \leqslant a$.

If $y \geqslant a$ then chosing $(u, v) \in \widetilde{P}(a, 0)$, with $u > y$, we get $(x, y)(u, v) = (x + u - y, v) \in M$. Hence $\widetilde{P}(x, 0) \subset M$. Thus, by 4.4, $x = a$. It follows that $(a, y) \in M$, for every $y \in G$, $y \geqslant a$.

Also, from 4.4, it results that it remains to study the case $0 \leqslant y < x$. In this situation, $(a, a)(x, y) = (a, a + y - x) \in M$. Then, by 4.4, as $(0, a)(x, y) = (0, a)(a, a + y - x)$, we get once again $x = a$. Hence $M = P(a, 0)$.

(v) If we chose $G = \mathbb{Z}$ and denote $1 = (0, 0)$, $p = (0, 1)$, $q = (1, 0)$, then $P(0, 0)$ is the bicyclic monoid. Thus, by (iv), it follows that in the bicyclic monoid every magnifier admits one and only one minimal subsemigroup.

(vi) $P(0, 0)$ is a $\mathcal{D}$-simple inverse semigroup with identity and its endomorphisms can be determined by using a method given by R. J. Warne in Theorem 1.1 of [22].

Thus, the endomorphisms of $P(0, 0)$ are defined by $\psi_{(\varphi, k)}(x, y) = (k + \varphi(x), k + \varphi(y))$, for every x, y in $G^+ = [0, +\infty[$, where φ is an endomorphism of G^+ and $k \in G$.

For instance, for $G = \mathbb{Z}$, as the endomorphisms of $\mathbb{Z}^+$ are of the form φ_l (where $\varphi_l(n) = nl$, for every $n \in \mathbb{Z}^+$), it follows that, in this case, the endomorphisms of $P(0, 0)$ are $\psi_{k, l}$, with k, l in $\mathbb{Z}^+$, where $\psi_{k, l}(m, n) = (k + ml, k + nl)$, for every m, n in $\mathbb{Z}^+$.

(vii) Consider $M = P(0, 0)$, $e = (0, 0)$, $u = (0, a)$, $v = (a, 0)$, where $a \in G$, $a > 0$. Then there exists a unique endomorphism ψ of M such that the conditions (β)-(ε) are fulfilled, namely $\psi(x, y) = (a + x, a + y)$, for every x, y in G^+. Therefore $\psi = \lambda_v \circ \varrho_u$.

4.9. We now deal with semigroups of the form $P(a, 0)$, with $a \in G$ and $a < 0$.

(i) If $(x, y) \in P(a, 0)$ then $(x, y)\, P(a, 0) = (x, 0)\, P(a, 0) = P(x, 0)$.

(ii) $\mathfrak{R}\mathfrak{I}(P(a, 0)) = \mathfrak{L}\mathfrak{M}(P(a, 0)) = \{(a, b) \mid b \in G^+\}$.

(iii) If $b \in G^+$ then $(a, b)\, P(b, 0) = P(a, 0)$. Hence, by 4.4, $P(b, 0)$ is a minimal right ideal associated with the left magnifier (a, b). Therefore, every left magnifier of $P(a, 0)$ is very good.

(iv) With similar arguments like in 4.8.(iv) it follows that $P(a, 0)$ is the unique minimal subsemigroup associated with the magnifier (a, b).

(v) If $G = \mathbb{Z}$ then $P(-1, 0)$ is isomorphic to the semigroup $\Sigma(a, p, q)$, presented in Section 1. For $P(-1, 0)$ one has $a = (-1, 0)$, $p = (0, 1)$ and $q = (1, 0)$.

Remark that 4.8 and 4.9 contain as special cases some results established by F. Migliorini (Theorem 10.2 of [11] and Section II of [17]).

The two foregoing cases 4.8.(iv) and 4.9.(iv) suggest us to consider the following:

PROBLEM. – *Characterize the semigroups having the property that every of their left magnifiers admits a unique associated minimal subsemigroup.*

We think that these semigroups can be characterized via an embedding of a semigroup of type $P(a, b)$.

5. – Semigroups with good but not very good magnifiers.

Throughout this section M is a subsemigroup of the underlying multiplicative semigroup of a ring $(R, +, \cdot)$.

Suppose that there exist e, u, v, elements in M, such that conditions (1) are fulfilled. Let $\psi : M \to M$ be a map and define $\varphi : M \to R$ by $\varphi(m) = \psi(m) - vmu$, for every $m \in M$.

LEMMA 5.1. – *The map ψ satisfies conditions (α)-(ε) if and only if $\varphi(m) + vmu \in M$ and φ fulfils the following five conditions:*

$$
\begin{array}{lll}
(\alpha'') & \varphi(mm') & = vmu\varphi(m') + \varphi(m)\,\varphi(m')\,; \\
(\beta'') & \varphi(e) & = u\varphi(v) \text{ and } \varphi(m)\,e = \varphi(m)\,; \\
(\gamma'') & \varphi(m)\,v & = 0\,; \\
(\delta'') & u\varphi(u) & = 0 \text{ and } u\varphi(vmu + \varphi(m)) = 0\,; \\
(\varepsilon'') & u\varphi(e) & = 0
\end{array}
$$

for every m, m' of M.

The map $\varphi = 0$ fulfils (α'')-(ε''), whence $\psi = \lambda_v \circ \varrho_u$ satisfies (α)-(ε). Thus, it remains to study the case $\varphi \neq 0$. Remark that $P(M, u, v, \psi) = \{m \in M \mid u\varphi(m) = 0\}$.

Let now consider the ring $\mathbb{CFM}_{\mathbb{N}}(R)$ of column finite matrices over R, where R is a commutative ring with 1 (see 1.14 of [1]). Properties of magnifiers of the underlying multiplicative semigroup of this ring have been established by N. Jacobson ([10], Theorem 4).

5.2. Consider as M the subsemigroup of the underlying multiplicative semigroup of the ring $\mathbb{CFM}_{\mathbb{N}}(R)$ containing the matrices $\boldsymbol{X} = (x_{ij})_{(i,j) \in \mathbb{N}^2}$ such that $x_{00} = 1$ and $x_{0j} = 0$, for every $j \in \mathbb{N}^*$. For every matrix $\boldsymbol{X}$ of M and every

$k \in \mathbb{N}$ let us designate

$$C_k(\boldsymbol{X}) = \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \end{pmatrix} \quad \text{and} \quad B(\boldsymbol{X}) = \begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ . & . & \dots \end{pmatrix}.$$

Notice that if $\boldsymbol{X}$ and $\boldsymbol{X}'$ are two matrices of M and $k \in \mathbb{N}^*$ then

$$\text{(3)} \qquad \begin{cases} C_0(\boldsymbol{XX}') = C_0(\boldsymbol{X}) + B(\boldsymbol{X})\, C_0(\boldsymbol{X}') \\ C_k(\boldsymbol{XX}') = B(\boldsymbol{X})\, C_k(\boldsymbol{X}') \\ B(\boldsymbol{XX}') \ = B(\boldsymbol{X})\, B(\boldsymbol{X}'). \end{cases}$$

The matrix $\boldsymbol{E}$ of M such that $C_0(\boldsymbol{E}) = \boldsymbol{0}$ and $B(\boldsymbol{E}) = (\delta_{ij})_{(i,j) \in (\mathbb{N}^*)^2}$ is the identity of M.

Consider the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ of M such that

$$C_0(\boldsymbol{U}) = \boldsymbol{0}, \qquad B(\boldsymbol{U}) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ . & . & . & . & \dots \end{bmatrix}$$

$$C_0(\boldsymbol{V}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \end{bmatrix}, \qquad B(\boldsymbol{V}) = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ . & . & . & \dots \end{bmatrix}.$$

Then $\boldsymbol{UV} = \boldsymbol{E}$ and $\boldsymbol{VU} \neq \boldsymbol{E}$, whence the elements $\boldsymbol{E}$, $\boldsymbol{U}$, $\boldsymbol{V}$ of M satisfy conditions (1). Notice that

$$C_0(\boldsymbol{VXU}) = \begin{bmatrix} 1 + x_{10} \\ x_{10} \\ x_{20} \\ \dots \end{bmatrix} \quad \text{and} \quad B(\boldsymbol{VXU}) = \begin{bmatrix} 0 & x_{11} & x_{12} & x_{13} & \dots \\ 0 & x_{11} & x_{12} & x_{13} & \dots \\ 0 & x_{21} & x_{22} & x_{23} & \dots \\ . & . & . & . & \dots \end{bmatrix}.$$

We want to find a map $\varphi : M \to R$ such that conditions (α'')-(ε'') hold. By

(γ''), it follows that $\varphi(\boldsymbol{X})$ has the form

$$\varphi(\boldsymbol{X}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ -\varphi_1(\boldsymbol{X}) & \varphi_1(\boldsymbol{X}) & -\varphi_1(\boldsymbol{X}) & 0 & 0 & \dots \\ -\varphi_2(\boldsymbol{X}) & \varphi_2(\boldsymbol{X}) & -\varphi_2(\boldsymbol{X}) & 0 & 0 & \dots \\ . & . & . & . & . & \dots \end{bmatrix}.$$

It remains to find $\varphi_i(\boldsymbol{X})$, for $i \in \mathbb{N}^*$.

The condition (α'') is equivalent to:

$$\text{(4)} \quad \begin{cases} \varphi_1(\boldsymbol{XX'}) = \varphi_1(\boldsymbol{X})[\varphi_1(\boldsymbol{X'}) - \varphi_2(\boldsymbol{X'})] + x_{11}\varphi_2(\boldsymbol{X'}) + x_{12}\varphi_3(\boldsymbol{X'}) + \dots \\ \varphi_{k+1}(\boldsymbol{XX'}) = \varphi_{k+1}(\boldsymbol{X})[\varphi_1(\boldsymbol{X'}) - \varphi_2(\boldsymbol{X'})] + x_{k1}\varphi_2(\boldsymbol{X'}) + x_{k2}\varphi_3(\boldsymbol{X'}) + \dots \end{cases}$$

where $k \in \mathbb{N}^*$.

From (ε''), it follows that $\varphi_{k+1}(\boldsymbol{E}) = \boldsymbol{0}$, for every $k \in \mathbb{N}^*$.

The first two equations of (4) give:

$$\varphi_1(\boldsymbol{XX'}) - \varphi_2(\boldsymbol{XX'}) = [\varphi_1(\boldsymbol{X}) - \varphi_2(\boldsymbol{X})] \cdot [\varphi_1(\boldsymbol{X'}) - \varphi_2(\boldsymbol{X'})],$$

for every $\boldsymbol{X}, \boldsymbol{X'}$ in M.

This equality holds if

$$\varphi_1(\boldsymbol{X}) - \varphi_2(\boldsymbol{X}) = \omega \in \operatorname{Idemp}(R), \qquad \text{for every } \boldsymbol{X} \in M.$$

If we put

$$D(\boldsymbol{X}) = \begin{pmatrix} \varphi_2(\boldsymbol{X}) \\ \varphi_3(\boldsymbol{X}) \\ \vdots \end{pmatrix}$$

then equations (4) become

$$\text{(4')} \qquad D(\boldsymbol{XX'}) = \omega D(\boldsymbol{X}) + B(\boldsymbol{X})\, D(\boldsymbol{X'}).$$

Remark that $\boldsymbol{U}\varphi(\boldsymbol{X}) = \boldsymbol{0}$ if and only if $D(\mathbf{X}) = \boldsymbol{0}$.

Using (3), we obtain that $D(\boldsymbol{X}) = \omega C_0(\boldsymbol{X})$ satisfies (4′). Hence if:

$$\varphi(\boldsymbol{X}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ -\omega(1+x_{10}) & \omega(1+x_{10}) & -\omega(1+x_{10}) & 0 & \dots \\ -\omega x_{10} & \omega x_{10} & -\omega x_{10} & 0 & \dots \\ -\omega x_{20} & \omega x_{20} & -\omega x_{20} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and

$$\psi(\boldsymbol{X}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ (1-\omega)(1+x_{10}) & \omega(1+x_{10}) & x_{11}-\omega(1+x_{10}) & x_{12} & \dots \\ (1-\omega)\,x_{10} & \omega x_{10} & x_{11}-\omega x_{10} & x_{12} & \dots \\ (1-\omega)\,x_{20} & \omega x_{20} & x_{21}-\omega x_{20} & x_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

the application φ satisfies the conditions (α'')-(ε''), whence the application ψ fulfils the conditions (α)-(ε). For this φ we have that:

$$\psi(\boldsymbol{E}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 1-\omega & \omega & 1-\omega & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then

- $\psi(\boldsymbol{E}) = \boldsymbol{E}$ if and only if $\omega = 1$
- $\psi(\boldsymbol{E}) = \boldsymbol{VU}$ if and only if $\omega = 0$.

Hence, if we take $\omega \in \mathrm{Idemp}(R)$, $\omega \neq 0$, $\omega \neq 1$, we obtain an endomorphism ψ of M which is of type (III) (see Section 3).

Also, for this ψ we have that $P(M, \boldsymbol{U}, \boldsymbol{V}, \psi) = \{\boldsymbol{X} \in M \mid \omega C_0(\boldsymbol{X}) = \boldsymbol{0}\}$.

As

$$R(M, \boldsymbol{E}, \boldsymbol{U}) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ x_{10} & x_{11} & x_{12} & x_{13} & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \;\middle|\; x_{1j} \in R, \text{ for every } j \in \mathbb{N} \right\}$$

it results that $P(M, \boldsymbol{U}, \boldsymbol{V}, \psi) \cap R(M, \boldsymbol{E}, \boldsymbol{U}) \neq \emptyset$.

Therefore, using Theorem 3.1, it follows that all the left magnifiers of the semigroup $\mathfrak{S}(M, \boldsymbol{U}, \boldsymbol{V}, \psi)$ are very good.

5.3. Now, using the example from 5.2, we construct a semigroup for which all the left magnifiers are good but none is very good.

Thus, in the construction done in 5.2, with the idempotent ω different from

0 and 1, let us consider: $\mathscr{B}$ the semigroup of M generated by $\boldsymbol{U}$ and $\boldsymbol{V}$, $\Gamma = \bigcup_{k \in \mathbb{N}} \psi^k(\mathscr{B})$ and M' the subsemigroup of M generated by Γ. As $\psi(\Gamma) \subset \Gamma$ we get that $\psi' = \psi_{|M'}$, the restriction of ψ to M', is an endomorphism of M' and ψ' fulfils the conditions (α)-(ε) from Lemma 2.1. We prove below that $P(M', \boldsymbol{U}, \boldsymbol{V}, \psi') \cap R(M', \boldsymbol{E}, \boldsymbol{U}) = \emptyset$.

For every $k \in \mathbb{N}$, denote $\varphi(\boldsymbol{V}^k) = \boldsymbol{\Lambda}_{k+1}$. Then, for every k, l, r in $\mathbb{N}$, $r \geqslant 3$:

$$\psi(\boldsymbol{V}^k \boldsymbol{U}^l) = \boldsymbol{V}^{k+1} \boldsymbol{U}^{l+1} + \boldsymbol{\Lambda}_{k+1}, \qquad \psi^2(\boldsymbol{V}^k \boldsymbol{U}^l) = \boldsymbol{V}^{k+2} \boldsymbol{U}^{l+2} + \boldsymbol{V} \boldsymbol{\Lambda}_{k+1} \boldsymbol{U} + \boldsymbol{\Lambda}_1$$

and generally,

$$\psi^r(\boldsymbol{V}^k \boldsymbol{U}^l) = \boldsymbol{V}^{k+r} \boldsymbol{U}^{l+r} + \boldsymbol{V}^{r-1} \boldsymbol{\Lambda}_{k+1} \boldsymbol{U}^{r-1} + \sum_{s=0}^{r-2} \boldsymbol{V}^s \boldsymbol{\Lambda}_1 \boldsymbol{U}^s.$$

Using the previous relations (3) we deduce that $I = \{\boldsymbol{X} \in M \mid C_1(\boldsymbol{X}) \in \mathbb{M}_{\mathbb{N}^* \times \{1\}}(R\omega)\}$ is a left ideal of M. Moreover, $\psi(M) \subset I$ and $\boldsymbol{U}^k \in I$, for every $k \in \mathbb{N}^*$. Also $\Gamma \setminus I = \{\boldsymbol{V}^k \mid k \in \mathbb{N}\}$.

Consider $\boldsymbol{X} = (x_{ij})_{(i,j) \in \mathbb{N}^2} \in R(M', \boldsymbol{E}, \boldsymbol{U})$. Then $x_{21} = 1$, whence $\boldsymbol{X}$ does not belong to I. Let s be the minimum positive integer such that $\boldsymbol{X} = \boldsymbol{X}_1 \dots \boldsymbol{X}_s$, where $\boldsymbol{X}_i \in \Gamma$, for every $i \in \{1, \dots, s\}$.

If $s = 1$ it follows that $\boldsymbol{X} = \boldsymbol{V}$.

Suppose $s \geqslant 2$. Since $\boldsymbol{X} \notin I$ it results that $\boldsymbol{X}_s = \boldsymbol{V}^k$, where $k \in \mathbb{N}^*$. On the other hand, because ψ' satisfies (γ) from Lemma 2.1, we obtain a contradiction with the minimality of s.

Hence $R(M', \boldsymbol{E}, \boldsymbol{U}) = \{\boldsymbol{V}\}$, therefore $P(M', \boldsymbol{U}, \boldsymbol{V}, \psi') \cap R(M', \boldsymbol{E}, \boldsymbol{U}) = \emptyset$.

Thus, applying Theorem 3.1 to the semigroup $S = \mathfrak{S}(M', \boldsymbol{U}, \boldsymbol{V}, \psi')$ we get that $\phi(\boldsymbol{E})$ is a good left magnifier for S but S contains no very good left magnifier. Notice also that $M' \setminus I = \{\boldsymbol{V}^k \mid k \in \mathbb{N}\}$.

According to Corollary 2.7, the semigroup S has no left identity and if $\boldsymbol{X} \in M'$ then $\phi(\boldsymbol{X})$ is a left magnifier of S if and only if there exists $\boldsymbol{X}' \in M'$ such that $\boldsymbol{U} \psi(\boldsymbol{X}) \boldsymbol{X}' = \boldsymbol{E}$. Thus $\psi(\boldsymbol{X}) \boldsymbol{X}' = \boldsymbol{V}$. As $\boldsymbol{V} \notin I$ it follows that $\boldsymbol{X}' \notin I$, whence $\boldsymbol{X}' = \boldsymbol{V}^k$, where $k \in \mathbb{N}^*$. We get that $\boldsymbol{X} \boldsymbol{V}^{k-1} = \boldsymbol{E}$. Then, using Corollary 2.7.(iv), it follows that $\phi(\boldsymbol{X})$ is a good magnifier of S. Hence all the left magnifiers of S are good.

5.4. The same construction as in 5.3 can be done with a bigger Γ, namely $\Gamma = \mathscr{B} \cup \psi(M)$. If we denote by M'' the subsemigroup of M generated by this Γ and consider $\psi'' = \psi_{|M''}$ then $R(M'', \boldsymbol{E}, \boldsymbol{U}) = \{\boldsymbol{V}\}$, $M'' \setminus I = \{\boldsymbol{V}^k \mid k \in \mathbb{N}\}$ and $P(M'', \boldsymbol{E}, \boldsymbol{V}, \psi'') \cap R(M'', \boldsymbol{E}, \boldsymbol{U}) = \emptyset$. Also, all the left magnifiers of the semigroup $\mathfrak{S}(M'', \boldsymbol{U}, \boldsymbol{V}, \psi'')$ are good but none is very good.

Others examples of semigroups for which all the left magnifiers are good but none is very good can be obtained using Examples 5.3 and 5.4 and the following result.

PROPOSITION 5.5. – *Let S and T be semigroups without left identity and a be a good left magnifier for S and b be a good left magnifier for T. Then*:

(i) $\mathfrak{LM}(S \times T) = \mathfrak{LM}(S) \times \mathfrak{LM}(T)$;

(ii) (a, b) *is a good left magnifier for the semigroup* $S \times T$;

(iii) (a, b) *is a very good left magnifier for* $S \times T$ *if and only if* a *is a very good left magnifier for* S *and* b *is a very good left magnifier for* T.

PROOF. – (i) is proved in [5].

(ii) If M is a minimal subsemigroup of S associated with a, and N a minimal subsemigroup of T associated with b, then $M \times N$ is a minimal subsemigroup of $S \times T$ associated with (a, b).

(iii) Let R be a minimal right ideal of $S \times T$ associated with (a, b) and let $(e, f) \in R$ such that $(a, b) = (a, b)(e, f)$. Then, by Lemma 2.5, $R = eS \times fT$. Hence eS (resp. fT) is a minimal right ideal of S (resp. T) associated with a (resp. b). ■

Using Proposition 5.5 we obtain that in the semigroups $\mathfrak{S}(M', \boldsymbol{U}, \boldsymbol{V}, \psi') \times \Sigma(a, p, q)$, $\mathfrak{S}(M', \boldsymbol{U}, \boldsymbol{V}, \psi') \times \mathfrak{S}(M'', \boldsymbol{U}, \boldsymbol{V}, \psi'')$ all the left magnifiers are good but none is very good.

Finally we mention some open:

PROBLEMS.

1) *Find a characterisation of semigroups for which all the magnifiers are good.*

2) *Does there exist semigroups which contain good and bad* (i.e. *not good*) *magnifiers* ([11], p. 74)?

3) *If a is a good magnifier for a semigroup S, what are the connections between the minimal subsemigroups associated to a in S*?

4) *If* M, e, u, v, ψ, *respectively* M', e', u', v', ψ' *satisfy* (1) *and* (α)-(ε) *find necessary and sufficient conditions in order that* $\mathfrak{S}(M, u, v, \psi)$ *and* $\mathfrak{S}(M', u', v', \psi')$ *be isomorphic.*

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