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Ernesto Buzano

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Some Remarks on the Weyl Asymptotics by the Approximate Spectral Projection Method.

ERNESTO BUZANO

Sunto. – *In questo lavoro studiamo il resto relativo della formula asintotica per gli autovalori di un operatore differenziale in $\boldsymbol{R}^n$, ottenuta mediante il metodo delle proiezioni spettrali approssimate ([3], Theorem 6.2). In un primo tempo diamo un controesempio di un operatore di Schrödinger con potenziale a crescita algebrica, per il quale il resto non è limitato. Quindi specifichiamo alcune condizioni addizionali da imporre all'operatore in modo da avere un resto infinitesimo.*

1. – Introduction.

The study of the asymptotic behavior of the eigenvalues of a differential operator in $\boldsymbol{R}^n$ with compact resolvent has been the subject of several papers, starting from the fifties. Among the various techniques employed to evaluate the remainder in the asymptotic formula, there is the so-called approximate spectral projection method. While this technique yields a weaker remainder estimate than the hyperbolic operator method, it can be applied to a broader number of cases.

The approximate spectral projection method was introduced by Tulovskiĭ and Shubin in [9] (see also [8]) and improved and extended by several authors. Concerning systems of differential operators in $\boldsymbol{R}^n$ a rather general result is due to Feĭgin [4, 5]. In the scalar case, i.e. for a single differential operator, the result has been improved by Dencker [3]. Let us describe the Feĭgin's result in the scalar case, with the Dencker's improvement.

We employ the following notation: given two functions $f, g \colon X \to \boldsymbol{R}$, and a subset $A \subset X$, we write

$$f(x) \prec g(x), \qquad \forall x \in A,$$

if there exists a constant $C = C(f, g, A)$ such that

$$f(x) \leqslant Cg(x), \qquad \forall x \in A.$$

We say that $\Lambda \in \mathcal{C}^1(\boldsymbol{R}^n, \boldsymbol{R})$ is a *weight function* if there exists

$$0 \leqslant \delta_0 < 1,$$

such that

$$1 < \Lambda(x), \tag{1}$$

$$|\nabla \Lambda(x)| < \Lambda(x)^{1+\delta_0}, \tag{2}$$

for all $x \in \boldsymbol{R}^n$.

Consider a differential operator of order m in $\boldsymbol{R}^n$:

$$Au(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, D_x^\alpha u(x)$$

with smooth coefficients. Assume that there exists

$$\delta_0 \leqslant \delta < 1$$

such that for each $\beta \in \boldsymbol{N}^n$ (1) we have

$$|\partial_x^\beta a_\alpha(x)| < \Lambda(x)^{m - |\alpha| + \delta|\beta|}, \qquad \forall x \in \boldsymbol{R}^n,$$

then A is a properly supported pseudo-differential operator:

$$Au(x) = (2\pi)^{-n} \int e^{(x-y)\cdot\xi}\, a\left(\frac{1}{2}(x+y),\, \xi\right) u(y)\, dy\, d\xi\,,$$

with *Weyl symbol*:

$$a(x, \xi) = \sum_{|\alpha| \leqslant m} \sum_{\beta \leqslant \alpha} \left(\frac{i}{2}\right)^{|\beta|} \binom{\alpha}{\beta} \partial_x^\beta a_\alpha(x)\, \xi^{\alpha-\beta}. \tag{3}$$

If a is real valued, and $a(x, \xi) \to \infty$ as $|x| + |\xi| \to \infty$, then A is a closed, self-adjoint operator in $L^2(\boldsymbol{R}^n)$, with discrete spectrum diverging to $+\infty$. Let $(\lambda_j)_{j \geqslant 1}$ be the sequence of the eigenvalues of A, repeated according to their multiplicity, and define the *counting function:*

$$\mathcal{N}(\tau) = \sum_{\lambda_j \leqslant \tau} 1\,.$$

$\mathcal{N}(\tau)$ is the number of eigenvaues less or equal to τ. The Feĭgin's result with the Dencker's improvement (see [3], Theorem 6.2 and Example 3.5), is the following.

THEOREM 1. – *Assume that a is real valued and that there exist*

$$l > 0\,, \qquad \delta_0 \leqslant \delta < \varrho \leqslant 1\,,$$

and $R \geqslant 0$, such that

$$a(x, \xi) > (1 + |x| + |\xi|)^l, \qquad \textit{for all } |x| + |\xi| \geqslant R$$

(1) $\boldsymbol{N} = \{0, 1, \ldots\}$.

and for each $\alpha, \beta \in \boldsymbol{N}^n$

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \prec a(x, \xi)(|\xi| + \Lambda(x))^{-\varrho|\alpha| + \delta|\beta|}, \tag{4}$$

for all $|x| + |\xi| \geqslant R$. *Then*

$$\mathcal{N}(\tau) = W(\tau)\Big\{1 + \mathcal{O}(R_\varepsilon(\tau))\Big\}, \qquad \textit{as } \tau \to \infty, \tag{5}$$

where

$$W(\tau) = (2\pi)^{-n} \int_{a(x, \xi) \leqslant \tau} dx\, d\xi\,,$$

$$R_\varepsilon(\tau) = \frac{W(t + \tau^{1-\varepsilon}) - W(t - \tau^{1-\varepsilon})}{W(\tau)}, \tag{6}$$

and

$$0 < \varepsilon < \frac{2(\varrho - \delta)}{3m}. \qquad \blacksquare \tag{7}$$

Observe that $2(\varrho - \delta)/3m < 1$, since $m \geqslant 1$, because $a(x, \xi)$ is a polynomial in ξ.

Under suitable non-degeneracy conditions, the asymptotic evaluation of $W(\tau)$ has been carried out by Boggiatto and Buzano in [1].

It is clear that *the asymptotic formula* (5) *makes sense only when the relative remainder* $R_\varepsilon(\tau)$ *vanishes as* $\tau \to \infty$. This is far from being true: in the next Section we give an example of a Schrödinger operator satisfying the hypotheses of Theorem 1, so that the asymptotic formula (5) is true, but

$$\limsup_{\tau \to \infty} R_\varepsilon(\tau) = \infty\,,$$

for every $\varepsilon < 1$.

In the third Section we specify some additional conditions to be imposed on the symbol a in order to obtain that $R_\varepsilon(\tau) \to 0$ as $\tau \to \infty$.

2. – A counterexample.

Consider two sequences of real numbers $(t_k)_{k \geqslant 0}$ and $(y_k)_{k \geqslant 0}$ which are strictly increasing and diverging to $+\infty$ as $k \to \infty$. Assume moreover that there exists $\eta > 0$, such that

$$t_{k+1} - t_k > 2\eta\,, \qquad \forall k \geqslant 0\,.$$

Set

$$\mu_k = \frac{y_{k+1} - y_k}{t_{k+1} - t_k}, \qquad \text{for } k \geqslant 0$$

and

$$\mu_{-1} = 0 .$$

Choose a function $\phi \in \mathcal{C}^\infty(\boldsymbol{R}, \boldsymbol{R})$ such that

$$\begin{aligned} &\phi(t) = 0 , && \text{for } t \leqslant 0 , \\ &0 < \phi(t) < 1 , && \text{for } 0 < t < 1 , \\ &\phi(t) = 1 , && \text{for } t \geqslant 1 , \\ &\phi'(t) > 0 , && \text{for } 0 < t < 1 . \end{aligned}$$

For each $k \geqslant 0$ set

$$\phi_k(t) = \begin{cases} 1 - \phi\left(\dfrac{t - t_k}{\eta}\right), & \text{if } \mu_{k-1} < \mu_k, \\ 0 , & \text{if } \mu_{k-1} \geqslant \mu_k, \end{cases}$$

$$\psi_k(t) = \begin{cases} 1 - \phi\left(\dfrac{t_{k+1} - t}{\eta}\right), & \text{if } \mu_k > \mu_{k+1}, \\ 0 , & \text{if } \mu_k \leqslant \mu_{k+1}. \end{cases}$$

Define $f \in \mathcal{C}^\infty(\boldsymbol{R}, \boldsymbol{R})$ as

$$f(t) = y_0,$$

for $t < t_0$ and

$$f(t) = y_k + \mu_k(t - t_k) + (\mu_{k-1} - \mu_k)(t - t_k)\, \phi_k(t) + (\mu_{k+1} - \mu_k)(t - t_{k+1})\, \psi_k(t),$$

for $t_k \leqslant t < t_{k+1}$ and $k \geqslant 0$.

It is elementary to prove the following

LEMMA 1. – *We have*

1) $f(t_k) = y_k$, *for all* $k \geqslant 0$,
2) $f'(t) > 0$, *for* $t > t_0$,
3) *for each* $l \in \boldsymbol{N}$, *there exists* $C_l > 0$ *such that*

$$|f^{(l)}(t)| \leqslant C_l \max\{\mu_{k-1}, \mu_k, \mu_{k+1}\},$$

for $t_k \leqslant t < t_{k+1}$ *and* $k \geqslant 0$. ■

Now we choose the sequences $(t_k)_{k\geqslant 0}$ and $(y_k)_{k\geqslant 0}$ in the following way. Fix

$$0<p<q\,,\qquad r>1\,, \tag{8}$$

and let by induction

$$\begin{aligned}
&t_0=0\,,\\
&t_1=(3/2)^{pq/n(rq-p)},\\
&t_{2k}=t_{2k-1}^{q/p},\\
&t_{2k+1}=t_{2k}^{r},\\
&y_0=1\,,\\
&y_{2k-1}=t_{2k-1}^{n/p},\\
&y_{2k}=t_{2k}^{n/q}+\frac{1}{2}\,t_{2k}^{n/kq}.
\end{aligned}$$

We have

$$t_{k+1}>t_k,\qquad y_{k+1}>y_k,$$

for all $k\geqslant 0$, and

$$t_k\to\infty\,,\qquad y_k\to\infty\,,\qquad t_{k+1}-t_k\to\infty\,,$$

as $k\to\infty$. Let f be the function associated to $(t_k)_{k\geqslant 0}$ and $(y_k)_{k\geqslant 0}$ and consider the potential

$$V(x)=f(|x|)\,.$$

By using Lemma 1 it is elementary to prove the following

PROPOSITION 1. – *We have*

1) $V\in\mathcal{C}^\infty(\boldsymbol{R}^n,\boldsymbol{R})$,
2) $(1+|x|)^{n/rq}\prec V(x)\prec(1+|x|)^{nr/p},\qquad \forall x\in\boldsymbol{R}^n$,
3) *for each* $\alpha\in\boldsymbol{N}^n$ *we have*

$$|\partial_x^\alpha V(x)|\prec V(x)^{1+\delta_1|\alpha|},\qquad \forall x\in\boldsymbol{R}^n,$$

with

$$\delta_1=\left[\left(\frac{n}{p}-1\right)\frac{rq}{n}-1\right]_+.\qquad\blacksquare$$

From this Proposition it follows in a standard way

PROPOSITION 2. – *Assume that*

$$\left(\frac{n}{p}-1\right)\frac{rq}{n}<\frac{3}{2}, \tag{9}$$

then

1) $\Lambda(x)=V(x)^{1/2}$ *is a weight function, i.e. it satisfies* (1) *and* (2) *with*

$$\delta_0=2\delta_1<1,$$

2) *The Schrödinger operator* $A=-\Delta+V(x)$ *is a differential operator of second order with Weyl symbol* $a(x,\xi)=|\xi|^2+V(x)$ *which satifies the hypotheses of Theorem* 1 *with*

$$l=\min\left\{1,\frac{n}{2rq}\right\},\qquad \varrho=1,\qquad \delta=\delta_0=2\delta_1<1. \qquad \blacksquare$$

From this Proposition and Theorem 1 we obtain that

$$\mathcal{N}(\tau)=W(\tau)\left\{1+\mathcal{O}(R_\varepsilon(\tau))\right\},\qquad \text{as } \tau\to\infty,$$

with

$$W(\tau)=(2\pi)^{-n}\int_{|\xi|^2+V(x)\leqslant\tau}dx\,d\xi=\frac{\omega_n}{(2\pi)^n}\int(\tau-V(x))_+^{n/2}\,dx, \tag{10}$$

R_ε given by (6), and

$$0<\varepsilon<\frac{1-2\delta_1}{3}.$$

In (10), we denote by ω_n the volume of the unit ball in $\boldsymbol{R}^n$.

PROPOSITION 3. – *Let*

$$\tau_k=t_{2k-1}^{n/p}=y_{2k-1},\qquad \textit{for } k\geqslant 1.$$

Then, if

$$q-p>\frac{n}{2}, \tag{11}$$

we have that

$$\lim_{k\to\infty}R_\varepsilon(\tau_k)=\infty,$$

for any $0<\varepsilon<1$.

PROOF. – Set

$$M(\tau)=\int_{V(x)\leqslant\tau}dx$$

$$=n\omega_n\int_0^{f^{-1}(\tau)}t^{n-1}\,dt$$

$$=\omega_n(f^{-1}(\tau))^n.$$

Hence

$$f^{-1}(\tau)=\left(\frac{1}{\omega_n}M(\tau)\right)^{1/n}$$

and

$$f(t)=M^{-1}(\omega_n t^n).$$

Thus we have

$$W(\tau)=\frac{\omega_n}{(2\pi)^n}\int(\tau-V(x))_+^{n/2}\,dx$$

$$=\frac{n\omega_n^2}{(2\pi)^n}\int_0^{f^{-1}(\tau)}(\tau-f(t))^{n/2}t^{n-1}\,dt$$

$$=\frac{n\omega_n^2}{(2\pi)^n}\int_0^{f^{-1}(\tau)}(\tau-M^{-1}(\omega_n t^n))^{n/2}t^{n-1}\,dt$$

$$=\frac{\omega_n}{(2\pi)^n}\int_0^{M(\tau)}(\tau-M^{-1}(u))^{n/2}\,du\,.$$

It follows that

$$R_\varepsilon(\tau) = \frac{W(\tau+\tau^{1-\varepsilon}) - W(\tau-\tau^{1-\varepsilon})}{W(\tau)}$$

$$= \frac{\int_0^{M(\tau+\tau^{1-\varepsilon})} (\tau+\tau^{1-\varepsilon}-M^{-1}(u))^{n/2}du - \int_0^{M(\tau-\tau^{1-\varepsilon})} (\tau-\tau^{1-\varepsilon}-M^{-1}(u))^{n/2}du}{\int_0^{M(\tau)} (\tau-M^{-1}(u))^{n/2}du}$$

$$\geqslant \frac{\int_{M(\tau-\tau^{1-\varepsilon})}^{M(\tau+\tau^{1-\varepsilon})} (\tau+\tau^{1-\varepsilon}-M^{-1}(u))^{n/2}\,du}{\tau^{n/2}M(\tau)}$$

$$\geqslant \frac{\int_{M(\tau)}^{M(\tau+(1/2)\,\tau^{1-\varepsilon})} (\tau+\tau^{1-\varepsilon}-M^{-1}(u))^{n/2}\,du}{\tau^{n/2}M(\tau)}$$

$$\geqslant \frac{(\frac{1}{2}\tau^{1-\varepsilon})^{n/2}\{M(\tau+\frac{1}{2}\tau^{1-\varepsilon})-M(\tau)\}}{\tau^{n/2}M(\tau)}$$

$$\geqslant \frac{M(\tau+\frac{1}{2}\tau^{1-\varepsilon})-M(\tau)}{2^{n/2}\tau^{n\varepsilon/2}M(\tau)}\,.$$

Now observe that

$$\tau_k + \frac{1}{2}\tau_k^{1/k} = t_{2k-1}^{n/p} + \frac{1}{2}t_{2k-1}^{n/kp}$$

$$= t_{2k}^{n/q} + \frac{1}{2}t_{2k}^{n/kq}$$

$$= y_{2k}.$$

It follows that

$$M\left(\tau_k + \frac{1}{2}\tau_k^{1-\varepsilon}\right) \geqslant M\left(\tau_k + \frac{1}{2}\tau_k^{1/k}\right)$$
$$= \omega_n (f^{-1}(y_{2k}))^n$$
$$= \omega_n t_{2k}^n$$
$$= \omega_n t_{2k-1}^{nq/p}, \qquad \text{for} \quad k \geqslant \frac{1}{1-\varepsilon},$$

and

$$M(\tau_k) = \omega_n (f^{-1}(y_{2k-1}))^n$$
$$= \omega_n t_{2k-1}^n .$$

Therefore

$$R_\varepsilon(\tau_k) \geqslant \frac{M(\tau_k + \frac{1}{2}\tau_k^{1/k}) - M(\tau_k)}{2^{n/2}\tau_k^{(n/2)(1-1/k)} M(\tau_k)}$$
$$= \frac{\omega_n t_{2k-1}^{nq/p} - \omega_n t_{2k-1}^n}{2^{(n/2)} t_{2k-1}^{(n/p)(n/2)(1-1/k)} \omega_n t_{2k-1}^n}$$
$$= t_{2k-1}^{(n/p)(q-p-(n/2)(1-1/k))} 2^{-n/2}(1 - t_{2k-1}^{n(1-q/p)}) \longrightarrow \infty ,$$

as $k \longrightarrow \infty$. ■

It is easy to ascertain that the inequalities (8), (9), and (11) are consistent. In this way we have shown that there are operators satisfying the hypotheses of Theorem 1, for which the relative remainder $R_\varepsilon(\tau)$ in the asymptotic formula (5) is unbounded as $\tau \longrightarrow \infty$.

3. – Some operators with vanishing remainder in the asymptotic formula.

Throughout this section we consider a differential operator A with Weyl symbol $a(x, \xi)$ satisfying all the hypotheses of Theorem 1, so that the asymptotic formula (5) is true. We employ the following notation: given $\nu \in \boldsymbol{R}_+^n$ [2], $t \in \boldsymbol{R}_+$, and $\xi \in \boldsymbol{R}^n$, we set

$$t^\nu \xi = (t^{\nu_1}\xi_1, \ldots, t^{\nu_n}\xi_n).$$

[2] $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} \colon x > 0\}$.

PROPOSITION 4. – *Let*

$$V(x) = a(x,\, 0)\,,$$

$$M(\tau) = \int_{V(x) \leqslant \tau} dx\,,$$

and

$$a_0(x,\, \xi) = a(x,\, \xi) - a(x,\, 0)\,.$$

Assume there exist

$$\nu,\, \tilde{\nu} \in \boldsymbol{R}^n_+, \qquad C \geqslant 1\,, \qquad 1 > t_0 > 0\,, \qquad \tau_0 > 0\,,$$

and a measurable function

$$f\colon \boldsymbol{R}^n \to \boldsymbol{R}\,,$$

such that

$$a_0(x,\, t^\nu \xi) \geqslant C^{-1} t a_0(x,\, \xi), \qquad \text{for } t \in (0,\, t_0),\, x,\, \xi \in \boldsymbol{R}^n, \tag{12}$$

$$a_0(x, (1+t)^{\tilde{\nu}} \xi) \geqslant (1 + C^{-1} t)\, a_0(x,\, \xi), \qquad \text{for } t \in (0,\, t_0),\, x,\, \xi \in \boldsymbol{R}^n, \tag{13}$$

$$V(x) \geqslant 0, \qquad \text{for } x \in \boldsymbol{R}^n, \tag{14}$$

$$0 \leqslant f(\xi) \leqslant a_0(x,\, \xi), \qquad \text{for } x,\, \xi \in \boldsymbol{R}^n, \tag{15}$$

$$a_0(x,\, \xi) + V(x) \leqslant C(f(\xi) + V(x)), \qquad \text{for } x,\, \xi \in \boldsymbol{R}^n, \tag{16}$$

$$M(2\tau) \leqslant CM(\tau), \qquad \text{for } \tau \geqslant \tau_0. \tag{17}$$

Then

$$R_\varepsilon(\tau) = \mathcal{O}(\tau^{-|\nu|\varepsilon/(|\nu|+1)}), \qquad \text{as } \tau \to \infty\,, \tag{18}$$

and therefore

$$\mathcal{N}(\tau) = W(\tau)\{1 + \mathcal{O}(\tau^{-|\nu|\varepsilon/(|\nu|+1)})\}, \qquad \text{as } \tau \to \infty\,.$$

REMARK. – Condition (17) is the Tauberian condition of Rozenbljum, see [7].

PROOF. – The estimate (18) is a consequence of the following Lemma which is a reworking of a Lemma of Rozenbljum ([7], Lemma 1.1).

LEMMA 2. – *There exist*

$$K > 0\,, \qquad 0 < \theta_0 \leqslant 1, \qquad \tau_1 \geqslant \tau_0,$$

such that

$$W((1+\theta)\,\tau) \leqslant (1 + K\theta^{|\nu|/(|\nu|+1)})\,W(\tau)\,, \tag{19}$$

$$W((1-\theta)\,\tau) \geqslant (1 - K\theta^{|\nu|/(|\nu|+1)})\,W(\tau)\,, \tag{20}$$

for $0 < \theta \leqslant \theta_0$ *and* $\tau \geqslant \tau_1$.

PROOF. – Set

$$W((1+\theta)\,\tau) = W_1 + W_2$$

with

$$W_1 = (2\pi)^{-n} \int\limits_{\substack{a_0 + V \leqslant (1+\theta)\,\tau \\ V \leqslant (1-\theta^s)\,\tau}} dx\,d\xi\,,$$

$$W_2 = (2\pi)^{-n} \int\limits_{\substack{a_0 + V \leqslant (1+\theta)\,\tau \\ (1-\theta^s)\,\tau < V}} dx\,d\xi\,,$$

where

$$s = \frac{1}{|\nu| + 1}\,.$$

Let us estimate W_1. On the domain of integration we have:

$$\theta^s \tau \leqslant \tau - V\,.$$

Hence

$$a_0 \leqslant \tau - V + \theta\tau \leqslant (1 + \theta^{1-s})(\tau - V)\,.$$

Therefore

$$W_1 \leqslant (2\pi)^{-n} \int\limits_{a_0 \leqslant (1+\theta^{1-s})(\tau - V)} dx\,d\xi\,.$$

Let

$$\xi = (1 + C\theta^{1-s})^{\tilde{\nu}}\,\eta\,,$$

where C is the constant which appears in (12) and (13). We have

$$a_0(x,\,\xi) \geqslant (1 + \theta^{1-s})\,a_0(x,\,\eta)$$

and

$$W_1 \leqslant (2\pi)^{-n}(1 + C\theta^{1-s})^{|\tilde{\nu}|} \int_{a_0(x,\,\eta) \leqslant \tau - V(x)} dx\, d\eta$$

$$= (1 + C\theta^{1-s})^{|\tilde{\nu}|}\, W(\tau)\,.$$

Now we estimate W_2. On the domain of integration we have

$$V \leqslant (1+\theta)\,\tau - a_0 \leqslant 2\tau\,,$$

$$a_0 \leqslant (1+\theta)\,\tau - V \leqslant (\theta + \theta^s)\,\tau \leqslant 2\theta^s\tau\,.$$

Therefore

$$W_2 \leqslant (2\pi)^{-n} \int_{\substack{a_0 \leqslant 2\theta^s\tau \\ V \leqslant 2\tau}} dx\, d\xi\,.$$

Let

$$\xi = (4C^2\theta^s)^{\nu}\,\eta\,,$$

then, from (12) we have

$$a_0(x,\,\xi) \geqslant 4C\theta^s\, a_0(x,\,\eta)\,,$$

and

$$W_2 \leqslant (2\pi)^{-n}(4C^2\theta^s)^{|\nu|} \int_{\substack{a_0(x,\,\eta) \leqslant \tau/2C \\ V(x) \leqslant 2\tau}} dx\, d\eta$$

$$\leqslant (2\pi)^{-n} 4^{|\nu|}\, C^{2|\nu|}\, \theta^{s|\nu|} \int_{\substack{f(\eta) \leqslant \tau/2C \\ V(x) \leqslant 2\tau}} dx\, d\eta$$

$$= (2\pi)^{-n} 4^{|\nu|}\, C^{2|\nu|}\, \theta^{s|\nu|} \int_{V(x) \leqslant 2\tau} dx \int_{f(\eta) \leqslant \tau/2C} d\eta$$

$$\leqslant (2\pi)^{-n} 4^{|\nu|}\, C^{2|\nu|}\, \theta^{s|\nu|}\, C^N \int_{V(x) \leqslant \tau/2C} dx \int_{f(\eta) \leqslant \tau/2C} d\eta$$

$$\leqslant (2\pi)^{-n} 4^{|\nu|}\, C^{2|\nu| + N}\, \theta^{s|\nu|} \int_{C(f+V) \leqslant \tau} dx\, d\eta$$

$$\leqslant (2\pi)^{-n} 4^{|\nu|}\, C^{2|\nu| + N}\, \theta^{s|\nu|} \int_{a_0 + V \leqslant \tau} dx\, d\eta$$

$$= 4^{|\nu|}\, C^{2|\nu| + N}\, \theta^{s|\nu|}\, W(\tau), \qquad \text{for } \tau \geqslant 2^N \tau_0,$$

where N is such that

$$2^N \geqslant 2C\,,$$

i.e.

$$M(2\tau) \leqslant CM(\tau) \leqslant \ldots \leqslant C^N M(\tau/2^N) \leqslant C^N M(\tau/2C)\,.$$

Now observe that

$$s|\nu| = \frac{|\nu|}{|\nu|+1} = 1 - s\,,$$

and that, for θ_0 sufficiently small, we have

$$(1+\theta^{1-s})^{|\tilde{\nu}|} \leqslant 1 + (|\tilde{\nu}|+1)\theta^{|\nu|/(|\nu|+1)}, \qquad \text{for } 0<\theta\leqslant\theta_0.$$

This proves the estimate (19) with $K \geqslant K'$, where K' is large enough. In order to prove (20), we observe that we can choose θ_0 so small that

$$\begin{aligned} W(\tau) &= W\left(\left(1+\frac{\theta}{1-\theta}\right)(1-\theta)\,\tau\right) \\ &\leqslant \left[1+K'\left(\frac{\theta}{1-\theta}\right)^{|\nu|/(|\nu|+1)}\right] W((1-\theta)\,\tau)\,, \end{aligned}$$

for $0<\theta\leqslant\theta_0$ and $\tau \geqslant \tau_1 = (1-\theta_0)^{-1}2^N\tau_0$. But this implies

$$\begin{aligned} W((1-\theta)\,\tau) &\geqslant \left(1+K'\left(\frac{\theta}{1-\theta}\right)^{|\nu|/(|\nu|+1)}\right)^{-1} W(\tau) \\ &\geqslant (1-K''\theta^{|\nu|/(|\nu|+1)})\,W(\tau)\,, \end{aligned}$$

for a suitable $K'' \geqslant K'$. Thus (20) is proven with $K \geqslant K''$. ■

Now we can prove estimate (18). It suffices to put

$$\theta = \tau^{-\varepsilon}$$

into (19) and (20). We obtain

$$W(\tau+\tau^{1-\varepsilon}) \leqslant (1+K\tau^{-|\nu|\varepsilon/(|\nu|+1)})\,W(\tau)\,,$$

$$W(\tau-\tau^{1-\varepsilon}) \geqslant (1-K\tau^{-|\nu|\varepsilon/(|\nu|+1)})\,W(\tau)\,,$$

and therefore

$$R_\varepsilon(\tau) \leqslant 2K\tau^{-|\nu|\varepsilon/(|\nu|+1)}, \qquad \text{for } \tau\geqslant\tau_1. \quad ■$$

Now we show two classes of potentials which meet the Tauberian condition (17).

PROPOSITION 5. – *If there exixt* $\theta > 0$ *and* $R \geqslant 0$ *such that*

$$V(\theta^{-1}x) \geqslant (1+\theta)\, V(x), \qquad \forall |x| \geqslant R,$$

then $M(\tau)$ *satisfies* (17).

PROOF. – Let

$$\tau_1 = \max\left\{ \frac{1+\theta}{2} V(x) \colon |x| \leqslant R \right\},$$

then we have

$$\begin{aligned}
M(2\tau) &= \theta^{-n} \int_{V(\theta^{-1}x) \leqslant 2\tau} dx \\
&\leqslant \theta^{-n} \left\{ \int_{|x| \leqslant R} dx + \int_{\substack{V(\theta^{-1}x) \leqslant 2\tau \\ |x| > R}} dx \right\} \\
&\leqslant \theta^{-n} \left\{ \int_{|x| \leqslant R} dx + \int_{\substack{V(x) \leqslant (2/(1+\theta))\,\tau \\ |x| > R}} dx \right\} \\
&= \theta^{-n} \int_{V(x) \leqslant (2/(1+\theta))\,\tau} dx \\
&= \theta^{-n} M\left(\frac{2}{1+\theta} \tau \right), \qquad \text{for } \tau \geqslant \tau_1 .
\end{aligned}$$

Now we can iterate:

$$M(2\tau) \leqslant \theta^{-nN} M\left(\frac{2}{(1+\theta)^N} \tau \right),$$

for

$$\tau \geqslant \tau_N = \max\left\{ \frac{(1+\theta)^N}{2} V(x) \colon |x| \leqslant R \right\}.$$

Then, if we choose N such that $(1+\theta)^N/2 \leqslant 1$, we obtain the result. ■

PROPOSITION 6. – *Assume there exists a real valued polynomial $g(x)$, three constants $L \geqslant 1$, $R \geqslant 0$, $\tau_1 > 0$ and a strictly increasing function $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$, such that*

$$\lim_{\tau \to \infty} \phi(\tau) = \infty ,$$

$$\phi(2\tau) \leqslant L\phi(\tau), \qquad \text{for } \tau \geqslant \tau_1, \tag{21}$$

$$L^{-1} g(x) \leqslant \phi(V(x)) \leqslant L g(x), \qquad \text{for } |x| \geqslant R . \tag{22}$$

Then $M(\tau)$ meets (17).

PROOF. – Recall the following estimate due to Nilsson [6], Theorem 1:

THEOREM 2. – *Given a real valued polynomial $g(x)$ such that $g(x) \to \infty$, as $|x| \to \infty$, there exist $C \geqslant 1$, $\tau' > 0$, $s > 0$ and $k \in \boldsymbol{N}$ such that*

$$C^{-1} \tau^s (\log \tau)^k \leqslant \int_{g(x) \leqslant \tau} dx \leqslant C\tau^s (\log \tau)^k, \qquad \text{for } \tau \geqslant \tau' . \qquad \blacksquare \tag{23}$$

Because $\phi(\tau) \to \infty$ as $\tau \to \infty$, there exists $\tau_0 \geqslant \tau'$, τ_1, such that

$$L^{-1} \phi(\tau) \geqslant \tau'$$

for $\tau \geqslant \tau_0$. Moreover we can take τ_0 so large that

$$V(x) \leqslant \tau \quad \text{and} \quad g(x) \leqslant L^{-1} \phi(\tau),$$

for

$$|x| \leqslant R \quad \text{and} \quad \tau \geqslant \tau_0 .$$

Because ϕ is strictly increasing we have

$$M(2\tau) = \int_{V(x) \leqslant 2\tau} dx = \int_{\phi(V(x)) \leqslant \phi(2\tau)} dx . \tag{24}$$

For $\tau \geqslant \tau_0$, we have

$$\int_{\phi(V) \leqslant \phi(2\tau)} dx = \int_{|x| \leqslant R} dx + \int_{\substack{\phi(V) \leqslant \phi(2\tau) \\ |x| > R}} dx . \tag{25}$$

By (21) and (22) we have

$$\int_{\substack{\phi(V) \leqslant \phi(2\tau) \\ |x| > R}} dx \leqslant \int_{\substack{\phi(V) \leqslant L\phi(\tau) \\ |x| > R}} dx \tag{26}$$

$$\leqslant \int_{\substack{g \leqslant L^2 \phi(\tau) \\ |x| > R}} dx\,, \qquad \text{for } \tau \geqslant \tau_0.$$

By (24), (25), and (26) we have

$$M(2\tau) \leqslant \int_{g \leqslant L^2 \phi(\tau)} dx\,, \qquad \forall \tau \geqslant \tau_0. \tag{27}$$

Now $g(x) \to \infty$ as $|x| \to \infty$, hence by (23) we have

$$\int_{g \leqslant L^2 \phi(\tau)} dx \prec (L^{-1}\phi(\tau))^s (\log (L^{-1}\phi(\tau)))^k$$

$$\prec \int_{g \leqslant L^{-1}\phi(\tau)} dx, \qquad \forall \tau \geqslant \tau_0.$$

Therefore, from (27) we have

$$M(2\tau) \prec \int_{g \leqslant L^{-1}\phi(\tau)} dx$$

$$\leqslant \int_{|x| \leqslant R} dx + \int_{\substack{L^{-1}\phi(V) \leqslant L^{-1}\phi(\tau) \\ |x| > R}} dx$$

$$= M(\tau)\,. \qquad \blacksquare$$

We end the paper with a simple example in $\boldsymbol{R}^2$. Consider

$$A = D_x^6 + D_x^4 D_y^2 + D_y^4 + (1 + x^2 + y^2 + x^2 y^2)^{1/2}.$$

The Weyl symbol is $a = a_0 + V$ with

$$a_0(\xi, \eta) = \xi^6 + \xi^4 \eta^2 + \eta^4,$$

and

$$V(x, y) = (1 + x^2 + y^2 + x^2 y^2)^{1/2}.$$

a_0 is multi-quasi-elliptic, (see [2], page 62), so

$$|\partial_\zeta^\alpha a_0(\zeta)| \prec a_0(\zeta)^{1-(1/8)|\alpha|},$$

moreover

$$\partial_z^\alpha V(z)| \prec V(z),$$

with

$$\zeta = (\xi, \eta), \qquad z = (x, y).$$

It follows that

$$a(z, \zeta) \leqslant (|\zeta| + \Lambda(z))^6,$$

$$|\partial_\zeta^\alpha \partial_z^\beta a(z, \zeta)| \prec a(z, \zeta)(|\zeta| + \Lambda(z))^{-(1/2)|\alpha|},$$

where

$$\Lambda(x, y) = (1 + x^2 + y^2 + x^2 y^2)^{1/12}.$$

Moreover, if we set

$$\nu = \left(\frac{1}{8}, \frac{1}{4}\right), \qquad \tilde{\nu} = \left(\frac{1}{6}, \frac{1}{4}\right),$$

then we have

$$\begin{aligned} a_0(t^\nu \zeta) &= t^{3/4} \xi^6 + t\xi^4 \eta^2 + t\eta^4 \\ &\geqslant t a_0(\zeta), \end{aligned}$$

and

$$\begin{aligned} a_0((1+t)^{\tilde{\nu}} \zeta) &= (1+t)\xi^6 + (1+t)^{7/6} \xi^4 \eta^2 + (1+t)\eta^4 \\ &\geqslant (1+t)\, a_0(\zeta), \end{aligned}$$

for $0 < t < 1$.

Finally, it is obvious to see that V satisfies the hypotheses of Proposition 6 with $g(z) = (1+x^2)(1+y^2)$, $\phi(\tau) = \sqrt{\tau}$, $L = \sqrt{2}$, and $\tau_1 = 1$. Therefore, by Propositions 4 and 6 we have

$$\mathcal{N}(\tau) = W(\tau)\{1 + \mathcal{O}(\tau^{-3\varepsilon/11})\}, \qquad \text{as } \tau \to \infty,$$

where

$$0 < \varepsilon < \frac{1}{18}.$$

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Ernesto Buzano: Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy

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