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On Weighted Inductive Limits of Non-Archimedean Spaces of Continuous Functions.

A. K. KATSARAS - V. BENEKAS

Sunto. – *Si studiano alcune proprietà di un certo limite induttivo di spazi non-archimedei di funzioni continue. In particolare, si esamina la completezza di questo limite induttivo e si indaga il problema di quando lo spazio coincide con il proprio inviluppo proiettivo.*

Introduction.

Weighted spaces of continuous functions were introduced in the complex scalar case by Nachbin in [24] and in the vector case by Prolla in [25]. Several other authors have continued the investigation of such spaces. Papers [1]-[15], [18]-[20], [24], [25] and many others refer to such spaces. Carneiro introduced in [16] the non-Archimedean weighted spaces. Some problems related to p-adic weighted spaces were studied in [21]-[23].

In this paper, for a decreasing sequence $\mathcal{V} = (v_n)$ of strictly positive upper-semicontinuous functions on a topological space X, we study the weighted inductive limit $\mathcal{V}C(X)$ and its projective hull $C\overline{V}(X)$, where $\overline{V}$ is the maximal Nachbin family associated with $\mathcal{V}$. It is shown that $\mathcal{V}C(X)$ is the bornological space associated with $C\overline{V}(X)$ and we examine the question of when these two spaces coincide topologically. If $\mathcal{V}$ is regularly decreasing, we prove that the topologies of $\mathcal{V}C(X)$ and $C\overline{V}(X)$ coincide on bounded sets and that the two spaces have the same compactoid sets. In case $\mathcal{V} \subset |C(X)|$, it is proved that $C\overline{V}(X)$ is bornological iff it is quasibarrelled. We also look at the problem of whether $\mathcal{V}C(X)$ is complete.

1. – Preliminaries.

Throughout this paper, $\boldsymbol{K}$ will stand for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\boldsymbol{K}$, we will mean a non-Archimedean seminorm and a locally convex space over $\boldsymbol{K}$ will be a non-Archimedean locally convex space. In case of a vector space G over the field of real numbers $\boldsymbol{R}$, the notions of seminorm on G and locally convex topology on G are the usual ones. For a subset A of a locally convex space E, we will denote by A^o the polar of A, in the topological dual space E' of E, and by A^{oo} the bipolar of A. The edged hull A^e, of an absolutely convex subset A of a locally convex space over $\boldsymbol{K}$, is defined by $A^e = A$, if the valuation of $\boldsymbol{K}$ is discrete, and

$$A^e = \cap\{\lambda A : \lambda \in \boldsymbol{K},\ |\lambda| > 1\}$$

if the valuation is dense (see [26]). As it is shown in [26], A^{oo} coincides with the edged hull of the weak closure of A (if A is absolutely convex). The definition of the inductive limit $\varinjlim E_n$, of a sequence (E_n) of non-Archimedean locally convex spaces, is analogous to the one in the classical case (see [18]). For all unexplained terms, concerning non-Archimedean spaces, we will refer to [26] or [27].

Let now X be a topological space. The space of all continuous $\boldsymbol{K}$-valued functions on X will be denoted by $C(X)$ while $|C(X)|$ will be the set $\{|f| : f \in C(X)\}$. If v is a non-negative real function on X, then for $f \in \boldsymbol{K}^X$ or $f \in \boldsymbol{R}^X$ we define $q_v(f)$ by

$$q_v(f) = \sup\{v(x)\,|f(x)| : x \in X\}.$$

Recall that a Nachbin family on X is a family V of non-negative upper-semicontinuous (u.s.c.) real functions on X such that: a) V is directed in the sense that if v_1, v_2 are in V and $\alpha > 0$, then there exists $v \in V$ with $v_1, v_2, \alpha v_1 \leq v$ (pointwise on X). b) For each $x \in X$ there exists v in V with $v(x) > 0$. The weighted space $CV(X)$ is the space of all f in $C(X)$ such that $q_v(f) < \infty$ for all v in V. We will consider on $CV(X)$ the locally convex topology τ_V generated by the seminorms q_v, $v \in V$.

2. – The spaces $C\overline{V}(X)$ and $\mathcal{V}C(X)$.

Let $\mathcal{V} = (v_n)$ be a decreasing sequence of strictly positive upper-semicontinuous functions on X. By $\overline{V} = \overline{V}(\mathcal{V})$ we will denote the set of all non-negative u.s.c. functions w on X for which there exists a sequence (α_n) of positive real numbers such that $w \leq \inf_n \alpha_n v_n$. Clearly $\overline{V}$ is a Nachbin family. We will consider on $C\overline{V}(X)$ the Nachbin topology $\tau_{\overline{V}}$. Also, for each n in the set of natural numbers $\boldsymbol{N}$, we will let $Cv_n(X)$ denote the space of all f in $C(X)$ such that $q_{v_n}(f) < \infty$. With the norm q_{v_n}, $Cv_n(X)$ becomes a non-Archimedean normed space. We will denote by $\mathcal{V}C(X)$ the inductive limit $\varinjlim Cv_n(X)$. We will only sketch the proof of the next proposition since it is analogous to the one in the classical case.

PROPOSITION 2.1. – a) $C\overline{V}(X) = \mathcal{V}C(X)$ *algebraically.* b) *A subset* B *of* $C\overline{V}(X)$ *is bounded iff there exists an* n *such that* B *is a bounded subset of* $Cv_n(X)$.

PROOF. – Clearly $\mathcal{V}C(X)$ is continuously injected into $C\overline{V}(X)$. Assume that some bounded subset B of $C\overline{V}(X)$ is not bounded in any $Cv_n(X)$. Inductively, we may choose a sequence (x_n) of distinct elements of X and a sequence (f_n) in B such that $v_n(x_n)\,|f_n(x_n)| > n$, for all n. If

$$\beta_n = \max\{v_j(x_j)/v_n(x_j) : j = 1, 2, \dots, n\},$$

then $\bar{v} = \inf \beta_n v_n$ is in $\overline{V}$ and $\bar{v}(x_n)\,|f_n(x_n)| \geqslant n$, for all n, which is a contradiction. It is clear now that the result follows.

Since $\mathcal{V}C(X)$ is bornological (as an inductive limit of bornological spaces), we have the following

COROLLARY 2.2. – 1) $\mathcal{V}C(X)$ *is the bornological space associated with* $C\overline{V}(X)$.

2) *If* $D_n = \{f \in C(X) : q_{v_n}(f) \leqslant n\}$, *then* (D_n) *is a fundamental sequence of absolutely convex bounded sets for both* $C\overline{V}(X)$ *and* $\mathcal{V}C(X)$.

PROPOSITION 2.3. – *If* $h_1, h_2, \dots, h_n$ *are in* $C(X)$, *then there are* f, g *in* $C(X)$ *such that* $|f(x)| = \max_k |h_k(x)|$ *and* $|g(x)| = \min_k |h_k(x)|$, *for all* x *in* X.

PROOF. – Using induction on n, it suffices to prove our result for $n = 2$. Assume first that there is no x in X with $h_1(x) = h_2(x) = 0$ and let $A = \{x : |h_1(x)| \leqslant |h_2(x)|\}$. Clearly A is closed. Also, A is open. Indeed, let $x_0 \in A$. The sets

$$A_1 = \{x : |h_1(x)| \leqslant |h_1(x_0)|\} \quad \text{and} \quad A_2 = \{x : |h_2(x) - h_2(x_0)| < |h_2(x_0)|\}$$

are open and $x_0 \in A_1 \cap A_2 \subset A$, which shows that A is open. If now $f = h_2$ on A and $f = h_1$ on the complement A^c of A, then f is continuous and $|f| = \max\{|h_1|, |h_2|\}$. Similarly, we may take $g = h_1$ on A and $g = h_2$ on A^c. In the general case, let $Y = \{x \in X : |h_1(x)| + |h_2(x)| \neq 0\}$ and let $g_i = h_i|_Y$, $i = 1, 2$. By the first part of our proof, there are $\tilde{f}$, $\tilde{g}$ in $C(Y)$ with $|\tilde{f}| = \max\{|g_1|, |g_2|\}$, $|\tilde{g}| = \min\{|g_1|, |g_2|\}$. Define f, g on X by taking $f = \tilde{f}$ and $g = \tilde{g}$ on Y while $f = g = 0$ on Y^c. Since Y is an open subset of X, the functions f, g are continuous at each point of Y. Also, they are continuous at each point $x \in Y^c$. Indeed, let (x_δ) be a net in X converging to x. Since h_1, h_2 are continuous, given $\varepsilon > 0$, there exists δ_0 such that $|h_1(x_\delta)|, |h_2(x_\delta)| < \varepsilon$, if $\delta \geqslant \delta_0$, and so $|f(x_\delta)|, |g(x_\delta)| < \varepsilon$ if $\delta \geqslant \delta_0$. This clearly completes the proof.

PROPOSITION 2.4. – *Let* $\{f, h_1, h_2, \ldots, h_n\} \subset C(X)$ *be such that*

$$|f| \leqslant \max\{|h_1|, |h_2|, \ldots, |h_n|\}.$$

Then, there exist $f_1, \ldots, f_n$ *in* $C(X)$ *with* $|f_k| \leqslant |h_k|$ *and* $f = f_1 + \ldots + f_n$.

PROOF. – In view of the preceeding Proposition, there exists g in $C(X)$ with $|g| = \max\{|h_1|, \ldots, |h_{n-1}|\}$. It follows from this that it suffices to prove the result for $n = 2$. Let $Y = \{x \in X : |h_1(x)| + |h_2(x)| \neq 0\}$. Assume first that $Y = X$ and let A be as in the proof of the preceeding Proposition. If f_2 is defined on X by $f_2 = f$ on A and $f_2 = h_2$ on A^c, then f_2 is continuous and $|f_2| \leqslant |h_2|$. Also, if $f_1 = f - f_2$, then $|f_1| \leqslant |h_1|$, which proves the result when $Y = X$. If $Y \neq X$, let $g_i = h_i|_Y$, $i = 1, 2$, $g = f|_Y$. By the first case, there are $w_i \in C(Y)$, $|w_i| \leqslant |g_i|$, $g = w_1 + w_2$. Extend w_i to a function f_i on all of X by taking $f_i = 0$ on Y^c. Then f_i is continuous on X, $|f_i| \leqslant |h_i|$, $f = f_1 + f_2$.

PROPOSITION 2.5. – *Assume that* X *is a zero-dimensional locally compact σ-compact topological space and that* $\mathcal{V} \subset |C(X)|$. *Then, for each* $\bar{v} \in \bar{V}$ *there exists* $\bar{w} \in \bar{V} \cap |C(X)|$ *strictly positive with* $\bar{v} \leqslant \bar{w}$.

PROOF. – Let (α_n) be a sequence of positive numbers such that $\bar{v} \leqslant \inf_n \alpha_n v_n$. Assume first that X is compact. Since $\bar{v}$ is u.s.c., there exists a non-zero element μ of $\boldsymbol{K}$ with $|\mu| \geqslant \sup_{x \in X} \bar{v}(x)$. Let $g \in C(X)$ with $g(x) = \mu$ for all x in X. Then $|g| \in \bar{V}$ since $\inf_{x \in X} v_n(x) > 0$ for all n, which proves the result in this case. Consider next the case when X is not compact. Our hypothesis on X implies that there exists an infinite sequence (Y_n) of clopen compact subsets of X covering X and such that each Y_n is a proper subset of Y_{n+1}. For each n, let $h_n \in C(X)$ with $|h_n| = v_n$. Choose inductively a sequence (μ_n) in $\boldsymbol{K}$ with $|\mu_n| \geqslant \alpha_n$ and

$$|\mu_n| \inf\{v_n(x) : x \in Y_n\} \geqslant \sup\{|\mu_j| v_j(x) : x \in Y_{n-1},\ j = 1, \ldots, n-1\}$$

for $n \geqslant 2$. In view of Proposition 2.3, there exists g_n in $C(X)$ with $|g_n| = \min_{1 \leqslant k \leqslant n} |\mu_k h_k|$. Let ϕ_n be the $\boldsymbol{K}$-characteristic function of $Y_n \backslash Y_{n-1}$, where $Y_0 = \emptyset$, and let $g = \sum_n \phi_n g_n$. Clearly g is continuous. Also, $|g| \leqslant |\mu_n| v_n$ for all n. Indeed, let $x \in X$, $x \in Y_m \backslash Y_{m-1}$. Then $g(x) = g_m(x)$. If $m \geqslant n$, then $|g_m(x)| \leqslant |\mu_n| v_n(x)$, while for $m < n$ we have $|g_m(x)| \leqslant |\mu_m h_m(x)| \leqslant |\mu_n| v_n(x)$, which proves that $|g| \leqslant |\mu_n| v_n$. Finally, $|g| \geqslant \bar{v}$. In fact, let $x \in Y_m \backslash Y_{m-1}$. There exists $1 \leqslant k \leqslant m$ such that $|g(x)| = |g_m(x)| = |\mu_k| v_k(x) \geqslant \alpha_k v_k(x) \geqslant \bar{v}(x)$. This clearly completes the proof.

LEMMA 2.6. – *If v is a strictly positive u.s.c. function on X, then for each $\lambda \in \boldsymbol{K}$, with $|\lambda|>1$, there exists $\phi \in \boldsymbol{K}^X$ such that $|\phi|$ is u.s.c. and $|\phi| \leqslant v < |\lambda\phi|$.*

PROOF. – For each integer m, let $D_m = \{x : v(x) \geqslant |\lambda|^m\}$. Then D_m is closed and $\cup\{D_m : m \text{ integer}\} = X$. Define $\phi : X \to \boldsymbol{K}$ by taking $\phi(x) = \lambda^m$ if $x \in D_m \setminus D_{m+1}$. Then $|\phi|$ is u.s.c. Indeed, for ε a real number, set $B_\varepsilon = \{x : |\phi(x)| \geqslant \varepsilon\}$. If $\varepsilon \leqslant 0$, then $B_\varepsilon = X$. Assume that $\varepsilon > 0$ and let m be such that $|\lambda|^m < \varepsilon \leqslant |\lambda|^{m+1}$. Then $B_\varepsilon = D_{m+1}$. Thus B_ε is closed, for all ε, and so $|\phi|$ is u.s.c. Also, $|\phi| \leqslant v < |\lambda\phi|$. Indeed, if $x \in D_m \setminus D_{m+1}$, then $|\phi(x)| = |\lambda|^m \leqslant v(x) < |\lambda|^{m+1} = |\lambda\phi(x)|$.

PROPOSITION 2.7. – *Assume that the valuation of $\boldsymbol{K}$ is dense and let $\mathcal{V} = (v_n)$ be a decreasing sequence of strictly positive functions on X. If λ is any element of $\boldsymbol{K}$ with $|\lambda| > 1$, then there exists a decreasing sequence $\mathcal{W} = (w_n)$ of strictly positive u.s.c. functions on X such that:*

(a) *$w_n(X) \subset |\boldsymbol{K}|$, for all $n \in \boldsymbol{N}$.*

(b) *The maximal Nachbin families $\overline{V}$, $\overline{W}$, which correspond to $\mathcal{V}$ and $\mathcal{W}$ respectively, coincide.*

(c) *$w_n \leqslant v_n < |\lambda| w_n$, for all n.*

(d) *$Cv_n(X) = Cw_n(X)$ topologically.*

PROOF. – Choose inductively a sequence (λ_n) in $\boldsymbol{K}$ with $|\lambda_n| > 1$ and $|\lambda_1 \lambda_2 \ldots \lambda_n| < |\lambda|$, for all n. By the preceeding Lemma, there exists a sequence (ϕ_n) in $\boldsymbol{K}^X$ such that $|\phi_n|$ is u.s.c. and $|\phi_n| \leqslant v_n < |\lambda_n \phi_n|$ for all n. Let $w_1 = |\phi_1|$ and $w_{n+1} = |\lambda_1 \lambda_2 \ldots \lambda_n|^{-1} |\phi_{n+1}|$ for all $n \in \boldsymbol{N}$. Since

$$w_{n+1} \leqslant |\lambda_1 \lambda_2 \ldots \lambda_n|^{-1} v_{n+1} \leqslant |\lambda_1 \lambda_2 \ldots \lambda_n|^{-1} v_n \leqslant |\lambda_1 \lambda_2 \ldots \lambda_{n-1}|^{-1} |\phi_n| = w_n,$$

the sequence (w_n) is decreasing and clearly $w_n(X) \subset |\boldsymbol{K}|$. Also, $w_n \leqslant v_n \leqslant |\lambda_1 \lambda_2 \ldots \lambda_n| w_n < |\lambda| w_n$.

This proves (a) and (c), while (b) and (d) follow easily from (c).

PROPOSITION 2.8. – *Assume that the valuation of $\boldsymbol{K}$ is discrete and let $\mathcal{V} = (v_n)$ be a decreasing sequence of strictly positive u.s.c. functions on X. If $\varrho > 1$ is the generator of the value group of $\boldsymbol{K}$, then there exists a decreasing sequence $\mathcal{W} = (w_n)$ of strictly positive functions on X such that:*

(a) *$w_n(X) \subset |\boldsymbol{K}|$ for each n.*

(b) *$w_n \leqslant v_n < \varrho w_n$.*

(c) *The maximal Nachbin families* $\overline{V}$, $\overline{W}$ *which correspond to* $\mathcal{V}$ *and* $\mathcal{W}$, *respectively, coincide.*

(d) $Cv_n(X) = Cw_n(X)$ *topologically.*

PROOF. – There exist $\lambda \in \boldsymbol{K}$ with $|\lambda| = \varrho$. By Lemma 2.6, for each $n \in \boldsymbol{N}$, there exist $\phi_n \in \boldsymbol{K}^X$ such that $|\phi_n|$ is u.s.c. and $|\phi_n| \leqslant v_n < \varrho|\phi_n|$. Take $w_n = |\phi_n|$. The sequence $\mathcal{W} = (w_n)$ is decreasing since $|\phi_{n+1}(x)| \leqslant v_{n+1}(x) \leqslant v_n(x) < \varrho|\phi_n(x)|$ and so $|\phi_{n+1}(x)| \leqslant |\phi_n(x)|$. This proves (a) and (b). It is easy to see that (c) and (d) follow easily from the fact that $w_n \leqslant v_n < \varrho w_n$ for all n.

3. – Completeness of $\mathcal{V}C(X)$.

Let $\mathcal{V} = (v_n)$ and $\overline{V}$ be as in the preceeding section. For each n, let

$$l_\infty(v_n, \boldsymbol{K}) = \{f \in \boldsymbol{K}^X \colon q_{v_n}(f) < \infty\}$$

and

$$l_\infty(v_n, \boldsymbol{R}) = \{u \in \boldsymbol{R}^X \colon q_{v_n}(u) < \infty\}.$$

If we consider on $l_\infty(v_n, \boldsymbol{R})$ the norm q_{v_n}, then it becomes a Banach space. Similarly, $l_\infty(v_n, \boldsymbol{K})$ with the non-Archimedean norm q_{v_n} is a non-Archimedean Banach space. We will denote by $k_\infty(\mathcal{V}, \boldsymbol{R})$ the inductive limit $\varinjlim l_\infty(v_n, \boldsymbol{R})$. Similarly, we define

$$k_\infty(\mathcal{V}, \boldsymbol{K}) = \varinjlim l_\infty(v_n, \boldsymbol{K}).$$

Let

$$K_\infty(\overline{V}, \boldsymbol{K}) = \{f \in \boldsymbol{K}^X \colon q_{\bar{v}}(f) < \infty \text{ for all } \bar{v} \in \overline{V}\}$$

and

$$K_\infty(\overline{V}, \boldsymbol{R}) = \{u \in \boldsymbol{R}^X \colon q_{\bar{v}}(u) < \infty \text{ for all } \bar{v} \in \overline{V}\}.$$

On each of the spaces $K_\infty(\overline{V}, \boldsymbol{K})$ and $K_\infty(\overline{V}, \boldsymbol{R})$ we consider the locally convex topology generated by the seminorms $q_{\bar{v}}$, $\bar{v} \in \overline{V}$. It is well known that $K_\infty(\overline{V}, \boldsymbol{R}) = k_\infty(\mathcal{V}, \boldsymbol{R})$ algebraically. Also, in view of Proposition 2.1, $K_\infty(\overline{V}, \boldsymbol{K}) = k_\infty(\mathcal{V}, \boldsymbol{K})$ algebraically and they have the same bounded sets. Also, the topology $\tau_{\overline{V}}$ of $K_\infty(\overline{V}, \boldsymbol{K})$ is coarser than the inductive topology of $k_\infty(\mathcal{V}, \boldsymbol{K})$.

PROPOSITION 3.1. – *Let* (f_a) *be a net in* $k_\infty(\mathcal{V}, \boldsymbol{K})$. *Then:* 1) $f_a \to 0$ *in* $k_\infty(\mathcal{V}, \boldsymbol{K})$ *iff* $|f_a| \to 0$ *in* $k_\infty(\mathcal{V}, \boldsymbol{R})$. 2) *If* (f_a) *is a Cauchy net in* $k_\infty(\mathcal{V}, \boldsymbol{K})$, *then* $(|f_a|)$ *is a Cauchy net in* $k_\infty(\mathcal{V}, \boldsymbol{R})$.

PROOF. – 1) Assume that $f_\alpha \to 0$ and let W be an absolutely convex neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{R})$. For each n, there exists a positive number ε_n such that

$$W_n = \{u \in \boldsymbol{R}^X : q_{v_n}(u) \leqslant \varepsilon_n\} \subset W .$$

Set

$$D_k = \{f \in \boldsymbol{K}^X : q_{v_k}(f) \leqslant \varepsilon_k/2^k\}$$

and

$$D = \bigcup_n \sum_{k=1}^{n} D_k .$$

Since D is a convex neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{K})$, there exists α_0 such that $f_\alpha \in D$ if $\alpha \geqslant \alpha_0$. Let $f = f_1 + \ldots + f_n$ with $f_k \in D_k$, $k = 1, \ldots, n$. Then $|f| \leqslant \max\{|f_1|, \ldots, |f_n|\}$. There are $u_1, \ldots, u_n$ in $\boldsymbol{R}^X$, with $0 \leqslant u_i \leqslant |f_i|$, such that $|f| = u_1 + \ldots + u_n$. Since $2^k u_k \in W_k$, it follows that $|f| = \sum_{k=1}^{n} 2^{-k}(2^k u_k) \in W$ since W is absolutely convex. Thus, $|f_\alpha| \in W$, for $\alpha \geqslant \alpha_0$, which proves that $|f_\alpha| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$. Conversely, assume that $|f_\alpha| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$ and let W_0 be a convex neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{K})$. Let $d_n > 0$ be such that

$$Z_n = \{f \in l_\infty(v_n, \boldsymbol{K}) : q_{v_n}(f) \leqslant d_n\} \subset W_0 .$$

The set $W = \bigcup_n \sum_{k=1}^{n} Z_k$ (which is contained in W_0) is a convex neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{K})$.

Claim I: The set W is solid, i.e. if $f \in W$ and if $g \in \boldsymbol{K}^X$ with $|g| \leqslant |f|$, then $g \in W$. Indeed, $f = \sum_{k=1}^{n} f_k$ with $f_k \in Z_k$. In view of Proposition 2.4, there are $g_k \in \boldsymbol{K}^X$, $|g_k| \leqslant |f_k|$, $g = g_1 + \ldots + g_n$, and so $g \in W$ since $g_k \in Z_k$.

Claim II: The set

$$D = \{u \in \boldsymbol{R}^X : \exists f \in W \text{ with } |u| \leqslant |f|\}$$

is a neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{R})$. Indeed, let $\lambda \in \boldsymbol{K}$, with $|\lambda| > 1$, and let $u \in \boldsymbol{R}^X$ with $q_{v_n}(u) \leqslant |\lambda|^{-1} d_n$. For each $x \in X$, there exists $\mu_x \in \boldsymbol{K}$ with $|\mu_x| \leqslant u(x) \leqslant |\lambda \mu_x|$. If $f \in \boldsymbol{K}^X$, $f(x) = \lambda \mu_x$, then $f \in Z_n$ and so $u \in D$ since $|u| \leqslant |f|$. Since we can prove that, for any f, g in W, there exists h in W with $|h| = \max\{|f|, |g|\}$, it follows easily that is absolutely convex and so D is a neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{R})$.

Since now $|f_\alpha| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$, there exists α_0 such that $|f_\alpha| \in D$ if $\alpha \geqslant \alpha_0$. For each such α, there exists $f \in W$ with $|f_\alpha| \leqslant |f|$ and so $f_\alpha \in W$ by claim I. This completes the proof of 1).

2) Let $(f_\alpha)_{\alpha \in A}$ be a Cauchy net in $k_\infty(\mathcal{V}, \boldsymbol{K})$ and order $\Delta = A \times A$ by $(\alpha_1, \beta_1) \geqslant (\alpha, \beta)$ iff $\alpha_1 \geqslant \alpha$ and $\beta_1 \geqslant \beta$. For $\delta = (\alpha, \beta) \in \Delta$, set $g_\delta = f_\alpha - f_\beta$.

Then $g_\delta \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{K})$ and so $|g_\delta| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$. Since $||f_\alpha| - |f_\beta|| \leqslant |g_\delta|$ and since $k_\infty(\mathcal{V}, \boldsymbol{R})$ has a base at zero consisting of solid sets, it follows that the net $(|f_\alpha|)$ is Cauchy in $k_\infty(\mathcal{V}, \boldsymbol{R})$.

THEOREM 3.2. – *The space $k_\infty(\mathcal{V}, \boldsymbol{K})$ is complete.*

PROOF. – Since for each $x \in X$ there exists $\bar{v}$ in $\bar{V}$ with $\bar{v}(x) > 0$, it follows easily that $K_\infty(\bar{V}, \boldsymbol{K})$ is complete. Let now (f_α) be a Cauchy net in $k_\infty(\mathcal{V}, \boldsymbol{K})$. Then (f_α) is $\tau_{\bar{V}}$-Cauchy and hence (f_α) is $\tau_{\bar{V}}$-convergent to some $f \in \boldsymbol{K}^X$ since $K_\infty(\bar{V}, \boldsymbol{K})$ is complete. Let $g_\alpha = f_\alpha - f$. Then (g_α) is a Cauchy net in $k_\infty(\mathcal{V}, \boldsymbol{K})$ and so $(|g_\alpha|)$ is a Cauchy net in $k_\infty(\mathcal{V}, \boldsymbol{R})$ by the preceeding Proposition. But $k_\infty(\mathcal{V}, \boldsymbol{R})$ is complete (see [10]). Thus, there exists $u \in \boldsymbol{R}^X$ such that $|g_\alpha| \to u$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$. Since the topology of $k_\infty(\mathcal{V}, \boldsymbol{R})$ is finer than the topology of simple convergence, we have that $|g_\alpha(x)| \to u(x)$ for all $x \in X$. Also, since for each $x \in X$ there exists $\bar{v} \in \bar{V}$ with $\bar{v}(x) \geqslant 1$, the topology $\tau_{\bar{V}}$ of $K_\infty(\mathcal{V}, \boldsymbol{K})$ is finer than the topology of pointwise convergence. Since (g_α) is $\tau_{\bar{V}}$-convergent to zero, we have that $g_\alpha(x) \to 0$ for each x. It follows that $u = 0$. Thus $|g_\alpha| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$ and so $g_\alpha \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{K})$ by Proposition 3.1. This completes the proof.

PROPOSITION 3.3. – *If $k_\infty(\mathcal{V}, \boldsymbol{R}) = K_\infty(\bar{V}, \boldsymbol{R})$ topologically, then $k_\infty(\mathcal{V}, \boldsymbol{K}) = K_\infty(\bar{V}, \boldsymbol{K})$ topologically.*

PROOF. – Let (f_α) be a net in $K_\infty(\bar{V}, \boldsymbol{K})$ which converges to zero. Then $|f_\alpha| \to 0$ in $K_\infty(\bar{V}, \boldsymbol{R})$. By our hypothesis, $|f_\alpha| \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{R})$ and so $f_\alpha \to 0$ in $k_\infty(\mathcal{V}, \boldsymbol{K})$ in view of Proposition 3.1. Since the topology of $K_\infty(\bar{V}, \boldsymbol{K})$ is always coarser than the one of $k_\infty(\mathcal{V}, \boldsymbol{K})$, the result follows.

PROPOSITION 3.4. – *If each v_n is continuous, then $\mathcal{V}C(X)$ is a closed subset of $k_\infty(\mathcal{V}, \boldsymbol{K})$.*

PROOF. – Let f be in the closure of $\mathcal{V}C(X)$ in $k_\infty(\mathcal{V}, \boldsymbol{K})$ and let $x_0 \in X$. We will prove that f is continuous at x_0. Let $\varepsilon > 0$ be given and set

$$D_n = \{g \in \boldsymbol{K}^X : q_{v_n}(g) \leqslant \varepsilon v_n(x_0)/2\}$$

$$D = \bigcup_m \sum_{k=1}^m D_k .$$

Since D is a neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{K})$, there exists $g \in \mathcal{V}C(X)$ with $f - g \in D$ and so $f - g \in \sum_{k=1}^m D_k$ for some m. Let

$$Z = \left[\bigcap_{k=1}^m \{x \in X : v_k(x) > v_k(x_0)/2\}\right] \cap \{x : |g(x) - g(x_0)| < \varepsilon\} .$$

Then, Z is a neighborhood of x_0 in X. Let $h_k \in D_k$, $f - g = h_1 + \ldots + h_m$. Since $|h_k| \leq \varepsilon v_k(x_0)/(2v_k)$, we have that $|h_k(x)| < \varepsilon$ for each $x \in Z$ and so $|f(x) - g(x)| < \varepsilon$ for $x \in Z$. Thus

$$|f(x) - f(x_0)| \leq \max\{|f(x) - g(x)|, \ |g(x) - g(x_0)|, \ |g(x_0) - f(x_0)|\} < \varepsilon$$

at every point of Z. This clearly completes the proof.

Recall that a topological space X is said to be ultranormal if any two disjoint closed subsets of X can be separated by disjoint clopen sets.

LEMMA 3.5. – *Let X be ultranormal and let $\{A_1, \ldots, A_n\}$ be a finite open cover of X. Then, there are pointwise disjoint clopen subsets $D_1, \ldots, D_n$ of X with $D_k \subset A_k$ and $\bigcup_{k=1}^{n} D_k = X$.*

PROOF. – We will use induction on n. For $n = 2$, the result follows directly from the definition of ultranormality. Assume that the result holds for $n = m$ and let $n = m + 1$. There are disjoint clopen sets D_{m+1}, D with $D_{m+1} \subset A_{m+1}$, $D \subset \bigcup_1^m A_k$, $D_{m+1} \cup D = X$. Set

$$B_k = (D \cap A_k) \cup D_{m+1}, \quad k = 1, 2, \ldots, m .$$

By our induction hypothesis, there are pairwise disjoint clopen sets $C_1, \ldots, C_m$ with $C_k \subset B_k$ and $\bigcup_1^m C_k = X$. Now it suffices to take $D_k = C_k \cap D_{m+1}^c$ for $k = 1, \ldots, m$.

PROPOSITION 3.6. – *Assume that one of the following two conditions holds*

(a) *For each n, there exists $h_n \in C(X)$ with $v_n = |h_n|$.*

(b) *X is ultranormal.*

Then: 1) *The family of all subsets of $\mathcal{V}C(X)$ of the form*

$$W_\alpha = \bigcup_m \left\{ f \in C(X) : \sup_x \min_{1 \leq k \leq m} \alpha_k v_k(x) |f(x)| \leq 1 \right\},$$

where α runs through the family of all sequences $\alpha = (\alpha_n)$ of positive numbers, is a base at zero in $\mathcal{V}C(X)$.

2) *$\mathcal{V}C(X)$ is a topological subspace of $k_\infty(\mathcal{V}, \boldsymbol{K})$.*

PROOF. – 1) We first observe that W_α is a neighborhood of zero in $\mathcal{V}C(X)$. On the other hand, let W be a convex neighborhood of zero in $\mathcal{V}C(X)$. For each n, there exists a non-zero μ_n in $\boldsymbol{K}$ such that

$$W_n = \{ f \in C(X) : q_{v_n}(f) \leq |\mu_n|^{-1} \} \subset W .$$

Let $|\lambda| > 1$, $\alpha_n = |\lambda\mu_n|$, $\alpha = (\alpha_n)$ and $f \in W_\alpha$. There exists m such that

$$\sup_x \min_{1 \leqslant k \leqslant m} \alpha_k v_k(x) |f(x)| \leqslant 1 .$$

If $x \in X$, then there exists k, $1 \leqslant k \leqslant m$, with $\alpha_k v_k(x) |f(x)| \leqslant 1$ and so $|\mu_k| v_k |f(x)| \leqslant |\lambda|^{-1} < 1$. Each of the sets

$$A_k = \{x : |\mu_k| v_k(x) |f(x)| < 1\}$$

is open and $\bigcup_1^m A_k = X$. In case (a) each A_k is clopen while in case (b) there are pairwise disjoint clopen sets $B_1, \ldots, B_m$, $B_k \subset A_k$, $\bigcup_1^m B_k = X$. In both cases, there are pairwise disjoint clopen sets $D_1, \ldots, D_m$ covering X with $D_k \subset A_k$. Let $f_k = \phi_k f$, where ϕ_k is the $\boldsymbol{K}$-characteristic function of D_k. Then $f_k \in W_k$ and so $f = f_1 + \ldots + f_m \in W$, which proves that $W_\alpha \subset W$.

2) It follows easily from 1).

Combining Theorem 3.2 with Propositions 3.4 and 3.6, we get the following

THEOREM 3.7. – *Assume that one of the following two conditions holds*:

(a) *For each* n, *there exists* $h_n \in C(X)$ *with* $|h_n| = v_n$.

(b) X *is ultranormal and each* v_n *is continuous.*

Then: 1) $\mathcal{V}C(X)$ *is a closed topological subspace of* $k_\infty(\mathcal{V}, \boldsymbol{K})$.

2) $\mathcal{V}C(X)$ *is complete.*

4. – Bornological $C\overline{V}(X)$ spaces.

Let $\mathcal{V} = (v_n)$ and $\overline{V}$ be as in section 2. By Corollary 2.2, $\mathcal{V}C(X)$ is the bornological space associated with $C\overline{V}(X)$. In this section, we will look at the question of when $C\overline{V}(X)$ is bornological, i.e. when $\mathcal{V}C(X) = C\overline{V}(X)$ topologically. We recall the following

DEFINITION ([11]). – The sequence $\mathcal{V}$ satisfies condition (D) if there exists an increasing sequence $J = (X_n)$ of non-empty subsets of X such that:

(NJ) For each $n \in \boldsymbol{N}$, there exists $m \geqslant n$ in $\boldsymbol{N}$ such that $\inf_{x \in X_n} v_k(x)/v_m(x) > 0$ for all $k > m$.

(MJ) For each $n' \in \boldsymbol{N}$ and each subset Y of X which is not contained in any X_m, there exists $m' = m'(n', Y)$ such that $\inf_{x \in Y} v_{m'}(x)/v_{n'}(x) = 0$.

PROPOSITION 4.1 (*see* [1]). – *The following are equivalent*:

1) $\mathcal{V}$ *satisfies condition* (D).

2) *For each sequence* (λ_n) *of non-zero elements of* $\boldsymbol{K}$, *there exists* $\overline{v}$ *in* $\overline{V}$ *such that, for each non-zero* μ *in* $\boldsymbol{K}$ *and each* $m \in \boldsymbol{N}$, *there exists* n *such that*

$$(\,*\,) \qquad \min\{|\mu|/v_m,\ 1/\overline{v}\} \leq \max_{1 \leq k \leq n} |\lambda_k|/v_k .$$

PROOF. – In view of Proposition 2.7, we may assume that, for each n, $v_n = |\phi_n|$ for some $\phi_n \in \boldsymbol{K}^X$. Assume that $\mathcal{V}$ satisfies condition (D), then $K_\infty(\overline{V}, \boldsymbol{R}) = k_\infty(\mathcal{V}, \boldsymbol{R})$ topologically, by the main Theorem in [1], and so $\boldsymbol{K}_\infty(\overline{V}, \boldsymbol{K}) = k_\infty(\mathcal{V}, \boldsymbol{K})$ topologically by Proposition 3.3. Let now (λ_n) be a sequence of non-zero elements of $\boldsymbol{K}$. Set

$$D_n = \{f \in l_\infty(v_n, \boldsymbol{K}) : q_{v_n}(f) \leq |\lambda_n|\}, \qquad W_n = \sum_{k=1}^{n} D_k .$$

The set $W = \bigcup_n W_n$ is a convex neighborhood of zero in $k_\infty(\mathcal{V}, \boldsymbol{K})$ and thus it is also a neighborhood of zero in $K_\infty(\overline{V}, \boldsymbol{K})$. Let $\overline{v} \in \overline{V}$ be such that

$$\{f \in K_\infty(\overline{V}, \boldsymbol{K}) : q_{\overline{v}}(f) \leq 1\} \subset W .$$

Given $\lambda \in \boldsymbol{K}$ with $|\lambda| > 1$, there exists $\phi \in \boldsymbol{K}^X$ such that $|\phi|$ is u.s.c. and $|\phi| \leq \overline{v} \leq |\lambda\phi|$ (by Lemma 2.6). Taking $|\lambda\phi|$ in place of $\overline{v}$, we may assume that $\overline{v} = |\lambda\phi|$. Let $A = \{x : \overline{v}(x) \leq v_m(x)/|\mu|\}$ and take $h \in \boldsymbol{K}^X$, $h(x) = \mu/\phi_m(x)$ if $x \in A$ and $h(x) = [\lambda\phi(x)]^{-1}$ if $x \notin A$. Then $|h| = \min\{|\mu|/v_m, 1/\overline{v}\}$. (Note that, if $\overline{v}(x) = 0$, we take $1/\overline{v}(x) = \infty$). Now $q_{\overline{v}}(h) \leq 1$ and so $h \in W_n$ for some n. Let $f_k \in D_k$ be such that $h = \sum_{k=1}^{n} f_k$. Then $|h| \leq \max_{1 \leq k \leq n} |\lambda_k|/v_k$ since $|f_k| \leq |\lambda_k|/v_k$. Conversely, assume that (2) is satisfied. Since $\max_{1 \leq k \leq n} |\lambda_k|/v_k \leq \sum_{k=1}^{n} |\lambda_k|/v_k$, it follows that $\mathcal{V}$ satisfies condition (D) by the main Theorem in [1].

THEOREM 4.2. – *Assume that* $\mathcal{V} \subset |C(X)|$. *Then*:

(a) *If* $\mathcal{V}$ *satisfies condition* (D), *then* $C\overline{V}(X)$ *is bornological.*

(b) *If, for each* $\overline{v} \in \overline{V}$, *there exists* $\tilde{v} \in \overline{V} \cap |C(X)|$ *with* $\tilde{v} \geq \overline{v}$, *then* $C\overline{V}(X)$ *is bornological iff* $\mathcal{V}$ *satisfies condition* (D). *In particular, if* X *is a zero-dimensional locally compact* σ*-compact space, then* $C\overline{V}(X)$ *is bornological iff condition* (D) *is satisfied.*

PROOF. – (a) Assume that $\mathcal{V}$ satisfies condition (D) and let W be a convex neighborhood of zero in $\mathcal{V}C(X)$. For each n, there exists a non-zero element

λ_n of $\boldsymbol{K}$ such that $\lambda_n A_n \subset W$, where

$$A_n = \{ f \in Cv_n(X) : q_{v_n}(f) \leqslant 1 \} .$$

Set $A = \bigcup_n \sum_{k=1}^{m} \lambda_k A_k$. By the preceeding Proposition, there exists $\overline{v} \in \overline{V}$ such that, for each non-zero μ in $\boldsymbol{K}$ and each m, there exists n such that ($*$) holds. We claim that

$$\{ f \in C\overline{V}(X) : q_{\overline{v}}(f) \leqslant 1 \} \subset W .$$

Indeed, let $q_{\overline{v}}(f) \leqslant 1$. Since A is absorbing, there exist a positive integer m and $\mu \neq 0$ such that $f \in \mu A_m$. Let n be such that ($*$) holds. Then

$$|f| \leqslant \min \{ |\mu|/v_m , \ 1/\overline{v} \} \leqslant \max_{1 \leqslant k \leqslant n} |\lambda_k|/v_k .$$

Since $\nabla \subset |C(X)|$, there are (by Proposition 2.4), $f_k \in A_k$, $k = 1, \ldots, n$, such that $f = \lambda_1 f_1 + \ldots + \lambda_n f_n$ and so $f \in A \subset W$. This clearly completes the proof of (a).

(b) Assume that each $\overline{v} \in \overline{V}$ is dominated by some element of $\overline{V} \cap |C(X)|$ and that $C\overline{V}(X)$ is bornological. Let (λ_n) be a sequence of non-zero elements of $\boldsymbol{K}$ and set

$$Z = \bigcup_n \sum_{k=1}^{n} \lambda_k A_k ,$$

where A_k is as above. Since A is a convex neighborhood of zero in $\nabla C(X)$, our hypothesis implies that there exists $\overline{v} \in \overline{V} \cap |C(X)|$ such that

$$H = \{ f \in C\overline{V}(X) : q_{\overline{v}}(f) \leqslant 1 \} \subset Z .$$

Let now $\mu \neq 0$ and let m be a positive integer. By Proposition 2.3, there exists $f \in C(X)$ such that $|f| = \min \{ |\mu|/v_m , \ 1/\overline{v} \}$. Then, $f \in H$ and so $f \in Z$. Let n be such that $f = \sum_{k=1}^{n} \lambda_k f_k$ with $f_k \in A_k$. Since $|\lambda_k f_k| \leqslant |\lambda_k|/v_k$, we have that

$$|f| \leqslant \max_{1 \leqslant k \leqslant n} |\lambda_k|/v_k .$$

Now the result follows from the preceeding Proposition.

PROPOSITION 4.3. – *Let V be a Nachbin family on the zero-dimensional topological space X and assume that, for each $x \in X$, there exists $f \in CV(X)$ with $f(x) \neq 0$. If u, v are non-negative u.s.c. functions on X such that*

$$\{ f \in CV(X) : q_v(f) \leqslant 1 \} \subset \{ f \in CV(X) : q_u(f) \leqslant 1 \} ,$$

then, for each $\lambda \in \boldsymbol{K}$ with $|\lambda| > 1$, we have $u \leqslant |\lambda| v$.

PROOF. – Assume that $u(x_0) > |\lambda| v(x_0)$. Choose $\mu \in \boldsymbol{K}$ with $|\mu| < u(x_0) \leq |\lambda\mu|$ and so $v(x_0) < |\mu| < u(x_0)$. Since v is u.s.c. and X zero-dimensional, there exists a clopen neighborhood Z of x_0 in X such that $v(x) < |\mu|$ if $x \in Z$. By our hypothesis, there exists $f \in CV(X)$ with $f(x_0) \neq 0$. The set

$$A = \{x : |f(x)| \geq |f(x_0)|\}$$

is clopen. Define g on X by $g(x) = \mu^{-1}$ if $x \in Z \cap A$, $g(x) = [\mu f(x_0)]^{-1} f$ if $x \in Z \cap A^c$ and $g(x) = 0$ if $x \notin Z$. Then g is continuous and $g \in CV(X)$ since $|g| \leq |\mu f(x_0)|^{-1} |f|$. Moreover, $q_v(g) \leq 1$, which is a contradiction since $u(x_0)|g(x_0)| > 1$.

LEMMA 4.4. – *If X is zero-dimensional, then for each x in X there exists f in $C\overline{V}(X)$ with $f(x) \neq 0$.*

PROOF. – For $x_0 \in X$, the set

$$D = \{x \in X : v_1(x) < 2 v_1(x_0)\}$$

is open. Let Z be a clopen neighborhood of x_0 contained in D. Now it sufficies to take as f the $\boldsymbol{K}$-characteristic function of Z in X.

PROPOSITION 4.5. – *If X is zero-dimensional, then $\mathcal{V}$ satisfies condition (D) iff bounded subsets of $C\overline{V}(X)$ are metrizable.*

PROOF. – By [2, Corollary I.2.6], condition (D) is equivalent to the following condition:

(1) There exists an increasing sequence $(\overline{v}_n)$ in $\overline{V}$ such that, for every $n \in \boldsymbol{N}$ and every $\overline{v} \in \overline{V}$, there exists m such that $\overline{v} \leq \sup\{\overline{v}_m, v_n\}$.

Assume now that (D) is satisfied and let $(\overline{v}_m)$ be as in (1). If B is a bounded absolutely convex subset of $C\overline{V}(X)$, then there exists n such that

$$B \subset D = \{f \in C(X) : q_{v_n}(f) \leq n\}.$$

Given $\overline{v} \in \overline{V}$ let m be such that $n\overline{v} \leq \sup\{\overline{v}_m, v_n\}$. Now

$$W = \{f \in C\overline{V}(X) : q_{\overline{v}_m}(f) \leq 1\} \cap B \subset \{f : q_{\overline{v}}(f) \leq 1\}.$$

This clearly proves that B is a metrizable subset of $C\overline{V}(X)$. Conversely, assume that each bounded subset of $C\overline{V}(X)$ is metrizable. For each n, the set $B_n = \{f \in C(X) : q_{v_n}(f) \leq 1\}$ is bounded in $C\overline{V}(X)$ and thus there exists a sequence $(\overline{v}_{n,m})_{m \in \boldsymbol{N}}$ in $\overline{V}$ such that the sets

$$D_{n,m} = B_n \cap \{f \in C\overline{V}(X) : q_{\overline{v}_{n,m}}(f) \leq 1\}, \qquad m \in \boldsymbol{N},$$

is a base at zero in B_n. Let $(\overline{w}_k)$ be any arrangement of the double sequence $(\overline{v}_{n,m})_{n,m}$, and set $\overline{v}_m = \max\{\overline{w}_1, \ldots, \overline{w}_m\}$. Then $(\overline{v}_m)$ is an increasing sequence in $\overline{V}$. Given $n \in \boldsymbol{N}$ and $\bar{v} \in \overline{V}$, there exists m such that

$$B_n \cap \{f \in C\overline{V}(X) : q_{\bar{v}_m}(f) \leq 1\} \subset \{f : q_{\bar{v}}(f) \leq |\lambda|^{-1}\},$$

where $|\lambda| > 1$. If $w = \max\{v_n, \bar{v}_m\}$, then

$$\{f \in C\overline{V}(X) : q_w(f) \leq 1\} \subset \{f : q_{\bar{v}}(f) \leq |\lambda|^{-1}\}.$$

In view of Lemma 4.4 and Proposition 4.3, we have that $\bar{v} \leq w$. Thus condition (1) is satisfied and so $\mathcal{V}$ satisfies condition (D). This completes the proof.

PROPOSITION 4.6. – *If* $\mathcal{V} \subset |C(X)|$, *then* $C\overline{V}(X)$ *is bornological iff it is quasi-barrelled.*

PROOF. – Assume that $C\overline{V}(X)$ is quasi-barrelled and let W be an absolutely convex bornivorous subset of $C\overline{V}(X)$. Let $B_n = \{f \in C(X) : q_{v_n}(f) \leq 1\}$. Each B_n is bounded in $C\overline{V}(X)$ and so $\mu_n B_n \subset W$ for some non-zero μ_n in $\boldsymbol{K}$. Let $D_n = \bigcup_{k=1}^{n} \mu_k B_k$. Since $\mathcal{V} \subset |C(X)|$, using Proposition 2.4 we get that

$$D_n = \{f \in C(X) : |f| \leq \max_{1 \leq k \leq n} |\mu_k| / v_k\}.$$

Since, for each $x \in X$, there exists $\bar{v}$ in $\overline{V}$ with $\bar{v}(x) > 0$, it follows that the weak topology of $C\overline{V}(X)$ is finer than the topology of pointwise convergence and so D_n is weakly closed in $C\overline{V}(X)$. If $D = \bigcup_{n=1}^{\infty} D_n$, then $\overline{D}$ is a neighborhood of zero in $C\overline{V}(X)$ since $C\overline{V}(X)$ is quasi-barrelled. We claim that

$$\overline{D} \subset \bigcup_{n=1}^{\infty} D_n^e \subset \lambda D,$$

for $|\lambda| > 1$. Indeed, assume that some $f \in \overline{D}$ is not in any D_n^e. Since $D_n^e = D_n^{oo}$, there exists $\phi_n \in D_n^o$ with $|\phi_n(f)| > 1$. The set $H = \{\phi_n : n \in \boldsymbol{N}\}$ is strongly bounded in the dual space of $C\overline{V}(X)$. Indeed, let B be a bounded subset of $C\overline{V}(X)$. There exist $m \in \boldsymbol{N}$ and $\mu \neq 0$ such that $B \subset \mu B_m$. If $f \in B$ and $n \geq m$, then $f \in \mu_m^{-1} D_n$ and so $|\phi_n(f)| \leq |\mu_m^{-1}|$. This clearly proves that H is absorbed by B^o and so H is strongly bounded. Since $C\overline{V}(X)$ is quasi-barrelled, it follows that H is equicontinuous and so its polar H^o in $C\overline{V}(X)$ is a neighborhood of zero. Since $f \in \overline{D}$, there exists n and $g \in D_n$ such that $f - g \in H^o$, which is a contradiction since $|\phi_n(g)| \leq 1$ and $|\phi_n(f)| > 1$. This contradiction proves that λD (and hence D) is a neighborhood of zero in $C\overline{V}(X)$ and so W is also a neighborhood of zero. Hence the result follows.

For a locally convex topology on a vector space E over $\boldsymbol{K}$, we will denote by τ^b the associated barrelled topology.

PROPOSITION 4.7. – *If* $\mathcal{V} \subset |C(X)|$, *then* $\mathcal{V}C(X)$ *coincides with the barrelled space associated with* $C\overline{V}(X)$.

PROOF. – Let $\tau = \tau_{\overline{V}}$ be the topology of $C\overline{V}(X)$ and let τ_{ind} be the inductive topology of $\mathcal{V}C(X)$. Since v_n is continuous, it follows easily that $Cv_n(X)$ is a Banach space and $\mathcal{V}C(X)$ is barrelled. Since $\tau \leqslant \tau_{\text{ind}}$, it follows that $\tau \leqslant \tau^b \leqslant \tau_{\text{ind}}$. If $B_n = \{f \in C(X) : q_{v_n}(f) \leqslant 1\}$ then each $\sum_{k=1}^{n} \lambda_k B_k$ is weakly closed in $C\overline{V}(X)$ and hence it is weakly closed in $(C\overline{V}(X), \tau^b) = G$. With the same argument as in the proof of Proposition 4.6, it follows that G is bornological and so $\tau^b = \tau_{\text{ind}}$. Thus the result follows.

Recall that an inductive limit $E = \varinjlim E_n$ is called strongly boundedly retractive if: a) It is regular, i.e. each bounded subset of E is a bounded subset of some E_n.

b) For each n, there exists $m \geqslant n$ such that for each bounded subset B of E_n we have that $\tau|_B = \tau_m|_B$, where τ is the inductive topology of E and τ_m the topology of E_m.

We also recall the following definition:

The sequence $\mathcal{V} = (v_n)$ is regularly decreasing (see [12], Definition 2.1) if, given $n \in \mathbf{N}$, there exists $m \geqslant n$ such that, for every $\varepsilon > 0$ and every $k \geqslant m$, we can find $\delta = \delta(k, \varepsilon) > 0$ with $v_k(x) \geqslant \delta v_n(x)$ whenever $v_m(x) \geqslant \varepsilon v_n(x)$.

PROPOSITION 4.8 ([12], *Proposition* 2.2). – *The sequence* $\mathcal{V} = (v_n)$ *is regularly decreasing iff the following condition is satisfied*:

(wV) *For every* $n \in \mathbf{N}$, *there exists* $m \geqslant n$ *so that, for every* $\varepsilon > 0$, *there is some* $\bar{v} \in \overline{V}$ *such that* $v_m(x) \leqslant \varepsilon v_n(x)$ *whenever* $\bar{v}(x) < v_m(x)$.

PROPOSITION 4.9. – *If* $\mathcal{V} = (v_n)$ *is regularly decreasing, then for each* $n \in \mathbf{N}$, *there exists* $m \geqslant n$ *so that, on each bounded subset* A *of* $Cv_n(X)$, *the topology induced on* A *by the topology* τ_m *of* $Cv_m(X)$ *coincides with the topology induced on* A *by the topology* $\tau_{\overline{V}}$ *of* $C\overline{V}(X)$ *and by the topology* τ_{ind} *of* $\mathcal{V}C(X)$.

PROOF. – Let $n \in \mathbf{N}$ and let $m \geqslant n$ be as in the condition (wV). Let A be an absolutely convex bounded subset of $Cv_n(X)$. Let τ_m be the topology of $Cv_m(X)$, $\tau_{\overline{V}}$ the topology of $C\overline{V}(X)$ and τ_{ind} the inductive topology of $\mathcal{V}C(X)$. Clearly $\tau_{\overline{V}}|_A \leqslant \tau_{\text{ind}}|_A \leqslant \tau_m|_A$. On the other hand, let $\varepsilon > 0$ and let

$$W = \{f \in C(X) : q_{v_m}(f) \leqslant \varepsilon\}.$$

Let $d \geqslant \sup_{f \in A} q_{v_n}(f)$. By condition ($wV$), there exists $\bar{v} \in \overline{V}$ such that $v_m(x) \leqslant (\varepsilon/d)\, v_n(x)$ if $\bar{v}(x) < v_m(x)$. Set

$$D = \{f \in C\overline{V}(X) : q_{\overline{V}}(f) \leqslant \varepsilon\}.$$

We will finish the proof by showing that $D \cap A \subset W$. Indeed, let $f \in D \cap A$ and assume that $f \notin W$. Then, there exists $x \in X$ with $|f(x)|\, v_m(x) > \varepsilon$. Since

$|f(x)|\,\overline{v}(x) \leqslant q_{\overline{v}}(f) \leqslant \varepsilon$, we have that $\overline{v}(x) < v_m(x)$ and so $v_m(x) \leqslant (\varepsilon/d)\, v_n(x)$. Thus

$$v_m(x)\,|f(x)| \leqslant \frac{\varepsilon}{d} v_n(x)\,|f(x)| \leqslant \frac{\varepsilon}{d} q_{v_n}(f) \leqslant \varepsilon\,,$$

a contradiction. This completes the proof.

PROPOSITION 4.10. – *Assume that* $\mathcal{V} = (v_n)$ *is regularly decreasing. Then*:

1) *On each bounded subset of* $\mathcal{V}C(X)$ *the topologies of* $C\overline{V}(X)$ *and* $\mathcal{V}C(X)$ *coincide.*

2) $\mathcal{V}C(X)$ *and* $C\overline{V}(X)$ *have the same compactoid sets.*

3) *The inductive limit* $\mathcal{V}C(X) = \varinjlim Cv_n(X)$ *is compactoid regular.*

4) *The inductive limit* $\varinjlim Cv_n(X)$ *is strongly boundedly retractive.*

PROOF. – 1) It follows from the preceeding Proposition in view of Proposition 2.1.

2) Let A be a compactoid subset of $C\overline{V}(X)$. We may assume that A is absolutely convex. There exists n such that A is a bounded subset of $Cv_n(X)$. By the preceeding Proposition, we have that $\tau_{\overline{V}} = \tau_{\text{ind}}$ on A. Since A is absolutely convex and $\tau_{\overline{V}}$-compactoid, it follows that it is also τ_{ind}-compactoid.

3) Let A be an absolutely convex τ_{ind}-compactoid. There exists n such that A is a compactoid subset of $Cv_n(X)$. By the preceeding Proposition, there exists $m \geqslant n$ such that $\tau_m = \tau_{\text{ind}}$ on A. It follows that A is a compactoid subset of $Cv_m(X)$.

4) It follows from the preceeding Proposition.

PROPOSITION 4.11. – *If X is zero-dimensional, then* $\mathcal{V} = (v_n)$ *is regularly decreasing iff the following condition is satisfied: For each* $n \in \boldsymbol{N}$, *there exists* $m \geqslant n$ *such that* $\tau_{\overline{V}} = \tau_m$ *on each bounded subset of* $Cv_n(X)$.

PROOF. – In view of Proposition 2.8, we may assume that, in case the valuation of $\boldsymbol{K}$ is discrete, we have that $v_n(X) \subset |\boldsymbol{K}|$ for all n. Suppose now that the condition is satisfied and let $n \in \boldsymbol{N}$. Choose $m \geqslant n$ as in the condition. Given $\varepsilon > 0$, there exists $\overline{v} \in \overline{V}$ such that

$$(*) \quad \{f \in C\overline{V}(X) : q_{\overline{v}}(f) \leqslant 1\} \cap \{f : q_{v_n}(f) \leqslant 1/\varepsilon\} \subset \{f : q_{v_m}(f) < 1\}\,.$$

Assume now that, for some x_0 in X, we have $\overline{v}(x_0) < v_m(x_0)$ and $\varepsilon v_n(x_0) < v_m(x_0)$. Choose $\mu \in \boldsymbol{K}$ such that $\overline{v}(x_0) < |\mu| \leqslant v_m(x_0)$ and $\varepsilon v_n(x_0) < |\mu| \leqslant v_m(x_0)$. The set

$$O = \{x \in X : \overline{v}(x) < |\mu|\} \cap \{x : \varepsilon v_n(x) < |\mu|\}$$

is open. There is a clopen neighborhood A of x_0 contained in O. Let $f = \mu^{-1}\phi$, where ϕ is the $\boldsymbol{K}$-characteristic function of A. Then $q_{\bar{v}}(f) \leqslant 1$ and $q_{v_n}(f) \leqslant 1/\varepsilon$ while $q_{v_m}(f) \geqslant v_m(x_0)/|\mu| \geqslant 1$, which contradicts (*). Thus $\mathcal{V}$ satisfies condition (wV) and so it is regularly decreasing by Proposition 4.8. This and Proposition 4.9 complete the proof.

PROPOSITION 4.12. – *Let X be locally compact and assume that $\inf_{x \in Y} v_n(x) > 0$, for each $n \in \boldsymbol{N}$ and each non-empty compact subset Y of X. If $\mathcal{V} = (v_n)$ is regularly decreasing, then $C\overline{V}(X)$ and $\mathcal{V}C(X)$ are quasi-complete.*

PROOF. – Since every point of X has a compact neighborhood and since $\inf_{x \in Y} v_n(x) > 0$ for each non-empty compact subset Y of X, it follows easily that each $Cv_n(X)$ is a Banach space. Let now A be a closed absolutely convex bounded subset of $C\overline{V}(X)$. There exists n such that A is a bounded subset of $Cv_n(X)$. Since $\mathcal{V}$ is regularly decreasing, there exists (by Proposition 4.4) an $m \geqslant n$ such that $\tau_{\overline{V}}$, τ_{ind} and τ_m induce the same topology on A. Let now (f_δ) be a $\tau_{\overline{V}}$-Cauchy net in A. Then, (f_δ) is τ_m-Cauchy and so $f_\delta \to f$, with respect to τ_m, for some $f \in Cv_m(X)$. Then $f_\delta \to f$ with respect to $\tau_{\overline{V}}$ and $f \in A$ since A is $\tau_{\overline{V}}$-closed. This proves that $C\overline{V}(X)$ is quasi-complete. The proof of $\mathcal{V}C(X)$ is analogous.

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