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Approximate Quantities, Hyperspaces and Metric Completeness.

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Sunto. – *Mostriamo che se (X, d) è uno spazio metrico completo, allora è completa anche la metrica D, indotta in modo naturale da d sul sottospazio degli insiemi sfocati («fuzzy») di X dati dalle quantità approssimate. Come è ben noto, D è una metrica molto interessante nella teoria dei punti fissi di applicazioni sfocate, poiché permette di ottenere risultati soddisfacenti in questo contesto.*

1. – Introduction.

Recall the following known concepts.

Let (X, d) be a metric space. A fuzzy set in X is a function from X into $[0,1]$.

Let A be a fuzzy set in the metric space (X, d) and let $x \in X$. Then $A(x)$ is called the grade of membership of x in A. The r-level set of A, denoted by A_r, is defined by

$$A_r = \{x \in X : A(x) \geq r\} \qquad \text{if } r \in (0, 1],$$

and

$$A_0 = \overline{\{x \in X : A(x) > 0\}}.$$

A fuzzy set A is called an approximate quantity if for each $r \in [0, 1]$, A_r is compact, and $\sup_{x \in X} A(x) = 1$ (compare [4]).

We shall denote by $\mathcal{A}(X)$ the set of all approximate quantities in (X, d).

The metric d induces, in a natural way, a metric D on $\mathcal{A}(X)$ defined by

$$D(A, B) = \sup_{r \in [0, 1]} H_d(A_r, B_r),$$

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for all $A, B \in \mathcal{A}(X)$, where H_d denotes the Hausdorff metric of d on the set $\mathcal{K}_0(X)$ of all nonempty compact subsets of (X, d). $(\mathcal{K}_0(X), H_d)$ is called the Hausdorff metric hyperspace of (X, d).

A fuzzy mapping from a metric space (X, d) into $\mathcal{A}(X)$ is simply a mapping F on X such that $F(x) \in \mathcal{A}(X)$, for all $x \in X$.

It is well known that the metric space $(\mathcal{A}(X), D)$ provides an appropriate setting to extend many important fixed point theorems on complete metric spaces to fuzzy mappings (see, for instance, [4], [1], [5]). In such extensions the requirement that the metric space (X, d) to be complete is clearly essential. The purpose of this note is to prove that, in fact, completeness of (X, d) is inherited by $(\mathcal{A}(X), D)$. Consequently, many fixed point theorems for contractive self-mappings in complete metric spaces remain valid for contractive mappings from $\mathcal{A}(X)$ into itself, under the assumption that (X, d) is complete (see, for instance, Theorem 2 below).

2. – The main result.

We start this section with an easy but useful observation.

LEMMA. – *Let (X,d) be a metric space and let $A \in \mathcal{A}(X)$. Fix $r \in (0, 1]$ and let $(r_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence in $(0, 1]$ such that $r_k \to r$, and $(y_k)_{k \in \mathbb{N}}$ be a sequence in X such that $y_k \in A_{r_k}$ for all $k \in \mathbb{N}$. Then $(y_k)_{k \in \mathbb{N}}$ has a cluster point in A_r.*

PROOF. – Since $(y_k)_{k \in \mathbb{N}}$ is in A_{r_1}, there is a subsequence $(z_{k1})_{k \in \mathbb{N}}$ of $(y_k)_{k \in \mathbb{N}}$ which converges to a point $x_1 \in A_{r_1}$. Similarly, $(z_{k1})_{k \in \mathbb{N}}$ has a subsequence $(z_{k2})_{k \in \mathbb{N}}$ which converges to a point $x_2 \in A_{r_2}$. Then $x_1 = x_2$. By proceeding inductively, for each $m \geq 2$, the sequence $(z_{km})_{k \in \mathbb{N}}$ admits a subsequence $(z_{k(m+1)})_{k \in \mathbb{N}}$ which converges to a point $x_{m+1} \in A_{r(m+1)}$. So, $x_{m+1} = x_m = \ldots = x_1$. We conclude that x_1 is a cluster point of $(y_k)_{k \in \mathbb{N}}$ such that $x_1 \in \bigcap_{k=1}^{\infty} A_{r_k}$. Hence $x_1 \in A_r$. This completes the proof. ■

Our main result is the following

THEOREM 1. – *Let (X, d) be a metric space. Then $(\mathcal{A}(X), D)$ is complete if and only if (X, d) is complete.*

PROOF. – Suppose that (X, d) is complete. It is well known that, then, the Hausdorff metric hyperspace $(\mathcal{K}_0(X), H_d)$ is complete (see, for instance, [3, Theorem 2.4.4]).

Let $(A_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in the metric space $(\mathcal{A}(X), D)$. Then for each $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that for $n, m \geq n_\varepsilon$, $D(A_n, A_m) < \varepsilon/2$. So $H_d((A_n)_r, (A_m)_r) < \varepsilon/2$, whenever $r \in [0, 1]$ and $n, m \geq n_\varepsilon$.

Now define, for each $r \in [0, 1]$,

$$C(r) = \{x \in X : \text{there is a sequence } (x_n)_{n \in \mathbb{N}} \text{ with } x_n \in (A_n)_r \text{ and } x_n \to x\}.$$

Then, the classical proof that $(\mathcal{K}_0(X), H_d)$ is complete, actually shows that $C(r) \in \mathcal{K}_0(X)$ and that for each $\varepsilon > 0$, $H_d(C(r), (A_n)_r) \leqslant \varepsilon$ whenever $r \in [0, 1]$ and $n \geqslant n_\varepsilon$. ([3, proof of Theorem 2.4.4].)

We claim that

$$(*) \qquad C(r) \subseteq C(s) \text{ whenever } r \geqslant s;\ r, s \in [0, 1].$$

Indeed, if $r \geqslant s$, and $x \in C(r)$, then there is a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in (A_n)_r$ and $x_n \to x$. Since $(A_n)_r \subseteq (A_n)_s$, we deduce, by the definition of $C(s)$, that $x \in C(s)$.

Now define a fuzzy set B in X, as follows:

$$B(x) = 0 \text{ if } x \notin \bigcup_{r \in [0, 1]} C(r)$$

and

$$B(x) = \sup\{r \in [0, 1] : x \in C(r)\}, \text{ otherwise}.$$

(Note, that by $(*)$, $\bigcup_{r \in [0,1]} C(r) = C(0)$.)

We shall prove that $B \in \mathcal{A}(X)$ and that $D(B, A_n) \to 0$.

First note that $\sup_{x \in X} B(x) = 1$ because $C(r) \neq \emptyset$ for all $r \in [0, 1]$, and thus $B_1 \neq \emptyset$, in particular.

Next we show that B_r is compact in (X, d) for all $r \in [0, 1]$:

Fix $r \in (0, 1]$ and take a sequence $(y_n)_{n \in \mathbb{N}}$ in B_r. We may suppose two cases:

Case 1. There exists a subsequence $(y_{n(k)})_{k \in \mathbb{N}}$ of (y_n) such that each $y_{n(k)}$ is in some $C(r_k)$ with $r_k \geqslant r$.

Case 2. Without loss of generality, $y_n \notin C(r)$ for all $n \in \mathbb{N}$.

In the Case 1, $y_{n(k)} \in C(r)$ for all $k \in \mathbb{N}$, by condition $(*)$. Since $C(r)$ is compact, the sequence $(y_{n(k)})_{k \in \mathbb{N}}$ has a cluster point $y_0 \in C(r)$. So $B(y_0) \geqslant r$. We conclude that $y_0 \in B_r$.

In the Case 2, the fact that $y_n \in B_r$ implies that $y_n \in C(t)$ for all $n \in \mathbb{N}$ and all $t \in [0, r)$. Choose a strictly increasing sequence (r_n) in $[0, r)$ such that $r_n \to r$. Since $(y_n)_{n \in \mathbb{N}}$ is in $C(r_1)$, there is a subsequence $(z_{n1})_{n \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ which converges to a point $x_1 \in C(r_1)$. Similarly, there is a subsequence $(z_{n2})_{n \in \mathbb{N}}$ of $(z_{n1})_{n \in \mathbb{N}}$ which converges to a point $x_2 \in C(r_2)$. So $x_1 = x_2$. By proceding inductively, for $m \geqslant 2$, the sequence $(z_{nm})_{n \in \mathbb{N}}$ admits a subsequence $(z_{n(m+1)})_{n \in \mathbb{N}}$ which converges to a point $x_{m+1} \in C(r_{m+1})$. So $x_{m+1} = x_m = \ldots = x_1$. Therefore x_1 is a cluster point of $(y_n)_{n \in \mathbb{N}}$ and, by $(*)$, $x_1 \in \bigcap_{t \in [0, r)} C(t)$. Thus, $x_1 \in B_r$.

We have shown that for each $r \in (0, 1]$, B_r is compact. In order to prove that B_0 is compact, take any sequence $(y_n)_{n \in \mathbb{N}}$ in B_0. Then, for each $n \in \mathbb{N}$ there is a sequence $(z_{nk})_{k \in \mathbb{N}}$ such that $d(y_n, z_{nk}) \to 0$ and $B(z_{nk}) > 0$ for all $n, k \in \mathbb{N}$. By ($*$), each z_{nk} is in $C(0)$, and thus each sequence $(z_{nk})_{k \in \mathbb{N}}$ has a cluster point $x_n \in C(0)$. Hence $x_n = y_n$ for all $n \in \mathbb{N}$. Let $y_0 \in C(0)$ be a cluster point of $(y_n)_{n \in \mathbb{N}}$. For each $\delta > 0$ there is an $n \in \mathbb{N}$ such that $d(y_0, y_n) < \delta/2$, so there is a $k \in \mathbb{N}$ such that $d(y_n, z_{nk}) < \delta/2$. Thus $d(y_0, z_{nk}) < \delta$, which shows that, in fact, $y_0 \in B_0$. We conclude that B_0 is compact.

Consequently $B \in \mathcal{A}(X)$.

Next we show that for each $\varepsilon > 0$, $D(B, A_n) \leqslant \varepsilon$ whenever $n \geqslant n_\varepsilon$.

Let $\varepsilon > 0$ be given and let $n \geqslant n_\varepsilon$.

Take, first, any $r \in (0, 1]$ and recall that

$$H_d(B_r, (A_n)_r) = \max\left\{\sup_{b \in B_r} d(b, (A_n)_r),\ \sup_{a \in (A_n)_r} d(B_r, a)\right\}.$$

Let $b \in B_r$. If $b \in C(r)$, then $d(b, (A_n)_r) \leqslant \varepsilon$, because, as we observed above, $H_d(C(s), (A_n)_s) \leqslant \varepsilon$ for all $s \in [0, 1]$. If $b \notin C(r)$, since $r > 0$ we deduce that $b \in C(t)$ for every $t \in [0, r)$. Choose an increasing sequence $(r_k)_{k \in \mathbb{N}}$ in $[0, r)$ such that $r_k \to r$. Then, for each $k \in \mathbb{N}$ there is an $a_k \in (A_n)_{r_k}$ such that $d(b, a_k) \leqslant \varepsilon$ since each $(A_n)_{r_k}$ is compact. By the above lemma, the sequence $(a_k)_{k \in \mathbb{N}}$ has a cluster point $a \in (A_n)_r$. Hence, $d(b, a) \leqslant \varepsilon$, and, thus, $d(b, (A_n)_r) \leqslant \varepsilon$. Therefore, $\sup_{b \in B_r} d(b, (A_n)_r) \leqslant \varepsilon$.

Now let $a \in (A_n)_r$. Then $d(C(r), a) \leqslant \varepsilon$, so, by the compactness of $C(r)$, $d(c, a) \leqslant \varepsilon$ for some $c \in C(r)$. But $C(r) \subseteq B_r$ because $r > 0$. Hence, $\sup_{a \in (A_n)_r} d(B_r, a) \leqslant \varepsilon$.

Consequently, for each $r \in (0, 1]$ and each $n \geqslant n_\varepsilon$, $H_d(B_r, (A_n)_r) \leqslant \varepsilon$.

Now suppose that $r = 0$. Let $b \in B_0$. Then $b \in C(0)$ (indeed, since $b \in B_0$, there is a sequence $(z_k)_{k \in \mathbb{N}}$ such that $d(b, z_k) \to 0$ and $B(z_k) > 0$ for each $k \in \mathbb{N}$. Thus, each z_k is in $C(0)$. So the sequence $(z_k)_{n \in \mathbb{N}}$ has a cluster point $z \in C(0)$. We deduce that $b = z$). Hence, $d(b, (A_n)_0) \leqslant \varepsilon$, because $H_d(C(0), (A_n)_0) \leqslant \varepsilon$, and, thus, $\sup_{b \in B_0} d(b, (A_n)_0) \leqslant \varepsilon$.

Let $a \in (A_n)_0$. Then, there is a sequence $(a_k)_{k \in \mathbb{N}}$ such that $d(a, a_k) \to 0$ and $A_n(a_k) > 0$ for all $k \in \mathbb{N}$. Therefore, for each $k \in \mathbb{N}$ there is an $r_k \in (0, 1]$ such that $a_k \in (A_n)_{r_k}$, so $d(c_k, a_k) \leqslant \varepsilon$ for some $c_k \in C(r_k)$. Since $C(r_k) \subseteq B_0$, we deduce that the sequence $(c_k)_{k \in \mathbb{N}}$ has a cluster point $b \in B_0$. By the triangle inequality, $d(b, a) \leqslant \varepsilon$. Hence, $d(B_0, a) \leqslant \varepsilon$, and, thus, $\sup_{a \in (A_n)_r} d(B_0, a) \leqslant \varepsilon$.

We conclude that for each $n \geqslant n_\varepsilon$, $D(B, A_n) \leqslant \varepsilon$. Hence $D(B, A_n) \to 0$, so $(\mathcal{A}(X), D)$ is a complete metric space.

Conversely, suppose that $(\mathcal{A}(X), D)$ is complete. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in (X, d). Then, for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that

$d(x_n, x_m) < \varepsilon$ for all $n, m \geqslant n_\varepsilon$. For each $n \in \mathbb{N}$ denote by A_n the characteristic function of x_n. Then each A_n is in $\mathcal{A}(X)$, and $D(A_n, A_m) \leqslant \varepsilon$ for all $n, m \geqslant n_\varepsilon$. So, the sequence $(A_n)_{n \in \mathbb{N}}$ is convergent to an approximate quantity A. Choose an $a \in X$ such that $A(a) = 1$. It immediately follows that $d(a, x_n) \to 0$. Therefore (X, d) is complete. ■

Like illustration we state the following fixed point theorem as an immediate consequence of Theorem 1 and a well-known fixed point theorem of L. B. Ciric [2].

THEOREM 2. – *Let (X, d) be a complete metric space and let F be a mapping from $\mathcal{A}(X)$ into itself such that there is a constant h, $0 \leqslant h < 1$, such that for all $A, B \in \mathcal{A}(X)$,*

$$D(F(A), F(B)) \leqslant$$
$$h \max \{D(A, B), D(A, F(A)), D(B, F(B)), D(A, F(B)), D(F(A), B)\}.$$

Then F has a (unique) fixed point.

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