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Blow-up and Global Existence of a Weak Solution for a sine-Gordon Type Quasilinear Wave Equation.

JOÃO-PAULO DIAS - MÁRIO FIGUEIRA

Sunto. – *Si considera il problema di Cauchy per l'equazione* (*cf.* [1]):

$$\phi_{tt} - \phi_{xx} - \phi_x^2 \phi_{xx} + \sin \phi = 0 , \quad (x, t) \in \boldsymbol{R} \times \boldsymbol{R}_+ .$$

Nella prima parte di questo articolo si dimostra, per dati iniziali particolari, un risultato di «blow-up» della soluzione classica locale (*in tempo*), *seguendo le idee introdotte in* [8], [2] *ed* [4]. *Nella seconda parte, viene utilizzato il metodo di compattezza per compensazione* (*cf.* [13], [10] *ed* [5]) *ed una estensione del principio delle regioni invarianti* (*cf.* [12]) *per dimostrare l'esistenza di una soluzione debole globale entropica.*

1. – A blow-up result.

We consider the equation (cf. [1] for a physical motivation)

$$\phi_{tt} - \phi_{xx} - \phi_x^2 \phi_{xx} + \sin \phi = 0 , \quad (x, t) \in \boldsymbol{R} \times \boldsymbol{R}_+ \tag{1.1}$$

with the initial data

$$\phi(x, 0) = \phi_0(x) , \quad \phi_t(x, 0) = \phi_1(x) , \quad x \in \boldsymbol{R} . \tag{1.2}$$

Since $F(\phi) = -\sin \phi$ verifies $F'(0) = -1$ it is possible to find $T > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0[$ there exists $\phi_{0\varepsilon} \in C^2(\boldsymbol{R}) \cap W^{2, \infty}(\boldsymbol{R})$ and $\phi_{1\varepsilon} \in C^1(\boldsymbol{R}) \cap W^{1, \infty}(\boldsymbol{R})$, nonconstant periodic initial data with period T, such that $\|\phi_{0\varepsilon}\|_{W^{2, \infty}} + \|\phi_{1\varepsilon}\|_{W^{1, \infty}} \underset{\varepsilon \to 0}{\longrightarrow} 0$ and such that the corresponding solution ϕ_ε of the Cauchy problem for equation (1.1) has the form $\phi_\varepsilon(x, t) = \phi_{0\varepsilon}(x - \omega_\varepsilon t)$ for a suitable $\omega_\varepsilon \in \boldsymbol{R}_+$ with $\{\omega_\varepsilon\}$ bounded, and so ϕ_ε is defined globally (cf.[4], Theo. 1.1).

But in general, for $\phi_0 \in C^2(\boldsymbol{R}) \cap W^{2, \infty}(\boldsymbol{R})$ and $\phi_1 \in C^1(\boldsymbol{R}) \cap W^{1, \infty}(\boldsymbol{R})$ we only can prove the existence (and uniqueness) of a local (in time) solution $\phi \in C^2(\boldsymbol{R} \times [0, T'])$ of the Cauchy problem, for a certain $T' > 0$. Furthermore, $\phi \in L^\infty([0, T'], W^{2, \infty}(\boldsymbol{R})) \cap W^{1, \infty}([0,T'], W^{1, \infty}(\boldsymbol{R})) \cap W^{2,\infty}([0,T'], L^\infty(\boldsymbol{R}))$.

To see that let us put $u = \phi_x$, $v = \phi_t$. The equation (1.1) reduces to the system

$$\tag{1.3} \begin{cases} \phi_t = v \\ u_t - v_x = 0 \\ v_t - (1+u^2)\, u_x = -\sin\phi\,. \end{cases}$$

If we introduce the Riemann invariants

$$\tag{1.4} \begin{cases} l = v + \int_0^u \sqrt{1+\xi^2}\, d\xi \\ r = v - \int_0^u \sqrt{1+\xi^2}\, d\xi \end{cases}$$

we can write the system (1.3) as follows

$$\tag{1.5} \begin{cases} \phi_t = \dfrac{1}{2}(l+r) \\ l_t - \sqrt{1+u^2}\, l_x = -\sin\phi \\ r_t + \sqrt{1+u^2}\, r_x = -\sin\phi \end{cases}$$

with the initial conditions

$$\tag{1.6} \begin{gathered} \phi(x,0) = \phi_0(x), \quad l(x,0) = l_0(x), \quad r(x,0) = r_0(x) \\ \phi_0,\, l_0,\, r_0 \in C^1(\boldsymbol{R}) \cap W^{1,\infty}(\boldsymbol{R}). \end{gathered}$$

By the results in [6] and [7] (see also [9]) the Cauchy problem (1.5), (1.6) has a local (in time) unique solution $(\phi, l, r) \in (C^1(\boldsymbol{R}\times[0, T']))^3$ for a certain $T' > 0$ such that $(\phi, l, r) \in (L^\infty([0, T'], W^{1,\infty}(\boldsymbol{R})) \cap W^{1,\infty}([0, T'], L^\infty(\boldsymbol{R})))^3$.

Now we look for a blow-up result for the solution ϕ of our Cauchy problem. Let $T^* = \sup T'$ be such that $[0, T']$ is an interval of (local) existence of ϕ. For each $\alpha \in \boldsymbol{R}$ we define the left (resp. right) characteristic curve starting in α by

$$\tag{1.7} \frac{dx_1}{dt} = -\sqrt{1+u^2}, \quad x_1(0) = \alpha \quad \left(\text{respect. } \frac{dx_2}{dt} = \sqrt{1+u^2},\ x_2(0) = \alpha\right).$$

From (1.5), (1.7) we easily deduce,

$$\tag{1.8} \begin{aligned} \|r(.\,,t)\|_{L^\infty(\boldsymbol{R})} &\leq \|r_0\|_{L^\infty(\boldsymbol{R})} + t \\ \|l(.\,,t)\|_{L^\infty(\boldsymbol{R})} &\leq \|l_0\|_{L^\infty(\boldsymbol{R})} + t \end{aligned} \qquad t \in [0, T^*[$$

Hence, from $\phi(x, t) = \phi_0(x) + \int_0^t v(x, \tau)\, d\tau = \phi_0(x) + \frac{1}{2}\int_0^t (l + r)(x, \tau)\, d\tau$, we obtain,

$$\|\phi(.\,, t) - \phi_0\|_{L^\infty(\boldsymbol{R})} \leq \frac{1}{2} c_0 t + \frac{1}{2} t^2, \qquad t \in [0, T^*[\,, \tag{1.9}$$

where $c_0 = \|r_0\|_{L^\infty(\boldsymbol{R})} + \|l_0\|_{L^\infty(\boldsymbol{R})}$. From (1.4) we deduce

$$l - r = f(u) = 2\int_0^u \sqrt{1 + \xi^2}\, d\xi = u\sqrt{1 + u^2} + \operatorname{arc\,sinh} u$$

and so

$$\|u(.\,, t)\|_{L^\infty(\boldsymbol{R})} \leq (\|l(.\,, t)\|_{L^\infty(\boldsymbol{R})} + \|r(.\,, t)\|_{L^\infty(\boldsymbol{R})})^{1/2} \leq (c_0 + 2t)^{1/2}, \qquad t \in [0, T^*[\,. \tag{1.10}$$

Now, assume

$$\frac{\pi}{2} < \inf_{x \in \boldsymbol{R}} \phi_0(x), \qquad \sup_{x \in \boldsymbol{R}} \phi_0(x) < \frac{3\pi}{2}. \tag{1.11}$$

Let $t_0 = t_0\left(\inf_{x \in \boldsymbol{R}} \phi_0(x), \sup_{x \in \boldsymbol{R}} \phi_0(x), c_0\right) \in]0, T^*[$ be such that (cf. (1.9))

$$\frac{\pi}{2} \leq \inf_{x \in \boldsymbol{R}} \phi(x, t), \qquad \sup_{x \in \boldsymbol{R}} \phi(x, t) \leq \frac{3\pi}{2}, \qquad t \in [0, t_0] \tag{1.12}$$

Hence, with $F(\phi) = -\sin\phi$, we have $F'(\phi)(x, t) = -\cos\phi(x, t) \geq 0$ for $(x, t) \in \boldsymbol{R} \times [0, t_0]$.

The following lemma has the same proof of lemma 1 in [2]:

LEMMA 1.1. – *Let us assume* (1.11) *and fix* $\alpha \in \boldsymbol{R}$. *Let*

$$t_0 = t_0\left(\inf_{x \in \boldsymbol{R}} \phi_0(x), \sup_{x \in \boldsymbol{R}} \phi_0(x), c_0\right) > 0$$

be such that (1.12) *holds. Assume that* $l_0(x) < -\delta < 0$ *and* $r_0(x) \geq 0$ *for* $x \in [\beta_0, \alpha_0]$ *where* $\beta_0 = -2(1 + c_0 + 2t_0)^{1/2} t_0 + \alpha$ *and* $\alpha_0 > \alpha$. *Then* $u(x_1(t), t) < -\varepsilon = f^{-1}(-\delta)$ *along the left characteristic defined by* (1.7), *for* $t \in [0, t_0]$.

Now, let us denote by $\cdot$ the derivative along the left characteristic curve $x_1(t)$ defined in (1.7) with $x_1(0) = \alpha$ fixed in lemma 1.1. Following the proof of theorem 2 in [2], if we put

$$q = (1 + u^2)^{1/4} l_x, \qquad b = (-\cos\phi)(1 + u^2)^{1/4} u$$

we obtain, along the characteristic curve,

$$\dot{q} - \frac{u}{2(1+u^2)^{5/4}} q^2 = b\,, \qquad t \in [0, t_0]\,. \tag{1.13}$$

Since by lemma 1.1 $u(x_1(t), t) < -\varepsilon$, we derive

$$K(t) = -\frac{1}{2}\,\frac{u}{(1+u^2)^{5/4}} \geqslant c(\varepsilon)\,\frac{1}{|u|^{3/2}}\,,$$

where $c(\varepsilon) = (1/2)(1/\varepsilon^2 + 1)^{-5/4}$, and so, by (1.10),

$$K(t) \geqslant c(\varepsilon)\,\frac{1}{(c_0 + 2t)^{3/4}}\,, \qquad t \in [0, t_0] \tag{1.14}$$

Since $b \leqslant 0$, if $q(0) = (1 + u^2(\alpha))^{1/4}\, l_{0x}(\alpha) < 0$ we derive, by a comparison result,

$$q(t) \leqslant \theta(t) = q(0)\left[1 + q(0)\int_0^t K(\tau)\,d\tau\right]^{-1}. \tag{1.15}$$

But, by (1.14),

$$1 + q(0)\int_0^t K(\tau)\,d\tau \leqslant 1 + l_{0x}(\alpha)\,c(\varepsilon)\int_0^t (c_0 + 2\tau)^{-3/4}\,d\tau\,.$$

Hence, if we take $l_{0x}(\alpha)$ such that

$$l_{0x}(\alpha) \leqslant -\left[c(\varepsilon)\int_0^{t_0} (c_0 + 2\tau)^{-3/4}\,d\tau\right]^{-1} \tag{1.16}$$

there exists $t_1 \leqslant t_0$ such that

$$\lim_{t \to t_1^-} q(t) \leqslant \lim_{t \to t_1^-} \theta(t) = -\infty$$

and so the solution blows up. We can state the following

THEOREM 1.2. – *Under the hypothesis of lemma* 1.1, *assume that* $l_{0x}(\alpha) = v_{0x}(\alpha) + \sqrt{1 + u_0^2(\alpha)}\,u_{0x}(\alpha)$ *verifies* (1.16). *Then there exists* $t_1 \leqslant t_0$ *such that*

$$\overline{\lim_{t \to t_1^-}}\,(\|\phi(.\,, t)\|_{W^{2,\infty}(\boldsymbol{R})} + \|\phi_t(.\,, t)\|_{W^{1,\infty}(\boldsymbol{R})} + \|\phi_{tt}(.\,, t)\|_{L^\infty(\boldsymbol{R})}) = +\infty\,.$$

2. – Global existence of an entropy weak solution.

We will write (1.3) in the form

$$\begin{cases} u_t - v_x = 0 \\ v_t - (1+u^2)\, u_x = -\sin\phi\,, \end{cases} \tag{2.1}$$

where $\phi(x, t) = \int_0^t v(x, \tau)\, d\tau + \phi_0(x)$, with the initial data

$$u(x, 0) = u_0(x) = \phi_{0x}(x)\,, \qquad v(x, 0) = v_0(x) = \phi_1(x)\,, \qquad x \in \boldsymbol{R}\,. \tag{2.2}$$

We will assume that (cf. [1] for a physical motivation)

$$\phi_0 \in C(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R}),\ \ \phi_0 \in L^1(\boldsymbol{R}_-),\ \ \phi_0 - 2\pi \in L^1(\boldsymbol{R}_+)\ \text{ and }\ u_0, v_0 \in H^1(\boldsymbol{R})\,. \tag{2.3}$$

This implies

$$1 - \cos\phi_0 \in L^1(\boldsymbol{R}) \quad \text{and} \quad \int_{\boldsymbol{R}} (1-\cos\phi_0)\, dx \leqslant \int_{\boldsymbol{R}_-} |\phi_0|\, dx + \int_{\boldsymbol{R}_+} |\phi_0 - 2\pi|\, dx\,.$$

As usual, we will say that a pair of smooth functions $\eta, q : \boldsymbol{R}^2 \to \boldsymbol{R}$ is a pair entropy/entropy flux for the system (2.1) if η is convex and $\nabla\eta(y).\ \nabla f(y) = \nabla q(y)$, $\forall y \in \boldsymbol{R}^2$, where

$$f = (f_1, f_2)\,, \qquad f_1(y_1, y_2) = -y_2\,, \qquad f_2(y_1, y_2) = -\left(y_1 + \frac{1}{3} y_1^3\right).$$

We will say that $U = (u, v) \in (L^\infty_{loc}(\boldsymbol{R} \times [0, +\infty[))^2$ is an entropy weak solution for the Cauchy problem (2.1), (2.2) in $\boldsymbol{R} \times [0, +\infty[$ if

$$\int_{\boldsymbol{R}\times[0,+\infty[} (u\varphi_t - v\varphi_x)\, dx\, dt + \int_{\boldsymbol{R}} u_0\, \varphi(.\,, 0)\, dx + \tag{2.4}$$

$$\int_{\boldsymbol{R}\times[0,+\infty[} \left(v\psi_t - u\psi_x - \frac{1}{3}u^3\psi_x - \sin\phi\,\psi\right) dx\, dt + \int_{\boldsymbol{R}} v_0\, \psi(.\,, 0)\, dx = 0$$

for each pair $\varphi, \psi \in C_c^\infty(\boldsymbol{R} \times [0, +\infty[)$ and

$$\frac{\partial}{\partial t}\eta(U) + \frac{\partial}{\partial x} q(U) + \nabla\eta(U).\,(0, \sin\phi) \leqslant 0 \qquad \text{in } \mathcal{D}'(\boldsymbol{R} \times]0, +\infty[) \tag{2.5}$$

for each pair entropy/entropy flux (η, q) for the system (2.1).

By applying the compensated compactness method of Murat and Tartar and following the ideas of DiPerna [5] and Dias-Figueira [3] we will prove the following result:

THEOREM 2.1. – *Let us assume* (2.3). *Then, there exists a function* $U = (u, v)$ *measurable in* $\boldsymbol{R} \times [0, +\infty[$ *such that* $U \in (L^\infty(S_T))^2$ *for each strip* $S_T = \boldsymbol{R} \times [0, T[$, $T > 0$, *which is an entropy weak solution for the Cauchy problem* (2.1), (2.2) *in* $\boldsymbol{R} \times [0, +\infty[$. *Moreover*

$$\|u\|_{L^\infty(S_T)} \leq c_1(1+T)^{1/2}, \qquad \|v\|_{L^\infty(S_T)} \leq c_1(1+T), \qquad T > 0,$$

where $c_1 > 0$ *depends only on* u_0 *and* v_0.

To prove the theorem 2.1 we consider first the regularised parabolic system in $\boldsymbol{R} \times [0, +\infty[$ for $U_\varepsilon = (u_\varepsilon, v_\varepsilon)$, $\varepsilon > 0$, with $\varrho_\varepsilon \in \mathcal{D}(\boldsymbol{R})$, $0 \leq \varrho_\varepsilon \leq 1$, $\varrho_\varepsilon = 1$ in $[-(1/\varepsilon), 1/\varepsilon]$, $\varrho_\varepsilon = 0$ in $\boldsymbol{R} \backslash]-(1/\varepsilon)-1, (1/\varepsilon)+1[$:

$$\text{(2.6)} \qquad \begin{cases} u_{\varepsilon t} - v_{\varepsilon x} = \varepsilon \Delta u_\varepsilon \\ v_{\varepsilon t} - (1 + u_\varepsilon^2)\, u_{\varepsilon x} = \varepsilon \Delta v_\varepsilon - \varrho_\varepsilon \sin(\varrho_\varepsilon \phi_\varepsilon), \end{cases}$$

where $\phi_\varepsilon(x, t) = \int_0^t v_\varepsilon(x, \tau)\, d\tau + \phi_{0\varepsilon}(x)$ with $\phi_{0\varepsilon} = \theta_\varepsilon * \phi_0$ for the «mollifiers» θ_ε, $\varepsilon > 0$. We have $\phi_{0\varepsilon} \in C^\infty(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$, $\phi_{0\varepsilon} \in L^1(\boldsymbol{R}_-)$, $\phi_{0\varepsilon} - 2\pi \in L^1(\boldsymbol{R}_+)$ with

$$\int_{\boldsymbol{R}_-} |\phi_{0\varepsilon}|\, dx \leq \int_{-\infty}^{\varepsilon} |\phi_0|\, dx, \quad \int_{\boldsymbol{R}_+} |\phi_{0\varepsilon} - 2\pi|\, dx \leq \int_{-\varepsilon}^{+\infty} |\phi_0 - 2\pi|\, dx,$$

$$\phi_{0\varepsilon} \xrightarrow[\varepsilon \to 0]{} \phi_0 \quad \text{in } L^\infty_{loc}(\boldsymbol{R}),$$

$$u_{0\varepsilon} = \phi_{0\varepsilon x} = \theta_\varepsilon * u_0 \in H^3(\boldsymbol{R}), \qquad u_{0\varepsilon} \xrightarrow[\varepsilon \to 0]{} u_0 \quad \text{in } H^1(\boldsymbol{R}).$$

We can also put $v_{0\varepsilon} = \theta_\varepsilon * v_0 \in H^3(\boldsymbol{R})$, $v_{0\varepsilon} \xrightarrow[\varepsilon \to 0]{} v_0$ in $H^1(\boldsymbol{R})$.

Now we study the problem in $U_\varepsilon = (u_\varepsilon, v_\varepsilon)$ for the system (2.6) with initial data $U_{0\varepsilon}$ given by

$$\text{(2.7)} \qquad u_\varepsilon(x, 0) = u_{0\varepsilon}(x) = \phi_{0\varepsilon x}(x), \quad v_\varepsilon(x, 0) = v_{0\varepsilon}(x), \quad x \in \boldsymbol{R}.$$

Let $X = (H^3(\boldsymbol{R}))^2$, $E = \{U \in C([0, T']; X) : \|U - G(.)\, U_{0\varepsilon}\|_{L^\infty(0, T'; X)} \leq M\}$ for a certain T' and $M > 0$ to be determined, where $G(t)$ is the semi-group associated to the heat system $U_t - \varepsilon \Delta U = 0$. For $U = (u, v) \in E$ let us put (for fixed $\varepsilon > 0$):

$$\text{(2.8)} \qquad (\bar{F} U)(t) = G(t)\, U_{0\varepsilon} + \int_0^t G(t-s)\, J(U(s))\, ds,$$

where

$$J(U) = \begin{pmatrix} v_x \\ u_x \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 u_x - \varrho_\varepsilon \sin(\varrho_\varepsilon \phi) \end{pmatrix}, \qquad \phi(.\,, s) = \int_0^s v(.\,, \tau)\, d\tau + \phi_{0\varepsilon}(.)\,.$$

By the L^2 estimates for the semi-group $G(t)$ (cf.[11]) we have

$$\text{(2.9)} \quad \left\| \int_0^t G(t-s)\, J(U(s))\, ds \right\|_{H^3(\boldsymbol{R})} \leqslant c(\varepsilon) \int_0^t \frac{1}{(t-s)^{1/2}} \|J(U(s))\|_{H^2(\boldsymbol{R})}\, ds\,.$$

With the help of this kind of estimate and since, for example,

$$\|\varrho_\varepsilon \sin(\varrho_\varepsilon \phi)(s)\|_{L^2(\boldsymbol{R})} \leqslant \|(\varrho_\varepsilon \phi_{0\varepsilon})\|_{L^2(\boldsymbol{R})} + s \sup_{0 \leqslant \tau \leqslant s} \|v(\tau)\|_{L^2(\boldsymbol{R})},$$

we easily find $T' > 0$ and $M > 0$ such that $\overline{F}$ is a strict contraction in E and so there is a unique fixed point U_ε which is a local solution of the Cauchy problem (2.6), (2.7) such that $U_\varepsilon \in C([0, T']; X) \cap C^1([0, T']; (H^1(\boldsymbol{R}))^2)$.

To prove the global existence for U_ε we multiply the first equation by $(u_\varepsilon + u_\varepsilon^3/3)$ and the second by v_ε and we integrate over $\boldsymbol{R}$. We get

$$\frac{\partial}{\partial t} \int_{\boldsymbol{R}} \left[\frac{1}{2} u_\varepsilon^2 + \frac{1}{2} v_\varepsilon^2 + \frac{1}{12} u_\varepsilon^4 + (1 - \cos(\varrho_\varepsilon \phi_\varepsilon)) \right] dx + \varepsilon \int_{\boldsymbol{R}} [u_{\varepsilon x}^2 + v_{\varepsilon x}^2 + u_\varepsilon^2 u_{\varepsilon x}^2]\, dx = 0\,.$$

Moreover, for $\varepsilon \leqslant 1$, we deduce

$$\int_{\boldsymbol{R}} (1 - \cos(\varrho_\varepsilon \phi_{0\varepsilon}))\, dx = \int_{\boldsymbol{R}_-} (\cos 0 - \cos(\varrho_\varepsilon \phi_{0\varepsilon}))\, dx + \int_{\boldsymbol{R}_+} (\cos 2\pi - \cos(\varrho_\varepsilon \phi_{0\varepsilon}))\, dx \leqslant$$

$$\int_{\boldsymbol{R}_-} |\varrho_\varepsilon \phi_{0\varepsilon}|\, dx + \int_0^{1/\varepsilon} (\cos 2\pi - \phi_{0\varepsilon})\, dx + \int_{1/\varepsilon}^{1/\varepsilon + 1} (\cos 2\pi - \cos(\varrho_\varepsilon \phi_{0\varepsilon}))\, dx \leqslant$$

$$\int_{-\infty}^{1} |\phi_0|\, dx + \int_{-1}^{+\infty} |\phi_0 - 2\pi|\, dx + 2\,.$$

Hence, we derive

$$\text{(2.10)} \quad \int_{\boldsymbol{R}} \left[\frac{1}{2} u_\varepsilon^2 + \frac{1}{2} v_\varepsilon^2 + \frac{1}{12} u_\varepsilon^4 + (1 - \cos(\varrho_\varepsilon \phi_\varepsilon)) \right] dx +$$

$$\varepsilon \int_0^t \int_{\boldsymbol{R}} [u_{\varepsilon x}^2 + v_{\varepsilon x}^2 + u_\varepsilon^2 u_{\varepsilon x}^2]\, dx\, d\tau \leqslant c_2$$

where c_2 does not depend on $\varepsilon \leqslant 1$ (energy inequality for the system (2.6)). Now, for each $\varepsilon \leqslant 1$, and by applying to the integral formula (2.8) the L^2 estimates for $G(t)$ $\left(\text{cf. [11], namely } \left\| G(t)\begin{pmatrix} 0 \\ u^2 u_x \end{pmatrix} \right\|_{L^2} \leqslant \frac{c(\varepsilon)}{t^{3/4}} \|u^2 u_x\|_{L^1} \leqslant \frac{c(\varepsilon)}{t^{3/4}} \|u\|_{L^4}^2 \|u_x\|_{L^2}\right)$, we derive, from Gronwall inequality,

$$\|U_\varepsilon(t)\|_{H^1(\boldsymbol{R})} \leqslant c_\varepsilon(t).$$

With the same technique we deduce

$$\|U_\varepsilon(t)\|_{H^3(\boldsymbol{R})} \leqslant c_\varepsilon(t)$$

and so, for each $\varepsilon \leqslant 1$, this proves the global existence of the solution

$$U_\varepsilon = (u_\varepsilon, v_\varepsilon) \in C([0, +\infty[; X) \cap C^1\big([0, +\infty[; (H^1(\boldsymbol{R}))^2\big)$$

for the Cauchy problem (2.6), (2.7).

Now let us introduce, for a fixed $\delta > 0$,

$$\text{(2.11)} \qquad \begin{cases} l_{\pm\varepsilon} = v_\varepsilon + \displaystyle\int_0^{u_\varepsilon} \sqrt{1+\xi^2}\, d\xi \mp (1+\delta)\, t \\ r_{\pm\varepsilon} = v_\varepsilon - \displaystyle\int_0^{u_\varepsilon} \sqrt{1+\xi^2}\, d\xi \pm (1+\delta)\, t\,. \end{cases}$$

The functions $(l_{+\varepsilon}, r_{+\varepsilon})$ and $(l_{-\varepsilon}, r_{-\varepsilon})$ verify, respectively, the system

$$\begin{cases} l_{\pm\varepsilon t} - \sqrt{1+u_\varepsilon^2}\, l_{\pm\varepsilon x} = \varepsilon \Delta l_{\pm\varepsilon} - \dfrac{\varepsilon u_\varepsilon}{\sqrt{1+u_\varepsilon^2}} u_{\varepsilon x}^2 - \varrho_\varepsilon \sin(\varrho_\varepsilon \phi_\varepsilon) \mp (1+\delta) \\ r_{\pm\varepsilon t} + \sqrt{1+u_\varepsilon^2}\, r_{\pm\varepsilon x} = \varepsilon \Delta r_{\pm\varepsilon} + \dfrac{\varepsilon u_\varepsilon}{\sqrt{1+u_\varepsilon^2}} u_{\varepsilon x}^2 - \varrho_\varepsilon \sin(\varrho_\varepsilon \phi_\varepsilon) \pm (1+\delta)\,. \end{cases}$$

Let be, for $(y_1, y_2, t) \in \boldsymbol{R}^2 \times [0, +\infty[$,

$$G_\pm(y_1, y_2, t) = y_2 + \int_0^{y_1} \sqrt{1+\xi^2}\, d\xi \mp (1+\delta)\, t\,,$$

$$F_\pm(y_1, y_2, t) = y_2 - \int_0^{y_1} \sqrt{1+\xi^2}\, d\xi \pm (1+\delta)\, t\,.$$

With $a_\varepsilon = \|l_{\pm\varepsilon}(.\,, 0)\|_{L^\infty(\boldsymbol{R})}$, $b_\varepsilon = \|r_{\pm\varepsilon}(.\,, 0)\|_{L^\infty(\boldsymbol{R})}$, $d_\varepsilon = \max(a_\varepsilon, b_\varepsilon)$, we have

$$|G_\pm(u_{0\varepsilon}(x), v_{0\varepsilon}(x), 0)| \leqslant d_\varepsilon, \quad |F_\pm(u_{0\varepsilon}(x), v_{0\varepsilon}(x), 0)| \leqslant d_\varepsilon, \quad \forall x \in \boldsymbol{R}\,.$$

By an adaptation of the proof of the theorem 14.7 in [12] (invariant regions) we will prove the following

PROPOSITION 2.2. – *We have for* $t \geqslant 0$, $x \in \boldsymbol{R}$,

$$-d_\varepsilon \leqslant G_-(u_\varepsilon(x,t), v_\varepsilon(x,t), t), \quad G_+(u_\varepsilon(x,t), v_\varepsilon(x,t), t) \leqslant d_\varepsilon,$$

$$-d_\varepsilon \leqslant F_+(u_\varepsilon(x,t), v_\varepsilon(x,t), t), \quad F_-(u_\varepsilon(x,t), v_\varepsilon(x,t), t) \leqslant d_\varepsilon.$$

PROOF. – Let us fix $\varepsilon \in]0, 1]$ and put

$$\sum(t) = \{(y_1, y_2) \in \boldsymbol{R}^2 : G_+(y_1, y_2, t) - d_\varepsilon \leqslant 0, F_-(y_1, y_2, t) - d_\varepsilon \leqslant 0,$$
$$-G_-(y_1, y_2, t) - d_\varepsilon \leqslant 0, -F_+(y_1, y_2, t) - d_\varepsilon \leqslant 0\}, \quad t \in [0, +\infty[.$$

We have for $U = (u, v) = (u_\varepsilon, v_\varepsilon) = U_\varepsilon$, $(u_0(x), v_0(x)) \in \sum(0)$, $x \in \boldsymbol{R}$. It is easy to see that if $(u(x,t), v(x,t)) \in \sum(t)$, $\forall x \in \boldsymbol{R}$, $\forall t \in [0, t_0]$ and, for example, $G_+(u(x_0, t_0), v(x_0, t_0), t_0) - d_\varepsilon = 0$ implies $(\partial/\partial t)\, G_+(u(x,t), v(x,t), t) < 0$ for $(x, t) = (x_0, t_0)$, then $(u(x,t), v(x,t), t) \in \sum(t)$, $\forall x \in \boldsymbol{R}$, $\forall t \geqslant 0$. Let

$$g(x,t) = G_+(u(x,t), v(x,t), t) - d_\varepsilon \quad (g \in C([0, +\infty[; C^2(\boldsymbol{R})) \cap C^1([0, +\infty[; C(\boldsymbol{R}))).$$

We have, with $\nabla G_+ = (\sqrt{1+u^2}, 1)$ and by (2.6),

$$\frac{\partial g}{\partial t} = \varepsilon \nabla G_+ \cdot \frac{\partial^2 U}{\partial x^2} + \sqrt{1+u^2}\, \nabla G_+ \cdot \frac{\partial U}{\partial x} - \varrho_\varepsilon \sin(\varrho_\varepsilon \phi_\varepsilon) - (1+\delta).$$

Hence

$$\frac{\partial g}{\partial t} < \varepsilon \nabla G_+ \cdot \frac{\partial^2 U}{\partial x^2} + \sqrt{1+u^2}\, \nabla G_+ \cdot \frac{\partial U}{\partial x}. \tag{2.12}$$

Now let us put $h(x) = g(x, t_0)$. We have $h(x_0) = 0$, $h'(x) = \nabla G_+ \cdot (\partial U/\partial x)(x, t_0)$. As in the proof of theorem 14.7 in [12] we can prove that $h'(x_0) = 0$ and $h''(x_0) \leqslant 0$. Now we have

$$0 \geqslant h''(x_0) = (U_x^T \nabla^2 G_+ U_x)(x_0, t_0) + \nabla G_+ \cdot \frac{\partial^2 U}{\partial x^2}(x_0, t_0),$$

where

$$\nabla^2 G_+ = \begin{pmatrix} \dfrac{u}{\sqrt{1+u^2}} & 0 \\ 0 & 0 \end{pmatrix}$$

Moreover

$$(U_x^T \nabla^2 G_+ U_x)(x_0, t_0) = \frac{u u_x^2}{\sqrt{1+u^2}}(x_0, t_0) \geqslant 0, \quad \text{since } u(x_0, t_0) \geqslant 0$$

because $(u(x_0, t_0), v(x_0, t_0)) \in \sum(t_0)$ and $g(x_0, t_0) = 0$. We derive

$$\nabla G_+ \cdot \frac{\partial^2 U}{\partial x^2}(x_0, t_0) \leqslant 0 \quad \text{and so, by (2.12),} \quad \frac{\partial g}{\partial t}(x_0, t_0) < 0 .$$

This concludes the proof of proposition 2.2. ∎

Now we easily derive (see § 1) from (2.11) and proposition 2.2, that there exists a constant $c_1 > 0$ depending only on u_0 and v_0, such that

$$(2.13) \quad \|u_\varepsilon(.\,, t)\|_{L^\infty(\boldsymbol{R})} \leqslant c_1(1+t)^{1/2}, \quad \|v_\varepsilon(.\,, t)\|_{L^\infty(\boldsymbol{R})} \leqslant c_1(1+t), \quad t \geqslant 0$$

Therefore, if we choose $T > 0$ and put $S_T = \{(x, t) : x \in \boldsymbol{R},\, t \in [0, T[\}$, there is $U_T = (u_T, v_T) \in (L^\infty(S_T))^2$ such that, for a sub-sequence still denoted by U_ε, we have

$$(2.14) \qquad U_\varepsilon \xrightarrow[\varepsilon \to 0]{} U_T \quad \text{in } (L^\infty(S_T))^2 \text{ weak*}$$

Let be (η, q) a pair entropy / entropy flux for the system (2.1). We derive from (2.6),

$$(2.15) \quad \eta(U_\varepsilon)_t + q(U_\varepsilon)_x = \varepsilon\eta(U_\varepsilon)_{xx} - \varepsilon U_{\varepsilon x}^T \nabla^2 \eta(U_\varepsilon)\, U_{\varepsilon x} + \nabla\eta(U_\varepsilon) \cdot (0, -\varrho_\varepsilon \sin(\varrho_\varepsilon \phi_\varepsilon))$$

From (2.10) and (2.13) we deduce that the first term in the right hand side of (2.15) lies in a compact set of $H^{-1}(\boldsymbol{R} \times]0, T[)$ and the second and third terms lie in bounded sets of $L^1(\boldsymbol{R} \times]0, T[)$ and $L^\infty(\boldsymbol{R} \times]0, T[)$, respectively. We conclude that for each bounded open set $\Omega \subset S_T$ the second member of (2.15) lies in a compact set of $H^{-1}(\Omega)$. We continue as in [5] to get, from the div-curl lemma of Murat [10], the Tartar's relation concerning all the pairs $(\eta_1, q_1), (\eta_2, q_2)$, where η_i is a convex entropy (q_i the corresponding flux) and the Young measures ν_{xt} associated to the sequence U_ε which converges in $(L^\infty(S_T))^2$ weak * (cf. [13], [5]). It follows, as in [5], that there exists a sub-sequence $U_{\varepsilon_k} (\varepsilon_k \to 0)$ from U_ε such that

$$(2.16) \qquad U_{\varepsilon_k} \xrightarrow[k \to \infty]{} U_T \quad \text{a.e. in } S_T .$$

Hence, by (2.13),

$$(2.17) \qquad \|u_T\|_{L^\infty(S_T)} \leqslant c_1(1+T)^{1/2}, \quad \|v_T\|_{L^\infty(S_T)} \leqslant c_1(1+T) .$$

Now, by a standard diagonalisation procedure, we can extract a sub-sequence from U_ε, say U_{ε_k}, $\varepsilon_k \to 0$, and $U = (u, v)$, measurable in $\boldsymbol{R} \times [0, \infty[$, such that $U \in (L^\infty(S_T))^2$ for each strip S_T and $U_{\varepsilon_k} \xrightarrow[k \to \infty]{} U$ a.e. in $\boldsymbol{R} \times [0, \infty[$.

Now let be $K = [a, b] \times [0, T]$, $a, b \in \boldsymbol{R}$, $0 < T < +\infty$. We have

$$\phi_\varepsilon(x, t) = \int_0^t v_\varepsilon(x, \tau)\, d\tau + \phi_{0\varepsilon}(x)$$

and from (2.13), (2.6) we deduce $\displaystyle\int_K \left(\int_0^t |v_{\varepsilon_k} - v|\, d\tau \right) dx\, dt \xrightarrow[k \to \infty]{} 0$. Hence,

$$\int_K |\varrho_{\varepsilon_k} \sin(\varrho_{\varepsilon_k} \phi_{\varepsilon_k}) - \sin\phi|\, dx\, dt \xrightarrow[k \to \infty]{} 0 ,$$

where $\phi(x, t) = \displaystyle\int_0^t v(x, \tau)\, d\tau + \phi_0(x)$. This implies, with (2.13) and (2.16), that we can deduce from (2.6) that $U = (u, v)$ satisfies (2.4) and verify an inequality of type (2.17) for each $T > 0$. Moreover, from (2.15) (for $\varepsilon = \varepsilon_k$) we derive

$$\eta(U_{\varepsilon_k})_t + q(U_{\varepsilon_k})_x \leq \varepsilon_k \eta(U_{\varepsilon_k})_{xx} + \nabla\eta(U_{\varepsilon_k}) \cdot (0, \ -\varrho_{\varepsilon_k} \sin(\varrho_{\varepsilon_k} \phi_{\varepsilon_k})) .$$

Passing to the limit $\varepsilon_k \to 0$ we easily derive (2.5) and this achieves the proof of theorem 2.1. ■

REMARK. – After the submission of this paper, Prof. P. Marcati has pointed out to us that in his joint paper with R. Natalini, *Global weak ...*, *J. Math. Soc. Japan*, **50** (1998), 433-449, a similar result to Theorem 2.1 has been proved by a different method.

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