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Sunto. – *In questo lavoro viene trovata un'espressione esplicita per i rappresentanti dei laterali di sottogrupi parabolici di gruppi di Coxeter aventi lunghezza minima: dato un sistema di Coxeter* $(\boldsymbol{W}, S)$ *ed un suo sottogruppo parabolico* $(\boldsymbol{W}_I, I)$, *con* $I \subset S$, *si determina esplicitamente in ogni laterale* $\boldsymbol{W}_I w$ *di* $\boldsymbol{W}_I$ *un elemento avente lunghezza minima. Nella sezione 2 trattiamo i casi classici, i.e.* $\boldsymbol{W} = \boldsymbol{A}_n$, $\boldsymbol{B}_n$ *e* $\boldsymbol{D}_n$. *Dopo ciò, nella sezione 3, diamo una procedura per risolvere il problema nei restanti casi eccezionali, insieme a qualche esempio. Nell'ultima sezione, applichiamo i risultati ottenuti alle fattorizzazioni del polinomio di Poincaré di un gruppo di Coxeter. Le espressioni trovate sono utili per scrivere algoritmi che permettano il calcolo su computer della coomologia dei gruppi di Artin, come osservato alla fine dell'articolo.*

1. – Introduction.

Let $(\boldsymbol{W}, S)$ be an irreducible Coxeter system (see [1], [6]), that is, a group which is generated by the elements $s \in S$ subject only to relations of the kind

$$(ss')^{m(s, s')} = 1 , \tag{1}$$

where $m(s, s) = 1$, $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S. The parabolic subgroups of $\boldsymbol{W}$ are those subgroups which are conjugate to subgroups $\boldsymbol{W}_I$ generated by a subset of $I \subset S$. Let $I \subset S$ be such a subset. Then it is known that $(\boldsymbol{W}_I, I)$ is itself a Coxeter system and its length function is just the restriction of the length function of $\boldsymbol{W}$. Moreover, for any coset $\boldsymbol{W}w$ there is a unique $v \in \boldsymbol{W}w$ of minimal length.

In this paper, we provide explicitly the minimal length representatives of the cosets of $\boldsymbol{W}_I$ in $\boldsymbol{W}$. These expressions are the smallest in the lexicographic order, among all the reduced expressions for the minimal length representative of the given coset. We also provide some examples and applications. In section 2 we treat the classical cases, i.e. $\boldsymbol{W}$ a finite Coxeter group belonging to one of the three families $\boldsymbol{A}_n$, $\boldsymbol{B}_n$ and $\boldsymbol{D}_n$. Then, in section 3, we use some ideas from [5] to solve the exceptional cases. In the last section, we apply the results of section 2 to obtain an easy proof of the well-known formula for the Poincaré polynomial of $\boldsymbol{W}$. These results have been used in [10] to compute the cohomology of Artin groups: it follows from [3] that the knowledge of the reduced expressions of minimal coset representatives is a key ingredient in order to perform effective computations.

2. – Minimal length cosets representatives, classical cases.

First of all, some notations. In place of $S = \{s_1, \ldots, s_n\}$, we shall often write $S = \{1, \ldots, n\}$. If $\Gamma \subseteq \Gamma' \subseteq S$, then $\boldsymbol{W}_\Gamma$ will denote the parabolic subgroup generated by Γ and $\boldsymbol{W}_\Gamma^{\Gamma'}$ will denote a complete system of representatives for the cosets of $\boldsymbol{W}_\Gamma$ in $\boldsymbol{W}_{\Gamma'}$, each of which having minimal length in its coset, see [1]. This property is equivalent to $l(s_h w) > l(w)$ for each h in Γ.

We say that $w \in \boldsymbol{W}_\Gamma^{\Gamma'}$ is reduced mod $\boldsymbol{W}_\Gamma$. Elements which are reduced mod $\boldsymbol{W}_\Gamma$ for $\Gamma = \emptyset$, are simply called reduced. They are called normal if their expression in term of the generators $\{s_1, \ldots, s_n\}$ is the smallest with regard to the lexicographic ordering in $1, \ldots, n$.

In order to find a useful expression for the elements w in the set $\boldsymbol{W}_\Gamma^{\Gamma'}$, we first prove the following lemma, which holds in a more general context.

LEMMA 1. – *Let* $(\boldsymbol{W}, S)$ *be a Coxeter system, with*

$$S = (s_1, \ldots, s_n) \quad \text{and} \quad I = \{1, \ldots, n\}.$$

Let also H *and* K *be subsets of* I *and consider the three subgroups* $\boldsymbol{W}_H$, $\boldsymbol{W}_K$ *and* $\boldsymbol{W}_{H \cap K}$. *If*

$$\boldsymbol{W}_{H\cap K}^K = \{\alpha_j\}, \quad \boldsymbol{W}_{H\cap K}^H = \{\beta_k\}, \quad \boldsymbol{W}_K^I = \{\gamma_l\} \quad \text{and} \quad \boldsymbol{W}_H^I = \{\delta_m\},$$

then the set of minimal length representatives of the cosets of $\boldsymbol{W}_{H\cap K}$ *in* $\boldsymbol{W}$ *is*

$$\boldsymbol{W}_{H\cap K}^I = \{\alpha_j \gamma_l\} = \{\beta_k \delta_m\}.$$

PROOF. – It is enough to see that $\{\alpha_j \gamma_l\}$ is the set of representatives with minimal length for the cosets of $\boldsymbol{W}_{H\cap K}$ in $\boldsymbol{W}$.

By hypothesis, we have $l(s_i \alpha_j) > l(\alpha_j)$ for every $i \in H \cap K$ and $l(s_i \gamma_l) > l(\gamma_l)$ for every $i \in K$. Moreover, $l(v\gamma_l) > l(\gamma_l)$ for every $v \in \boldsymbol{W}_K$. We have to show that $l(s_i \alpha_j \gamma_l) > l(\alpha_j \gamma_l)$ for each i in $H \cap K$. Let us suppose the contrary, for a particular triple i, j, l. By the Exchange conditions (see [1] and [6]) we have (omitting the subscripts)

$$s\alpha\gamma = \begin{cases} \widehat{\alpha}\gamma \\ \alpha\widehat{\gamma} \end{cases}$$

where, if $\alpha = s_{i_1} \dots s_{i_m}$ then

$$\widehat{\alpha} \doteqdot s_{i_1} \dots s_{i_{h-1}} s_{i_{h+1}} \dots s_{i_m} \doteqdot s_{i_1} \dots s_{i_{h-1}} \widehat{s}_{i_h} s_{i_{h+1}} \dots s_{i_m}$$

for some $1 \leqslant h \leqslant m$ denotes the omission of one of the generators. We give a similar meaning to $\widehat{\gamma}$.

In the first case, we obtain $s\alpha = \widehat{\alpha}$ and $l(s\alpha) < l(\alpha)$ with s in $\boldsymbol{W}_{H \cap K}$, against our hypothesis. So the second case must hold. Then $\widehat{\gamma} = \alpha^{-1} s \alpha \gamma$; but α, s and α^{-1} are all in $\boldsymbol{W}_K$ so that

$$\boldsymbol{W}_K \widehat{\gamma} = \boldsymbol{W}_K \alpha^{-1} s \alpha \gamma = \boldsymbol{W}_K \gamma \,.$$

This means that γ and $\widehat{\gamma}$ are in the same coset of $\boldsymbol{W}_K$ and $l(\widehat{\gamma}) < l(\gamma)$, which contradicts our choice of the γ's. ■

With the aid of this lemma, we can solve the problem of finding canonical expressions for the elements in the sets $\boldsymbol{W}_K^I$ in the case of finite Coxeter groups: the lemma tells us that it is enough to restrict our attention to the principal case, in which $\boldsymbol{W} = \langle s_1, \dots, s_n \rangle$, $I = \{1, \dots, n\}$ and $K = I \setminus \{i\}$. We treat separately the three cases $\boldsymbol{W} = \boldsymbol{A}_n$, $\boldsymbol{B}_n$ and $\boldsymbol{D}_n$.

Let us first consider $\boldsymbol{W} = \boldsymbol{A}_n$. We use the canonical presentation for this group:

$$\boldsymbol{W} = \langle s_1, \dots, s_n \,|\, (s_j s_{j'})^{m(j, j')} = 1 \rangle .$$

with $m(j, j) = 1$, $m(j, j') = 2$ for $|j - j'| > 1$ and $m(j, j+1) = 3$ for $j = 1, \dots, n-1$. We also define $w_j(l_j) \doteqdot s_j s_{j-1} \dots s_{j - l_j + 1}$ (provided that $l_j \leqslant j$).

THEOREM 2. – *If $\boldsymbol{W} = \boldsymbol{A}_n$ then, with the preceding notations,*

$$\boldsymbol{W}^I_{I \setminus \{i\}} = \{ w_i(l_i) \dots w_n(l_n) \,|\, 0 \leqslant l_j \leqslant j \text{ and } l_{j+1} \leqslant l_j \} .$$

PROOF. – We first consider the case $i = n$. Let $w = s_n \dots s_1 = w_n(n)$. We claim that it is one of the elements of $\boldsymbol{W}^I_{I \setminus \{n\}}$. To prove this, we have to show that $l(s_j w) > l(w)$ if $j \neq n$. If $l(s_j w) < l(w)$, then

$$s_j w = s_n \dots \widehat{s}_h \dots s_1$$

and

$$s_j s_n \dots s_h = s_n \dots s_{h+1} .$$

If $j \leqslant h - 1$ then

$$s_n \dots s_{h+1} s_j s_h = s_n \dots s_{h+1} ,$$

which implies $s_j s_h = 1$, absurd.

So $j \geqslant h$ must hold and we get

$$s_j s_{j+1} s_j \dots s_h = s_{j+1} s_j \dots s_{h+1}$$

which means

$$s_{j+1} s_j s_{j+1} \dots s_h = s_{j+1} s_j \dots s_{h+1}.$$

Now, s_{j+1} commutes with $s_{j-1}, \dots, s_h$ so that we obtain $s_{j+1} s_h = 1$, which is incompatible with the relations defining our group.

We have thus found one of the elements of $\boldsymbol{W}^I_{I \setminus \{n\}}$. But if w is reduced mod $\boldsymbol{W}_{I \setminus \{n\}}$ then it is easy to see that so is also each of its substrings $s_n s_{n-1} \dots s_j$. In this way we account for $n+1$ elements in $\boldsymbol{W}^I_{I \setminus \{n\}}$. On the other hand, this set has exactly $n+1$ elements, so that we have found all of them.

We now consider the general case $K = I \setminus \{i\}$, with $i < n$. We use lemma 1 with $H = I \setminus \{n\}$. So we are given $\boldsymbol{W}^I_H = \{\delta_m\}$ and $\boldsymbol{W}^H_{H \cap K} = \{\beta_k\}$. Omitting the subscript, we must see for which $\beta\delta$ it holds $\beta\delta \in \boldsymbol{W}^I_K$ (for, since

$$\boldsymbol{W}^I_{H \cap K} = \{\alpha_j \gamma_l\} = \{\beta_k \delta_m\},$$

then $\{\gamma_l\}$ is contained in $\boldsymbol{W}^I_K$).

We know the δ's by the preceding case and the β's are known by induction:

$$\{\delta_m\} = \{w_n(l_n) \text{ for } 0 \leqslant l_n \leqslant n\},$$
$$\{\beta_k\} = \{w_i(l_i) \dots w_{n-1}(l_{n-1}) \text{ for } 0 \leqslant l_j \leqslant j \text{ and } l_{j+1} \leqslant l_j\}.$$

We have to show that $\beta\delta$ is reduced mod $\boldsymbol{W}_K$ if, and only if, $l_n \leqslant l_{n-1}$.

Suppose $l_n > l_{n-1}$. Then we must show that there exists a $j \neq i$ such that

$$l(s_j \beta\delta) < l(\beta\delta),$$

that is, $s_j \beta\delta$ is not reduced.

Since $\beta\delta$ is reduced mod $\boldsymbol{W}_{H \cap K}$, it is clear that this can eventually hold only for $j = n$. So let us look closer at $s_n \beta\delta$. In the expression of $w_j(l_j)$ there is no occurrence of s_n nor s_{n-1}, when $j \leqslant n-1$, thus it is enough to show that $s_n w_{n-1}(l_{n-1}) w_n(l_n)$ is not reduced. Moreover, given that $l_n > l_{n-1}$, it is enough to prove that $s_n w_{n-1}(l_{n-1}) w_n(l_{n-1}+1)$ is not reduced. We have then restricted the problem to proving that expressions of the form

$$s_n s_{n-1} \dots s_h s_n \dots s_h$$

are not reduced. We prove this by (reverse) induction on h.

If $h = n$, (i.e., $l_{n-1} = 0$) $s_n s_n = 1$ is not reduced.

If $0 < h < n$, then from

$$s_h s_n \dots s_h = s_n \dots s_{h+2} s_h s_{h+1} s_h$$
$$= s_n \dots s_{h+2} s_{h+1} s_h s_{h+1}$$

follows

$$s_n s_{n-1} \dots s_h s_n \dots s_h = s_n s_{n-1} \dots s_{h+1} s_n \dots s_{h+1} s_h s_{h+1}$$

and, by the inductive hypothesis, $s_n s_{n-1} \dots s_{h+1} s_n \dots s_{h+1}$ is not reduced. We have thus proved that $\{\gamma_l\} \subseteq \{\beta_k \delta_m \,|\, l_n \leqslant l_{n-1}\}$.

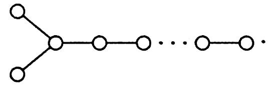
On the other side, the first set has $\binom{n+1}{i}$ elements, whereas the second has as many elements as the set of non-increasing applications $(l_i, \dots, l_n)$ from $\{i, \dots, n\}$ to $\{0, \dots, i\}$. The latters are exactly $\binom{n+1}{i}$ and so $\{\gamma_l\} = \{\beta_k \delta_m \,|\, l_n \leqslant l_{n-1}\}$. ■

REMARK 3. – In the canonical homomorphism $\boldsymbol{A}_n \to S_{n+1}$, the permutations corresponding to the images of the elements that we have found in Theorem 2 are also called the «shuffle» permutations of $\{1, \dots, i\}$ and $\{i+1, \dots, n\}$.

Similar results hold in the cases $\boldsymbol{W} = \boldsymbol{B}_n, \boldsymbol{D}_n$, but the corresponding formulas are quite more complicated. We study first the case $\boldsymbol{W} = \boldsymbol{D}_n$. To simplify notations, we use a convention slightly different from usual: we consider a canonical numbering of its generators inverted with respect to the usual one. That is:

$$\boldsymbol{D}_n = \langle s_1, \dots, s_n \,|\, (s_1 s_3)^3 = 1, (s_1 s_i)^2 = 1 \text{ for } i \neq 3,$$
$$s_i^2 = 1 \ \forall i, (s_i s_{i+1})^3 = 1 \text{ for } i > 1$$
$$s_i s_j = s_j s_i \text{ for } |\mathrm{i} - \mathrm{j}| > 1, \mathrm{i}, \mathrm{j} > 1 \rangle$$

which correspond to the Coxeter graph



We also use elements $w_j(l_j)$ in the same way as in the proof of case $\boldsymbol{A}_n$, but they are defined in a different way. Namely, we set

$$w_j \doteqdot s_j s_{j-1} \dots s_4 s_3 s_1 s_2 s_3 s_4 \dots s_{n-1} s_n$$

and with $w_j(l_j)$ we indicate, this time, the (left) substring of w_j having length l_j, with $0 \leqslant l_j \leqslant n + j - 2$. We remark that in the definition of w_j it makes no difference whether we write $\dots s_1 s_2 \dots$ or $\dots s_2 s_1 \dots$.

THEOREM 4. – *If* $\boldsymbol{W} = \boldsymbol{D}_n$ *and* $i \geqslant 3$ *then, with the preceding notations,*

$$\boldsymbol{W}^I_{I \setminus \{i\}} = \{w_i(l_i) \ldots w_n(l_n)\}$$

with $l_i, \ldots, l_n$ *satisfying*

i) $0 \leqslant l_j \leqslant j + i - 2$

ii) $l_{j+1} \leqslant l_j + 1$

iii) $l_{j+1} \leqslant l_j$ *if* $l_j \leqslant j - 2$

iv) if $l_{j+1} = l_j + 1 = j$ *then* $w_j(l_j)$ *and* $w_{j+1}(l_{j+1})$ *must be chosen in such a way that one has* s_1 *as final element and the other has* s_2.

REMARK 5. – If $l_j = j - 1$ then, with our notation, $w_j(j-1)$ is not uniquely defined, but it has two distincts values: one ending with s_1 and the other with s_2.

PROOF. – In the following we shall use the following trivial formulas:

$$\begin{aligned} &s_h s_n \ldots s_{h+1} s_h = s_n \ldots s_{h+1} s_h s_{h+1} \qquad \text{for } h \geqslant 2, \\ &s_1 s_n \ldots s_3 s_1 = s_n \ldots s_3 s_1 s_3 \qquad \text{for } h = 1 \end{aligned} \tag{2}$$

and

$$s_h s_{h+1} s_h \ldots s_3 s_1 s_2 s_3 \ldots s_h s_{h+1} = s_{h+1} s_h \ldots s_3 s_1 s_2 s_3 \ldots s_h s_{h+1} s_h . \tag{3}$$

Again, we first show the result for $i = n$. To this aim, we consider $w_n \doteqdot w_n(2n-2)$ and we claim that if $j < n$ then $l(s_j w_n) > l(w_n)$. Suppose not. Then $s_j w_n = \widehat{w}_n$ for some $j < n$. Since, by (3), we have $s_j w_n = w_n s_j$, then we can assume that the cancellation in $\widehat{w}_n$ takes place within the firsts $n-1$ elements, otherwise we proceed as follows with the only change of considering left cosets instead of right cosets (we remark that all the theorems in this section have a symmetric version for left cosets which is proved in the same way as the ones we prove, so we can assume that theorem 2 is proved even for left cosets. Plainly, in this case, we have to consider $\widetilde{w}_j(l_j) \doteqdot s_{j-l_j+1} \ldots s_j$).

We remark that $\langle s_2, s_3, \ldots, s_n \rangle \cong \langle s_1, s_3, \ldots, s_n \rangle \cong \boldsymbol{A}_{n-1}$.

So suppose $\widehat{w}_n = s_n \ldots \widehat{s}_h \ldots s_3 s_2 s_1 s_3 \ldots s_n$, We consider first the case in which h has no restriction if $j > 2$ and if $j = 1$ or 2 then $h = j$ or $h > 2$. Then $s_j w_n = \widehat{w}_n$ becomes

$$\begin{aligned} &s_j s_n \ldots s_h = s_n \ldots s_{h+1} \qquad \text{if } h > 1, \\ &s_j s_n \ldots s_3 s_1 = s_n \ldots s_3 \qquad \text{if } h = 1, \end{aligned}$$

which gives a contradiction to the case $\boldsymbol{A}_{n-1}$. In the remaining cases, that is $j = 1$ or 2 and $h = 2$ or 1, then we get, for example,

$$s_1 s_n \ldots s_3 s_2 = s_n \ldots s_3$$

which is easily seen impossible, since it reduces to $s_1 s_3 s_2 = s_3$. So w_n is in $\boldsymbol{W}^I_{I\setminus\{n\}}$ and the same holds for each of its substring. Since there are exactly $2n$ of them (see remark 5), we are done in this case.

Let now $3 \leqslant i < n$. As in Theorem 2, we apply Lemma 1 with $K = I\setminus\{i\}$ and $H = I\setminus\{n\}$. With the notations of the lemma, using the preceding case and the inductive hypothesis we have to see when $l(s_j\beta\delta) > l(\beta\delta)$ for each $j \neq i$, where $\delta = w_n(l_n)$ and $\beta = w_i(l_i)\dots w_{n-1}(l_{n-1})$, assuming that $l_i, \dots, l_{n-1}$ satisfy conditions *i)*-*iv)*.

First of all, since $j \in H \cap K$ if, and only if, $j \neq i, n$, lemma 1 implies $l(s_j\beta\delta) > l(\beta\delta)$ for each such j. Thus it is enough to show that $l_i, \dots, l_n$ satisfy *i)*-*iv)* if, and only if, $l(s_n\beta\delta) > l(\beta\delta)$. We first prove the «if» part by proving that if $l_i, \dots, l_n$ do not satisfy *i)*-*iv)* then $l(s_n\beta\delta) < l(\beta\delta)$. We remark that, by induction and lemma 1, the given expression for $w_i(l_i)\dots w_n(l_n)$ is reduced and, since s_n commutes with $w_i(l_i), \dots, w_{n-2}(l_{n-2})$, it is enough to show that if $l_i, \dots, l_n$ do not satisfy *i)*-*iv)* then

$$l(s_n w_{n-1}(l_{n-1})\, w_n(l_n)) < l(w_{n-1}(l_{n-1})\, w_n(l_n))\,.$$

Moreover, since $l_i, \dots, l_{n-1}$ are assumed to satisfy *i)*-*iv)*, l_n is the only number for which one of the conditions does not hold. It is easy to see that if l_n does not fulfill *i)* then it does not fulfill one of *ii)* or *iii)*, thus it is enough to assume that l_n does not satisfy one of *ii)*-*iv)*. Finally, it is enough to study the case when l_n is the *smallest* value which does not fulfill the conditions.

— $l_{n-1} \leqslant n-3$.

We contradict *iii)* by requiring $l_n = l_{n-1}+1$. We prove what requested by induction on l_{n-1}. Since $s_n^2 = 1$, the case $l_{n-1} = 0$ is trivial. Now, by (2),

$$\begin{aligned} s_n w_{n-1}(l_{n-1})\, w_n(l_{n-1}+1) &= s_n s_{n-1} s_{n-2} \dots s_{n-l_{n-1}+1} s_{n-l_{n-1}} \\ &\qquad s_n s_{n-1} s_{n-2} \dots s_{n-l_{n-1}+1} s_{n-l_{n-1}} \\ &= s_n s_{n-1} s_{n-2} \dots s_{n-l_{n-1}+1} \\ &\qquad s_n s_{n-1} s_{n-2} \dots s_{n-l_{n-1}+1} s_{n-l_{n-1}} s_{n-l_{n-1}+1} \\ &= s_n w_{n-1}(l_{n-1}-1)\, w_n(l_{n-1})\, s_{n-l_{n-1}} s_{n-l_{n-1}+1}\,. \end{aligned}$$

By induction, we have $l(s_n w_{n-1}(l_{n-1}-1)\, w_n(l_{n-1})) < l(w_{n-1}(l_{n-1}-1)\cdot w_n(l_{n-1}))$, so we are done.

— $l_{n-1} \geqslant n-2$ and $l_n > l_{n-1}+1$.

Again, we proceed by induction, the first step being $l_{n-1} = n-2$. We consider only the case in which $w_{n-1}(l_{n-1})$ ends by s_1, the other case being similar. We have

$$s_n w_{n-1}(n-2)\, w_n(n) = s_n s_{n-1} \dots s_3 s_1 s_n s_{n-1} \dots s_3 s_1 s_2 .$$

In this case, s_1 commutes with all of the firsts $n-2$ terms of $w_n(n)$, so that

$$s_n w_{n-1}(n-2)\, w_n(n) = s_n w_{n-1}(n-3)\, w_n(n-3)\, s_1 s_3 s_1 s_2 .$$

But $s_1 s_3 s_1 s_2 = s_3 s_1 s_3 s_2$ which implies

$$\begin{aligned} s_n w_{n-1}(n-2)\, w_n(n) &= s_n w_{n-1}(n-3)\, w_n(n-3)\, s_1 s_3 s_1 s_2 \\ &= s_n w_{n-1}(n-3)\, w_n(n-2)\, s_1 s_3 s_2 \end{aligned}$$

and we are reduced to the preceding case.

Now we go on by induction. Again, it is enough to consider $l_n = l_{n-1} + 2$: then, using (3), it is easy to obtain

$$s_n w_{n-1}(l_{n-1})\, w_n(l_{n-1}+2) = s_n w_{n-1}(l_{n-1}-1)\, w_n(l_{n-1}+2)\, s_{l_{n-1}-n+3}$$

which is not reduced, mod $\boldsymbol{W}^I_{I\setminus\{i\}}$, by the inductive hypothesis.

— $l_{n-1} = n-1$ and $l_n = n$.

In this case, to contradict *iv*), we have to study $s_n s_{n-1} \dots s_3 s_2 s_n s_{n-1} \dots s_3 s_2$, for example; but $\langle s_2, s_3, \dots, s_n \rangle \cong \boldsymbol{A}_{n-1}$ and then, by theorem 2, this is not reduced.

Now we have to prove the converse: if $l_i, \dots, l_n$ satisfy the conditions of the theorem, then the element $w_i(l_i) \dots w_n(l_n)$ is reduced mod $\boldsymbol{W}^I_{I\setminus\{i\}}$. By lemma 1 we get that all the elements of this form are reduced mod $\boldsymbol{W}^I_{I\setminus\{i,n\}}$, so we must only show that

$$l(s_n w_i(l_i) \dots w_n(l_n)) > l(w_i(l_i) \dots w_n(l_n)) .$$

Suppose not. Then

$$s_n w_i(l_i) \dots w_n(l_n) = w_i(l_i) \dots \widehat{w}_h(l_h) \dots w_n(l_n)$$

where, as usual, hat means omission of one of the generators.

If $h < n$ then we get

$$s_n = w_i(l_i) \dots \widehat{w}_h(l_h)(w_h(l_h))^{-1} \dots (w_i(l_i))^{-1} \in \langle s_1, \dots, s_{n-1} \rangle ,$$

a contradiction. Then $h = n$ and we easily get

$$s_n w_{n-1}(l_{n-1})\, w_n(l_n) = w_{n-1}(l_{n-1})\, \widehat{w}_n(l_n) .$$

There are two subcases: either

$$s_n s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_j s_n s_{n-1} \ldots s_h =$$

$$= s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_j s_n s_{n-1} \ldots s_{h+1}$$

or

$$s_n s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_j s_n s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_h =$$

$$= s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_j s_n s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_{h-1}$$

according to the position in which occurs the cancellation of s_h (i.e., in the left or right part of $w_n(l_n)$), where s_h is the omitted generator in $\widehat{w}_n(l_n)$ (with $h \leqslant j$, in the second subcase).

In both cases, we obtain an equality of the type

$$s_n w_{n-1}(l_{n-1})\, w_n(l) = w_{n-1}(l_{n-1})\, w_n(l-1)$$

with $l < l_n$, which we can also write as

$$(4) \qquad s_n s_{n-1} w_{n-2}(l_{n-1}-1)\, s_n w_{n-1}(l-1) = s_{n-1} w_{n-2}(l_{n-1}-1)\, s_n w_{n-1}(l-2).$$

If $j \leqslant n-2$, we have that s_n commutes with $w_{n-2}(l_{n-1}-1)$ so that (4) reduces to

$$s_{n-1} w_{n-2}(l_{n-1}-1)\, w_{n-1}(l-1) = w_{n-2}(l_{n-1}-1)\, w_{n-1}(l-2).$$

If $j = n-1$ then we write (4) as

$$s_n s_{n-1} w_{n-2}(l_{n-1}-2)\, s_{n-1} s_n s_{n-1} w_{n-2}(l-2) =$$

$$= s_{n-1} w_{n-2}(l_{n-1}-2)\, s_{n-1} s_n s_{n-1} w_{n-2}(l-3)$$

and, using the relation $s_{n-1} s_n s_{n-1} = s_n s_{n-1} s_n$ and the fact that s_n commutes with the element $w_{n-2}(l_{n-1}-2)$, we get

$$s_{n-1} w_{n-2}(l_{n-1}-2)\, w_{n-1}(l-1) = w_{n-2}(l_{n-1}-2)\, w_{n-1}(l-2).$$

In both cases, we contradict the inductive hypothesis for $\boldsymbol{D}_{n-1} = \langle s_1, \ldots, s_{n-1} \rangle$.

The initial step for the induction is provided by the case $\boldsymbol{D}_4$, where it is easily seen, even by direct inspection, that our claim holds. ■

In the previous theorem we have left out the cases $i = 1, 2$. We deal here with the case $i = 1$ (the case $i = 2$ is similar). In this case it is easy to see, by an induction similar to the case $i = n$, that if $w_1(l) \doteqdot s_1 s_3 s_4 s_5 \ldots s_{l+1}$,

$w_2(l) \doteqdot s_2 s_3 s_4 s_5 \ldots s_{l+1}$ for $l = 2, \ldots, n-1$ and $w_i = s_i$ for $l = 1$, $i = 1, 2$ then

$$\boldsymbol{W}^I_{I \setminus \{1\}} = \{ w_1(l_1)\, w_2(l_2)\, w_1(l_3)\, w_2(l_4) \ldots w_{(3+(-1)^{h-1})/2}(l_{h-1})\, w_{(3+(-1)^h)/2}(l_h)$$
$$\text{such that } 0 \leqslant h \leqslant n-1 \text{ and } l_j > l_{j+1} \}\,.$$

We are left the last case: $\boldsymbol{W} = \boldsymbol{B}_n$. We make again the convention of inverting the indices, that is we consider the group associated with the graph

and we define

$$w_j \doteqdot s_j s_{j-1} \ldots s_2 s_1 s_2 \ldots s_{n-1} s_n$$

and with $w_j(l_j)$ we denote, in this last case, the substring of w_j composed by the firsts l_j tokens (from the left), for $0 \leqslant l_j \leqslant n + j - 1$. Moreover, we record for later reference the following trivial formula, which hold in case $\boldsymbol{B}_n$:

$$s_j s_{j+1} s_j \ldots s_2 s_1 s_2 \ldots s_j s_{j+1} = s_{j+1} s_j \ldots s_2 s_1 s_2 \ldots s_j s_{j+1} s_j\,. \tag{5}$$

THEOREM 6. – *If* $\boldsymbol{W} = \boldsymbol{B}_n$ *then, with the preceding notations,*

$$\boldsymbol{W}^I_{I \setminus \{i\}} = \{ w_i(l_i) \ldots w_n(l_n) \}$$

with $l_j, \ldots, l_n$ *satisfying*

i) $0 \leqslant l_j \leqslant j + i - 1$

ii) $l_{j+1} \leqslant l_j + 1$

iii) $l_{j+1} \leqslant l_j$ *if* $l_j \leqslant j - 1$.

PROOF. – In the proof we shall need remarks which are quite similar to the ones we made during the proof of theorem 4, so we will not go into details as before.

As usual, we first see what happens when $i = n$. Let us show that the element $w_n \doteqdot w_n(2n-1)$ is in $\boldsymbol{W}^I_{I \setminus \{i\}}$. Suppose not, so that $l(s_j w_n) < l(w_n)$ for some $j \neq n$; then $s_j w_n = \widehat{w}_n$. Since from (5) we get $s_j w_n = w_n s_j$, we can suppose in the same way as we made in theorem 4, that the cancellation in w_n is between its firsts n elements and let h be the omitted generator; thus $s_j w_n = \widehat{w}_n$ becomes

$$s_j s_n s_{n-1} \ldots s_2 s_1 = s_n s_{n-1} \ldots \widehat{s}_h \ldots s_2 s_1\,. \tag{6}$$

If $h > 1$, then we contradict the result for the case $\boldsymbol{A}_{n-1}$, so that $h = 1$ must hold. Now multiply (6) on the right by $s_1, \ldots, s_n$. It becomes $s_j = w_n$ and we easily get from this that

$$s_{j+1} = s_{j-1} s_{j-2} \ldots s_2 s_1 s_2 \ldots s_{j-2} s_{j-1}$$

which implies $s_{j+1} \in \langle s_1, \ldots, s_{j-1} \rangle$, a contradiction.

We have seen that w_n is reduced mod $\boldsymbol{W}_{I \backslash \{n\}}$ and then the same must hold for each of its (left) substrings. Since there are exactly $2n$ of these, which is the same number of the cosets of $\boldsymbol{W}_{I \backslash \{n\}}$ in $\boldsymbol{W}$, we have found all of them.

Now we can proceed with the general case. As we did in theorem 2, we apply lemma 1 with $K = I \backslash \{i\}$ and $H = I \backslash \{n\}$. The inductive hypothesis gives us the set of representatives

$$\boldsymbol{W}^H_{H \cap K} = \{ w_i(l_i) \ldots w_{n-1}(l_{n-1}) \},$$

where the $l_i, \ldots, l_{n-1}$ satisfy the stated conditions, and the case $i = n$ gives us

$$\boldsymbol{W}^I_H = \{ w_n(l_n) \mid 0 \leqslant l_n \leqslant 2n - 1 \}.$$

Applying lemma 1, we have to look when the element $w_i(l_i) \ldots w_n(l_n)$ is reduced mod $\boldsymbol{W}_K$, assuming that $l_i, \ldots, l_{n-1}$ satisfy the conditions of the theorem.

So let us suppose l_n does not satisfy one of the three conditions in the thesis. As in the proof of theorem 2, using induction we easily reduce to prove that

$$l(s_n w_{n-1}(l_{n-1}) \, w_n(l_n)) < l(w_{n-1}(l_{n-1}) \, w_n(l_n)).$$

If $l_{n-1} \leqslant n-2$, then we can proceed exactly as in the case $\boldsymbol{A}_n$, since we have a group isomorphism $\langle s_2, \ldots, s_n \rangle \cong \boldsymbol{A}_{n-1}$. Then $l_n \leqslant l_{n-1}$ must hold.

It is easy to see that, when $l_{n-1} \geqslant n-1$, it is enough to treat the case $l_n = l_{n-1} + 2$ (otherwise we can reduce to this). First let $l_{n-1} = n - 1$. Then

$$\begin{aligned} s_n s_{n-1} \ldots s_2 s_1 s_n s_{n-1} \ldots s_2 s_1 s_2 &= s_n s_{n-1} \ldots s_2 s_n s_{n-1} \ldots s_1 s_2 s_1 s_2 \\ &= s_n s_{n-1} \ldots s_2 s_n s_{n-1} \ldots s_2 s_1 s_2 s_1 \end{aligned}$$

and we are led to the preceding case.

Now let us suppose $l_{n-1} > n-1$; we apply (5):

$$s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_j s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_j s_{j+1} =$$

$$= s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_{j-1} s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_j s_{j+1} s_j$$

and we can invoke induction to conclude.

So far, we have proved that if $w_i(l_i) \dots w_n(l_n)$ is reduced mod $\boldsymbol{W}_K$, then $l_i, \dots, l_n$ must satisfy the conditions *i*), *ii*) and *iii*).

To prove the converse, we could count how many elements we obtain this way, but we could also proceed directly in the same way as in the case of $\boldsymbol{D}_n$: the proof goes on nearly in the same way. ■

3. – Exceptional cases.

As for the exceptional cases, we outline an algorithm which provides a complete list of minimal length coset representatives.

We recall that given an expression of $w \in \boldsymbol{W}$ by means of the reflections in S, there are algorithms which reduce w in normal reduced form, see [5]. When w is in such a form, we can write $w = xy$ in only one way if we require $x \in \boldsymbol{W}_\Gamma$, $x \notin \boldsymbol{W}_{\Gamma'}$, whenever $\Gamma \subset \Gamma' \subseteq S$ and, clearly, we have $y \in \boldsymbol{W}_\Gamma^{\Gamma'}$. We call y the Γ'-part of w.

As we have already remarked, if $w \in \boldsymbol{W}_\Gamma^{\Gamma'}$ and $w = s_{i_1} \dots s_{i_h}$, then we also have that $s_{i_1} \dots s_{i_j} \in \boldsymbol{W}_\Gamma^{\Gamma'}$ for all $j \leq h$. Moreover, we can suppose $\Gamma' = S$, $\Gamma = S \backslash \{i\}$.

Now, in order to get the list $\mathcal{L}$ of the elements of $\boldsymbol{W}_\Gamma^{\Gamma'}$, one can proceed as follows. Let $\mathcal{L} = \{\emptyset\}$, $l = 0$.

i) Pick an element $w \in \mathcal{L}$ of length l.

ii) For all $s \in S$, consider ws and reduce it in normal form. If its S-part y is not in $\mathcal{L}$, then let $\mathcal{L} = \mathcal{L} \cup \{y\}$.

iii) If in $\mathcal{L}$ there is another element of length l, then go to *i*). If all elements of length l have been considered, then let $l = l+1$; if for a given length l no element in *ii*) has been added to $\mathcal{L}$, then $\mathcal{L} = \boldsymbol{W}_\Gamma^{\Gamma'}$, else go to *i*).

We now give some examples obtained with the above procedure for some exceptional group. Elements in $\boldsymbol{W}_{I \backslash \{i\}}^I$ are represented by a (directed) tree: starting from the vertex on the left, go outward (with regard to this vertex) and multiply by the generator s_j if j is on the edge.

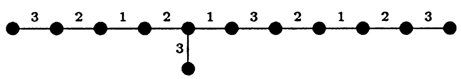
First of all, we remark that it is trivial to solve the case $\boldsymbol{W}=\boldsymbol{I}_2(m)$: it is clear that

$$\boldsymbol{W}_{\{1\}}^{\{1,2\}}=\{1,\, s_2,\, s_2 s_1,\, s_2 s_1 s_2,\, \ldots,\, \underbrace{s_2 s_1 s_2 \ldots}_{m-1 \text{ factors}}\},$$

that is, the whole (left) substring of the longest among the representatives. In the same way we obtain $\boldsymbol{W}_{\{2\}}^{\{1,2\}}$.

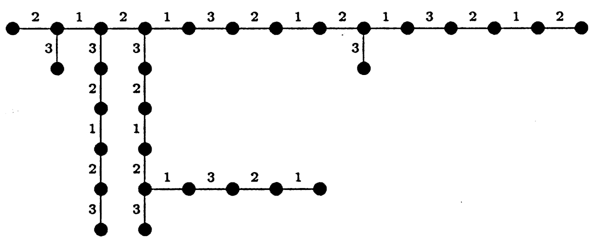
Now we come to $\boldsymbol{W}=\boldsymbol{H}_3$. We have

$\mathbf{W}_{\{1,2\}}^{\{1,2,3\}} =$



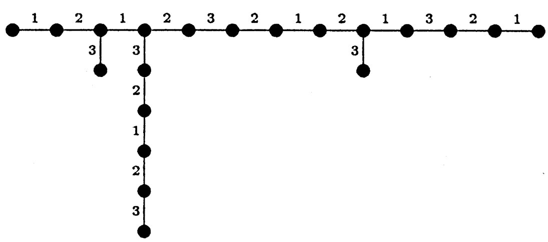
,

$\mathbf{W}_{\{1,3\}}^{\{1,2,3\}} =$



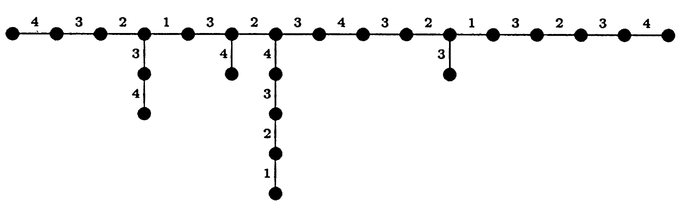
and

$\mathbf{W}_{\{2,3\}}^{\{1,2,3\}} =$

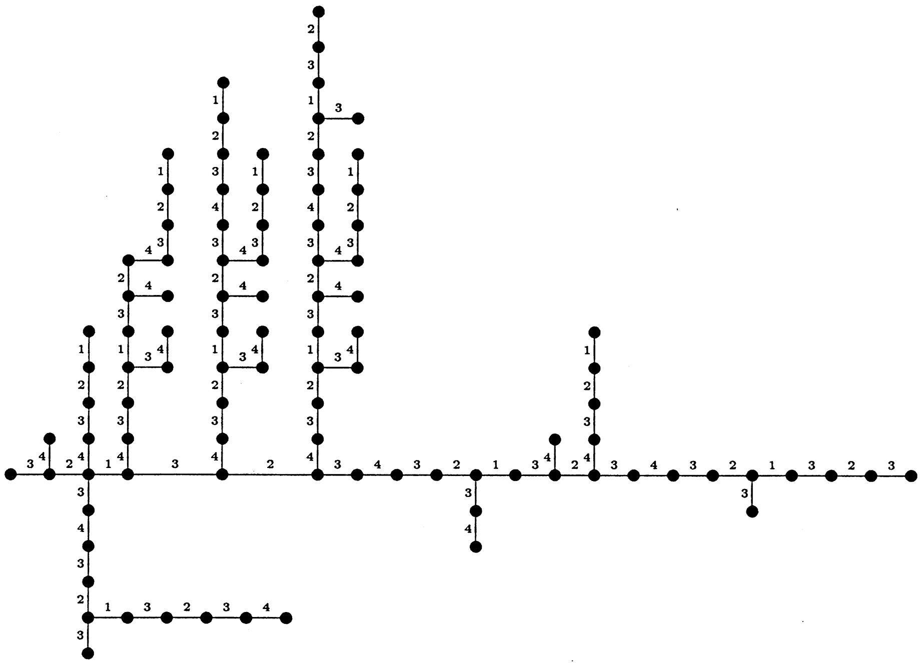


The last example we give is $\boldsymbol{W}=\boldsymbol{F}_4$. By the symmetry of the graph defining $\boldsymbol{F}_4$, it is clear that it suffices to provide $\boldsymbol{W}_{\{1,2,3\}}^{\{1,2,3,4\}}$ and $\boldsymbol{W}_{\{1,2,4\}}^{\{1,2,3,4\}}$. They are as follows:

$\mathbf{W}_{\{1,2,3\}}^{\{1,2,3,4\}} =$



whereas $W^{\{1,2,3,4\}}_{\{1,2,4\}}$ is given by the tree



4. – Factorization of the Poincaré polynomial.

In this section we show that the well-known formula for the factorization of the Poincaré polynomial of a finite irreducible Coxeter group (see [6]) is a simple corollary of the results found in section 2. Moreover, we remark that we shall use just the simplest case of Theorems 2, 4 and 6, that is the case with $i = n$, thus getting a simpler proof of the factorization theorem with respect to the standard ones (see [6], section 3.16 and the closing remark, [8] and [9].

PROPOSITION 7. – *If G is a finite Coxeter group and H is a parabolic subgroup, then*

$$W_G(x) = W_H(x) \sum_{r \in R} x^{l(r)},$$

where R is the set of minimal length representatives of the cosets of H in G.

PROOF. – Let R be defined as in the statement. Then every $w \in G$ can be uniquely written as $w = ru$ for some $r \in R$ and some $u \in H$ with $l(w) = l(r) +$

$l(u)$. In this way,

$$\begin{aligned} W_G(x) &= \sum_{w \in G} x^{l(w)} \\ &= \sum_{r \in R} \sum_{u \in H} x^{l(r)} x^{l(u)} \\ &= \sum_{r \in R} x^{l(r)} \sum_{u \in H} x^{l(u)} \\ &= \sum_{r \in R} x^{l(r)} W_H(x) \end{aligned}$$

as claimed. ■

The Poincaré polynomial can be further generalized in the case there is more than one length for the roots (see [1] for more definitions) by letting

$$W_G(x, y) \doteq \sum_{w \in G} x^{l'(w)} y^{l''(w)}$$

where $l'(w)$ is the number of long roots in a reduced expression for w while $l''(w)$ is the number of short roots.

Recalling the definition (and the value) of the *degrees* d_i from [6] or [1], we have the following

COROLLARY 8. – *If* $G = \boldsymbol{A}_n$ *or* $\boldsymbol{D}_n$ *then*

$$W_G(x) = \prod_{i=1}^{n} \frac{1 - x^{d_i}}{1 - x}$$

and, if $G = \boldsymbol{B}_n$, *then*

$$W_G(x, y) = \prod_{i=1}^{n} (1 + x^{i-1} y)(1 + x + \ldots + x^{i-1})$$

$\left(\textit{which reduces to } W_G(x) = \prod_{i=1}^{n} \frac{1 - x^{d_i}}{1 - x} \textit{ for } y = x\right).$

PROOF. – Let us use Proposition 7 with H being the parabolic subgroup generated by $s_1, \ldots, s_{n-1}$ (with the conventions used in section 2). Then, for $G = \boldsymbol{A}_n$, we have

$$R = \{1, s_n, s_n s_{n-1}, \ldots, s_n s_{n-1} \ldots s_1\} = \{\text{left substrings of } s_n s_{n-1} \ldots s_1\}$$

so that

$$\sum_{r \in R} x^{l(r)} = 1 + x + \ldots + x^n$$

and then, by Proposition 7 and an easy induction,

$$W_{\boldsymbol{A}_n} = \prod_{i=1}^{n} (1 + x + \ldots + x^i),$$

as claimed.

For $G = \boldsymbol{D}_n$, then

$$R = \{\text{left substrings of } s_n s_{n-1} \ldots s_3 s_2 s_1 s_3 \ldots s_{n-1} s_n\}$$

(remember to add also $s_n s_{n-1} \ldots s_3 s_1$) so that

$$W_{\boldsymbol{D}_n} = \prod_{i=1}^{n} (1 + x^{i-1})(1 + x + \ldots + x^{i-1});$$

noting that

$$(1 + x^{i-1})(1 + x + \ldots + x^{i-2}) = 1 + x + \ldots + x^{2i-3},$$

we get what claimed.

As to $G = \boldsymbol{B}_n$, it is trivial to see that Proposition 7 holds also for $W_{\boldsymbol{B}_n}(x, y)$. Thus, since

$$R = \{\text{left substrings of } s_n s_{n-1} \ldots s_3 s_2 s_1 s_2 s_3 \ldots s_{n-1} s_n\},$$

we get

$$\sum_{r \in R} x^{l'(r)} y^{l''(r)} = 1 + x + \ldots + x^{n-1} + x^{n-1} y + x^n y + \ldots + x^{2n-1} y$$

$$= (1 + x^{n-1} y)(1 + x + \ldots + x^{n-1})$$

and, again, we are done by means of an easy induction. ■

FURTHER REMARKS. – Consider an Artin group $\boldsymbol{G_W}$ associated to a Coxeter group $\boldsymbol{W}$ (see [2]). Using the results of section 2 it is easy to construct an algorithm to compute the cohomology of a $\boldsymbol{G_W}$-module were $\boldsymbol{W}$ is a finite irreducible Coxeter group, see [10].

Clearly, computations depend on a given representation $\varphi : \boldsymbol{G}_{\boldsymbol{A}_n} \to \text{Aut}\,(R)$ which has to be specified.

In table I we provide the result of the computations done for the standard representation in the linear group $\varphi : \boldsymbol{G}_{\boldsymbol{A}_n} \to GL(n+1, \mathbb{Z})$ by permutations: the generator s_i is mapped to the matrix which operates on the canonical base as the permutation $(i, i+1)$. Where no result is given, it means that computations go beyond the limits of the machine.

These results agree with what is known for such a representation, see [11]: the free part of the cohomology groups is always $\mathbb{Z} \times \mathbb{Z}$, but for the first and the last cohomology groups.

It is quite evident the usefulness of knowing a good deal of (computer generated) examples in order to have a better understanding of what goes on, so as to make realistic conjectures.

TABLE I.

	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
H^0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H^1	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
H^2	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
H^3		$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}$	*	*	*
H^4			$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2\times(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}^2\times(\mathbb{Z}/2\mathbb{Z})^2$	*	*	*
H^5				$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	*	*	*
H^6					$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}^2\times(\mathbb{Z}/6\mathbb{Z})^2$	*	*
H^7						$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/6\mathbb{Z}$	*	*
H^8							$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/6\mathbb{Z}$	*
H^9								$\mathbb{Z}$	$\mathbb{Z}^2\times\mathbb{Z}/30\mathbb{Z}$
H^{10}									$\mathbb{Z}$

REFERENCES

[1] N. BOURBAKI, *Groupes et Algèbres de Lie, Chap. 4-6*, Paris, Hermann (1968).
[2] E. BRIESKORN, *Sur les groupes de tresses*, Sém. Bourbaki, **401** (1971).
[3] C. DE CONCINI - M. SALVETTI, *Cohomology of Artin groups*, to appear in Math. Res. Lett.
[4] C. DE CONCINI - M. SALVETTI - F. STUMBO, *The top-cohomology of Artin Groups with coefficients in rank-1 local systems over* $\mathbb{Z}$, Topology and its applications, **78** (1997), 5-20.
[5] F. DU CLOUX, *Un algorithme de forme normale pour les groupes de Coxeter*, preprint, Ecole Politechnique, Palaiseau, 1990.
[6] J. E. HUMPHREYS, *Reflection groups and Coxeter groups*, Cambridge Un. Press, 1990.
[7] M. SALVETTI - F. STUMBO, *Artin groups associated to infinite Coxeter groups*, Discrete Mathematics, **163** (1997), 129-138.
[8] L. SOLOMON, *A decomposition of the group algebra of a finite Coxeter group*, J. of Algebra, **9** (1968), 220-239.
[9] R. STEINBERG, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., **80** (1968).
[10] F. STUMBO, *Coomologia dei gruppi di Artin*, Univ. di Pisa, *Ph. D. Thesis*, 1997.
[11] V. A. VASSILIEV, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Tran. of Math. Monog., AMS **98**, 1992.

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