Bollettino Unione Matematica Italiana

A. L. Gilotti, U. Tiberio

On the intersection of maximal non-supersoluble subgroups in a finite group

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.3, p. 691–698.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_3_691_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) 3-B (2000), 691-698

On the Intersection of Maximal Non-Supersoluble Subgroups in a Finite Group.

A. L. GILOTTI (*) - U. TIBERIO (**)

Sunto. – *Gli autori studiano il sottogruppo $\Sigma(G)$ intersezione dei sottogruppi massimali e non supersolubili di un gruppo finito G e le relazioni tra la struttura di G e quella di $\Sigma(G)$.*

Introduction.

Let G be a finite group. Let us denote by Σ the set of all maximal subgroups of G that are non supersoluble and by $\Sigma(G)$ the intersection of G with all members of that set.

$\Sigma(G)$ coincides with G if and only if Σ is empty, i.e. G is either supersoluble or minimal non supersoluble.

In this paper two kinds of problems are investigated.

The first concerns the characterization of $\Sigma(G)$ in various classes of groups.

The second studies the influence that properties of $\Sigma(G)$ can have on the structure of G.

Of both type are the results obtained by Shidov in 1971 ([1]), that studied the subgroup $H(G)$ intersection of all maximal and non nilpotent subgroups of G and proved that $H(G)$ is nilpotent in a non soluble group, while in the soluble groups it has a normal Sylow p-subgroup. Further if $H(G)$ is non nilpotent $G = QN$, where Q is a normal q-subgroup, q a prime, and N is nilpotent.

Of the first type are the results contained in a previous paper of the authors, in which the subgroup $H_p(G)$, intersection of all maximal and non p-nilpotent subgroups of G, for a fixed odd prime p, is studied. It is there proved that $H_p(G)$ is $\Phi(G)$ (the Frattini subgroup) in a finite non p-soluble group, while it has properties like Sylow's only if we restrict the attention to a special class of p-soluble groups (see [2]).

Similar results do not hold for $\Sigma(G)$. In particular, if G is non soluble, $\Sigma(G)$ not necessarily coincides with $\Phi(G)$ and it can be simple. For example in the

(*) Member of the G.N.S.A.G.A. of C.N.R.
(**) Research partially supported by ex 40%, 60% MURST funds.

group $G = \text{Aut}\,(SL(3,2))$ there is a unique maximal subgroup non supersoluble which is isomorphic to $SL(3,2)$ and coincides with $\Sigma(G)$ (see [3]).

Further if G is soluble, $\Sigma(G)$ is neither necessarily supersoluble nor it has a Sylow p-subgroup which is normal, as the Example 1 of [2] can show.

Restricting the class of soluble groups examined, the authors obtain the following results about the structure of $\Sigma(G)$

a) *If G is a soluble group such that $(|G|, \bar{r}(G)) = 1$ and $\Sigma(G) \neq 1$, then $\Sigma(G)$ has a Sylow p-subgroup which is normal in G.*

b) *If G is a soluble group and all maximal and supersoluble subgroups are conjugate then $\Sigma(G)$ has a normal p-subgroup Q and the factor group $\Sigma(G)/Q$ is nilpotent.*

About the second type of problems we concentrate our attention on the groups in which $\Sigma(G) \neq \Phi(G)$, in fact no particular implications can be expected if these two subgroups coincide.

If $\mathcal{S}$ is the saturated formation of supersoluble groups and $G_{\mathcal{S}}$ is the supersoluble residual of G, observe that $\Phi(G) \neq \Sigma(G)$ yields to $G_{\mathcal{S}} \subseteq \Sigma(G)$. This observation motivates the following results:

c) *If $\Sigma(G) = G_{\mathcal{S}}$ then $G = QN$ where $Q = \Sigma(G)$ is a normal q-subgroup and N is supersoluble and maximal in G. In particular $\Sigma(G)$ is nilpotent.*

Conversely we get

d) *If $\Sigma(G)$ is nilpotent (and $\Sigma(G) \neq \Phi(G)$) then $G = \Sigma(G)\,N$ where N is supersoluble and $\Sigma(G)/\Phi(G)$ is minimal normal subgroup of $G/\Phi(G)$ and $\Sigma(G) = G_{\mathcal{S}}\,\Phi(G)$.*

In particular, if G is primitive, $\Sigma(G)$ is the unique minimal normal subgroup of G.

Finally an example verifying c) and d) is exhibited.

Preliminaries.

All groups here considered are finite. The notation is standard (cfr. [4]). Given a finite group G, let us denote by Σ the set of all maximal and non supersoluble subgroups of G and by $\Sigma(G) = G \cap \Big(\bigcap_{M_i \in \Sigma} M_i\Big)$.

$\Sigma(G)$ is a characteristic subgroup of G and if Σ is empty, $\Sigma(G) = G$.

Further $\Sigma(G) = G$ happens if and only if G is either supersoluble or minimal non supersoluble and the structure of G is known (see [7]).

Let us recall the following:

DEFINITION 1 (cfr. [5, VI, 8.2]). – *Let G be a non identity soluble group and let*

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_m = G$$

be a chief series of G. Suppose that $|G_{i+1}/G_i| = p_i^{n_i}$ where p_i is a prime, $i = 1, \ldots, m$.

We define arithmetic rank $\bar{r}(G)$ of G the least common multiple of n_i's $\forall\, i = 1, \ldots, m$. If $G = 1$ we set $\bar{r}(G) = 0$.

LEMMA 1 (cfr. [5], 8.1, p. 711). – *Let G be a p-soluble group. Let V be a faithful irreducible G-module over $GF(p)$ of dimension n where $(n, |G|) = 1$. Then G is cyclic and $|G|$ divides $p^n - 1$.*

Let us observe the following easy general fact:

LEMMA 2. – *Let $\mathcal{X}$ be a quotient-closed class of finite groups and G be a finite group. Indicate by $C(G)$ the intersection of all maximal subgroups of G, that do not belong to $\mathcal{X}$. If none of these subgroups exist, $C(G)$ will be G itself.*

Then, if N is a normal subgroup of G

$$C(G)\, N/N \subseteq C(G/N)\,.$$

PROOF. – If M/N is a maximal subgroup of G/N not in $\mathcal{X}$, surely M is a maximal subgroup of G not in $\mathcal{X}$, since $\mathcal{X}$ is quotient-closed. Then the lemma follows. ■

REMARK 1. – *The class $\mathcal{X}$ of soluble groups such that $(|G|, \bar{r}(G)) = 1$ is quotient-closed.*

LEMMA 3. – *Let G be a finite group and let $\mathcal{X}$ be a class of finite groups subgroup and quotient-closed. Let $C(G)$ be the subgroup defined in Lemma 2. If $\Phi(G) \subsetneq C(G)$ then*

i) *$G = C(G)\, M$ where M is a maximal subgroup of G belonging to $\mathcal{X}$*

ii) *If G is soluble $G = QN$ where Q is a normal q-subgroup of G, q a prime, and N is a maximal subgroup of G belonging to $\mathcal{X}$.*

PROOF. – Since $\Phi(G) \neq C(G)$, there exists a maximal subgroup of G belonging to $\mathcal{X}$ such that $M \not\supseteq C(G)$.

It follows $G = C(G)\, M$ so i) is proved. Now assume G soluble. Denote $G/\Phi(G)$ by $\overline{G}$ and consequently every homomorphic image of a subgroup H of G in the homomorphism from G to $\overline{G}$ by $\overline{H}$. Hence $\overline{M} = M/\Phi(G)$, $\overline{C(G)} = C(G)/\Phi(G)$.

We have $\overline{G}/\overline{C(G)} \simeq G/C(G) \simeq M/(C(G) \cap M)$ so $\overline{G}/\overline{C(G)} \in \mathcal{X}$. Since $\overline{C(G)}$ is a non-trivial normal subgroup of $\overline{G}$, it contains a minimal normal subgroup $\overline{K}$ of $\overline{G}$. Suppose $|\overline{K}| = q^s$.

Further $\overline{G} = \overline{K}\,\overline{N}$, $\overline{K} \cap \overline{N} = 1$ where $\overline{N}$ is a maximal subgroup of $\overline{G}$. Since $N \not\supseteq C(G)$, N belongs to $\mathcal{X}$.

Let K be the complete preimage of $\overline{K}$ in the homomorphism $G \to \overline{G}$, then $K = Q\Phi(G)$, where Q is a Sylow q-subgroup of K.

By Frattini's argument $G = KN_G(Q) = \Phi(G)\, N_G(Q) = N_G(Q)$, so that Q is normal in G. Since $\overline{G} = \overline{K}\,\overline{N}$, we have $G = QN$ as wanted. ■

Results.

THEOREM 1. – *Let G be a soluble group such that $(|G|, \bar{r}(G)) = 1$. If $\Sigma(G) \neq 1$ then $\Sigma(G)$ has a Sylow q-subgroup which is normal.*

PROOF. – We can assume $\Phi(G) \subsetneqq \Sigma(G) \subsetneqq G$. In fact if $\Sigma(G) = \Phi(G)$ the theorem is trivially true. If $\Sigma(G) = G$, then G is either supersoluble or minimal non supersoluble and in both cases the theorem is also true (cfr. [7]).

Suppose G is a minimal counterexample to the theorem.

Claim: $\Phi(G) = 1$.

If $\Phi(G) \neq 1$, by Remark 1 and by minimality of G, $\Sigma(G/\Phi(G))$ has a normal Sylow p-subgroup. Let $\Delta/\Phi(G)$ be $\Sigma(G/\Phi(G))$. Then there exists a Sylow p-subgroup P of Δ such that $P\Phi(G) \lhd G$. By Frattini's argument, P is normal in G and $P \cap \Sigma(G)$ is normal in $\Sigma(G)$ and also in G. Let us prove that $P \cap \Sigma(G) \neq 1$.

If p divides $|\Sigma(G)|$ then surely $P \cap \Sigma(G) \neq 1$. So assume that p does not divide $|\Sigma(G)|$. Since $P \not\subseteq \Phi(G)$, P has a supplement H in G and we can assume that H is a maximal subgroup.

If H were supersoluble, we would obtain G/P supersoluble and since $G/\Sigma(G)$ is supersoluble by Lemma 3, we would get $G/P \cap \Sigma(G) \simeq G$ supersoluble, contradiction. It follows that H is a maximal non supersoluble subgroup of G. By $G = PH$ we have

$$G/\Phi(G) = (P/\Phi(G)/\Phi(G))(H/\Phi(G)) = (\Delta/\Phi(G))(H/\Phi(G)) .$$

By Lemma 3, $H/\Phi(G)$ is supersoluble, and hence so is $(G/\Phi(G))/(P\Phi(G)/\Phi(G))$. Since also $(G/\Phi(G))/(\Sigma(G)/\Phi(G))$ is supersoluble, we have $G/(P\Phi(G) \cap \Sigma(G))$ supersoluble.

But $P\Phi(G) \cap \Sigma(G) = (P \cap \Sigma(G))\Phi(G) = \Phi(G)$ since $P \cap \Sigma(G) = 1$. Thus $G/\Phi(G)$ is supersoluble and G supersoluble, a contradiction.

So we may assume that $\Phi(G) = 1$ and $1 \neq \Sigma(G) \subsetneqq G$.

Let Q^* be a minimal normal subgroup of G contained in $\Sigma(G)$ and let H ba a maximal subgroup of G complementing Q^*. Then $G = Q^* H$, $Q^* \cap H = 1$. If we call $G^* = G/\mathrm{Core}_G H$ we have G^* primitive (cfr. [6] p. 54).

So Q^* is a faithful and irreducible $GF(q)$-H^*-module (where $|Q^*| = q^s$). Since $(|G|, \bar{r}(G)) = 1$ the dimension of Q^* as a module is coprime with $|H^*|$. So by [5. Hilfsatz 8.1] H^* is cyclic and $|H^*|$ divides $|Q^*| - 1$. It follows that $(|Q^*|, |H^*|) = 1$.

But $\mathrm{Core}_G(H) \cap \Sigma(G) = 1$; in fact, Q^* is the unique minimal normal subgroup of G contained in $\Sigma(G)$, otherwise G would be supersoluble.

So, since $Q^* \cap (\mathrm{Core}_G(H) \cap \Sigma(G)) \subseteq Q^* \cap H = 1$ and $\mathrm{Core}_G(H) \cap \Sigma(G) \trianglelefteq G$, we have

$$\mathrm{Core}_G(H) \cap \Sigma(G) = 1 .$$

Then we have $\Sigma(G) \cap H \simeq (\Sigma(G) \cap H)\,\mathrm{Core}_G H/\mathrm{Core}_G(H)$ and so $|\Sigma(G) \cap H|$ coprime with $|Q^*|$.

Since $\Sigma(G) = \Sigma(G) \cap Q^* H = Q^* (\Sigma(G) \cap H)$ we obtain that Q^* is a Sylow q-subgroup of $\Sigma(G)$, as we wanted. ∎

Note that the hypothesis $(|G|, \bar{r}(G)) = 1$ can't be removed, as the example 1 of [2] shows.

By the same example, we can see the necessity of the hypothesis in the following:

THEOREM 2. – *Let G be a soluble group such that $\Phi(G) \subsetneqq \Sigma(G) \subsetneqq G$. Suppose that all maximal supersoluble subgroups af G are conjugate. Then $\Sigma(G) = QT$ where Q is a normal q-subgroup and T is nilpotent. In particular the Sylow q-subgroup of $\Sigma(G)$ is normal.*

PROOF. – By Lemma 3, $G = QM$ where M is a maximal supersoluble subgroup of G and Q is a q-subgroup of $\Sigma(G)$, normal in G. Let M_0 be a subgroup of M minimal w.r. to the condition that $G = QM_0$. Then $Q \cap M_0 \subseteq \Phi(M_0)$, as it is easily seen.

But then, by Dedekind's relation,

$$\Sigma(G) = \Sigma(G) \cap QM_0 = Q(\Sigma(G) \cap M_0) .$$

If $T = \Sigma(G) \cap M_0 \not\subseteq \Phi(M_0)$ there would exist a maximal subgroup H of M_0 such that $T \not\subseteq H$ and $M_0 = TH$. But then QH is a maximal subgroup of G. In fact if $QH \subseteq S \subseteq G$, we have $S = QM_0 \cap S = Q(M_0 \cap S)$. Since $M_0 \cap S \supseteq H$ and H is maximal in M_0, it follows that either $M_0 \cap S = H$ or $M_0 \cap S = M_0$. In the first case $S = QH$ and in the second case $S = G$. Therefore, QH is a maximal subgroup of G that cannot be conjugate to M, since M doesn't contain Q. Hence QH is not supersoluble and so $QH \supseteq \Sigma(G) \supseteq T$.

It would follow $G = QM_0 = QTH = QH$ a contradiction. Thus $T \subseteq \Phi(M_0)$ and T is nilpotent. ∎

COROLLARY 1. – *If G satisfies the same hypotheses of Theorem 2 then $\Sigma(G) = Q\Phi(M)$ where Q is a normal p-subgroup of G and M is a maximal supersoluble group G such that $G = QM$.*

PROOF. – By Lemma 3, $G = QM$ and by the same argument of Theorem 2 we have $\Sigma(G) \cap M \subseteq \Phi(M)$.

Obviously then $\Sigma(G) \subseteq Q\Phi(M)$, so it is enough to prove the reverse inclusion.

By the assumptions, if N is a maximal subgroup of G not conjugate to M, N is non supersoluble so $N \supseteq \Sigma(G) \supseteq Q$. By Dedekind's relation, $N = Q(N \cap M)$ and $N \cap M$ is maximal in M. Hence $N \cap M$ contains $\Phi(M)$. In particular $\Phi(M) \subseteq N$, for all maximal subgroups N of G which are not conjugate to M. Thus $\Phi(M) \subseteq \Sigma(G)$, $Q\Phi(M) \subseteq \Sigma(G)$ and so the equality. ■

Let $\mathcal{S}$ be the saturated formation of supersoluble groups and let $G_{\mathcal{S}}$ be the supersoluble residual of the group G.

Then by Lemma 3, we easily obtain that if $\Phi(G) \subsetneq \Sigma(G)$ and so it follows $G_{\mathcal{S}} \subseteq \Sigma(G)$.

This observation motivates hypothesis of the following:

THEOREM 3. – *Let G ba a soluble group such that $G_{\mathcal{S}} = \Sigma(G)$. Then $G = QN$ where Q is a normal q-subgroup, $Q = \Sigma(G)$ and N is a maximal and supersoluble subgroup of G. In particular $\Sigma(G)$ is nilpotent.*

PROOF. – We can assume $G_{\mathcal{S}} \neq 1$, otherwise G would be supersoluble and so $G = \Sigma(G) = 1$. It follows then $\Sigma(G) \supsetneq \Phi(G)$ otherwise $G/\Phi(G)$ supersoluble would imply G supersoluble, a contradiction.

Distinguish two cases: $\Sigma(G)$ nilpotent and $\Sigma(G)$ non nilpotent.

In the first case, there exists a Sylow q-subgroup Q of $\Sigma(G)$ such that $Q \not\subseteq \Phi(G)$. So by an argument used before we have $G = QM$ where M is maximal and supersoluble and Q is normal in G.

It follows $Q \supseteq G_{\mathcal{S}} = \Sigma(G)$ and the theorem is proved in this case.

So we may assume that $\Sigma(G)$ is not nilpotent. Choose a Sylow p-subgroup P_1 not normal in $\Sigma(G)$.

By Frattini's argument

$$G = \Sigma(G)\, N_G(P_1)\,.$$

If P is a Sylow p-subgroup of G such that $P_1 = P \cap \Sigma(G)$ we have $N_G(P) \subseteq N_G(P_1)$. So, if N is a maximal subgroup containing $N_G(P_1)$, N contains $N_G(P)$ and so $|G\colon N| = q^s$ where $q \neq p$ (s some integer).

Also $G = \Sigma(G)\, N$ so that N is supersoluble. Since q divides $|\Sigma(G)|$, $\Sigma(G)$ has a non trivial Sylow q-subgroup, Q. If Q is not normal in G,

$N_G(Q)$ is contained in a maximal subgroup H and, since $G = \Sigma(G)\, N_G(Q) = \Sigma(G)\, H$, H is supersoluble.

Since $\Sigma(G) = G_{\mathcal{S}}$, H and N are maximal supersoluble subgroups supplementing $G_{\mathcal{S}}$, so H and N are $\mathcal{S}$-projectors, therefore they are conjugate. But this contradicts the fact that $q \neq p$.

It follows $Q \trianglelefteq G$. Further $G = QN$. But, since N is supersoluble, this implies $Q \supseteq G_{\mathcal{S}}$ and since $\Sigma(G) \supseteq Q$ we obtain $\Sigma(G) = Q$ and the theorem is proved. ∎

Observe that the hypothesis in Theorem 3 leads to the nilpotency of $\Sigma(G)$. Now we invert such a situation and study soluble groups in which $\Sigma(G)$ is nilpotent, but not equal to $\Phi(G)$. Let us indicate by $F(G)$ the Fitting subgroup of G.

THEOREM 4. – *If $\Phi(G) \subsetneq \Sigma(G) \subseteq F(G) \subsetneq G$, then $\Sigma(G)/\Phi(G)$ is a chief factor of G complemented by a maximal supersoluble subgroup. In particular $\Sigma(G) = G_{\mathcal{S}}\, \Phi(G)$.*

PROOF. – Let $H/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$ contained in $\Sigma(G/\Phi(G))$. Then there exists a maximal subgroup M of G such that $G = HM$ and $H \cap M = \Phi(G)$. Since $G = \Sigma(G)M$, M is supersoluble, and G/H too.

If $K/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$ such that $K \neq H$ and $K \subseteq \Sigma(G)$, with the same reasoning we have G/K supersoluble. Then it would follow $G/(H \cap K)$ supersoluble and so $G/\Phi(G)$ supersoluble, contradiction since $\Sigma(G) \neq G$.

So $\Sigma(G)/\Phi(G)$ contains a unique minimal normal subgroup $H/\Phi(G)$ of $G/\Phi(G)$. Since $\Sigma(G)$ is nilpotent, this implies $\Sigma(G)/\Phi(G)$ a p-group, where p is a prime, and since $F(G)/\Phi(G)$ is a product of minimal normal subgroups of $G/\Phi(G)$, $\Sigma(G)/\Phi(G)$ is an elementary abelian p-group.

By this observation we note that

$$(\Sigma(G)/\Phi(G)) \cap (M/\Phi(G)) = (\Sigma(G) \cap M)/\Phi(G)$$

is normal in $G/\Phi(G)$.

Since $H \not\subseteq \Sigma(G) \cap M$ we have $\Sigma(G) \cap M = \Phi(G)$. So

$$\frac{\Sigma(G)}{\Phi(G)} = \frac{H}{\Phi(G)}\left(\frac{\Sigma(G) \cap M}{\Phi(G)}\right) = \frac{H}{\Phi(G)}\,.$$

So $\Sigma(G) = H$.

By the previous observation it also follow that the supersoluble residual $(G/\Phi(G))_{\mathcal{S}} = G_{\mathcal{S}}\,\Phi(G)/\Phi(G)$ coincides with $\Sigma(G)/\Phi(G)$ and so $G_{\mathcal{S}}\,\Phi(G) = \Sigma(G)$ as we wanted. ■

Example.

The following is an example of a group satisfying the hypothesis of Theorem 3 (and so also that of Theorem 4).

Let $H = SL(2, 2^4)$ and let G be the subgroup of H defined by:

$$G = \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \Bigg/ \begin{matrix} \alpha, \beta \in GF(2^4) \\ \alpha \neq 0 \end{matrix} \right\rangle.$$

Obviously G is the normalizer in H of a Sylow 2-subgroup of H and it is a maximal subgroup of H (see [3]). It is easy to see that $\Sigma(G)$ is equal to $G_{\mathcal{S}}$ and coincides with the Sylow 2-subgroup.

REFERENCES

[1] L. I. SHIDOV, *On maximal subgroups of finite groups*, Sibirsk. Mat. Zh. **12**, n. 3 (1971), 682-683.

[2] A. L. GILOTTI and U. TIBERIO, *On the intersection of a certain class of maximal subgroups of a finite group*, Arch. Math., **71** (1998), 89-94.

[3] J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER and R. A. WILSON, *Atlas of Finite Groups*, Oxford 1985.

[4] D. J. S. ROBINSON, *A Course in the Theory of Groups*, Berlin-Heidelberg-New York (1991).

[5] B. HUPPERT, *Endliche Gruppen I*, Berlin-Heidelberg-New York (1967).

[6] K. DOERK and T. HAWKES, *Finite Soluble Groups*, Berlin-New York (1992).

[7] K. DOERK, *Minimal nicht uberauflösbare, endliche Gruppen*, Math. Zeitschr., **91** (1966), 198-205.

A. L. Gilotti: Dipartimento di Matematica
P.za di P.ta S. Donato, Università di Bologna, 40127 Bologna
E-mail: gilotti@dm.unibo.it

U. Tiberio: Dipartimento di Matematica Applicata G. Sansone
Via di S. Marta 3, 50139 Firenze
E-mail: utiberio@dma.unifi.it

Pervenuta in Redazione
il 22 giugno 1999