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Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_3_691_0>
On the Intersection of Maximal Non-Supersoluble Subgroups in a Finite Group.

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Sunto. – Gli autori studiano il sottogruppo $\Sigma(G)$ intersezione dei sottogruppi massimali e non supersolubili di un gruppo finito $G$ e le relazioni tra la struttura di $G$ e quella di $\Sigma(G)$.

Introduction.

Let $G$ be a finite group. Let us denote by $\Sigma$ the set of all maximal subgroups of $G$ that are non supersoluble and by $\Sigma(G)$ the intersection of $G$ with all members of that set.

$\Sigma(G)$ coincides with $G$ if and only if $\Sigma$ is empty, i.e. $G$ is either supersoluble or minimal non supersoluble.

In this paper two kinds of problems are investigated.

The first concerns the characterization of $\Sigma(G)$ in various classes of groups.

The second studies the influence that properties of $\Sigma(G)$ can have on the structure of $G$.

Of both type are the results obtained by Shidov in 1971 ([1]), that studied the subgroup $H(G)$ intersection of all maximal and non nilpotent subgroups of $G$ and proved that $H(G)$ is nilpotent in a non soluble group, while in the soluble groups it has a normal Sylow $p$-subgroup. Further if $H(G)$ is non nilpotent $G = QS$, where $Q$ is a normal q-subgroup, $q$ a prime, and $S$ is nilpotent.

Of the first type are the results contained in a previous paper of the authors, in which the subgroup $H_p(G)$, intersection of all maximal and non $p$-nilpotent subgroups of $G$, for a fixed odd prime $p$, is studied. It is there proved that $H_p(G)$ is $\Phi(G)$ (the Frattini subgroup) in a finite non $p$-soluble group, while it has properties like Sylow’s only if we restrict the attention to a special class of $p$-soluble groups (see [2]).

Similar results do not hold for $\Sigma(G)$. In particular, if $G$ is non soluble, $\Sigma(G)$ not necessarily coincides with $\Phi(G)$ and it can be simple. For example in the

(*) Member of the G.N.S.A.G.A. of C.N.R.
(**) Research partially supported by ex 40%, 60% MURST funds.
group $G = \text{Aut}(SL(3, 2))$ there is a unique maximal subgroup non supersoluble which is isomorphic to $SL(3, 2)$ and coincides with $\Sigma(G)$ (see [3]).

Further if $G$ is soluble, $\Sigma(G)$ is neither necessarily supersoluble nor it has a Sylow $p$-subgroup which is normal, as the Example 1 of [2] can show.

Restricting the class of soluble groups examined, the authors obtain the following results about the structure of $\Sigma(G)$

a) If $G$ is a soluble group such that $(|G|, \overline{\tau}(G)) = 1$ and $\Sigma(G) \neq 1$, then $\Sigma(G)$ has a Sylow $p$-subgroup which is normal in $G$.

b) If $G$ is a soluble group and all maximal and supersoluble subgroups are conjugate then $\Sigma(G)$ has a normal $p$-subgroup $Q$ and the factor group $\Sigma(G)/Q$ is nilpotent.

About the second type of problems we concentrate our attention on the groups in which $\Sigma(G) \neq \Phi(G)$, in fact no particular implications can be expected if these two subgroups coincide.

If $\mathcal{S}$ is the saturated formation of supersoluble groups and $G_\mathcal{S}$ is the supersoluble residual of $G$, observe that $\Phi(G) \neq \Sigma(G)$ yields to $G_\mathcal{S} \subseteq \Sigma(G)$. This observation motivates the following results:

c) If $\Sigma(G) = G_\mathcal{S}$ then $G = QN$ where $Q = \Sigma(G)$ is a normal $q$-subgroup and $N$ is supersoluble and maximal in $G$. In particular $\Sigma(G)$ is nilpotent.

Conversely we get

d) If $\Sigma(G)$ is nilpotent (and $\Sigma(G) \neq \Phi(G)$) then $G = \Sigma(G)N$ where $N$ is supersoluble and $\Sigma(G)/\Phi(G)$ is minimal normal subgroup of $G/\Phi(G)$ and $\Sigma(G) = G_\mathcal{S}\Phi(G)$.

In particular, if $G$ is primitive, $\Sigma(G)$ is the unique minimal normal subgroup of $G$.

Finally an example verifying c) and d) is exhibited.

Preliminaries.

All groups here considered are finite. The notation is standard (cfr. [4]).

Given a finite group $G$, let us denote by $\Sigma$ the set of all maximal and non supersoluble subgroups of $G$ and by $\Sigma(G) = G \cap \bigcap_{M_i \in \Sigma} M_i$.

$\Sigma(G)$ is a characteristic subgroup of $G$ and if $\Sigma$ is empty, $\Sigma(G) = G$.

Further $\Sigma(G) = G$ happens if and only if $G$ is either supersoluble or minimal non supersoluble and the structure of $G$ is known (see [7]).

Let us recall the following:
Definition 1 (cfr. [5, VI, 8.2]). – Let $G$ be a non identity soluble group and let

$$1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_m = G$$

be a chief series of $G$. Suppose that $|G_{i+1}/G_i| = p_i^{n_i}$ where $p_i$ is a prime, $i = 1, \ldots, m$.

We define arithmetic rank $\overline{r}(G)$ of $G$ the least common multiple of $n_i$’s $\forall i = 1, \ldots, m$. If $G = 1$ we set $\overline{r}(G) = 0$.

Lemma 1 (cfr. [5], 8.1, p. 711). – Let $G$ be a $p$-soluble group. Let $V$ be a faithful irreducible $G$-module over $GF(p)$ of dimension $n$ where $(n, |G|) = 1$. Then $G$ is cyclic and $|G|$ divides $p^n - 1$.

Let us observe the following easy general fact:

Lemma 2. – Let $\mathcal{X}$ be a quotient-closed class of finite groups and $G$ be a finite group. Indicate by $C(G)$ the intersection of all maximal subgroups of $G$, that do not belong to $\mathcal{X}$. If none of these subgroups exist, $C(G)$ will be $G$ itself.

Then, if $N$ is a normal subgroup of $G$

$$C(G)N/N \subseteq C(G/N).$$

Proof. – If $M/N$ is a maximal subgroup of $G/N$ not in $\mathcal{X}$, surely $M$ is a maximal subgroup of $G$ not in $\mathcal{X}$, since $\mathcal{X}$ is quotient-closed. Then the lemma follows.

Remark 1. – The class $\mathcal{X}$ of soluble groups such that $(|G|, \overline{r}(G)) = 1$ is quotient-closed.

Lemma 3. – Let $G$ be a finite group and let $\mathcal{X}$ be a class of finite groups subgroup and quotient-closed. Let $C(G)$ be the subgroup defined in Lemma 2. If $\Phi(G) \subseteq_c C(G)$ then

i) $G = C(G)M$ where $M$ is a maximal subgroup of $G$ belonging to $\mathcal{X}$

ii) If $G$ is soluble $G = QN$ where $Q$ is a normal $q$-subgroup of $G$, $q$ a prime, and $N$ is a maximal subgroup of $G$ belonging to $\mathcal{X}$.

Proof. – Since $\Phi(G) \neq C(G)$, there exists a maximal subgroup of $G$ belonging to $\mathcal{X}$ such that $M \not\triangleleft C(G)$.

It follows $G = C(G)M$ so i) is proved. Now assume $G$ soluble. Denote $G/\Phi(G)$ by $\overline{G}$ and consequently every homomorphic image of a subgroup $H$ of $G$ in the homomorphism from $G$ to $\overline{G}$ by $\overline{H}$. Hence $M = M/\Phi(G)$, $\overline{C(G)} = C(G)/\Phi(G)$.
We have $G/C(G) = M/(C(G) \cap M)$ so $G/C(G) \in \mathcal{X}$. Since $C(G)$ is a non-trivial normal subgroup of $G$, it contains a minimal normal subgroup $\bar{K}$ of $G$. Suppose $|\bar{K}| = q^s$.

Further $G = KN_G = C(G) \cap M$ so $G/C(G) /EMX$. Since $C(G)$ is a non-trivial normal subgroup of $G$, we may assume that $F(G) \cong 1$ and $1 \leq S(G) \leq G$.

Let $Q^* be a minimal normal subgroup of $G$ contained in $S(G)$ and let $H$ ba

**Results.**

**Theorem 1.** Let $G$ be a soluble group such that $(|G|, \bar{\rho}(G)) = 1$. If $\Sigma(G) \neq 1$ then $\Sigma(G)$ has a Sylow $q$-subgroup which is normal.

**Proof.** We can assume $\Phi(G) \subseteq \Sigma(G) \subseteq G$. In fact if $\Sigma(G) = \Phi(G)$ the theorem is trivially true. If $\Sigma(G) = G$, then $G$ is either supersoluble or minimal non supersoluble and in both cases the theorem is also true (cfr. [7]).

Suppose $G$ is a minimal counterexample to the theorem.

**Claim:** $\Phi(G) = 1$.

If $\Phi(G) \neq 1$, by Remark 1 and by minimality of $G$, $\Sigma(G/\Phi(G))$ has a normal Sylow $p$-subgroup. Let $A/\Phi(G)$ be $\Sigma(G/\Phi(G))$. Then there exists a Sylow $p$-subgroup $P$ of $A$ such that $P/\Phi(G) \triangleleft G$. By Frattini's argument, $P$ is normal in $G$ and $P \cap \Sigma(G)$ is normal in $\Sigma(G)$ and also in $G$. Let us prove that $P \cap \Sigma(G) \neq 1$.

If $p$ divides $|\Sigma(G)|$ then surely $P \cap \Sigma(G) \neq 1$. So assume that $p$ does not divide $|\Sigma(G)|$. Since $P \not\subseteq \Phi(G)$, $P$ has a supplement $H$ in $G$ and we can assume that $H$ is a maximal subgroup.

If $H$ were supersoluble, we would obtain $G/P$ supersoluble and since $G/\Sigma(G)$ is supersoluble by Lemma 3, we would get $G/P \cap \Sigma(G) = G$ supersoluble, contradiction. It follows that $H$ is a maximal non supersoluble subgroup of $G$. By $G = PH$ we have

$$G/\Phi(G) = (P/\Phi(G)/\Phi(G))(H/\Phi(G)) \cong (A/\Phi(G))(H/\Phi(G)) .$$

By Lemma 3, $H/\Phi(G)$ is supersoluble, and hence so is $(G/\Phi(G))/(P\Phi(G)/\Phi(G))$. Since also $(G/\Phi(G))/(\Sigma(G)/\Phi(G))$ is supersoluble, we have $G/(P\Phi(G) \cap \Sigma(G))$ supersoluble.

But $P\Phi(G) \cap \Sigma(G) = (P \cap \Sigma(G))\Phi(G) = \Phi(G)$ since $P \cap \Sigma(G) = 1$. Thus $G/\Phi(G)$ is supersoluble and $G$ supersoluble, a contradiction.

So we may assume that $\Phi(G) = 1$ and $1 \neq \Sigma(G) \subseteq G$.

Let $Q^*$ be a minimal normal subgroup of $G$ contained in $\Sigma(G)$ and let $H$ ba
a maximal subgroup of $G$ complementing $Q^\ast$. Then $G = Q^\ast H$, $Q^\ast \cap H = 1$. If we call $G^\ast = G/\text{Core}_G H$ we have $G^\ast$ primitive (cfr. [6] p. 54).

So $Q^\ast$ is a faithful and irreducible $GF(q)-H^\ast$-module (where $|Q^\ast| = q^\ast$). Since $(|G|, \overline{\tau}(G)) = 1$ the dimension of $Q^\ast$ as a module is coprime with $|H^\ast|$. So by [5. Hilfsatz 8.1] $H^\ast$ is cyclic and $|H^\ast|$ divides $|Q^\ast| - 1$. It follows that $(|Q^\ast|, |H^\ast|) = 1$.

But $\text{Core}_G(H) \cap \Sigma(G) = 1$; in fact, $Q^\ast$ is the unique minimal normal subgroup of $G$ contained in $\Sigma(G)$, otherwise $G$ would be supersoluble.

So, since $Q^\ast \cap (\text{Core}_G(H) \cap \Sigma(G)) = \{Q^\ast \cap H = 1$ and $\text{Core}_G(H) \cap \Sigma(G) \leq G$, we have

$$\text{Core}_G(H) \cap \Sigma(G) = 1.$$ 

Then we have $\Sigma(G) \cap H = \langle \Sigma(G) \cap H \rangle \text{Core}_G H/\text{Core}_G(H)$ and so $|\Sigma(G) \cap H|$ coprime with $|Q^\ast|$.

Since $\Sigma(G) = \Sigma(G) \cap Q^\ast H = Q^\ast (\Sigma(G) \cap H)$ we obtain that $Q^\ast$ is a Sylow $q$-subgroup of $\Sigma(G)$, as we wanted. ■

Note that the hypothesis $(|G|, \overline{\tau}(G)) = 1$ can’t be removed, as the example 1 of [2] shows.

By the same example, we can see the necessity of the hypothesis in the following:

**Theorem 2.** Let $G$ be a soluble group such that $\Phi(G) \subseteq \Sigma(G) \subseteq G$. Suppose that all maximal supersoluble subgroups of $G$ are conjugate. Then $\Sigma(G) = QT$ where $Q$ is a normal $q$-subgroup and $T$ is nilpotent. In particular the Sylow $q$-subgroup of $\Sigma(G)$ is normal.

**Proof.** By Lemma 3, $G = QM$ where $M$ is a maximal supersoluble subgroup of $G$ and $Q$ is a $q$-subgroup of $\Sigma(G)$, normal in $G$. Let $M_0$ be a subgroup of $M$ minimal w.r. to the condition that $G = QM_0$. Then $Q \cap M_0 \subseteq \Phi(M_0)$, as it is easily seen.

But then, by Dedekind’s relation,

$$\Sigma(G) = \Sigma(G) \cap QM_0 = Q(\Sigma(G) \cap M_0).$$

If $T = \Sigma(G) \cap M_0 \not\subseteq \Phi(M_0)$ there would exist a maximal subgroup $H$ of $M_0$ such that $T \not\subseteq H$ and $M_0 = TH$. But then $QH$ is a maximal subgroup of $G$. In fact if $QH \subseteq S \subseteq G$, we have $S = QM_0 \cap S = Q(M_0 \cap S)$. Since $M_0 \cap S \not\subseteq H$ and $H$ is maximal in $M_0$, it follows that either $M_0 \cap S = H$ or $M_0 \cap S = M_0$. In the first case $S = QH$ and in the second case $S = G$. Therefore, $QH$ is a maximal subgroup of $G$ that cannot be conjugate to $M$, since $M$ doesn’t contain $Q$. Hence $QH$ is not supersoluble and so $QH \not\subseteq \Sigma(G) \not\subseteq T$.

It would follow $G = QM_0 = QTH = QH$ a contradiction. Thus $T \subseteq \Phi(M_0)$ and $T$ is nilpotent. ■
Corollary 1. – If $G$ satisfies the same hypotheses of Theorem 2 then $\Sigma(G) = Q\Phi(M)$ where $Q$ is a normal $p$-subgroup of $G$ and $M$ is a maximal supersoluble group $G$ such that $G = QM$.

Proof. – By Lemma 3, $G = QM$ and by the same argument of Theorem 2 we have $\Sigma(G) \cap M \subseteq \Phi(M)$.

Obviously then $\Sigma(G) \subseteq Q\Phi(M)$, so it is enough to prove the reverse inclusion.

By the assumptions, if $N$ is a maximal subgroup of $G$ not conjugate to $M$, $N$ is non supersoluble so $N \nsubseteq \Sigma(G) \nsubseteq Q$. By Dedekind’s relation, $N = Q(N \cap M)$ and $N \cap M$ is maximal in $M$. Hence $N \cap M$ contains $\Phi(M)$. In particular $\Phi(M) \subseteq N$, for all maximal subgroups $N$ of $G$ which are not conjugate to $M$. Thus $\Phi(M) \subseteq \Sigma(G)$, $Q\Phi(M) \subseteq \Sigma(G)$ and so the equality.

Let $S$ be the saturated formation of supersoluble groups and let $G_S$ be the supersoluble residual of the group $G$.

Then by Lemma 3, we easily obtain that if $\Phi(G) \nsubseteq \Sigma(G)$ and so it follows $G_S \subseteq \Sigma(G)$.

This observation motivates hypothesis of the following:

Theorem 3. – Let $G$ be a soluble group such that $G_S = \Sigma(G)$. Then $G = QN$ where $Q$ is a normal $q$-subgroup, $Q = \Sigma(G)$ and $N$ is a maximal and supersoluble subgroup of $G$. In particular $\Sigma(G)$ is nilpotent.

Proof. – We can assume $G_S \neq 1$, otherwise $G$ would be supersoluble and so $G = \Sigma(G) = 1$. It follows then $\Sigma(G) \nsubseteq \Phi(G)$ otherwise $G/\Phi(G)$ supersoluble would imply $G$ supersoluble, a contradiction.

Distinguish two cases: $\Sigma(G)$ nilpotent and $\Sigma(G)$ non nilpotent.

In the first case, there exists a Sylow $q$-subgroup $Q$ of $\Sigma(G)$ such that $Q \nsubseteq \Phi(G)$. So by an argument used before we have $G = QM$ where $M$ is maximal and supersoluble and $Q$ is normal in $G$.

It follows $Q \supseteq G_S = \Sigma(G)$ and the theorem is proved in this case.

So we may assume that $\Sigma(G)$ is not nilpotent. Choose a Sylow $p$-subgroup $P_1$ not normal in $\Sigma(G)$.

By Frattini’s argument

$$G = \Sigma(G) N_G(P_1).$$

If $P$ is a Sylow $p$-subgroup of $G$ such that $P_1 = P \cap \Sigma(G)$ we have $N_G(P) \subseteq N_G(P_1)$. So, if $N$ is a maximal subgroup containing $N_G(P_1)$, $N$ contains $N_G(P)$ and so $|G : N| = q^s$ where $q \neq p$ ($s$ some integer).

Also $G = \Sigma(G) N$ so that $N$ is supersoluble. Since $q$ divides $|\Sigma(G)|$, $\Sigma(G)$ has a non trivial Sylow $q$-subgroup, $Q$. If $Q$ is not normal in $G$,
$N_G(Q)$ is contained in a maximal subgroup $H$ and, since $G = \Sigma(G) N_G(Q) = \Sigma(G) H$, $H$ is supersoluble.

Since $\Sigma(G) = G_s$, $H$ and $N$ are maximal supersoluble subgroups supplementing $G_s$, so $H$ and $N$ are $S$-projectors, therefore they are conjugate. But this contradicts the fact that $q \neq p$.

It follows $Q \leq G$. Further $G = QN$. But, since $N$ is supersoluble, this implies $Q \supseteq G_s$ and since $\Sigma(G) \supseteq Q$ we obtain $\Sigma(G) = Q$ and the theorem is proved. \(\blacksquare\)

Observe that the hypothesis in Theorem 3 leads to the nilpotency of $\Sigma(G)$. Now we invert such a situation and study soluble groups in which $\Sigma(G)$ is nilpotent, but not equal to $\Phi(G)$. Let us indicate by $F(G)$ the Fitting subgroup of $G$.

**Theorem 4.** – If $\Phi(G) \subsetneq \Sigma(G) \subsetneq F(G)$, then $\Sigma(G)/\Phi(G)$ is a chief factor of $G$ complemented by a maximal supersoluble subgroup. In particular $\Sigma(G) = G_s \Phi(G)$.

**Proof.** – Let $H/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$ contained in $\Sigma(G/\Phi(G))$. Then there exists a maximal subgroup $M$ of $G$ such that $G = HM$ and $H \cap M = \Phi(G)$. Since $G = \Sigma(G)M$, $M$ is supersoluble, and $G/H$ too.

If $K/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$ such that $K \neq H$ and $K \subsetneq \Sigma(G)$, with the same reasoning we have $G/K$ supersoluble. Then it would follow $G/(H \cap K)$ supersoluble and so $G/\Phi(G)$ supersoluble, contradiction since $\Sigma(G) \neq G$.

So $\Sigma(G)/\Phi(G)$ contains a unique minimal normal subgroup $H/\Phi(G)$ of $G/\Phi(G)$. Since $\Sigma(G)$ is nilpotent, this implies $\Sigma(G)/\Phi(G)$ a $p$-group, where $p$ is a prime, and since $F(G)/\Phi(G)$ is a product of minimal normal subgroups of $G/\Phi(G)$, $\Sigma(G)/\Phi(G)$ is an elementary abelian $p$-group.

By this observation we note that

$$(\Sigma(G)/\Phi(G)) \cap (M/\Phi(G)) = (\Sigma(G) \cap M)/\Phi(G)$$

is normal in $G/\Phi(G)$.

Since $H \notin \Sigma(G) \cap M$ we have $\Sigma(G) \cap M = \Phi(G)$. So

$$\frac{\Sigma(G)}{\Phi(G)} = \frac{H}{\Phi(G)} \left( \frac{\Sigma(G) \cap M}{\Phi(G)} \right) = \frac{H}{\Phi(G)} .$$

So $\Sigma(G) = H$. 
By the previous observation it also follow that the supersoluble residual \((G/\Phi(G))_S = G_S \Phi(G)/\Phi(G)\) coincides with \(\Sigma(G)/\Phi(G)\) and so \(G_S \Phi(G) = \Sigma(G)\) as we wanted. ■

Example.

The following is an example of a group satisfying the hypothesis of Theorem 3 (and so also that of Theorem 4).

Let \(H = SL(2, 2^4)\) and let \(G\) be the subgroup of \(H\) defined by:

\[
G = \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha, \beta \in GF(2^4) \right\rangle.
\]

Obviously \(G\) is the normalizer in \(H\) of a Sylow 2-subgroup of \(H\) and it is a maximal subgroup of \(H\) (see [3]). It is easy to see that \(\Sigma(G)\) is equal to \(G_S\) and coincides with the Sylow 2-subgroup.

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Pervenuta in Redazione
il 22 giugno 1999