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Sunto. – *Sia* $X \subset \boldsymbol{P}^N$ *una varietà irriducibile n-dimensionale localmente Cohen-Macaulay,* $\boldsymbol{Q}$*-Gorenstein e non di tipo generale; assumiamo* $N = 6$, $2n = N + 2$ *e* $\dim(\mathrm{Sing}(X)) = 2n - N$. *In questo lavoro dimostriamo che* $\deg(X) \leqslant (N+1)^{N-n}$ *e quindi che l'insieme di tutte queste varietà è parametrizzato da un insieme finito di varietà algebriche.*

0. – Introduction.

The aim of this paper is to give a partial extension to singular varieties of a nice result of M. Schneider ([S]). Trivial examples (e.g. taking cones) show that it is essential to make some restrictions on the dimension of the singular locus and/or the nature of the singularities.

THEOREM 0.1. – *Fix integers n, N with* $N = 6$ *and* $2n = N + 2$. *There are only finitely many families of irreducible locally Cohen-Macaulay* $\boldsymbol{Q}$*-Gorenstein complex subvarieties X of* $\boldsymbol{P}^N$ *with* $\dim(X) = n$ *and* $\dim(\mathrm{Sing}(X)) = 2n - N$ *such that the Iitaka dimension* $\kappa(X, \omega_X)$ *of* ω_X *is at most* $n - 1$; *more precisely, for every such X we have* $\deg(X) \leqslant (N+1)^{N-n}$.

THEOREM 0.2. – *Fix integers n, N with* $N = 6$ *and* $2n = N + 2$. *There are only finitely many families of irreducible locally Cohen-Macaulay* $\boldsymbol{Q}$*-Gorenstein complex subvarieties X of* $\boldsymbol{P}^N$ *with* $\dim(X) = n$, *and* $\dim(\mathrm{Sing}(X)) = 2n - N$ *such that a desingularization of X is not of general type; more precisely, for every such X we have* $\deg(X) \leqslant (N+1)^{N-n}$.

Here if $\mathrm{Sing}(X) \neq 0$ $\dim(\mathrm{Sing}(X))$ is the maximal dimension of an irreducible components of $\mathrm{Sing}(X)$. We always assume $\mathrm{Sing}(X) \neq \emptyset$, otherwise the result was proved by M. Schneider in [S]. We will follow several of the steps of his proof. We stress that in the critical case $2n = N + 2$ of Theorems

0.1 and 0.2 we require only $\dim(\mathrm{Sing}(X)) = 1$, i.e. we do not require that X has only isolated singularities.

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1. – The proofs.

In both theorems the boundedness follows from the existence of the Chow variety of $\boldsymbol{P}^N$ or of the Hilbert scheme of $\boldsymbol{P}^N$ and the first assertion, i.e. an upper bound for $\deg(X)$. Since X is $\boldsymbol{Q}$-Gorenstein, there is a positive integer t such that $(\omega_X^{\otimes t})^{**}$ is a line bundle. By definition the Iitaka dimension $\kappa(X, \omega_X)$ of the pair $(X, \omega X)$ is the Iitaka dimension $\kappa(X, (\omega_{X^{\otimes t}})^{**})$ of the line bundle $(\omega_{X^{\otimes t}})^{**}$. Let $\pi : Z \to X$ be a desingularization of X. Since X is normal, we have $h^0(X, ((\omega_{X^{\otimes t}})^{**})^{\otimes m}) = h^0(X_{\mathrm{reg}}, (((\omega_{X^{\otimes t}})^{**})^{\otimes m}))\,|X_{\mathrm{reg}})$. Hence for all integers $m > 0$ we have $h^0(Z, \omega_{Z^{\otimes tm}}) \leqslant h^0(X, ((\omega_{X^{\otimes t}})^{**})^{\otimes m})$. Hence if $\kappa(X, \omega_X) < n$, then Z is not of general type. Hence Theorem 0.2 follows from Theorem 0.1. Now we will prove Theorem 0.1.

Proof of Theorem 0.1. We claim that the inequality $2\dim(X) = \mathrm{codim}(X) + 2$ implies that the restriction map $H^2_{DR}(\boldsymbol{P}^N) \to H^2_{DR}(X)$ is bijective; if X is assumed to be locally a complete intersection, this is [O], Th. 2.3; in the general case lift X to positive characteristic, use [HS], part (a) of Cor. 4.4, to obtain that $\boldsymbol{P}^N \setminus X$ has cohomological dimension $\leqslant 2(N-n) - 2$ and then apply [H2], Th. 7.1 at p. 86. By [FL], part (B) of Cor. 5.3, X is simply connected and hence $H^2(X, \boldsymbol{C})$ is dual to $H_2(X, \boldsymbol{C})$. By [H2], Th. 5.1 and the finiteness theorem 6.1, $H^2_{DR}(X)$ is dual to $H^{DR}_2(X)$. By [H], p. 89, $H^{DR}_2(X)$ is the Borel-Moore homology of X and hence by page 6 in A. Haeflinger's exposé in [B] we have $H^{DR}_2(X) \cong H_2(X, \boldsymbol{C})$. We obtain $H^2(X, \boldsymbol{C}) \cong \boldsymbol{C}$. Since X is $\boldsymbol{Q}$-Gorenstein, ω_X has a first Chern class $c_1(\omega_X)$ in $H^2(X, \boldsymbol{C})$ (or use the De Rham cohomology and [H2], p. 58). Call $f \cdot \boldsymbol{C}$ value of $c_1(\omega_X)$ obtained by the identification of $H^2(X, \boldsymbol{C})$ with $\boldsymbol{C}$. Notice that we have $f = 0$, otherwise by Seshadri criterion of ampleness ([H1], p. 37) we would have $(\omega_{X^{\otimes t}})^{**}$ ample and hence $\kappa(X, (\omega_{X^{\otimes t}})^{**}) = n$, contradicting our assumptions. Let M be a general linear subspace of $\boldsymbol{P}^N$ with $\mathrm{codim}(M) = \dim(X) - \mathrm{codim}(X)$. Set $Y := X \cap M$. By the assumption on $\dim(\mathrm{Sing}(X))$ and Bertini theorem $\mathrm{Sing}(X) \cap M = \emptyset$ and Y is smooth. Notice that Y has codimension $\dim(Y)$ in M and hence the normal bundle $N_{Y,M}$ of Y in M has rank $\dim(Y)$. Set $H := \boldsymbol{O}_Y(1)$. By the adjunction formula and the smoothness of X along Y the line bundle ω_Y has the same intersection products as $H^{\otimes(f + \dim(\mathrm{Sing}(X))}$ with all cohomology classes coming from $\boldsymbol{P}^N$ and with $c_1(\omega_X)$ and in particular with $c_1(N_Y, M) \cong H^{\otimes(N+1-\mathrm{codim}(M))} \otimes c_1(w_Y)^*$ and with $c_{N-n}(N_Y, M) \cong$

$\deg(Y)\,H^{N-n}$ (self-intersection formula ([F], Cor. 6.3)). Hence we may repeat verbatim the proofs in [S] and obtain all the statements of 0.1.

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