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# Some Problems for Measures on Non-Standard Algebraic Structures.

Maria Gabriella Graziano (*)

**Sunto.** – *Nell'ultimo ventennio tutta una serie di lavori è stata rivolta allo studio delle misure su strutture algebriche più generali delle algebre di Boole, come i poset e i reticoli ortomodulari, le effect algebras, le BCK-algebras. La teoria così ottenuta interessa l'analisi funzionale, il calcolo delle probabilità e la topologia, più recentemente la teoria delle decisioni. Si presentano alcuni risultati relativi a misure su strutture algebriche non-standard analizzando, in particolare, gli aspetti topologici e geometrici che intervengono nei teoremi di decomposizione e prolungamento.*

## 1. – Introduction.

The aim of this note is to review some classical problems and results arising in the study of measures on algebraic structures weaker than the ones usual in the classical probability theory, that is weaker than $\sigma$-algebras of sets or, more generally, than Boolean $\sigma$-algebras.

Before giving a general idea of a part of research activity recently made in this area, we cannot dispense with the attempt of briefly explain the motivations that, in the past, led physicists and mathematicians to the re-examination of the structure and the basic postulates of classical measure theory.

As incisively explained by Gudder [22], this re-examination is necessary if the theory is to describe accurately measurements of physical phenomena. Especially, the underlying algebraic structure of measure theory must be generalized to include the wider class of phenomena of quantum mechanics.

The non-classical aspects of the logic of quantum mechanics have been examined since the pioneer paper of Birkhoff and von Neumann, [7].

The mathematical formalism of the problem can be introduced as follows. Both in the classical and quantum case, states and propositions (or events) make possible the complete description of a mechanical system and predictions about its evolution. The set $\mathcal{L}$ of propositions has a natural partial order $\leqslant$ corresponding to the implication between events (if $a$, $b \in \mathcal{L}$, $a \leqslant b$ means that event $a$ implies event $b$) and an orthocomplementation * representing the

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negation of every event. The state of the system is, then, described by a probability measure $\mu : \mathcal{L} \to [0, 1]$ where $\mu(a)$ is interpreted as the probability that event $a$ occurs in the given state.

In connection with the ordered structure of $\mathcal{L}$, the passage from classical to quantum theory causes the loss of distributivity law. Indeed, in the classical Boolean case the validity in $\mathcal{L}$ of the identity

$$a = (a \wedge b) \vee (a \wedge b^*)$$

expresses the possibility of observing simultaneously events $a$ and $b$ in the following sense: it simply says that $a$ and $b$ occur at the same time or $a$ occurs while $b$ does not. This principle falls to be true in quantum physics. In this contest the celebrated Heisenberg indetermination principle states the impossibility of ascribing an exact trajectory to a quantum particle due to the impossibility of observing simultaneously the associated observable quantities. In fact, it is not possible to measure precisely a physical quantity (for example the motion quantity of the particle) without making undetermined the measure of other quantity (its position).

The characteristic feature of events of quantum mechanical system is the non-compatibility and their algebraic structure is not Boolean. Orthomodular posets and orthomodular lattices are then introduced as formalization of the natural algebraic structure of the set $\mathcal{L}$.

An orthomodular poset (OMP) is defined as a bounded partially ordered set $(\mathcal{L}, \leq, 0, 1)$ with an orthocomplementation $^*: \mathcal{L} \to \mathcal{L}$, that is a decreasing function such that $a^{**} = a$ and $a \wedge a^* = 0$ for any element $a \in \mathcal{L}$, satisfying the conditions:

1) if $a, b \in \mathcal{L}$ with $a$ and $b$ orthogonal (i.e. $a \leq b^*$), then $a \vee b$ exists in $\mathcal{L}$

2) if $a, b \in \mathcal{L}$ with $a \leq b$, then $b = a \wedge (a^* \wedge b)$ (orthomodular law).

The orthomodular law expresses the weaker form of distributivity that should be guaranteed also in the case of quantum phenomena. If the poset $(\mathcal{L}, \leq)$, in addition, is a lattice, then $\mathcal{L}$ is called an orthomodular lattice (OML). A distributive orthomodular lattice is a classical Boolean algebra.

The important example of non-Boolean orthomodular lattice that gives rise to the theory and represents its functional analytic aspect is the lattice $\mathcal{L}(H)$ of closed subspaces of an Hilbert space $H$. This structure is already present in the famous book of von Neumann on the mathematical foundations of quantum mechanic [28]. The set $\mathcal{L}(H)$ is ordered with the set inclusion and the usual orthogonal complementation $^\perp$ is the algebraic orthocomplementation. In the so called «hilbertian» formulation of quantum mechanics, to any state is associated a density operator and events are projection operators. The non-compatibility of two events $A$ and $B$ is then equivalent to non-commutativity of associated projections $P_A$ and $P_B$.

A case of OMP of special interest is represented by concrete logics. A concrete logic is defined as a family of subsets of a non-empty set $\Omega$ containing the empty set, which is closed with respect to the complementation and the finite disjoint union. Concrete logics have been recently used by decision theorists to describe sets of events which are unambiguous. In the wake of problems like the so-called Ellsberg paradox, generalizations of the classical theory of subjective expected utility have been provided allowing for ambiguity. Ambiguity is allowed to matter for choice as consequence of the imprecise information that a decision maker may have about some uncertain event. It turns out that the set of unambiguous events is in general not an algebra (examples show that the intersection of two unambiguous events may be ambiguous) but for preferences which have ambiguity attitude it can be simply characterized as a concrete logic called finite $\lambda$-system ([18], [19]).

The arrangement of the theory of OMPs and OMLs is essentially due to [5], [6], [22], [24], [37], [30], [32].

Other types of algebraic structures weaker than Boolean algebras come from the pursuit of different quantum mechanical constructions and from the attempts made in the past of generalize Hilbertian logics: we shall call them orthomodular structures (OMS). The name is justified since they all are partially ordered sets or lattices with (partially) defined sum $\oplus$ and subtraction $\ominus$ compatible with the partial order and satisfying the law

$$(*) \qquad a \leqslant b \Rightarrow b = a \oplus (b \ominus a)$$

which can be considered as a generalized form of orthomodular law, when $\oplus$ is the disjoint union and $\ominus$ is the intersection between orthogonal elements. Examples of orthomodular structures are given by Effect algebras (or Difference posets) introduced by [25] and containing as special case the set of all effects that is operators $T$ such that $0 \leqslant T \leqslant I$; BCK-algebras introduced by [23] which generalize in the commutative bounded case the MV-algebras of [8], [29]; Vitali spaces (commutative minimal clans, alternatively) introduced by [33] as a common abstraction of Boolean algebras and Riesz spaces. Relations among different types of orthomodular structures are investigated in [31], [16].

As said before, once introduced the algebraic structure describing the mechanical system, a complete knowledge of its evolution is given by the states defined on it. We call (finitely additive) measure on $\mathcal{L}$ any function $m : \mathcal{L} \to R$ such that

$$m(a \vee b) = m(a) + m(b), \quad \text{if } a \leqslant b^*$$

if $\mathcal{L}$ is an OMP,

$$m(a \oplus b) = m(a) + m(b), \quad \text{if } a \oplus b \text{ is defined}$$

in the case $\mathcal{L}$ is a more general OMS. The definition generalizes the usual definition of measure on Boolean algebras, since in the Boolean case two elements are disjoint if and only if they are orthogonal. If $m(1) = 1$, then $m$ is called a state. More generally, the measures may take values in abstract spaces such as Banach lattices, Riesz spaces, topological groups.

It is interesting to observe that in the case of Vitali spaces, commutative BCK-algebras, $\Delta$-l-semigroups, which are in particular lattices, every measure on $\mathcal{L}$ satisfy the modular law

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b), \quad \text{for all } a, b \in \mathcal{L}.$$

A function satisfying this property is usually called modular (or a valuation if it is real valued). In the case of Boolean algebras, modular functions null in zero are the same as measures. A special attention is devoted to modular functions defined on OMLs. Indeed, in this case, modular functions null in zero are measures but finite additivity does not entail modularity. For its peculiar properties this restricted class of measures on OMLs is often interesting by itself.

As in classical measure theory, in orthomodular theory one deals, between others, with the problems of decomposition, extension, common extension and convergence of sequence of measures. In the next two sections we focus on the first two questions presenting recent results concerning the decompositions of measures on OMSs and their extensions obtained by means of topological and geometrical tools. It is worth to point out that in some situations the algebraic structure of $\mathcal{L}$ is so general that it allows a unified treatment of different theories. For example, in the case of Vitali spaces, theories of classical probability measures, operators on function spaces, fuzzy measures can be carried out simultaneously.

## 2. – The problem of decomposition.

Assume to work in the general setting of measures taking values in a topological group $(G, +)$.

One of the main necessities dealing with a measure $\mu$ and a given property $\mathcal{P}$ is that of recognize the part of $\mu$ which «behaves well» with respect to $\mathcal{P}$ characterizing the remaining part. When the property $\mathcal{P}$ is that of absolute continuity with respect to a fixed measure $\eta$ or that of countable additivity, the characterization is possible if decompositions of Lebesgue and Hewitt-Yosida type hold, respectively. Indeed, in its classical formulation, the first decomposition theorem allows to write every bounded real-valued measure $\mu$ defined on a Boolean algebra as sum of two bounded measures $\lambda$ and $\nu$: $\lambda$ is absolutely continuous with respect to $\eta$ (we mean here the usual $\varepsilon - \delta$ absolute continuity), $\nu$ and $\eta$ are singular (that is any $\lambda$ absolutely continuous with respect to $\eta$ and less than or equal to $\nu$ is null). In the Hewitt-Yosida decomposition of $\mu$ the first measure $\lambda$ is countably additive while $\nu$ is purely finitely additive (that is singular with any countably additive measure). One classical way to obtain this type of decompositions is to split the order complete Riesz space of bounded measures in the direct sum of two suitable projection bands by a direct application of Riesz decomposition theorem (see [33], [12]).

When, as in our case, measures take values in a space $\mathcal{G}$ whose structure is so weak that band decomposition methods are not applicable, then different techniques are developed.

As observed by Drewnowski [15] at the beginning of seventies, the natural framework to obtain decompositions of measures with values in abstract spaces is the topological one. In the case of measures defined on Boolean algebras the tool is that of Frechét-Nikodym topologies. In the topological approach, substantially, properties of measures are viewed as properties of suitable associated topologies and in this context are investigated.

This idea is proved to be fruitful also for structures weaker than Boolean rings and algebras. The way to extend topological methods to the general context of OMSs is furnished by lattice uniformities. By lattice uniformity we mean any uniform structure on the space $\mathcal{L}$ such that the lattice operations $\vee$ and $\wedge$ are uniformly continuous. The induced topology is then a lattice topology. An extensive study of lattice uniformities on general lattices is present in [38].

We shall present the main ideas specialized in the case of Vitali spaces. This case is sufficiently general to include the results proved in [39] for Boolean rings, in [4] for l-groups, in [2] for MV-algebras an fuzzy measures.

Assume that the OMS $\mathcal{L}$ is a Vitali space, that is $\mathcal{L}$ is a partially ordered set $(\mathcal{L}, \leqslant)$ with a set of pairs of summable elements $S$, a partial addition $\oplus : S \to \mathcal{L}$ such that the resulting structure $(\mathcal{L}, S, \oplus, \leqslant)$ is a commutative lattice-ordered partial semigroup with the cancelation property and a difference property formulated as follows

$$\forall x, y \in \mathcal{L}\ \exists z \in \mathcal{L}\ \text{ s. t. } (x, z), (x \wedge y, z) \in S,\ x \oplus z = x \vee y,\ x \wedge y \oplus z = y\ .$$

By means of this property a (partial) subtraction $\ominus$ can be naturally introduced in $\mathcal{L}$ in such a way that the generalized orthomodular law $(*)$ is satisfied.

A suitable uniform structure on the space $\mathcal{L}$ is introduced in [20] as a generalization of Frechét-Nikodym topologies on Boolean rings and locally solid topologies on Riesz spaces. This uniformities, called Vitali space uniformities, must be lattice uniformities such that the partial operation $\oplus$ and $\ominus$ are uniformly continuous. Differently from the case of lattice uniformities on lattices in which the same topology may generate different uniformities; one can show that the uniformities of Vitali spaces are uniquely determined by their zero-neighbourood filter ([20], Theor. 3.6). This is the main point to obtain decomposition results for measures.

Once introduced a topological structure on the space $\mathcal{L}$, in a second step to any measure $\mu$ defined on $\mathcal{L}$ is associated a Vitali space uniformity $\mathcal{U}_\mu$ requiring that $\mathcal{U}_\mu$ is the weakest which makes $\mu$ uniformly continuous.

The main properties of measures that permits the extension of decomposition theorems are that of exhaustivity (which gives the usual boundedness in the real-valued case) and $\sigma$-order continuity.

A measure $\mu : \mathcal{L} \to \mathcal{G}$ is said to be exhaustive if for very monotone sequence $a_n$ $\mu(a_n)$ is Cauchy. A uniformity $\mathcal{U}$ on $\mathcal{L}$ is exhaustive if every monotone sequence $a_n$ is Cauchy. Then, it is possible to show that $\mu$ is exhaustive if and only if the associated topology $\mathcal{U}_\mu$ is exhaustive. Other useful definitions are that of ($\sigma$-)order continuity. The lattice uniformity $\mathcal{U}$ is said to be order-continuous (this is the Lebesgue-property for Riesz spaces) if every monotone net $a_\alpha$ order-converging to $a$, converges to $a$ in the topology induced from $\mathcal{U}$. If this property holds for monotone order-converging sequences, $\mathcal{U}$ is said to be $\sigma$-order continuous. A measure $\mu$ is $\sigma$-order continuous if for every monotone sequence $a_n$ order-converging to $a$, $\mu(a_n)$ converges to $\mu(a)$. As before, it is true that a measure $\mu$ is $\sigma$-order continuous if and only if $\mathcal{U}_\mu$ is $\sigma$-order continuous.

The notion of absolute continuity and singularity can be generalized naturally. We say that a measure $\mu$ is absolutely continuous with respect to a uniformity $\mathcal{U}$ (resp. with respect to a measure $\eta$) if the inclusion $\mathcal{U}_\mu \subseteq \mathcal{U}$ (resp. $\mathcal{U}_\mu \subseteq \mathcal{U}_\eta$) holds. We say that $\mu$ and $\mathcal{U}$ (resp. $\eta$) are singular if the infimum between $\mathcal{U}_\mu$ and $\mathcal{U}$ (resp. $\mathcal{U}_\mu$ and $\mathcal{U}_\eta$) taken in the lattice of all Vitali space uniformities is zero.

The general decomposition theorem we prove holds with respect to any Vitali space uniformity $\mathcal{U}$ and can be formulated as follows

THEOREM 2.1 ([20], Theor. 5.13). – *Let $\mathcal{G}$ be a complete commutative Hausdorff topological group and $\mu : \mathcal{L} \to \mathcal{G}$ be an exhaustive measure. For any Vitali space uniformity $\mathcal{U}$, there are unique measures $\lambda : \mathcal{L} \to \mathcal{G}$ and $\nu : \mathcal{L} \to \mathcal{G}$ such that $\mu = \lambda + \nu$, $\lambda$ is absolutely continuous with respect to $\mathcal{U}$, $\nu$ and $\mathcal{U}$ are singular, $\lambda$ and $\nu$ are singular.*

As consequence of preceding theorem, for a $\mathcal{G}$-valued measure $\eta$ on $\mathcal{L}$ and for $\mathcal{U} = \mathcal{U}_\eta$ we obtain a Lebesgue-type decomposition theorem. A particular case is the classical theorem proved in [36] for measures on Boolean rings. For a suitable chosen uniformity $\mathcal{U}$ ($\mathcal{U}$ is the supremum of all $\sigma$-order continuous uniformities) a decomposition of Hewitt-Yosida type follows ([20], Theor. 5.16).

The main idea that supports the proof can be summarized with the expression «completion principle» introduced in [39] in connection with Frechét-Nikodym topologies. The central result we use concerns the uniformly continuous extension of modular functions defined on general lattices.

The exhaustivity allows us to extend the measure $\mu$ defined on $\mathcal{L}$ to a measure $\tilde{\mu}$ defined on $\tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is the uniform completion of $\mathcal{L}$ with respect to its weakest exhaustive Vitali space uniformity.

It turns out that $\tilde{\mu}$ is uniformly continuous and completely additive, $\tilde{\mathcal{L}}$ is a complete lattice whose uniformity (obtained by completion) is order-continuous. In this advantageous situation, one obtains at a first step a decomposition for $\tilde{\mu}$ and, subsequently, derives a decomposition for $\mu$. The decomposition of $\tilde{\mu}$ is obtained through the splitting with respect to an element in the center of the lattice $\tilde{\mathcal{L}}$ which is suitably associated to the uniformity $\mathcal{U}$ and its complement ([21]).

In particular, by their nature, all the topological decompositions are characterized by uniqueness.

Similar techniques can be developed in the case of modular functions defined on OMLs. By means of lattice uniformities, a decomposition of general type can be proved for the group-valued case [40].

The situation is completely different in the case of measures defined on OMLs or OMPs even in the real-valued case. In fact, on one hand the loss of modularity and distributivity makes the topological methods inapplicable. On the other hand, using standard Boolean arguments the resulting decompositions are not unique. Indeed, even if the consideration is limited to the bounded case, the space of measures is a vector space but, in general, it is not a Riesz space (it is not a lattice). Therefore the Riesz decomposition theorem in the space of bounded measure is not directly applicable.

The lack of uniqueness is in some sense natural when the measures are defined on non-distributive ordered structures ([14], [27]). From this point of view, particular relevance assume all the results stating conditions which are sufficient for the uniqueness of decompositions.

In [13] and [14] general decompositions for states defined on OMPs (called $D$-decompositions) and their uniqueness are studied. The main idea consists in the use of geometrical results to study the structure of the states space. The D-decomposition has an evident intuitive appeal. To understand the way in which geometrical results may induce decompositions, we recall the following result of Shultz [35]:

*«the state space of an OMP is affinely homeomorphic to a compact convex subset of a locally convex Hausdorff topological vector space.»*

Clearly to decompose a measure in the sum of two measures is the same as writing a state as a convex combination of two states. Therefore, the problem of decomposition, in view of the previous result, can be reduced to the following geometric problem: every point in the state space must be shown to belong to a segment joining other two points.

In the $D$-decomposition, a subset $D$ of a convex compact set $C$ in the locally convex space is fixed (in general a face) and its complement $D^{\perp}$ is defined as the union of all faces which are disjoint from $D$. When an element of $C$ can be expressed as convex combination of an element of $D$ an element of $D^{\perp}$, the decomposition is called a D-decomposition. For an arbitrary set $D$, the $D$-decomposition need not exist in general. Examples in this sense can be found in [14].

Taking as convex compact set $C$ the state space of an OMP, then the $D$-decomposition can be applied by Shultz result. Choosing as set $D$ the family of all states on $\mathcal{L}$ which are absolutely continuous with respect to a fixed state $\eta$, its complement $D^{\perp}$ will be formed exactly by those states on $\mathcal{L}$ which are orthogonal with respect to $\eta$ (in the usual sense). For the chosen $D$ the $D$-decomposition exists and it is equal to the Lebesgue decomposition.

When $D$ is the set of all completely additive states, one obtains the Hewitt-Yosida decomposition.

The meaning of uniqueness in the case of $D$-decompositions is the following: if an element $a$ is convex combination $a = tb + (1-t)\,c$ where $b \in D$ and $c \in D^{\perp}$, then $t$ is unique and $b$ and $c$ are uniquely determined unless their coefficient is equal to zero.

As already observed, the decompositions are in general not unique. This should be clear looking to the following converse of Shultz result completely characterizing the state space ([35]):

*«Every compact convex subset of a locally convex Hausdorff topological linear space is affinely homeomorphic to the state space of an OMP».*

As consequence of the previous characterization, simple examples in finite dimensional spaces can be provided to show the lack of uniqueness. For example taking as set $C$ a square and as $D$ one edge of $C$, then $D^{\perp}$ is the opposite edge in the square, the $D$-decomposition exists for interior points but it is, clearly, not unique. A complete discussion of conditions sufficient for the uniqueness of $D$-decomposition and Hewitt-Yosida decomposition is present in [14].

Finally, decompositions of algebraic type are possible for measures defined on OMPs. This decompositions, proved in [9] for the restricted class of measures whose kernel is a $p$-ideal, are characterized by uniqueness.

## 3. – The problem of extension.

This section mainly concerns with the problem of extension and common extension of measures defined on OMSs. As it is well known, in the classical theory every measure defined on a sub-algebra of an algebra of sets can be extended to the whole algebra passing through an outer measure. The result can be transferred to general Boolean algebras using Stone representation theorem.

The attempts of applying standard techniques to a measure $\mu$ defined on a substructure of an OMP or OML $\mathcal{L}$ lead to different notions of $\mu$-measurability, but the results are only partial: generally the lattice structure of the $\mu$-measurable sets or, in some cases, the additivity of extension are lost. For example any state defined on a Boolean-subalgebra of an OMP can be extended to the whole OMP but the extension is a pseudo-state, that is a monotone increasing function preserving the additivity only on some special pairs of orthogonal elements (see [10]).

In this section we present some extension results for measures and modular functions defined on OMSs based on topological methods.

As for decomposition theorems, many authors studied the problem of extending classical measures defined on Boolean algebras and rings using Frechét-Nikodym topologies. As it is natural, some of this results can be extended to OMSs replacing topologies by suitable lattice uniformities.

In the first part we shall deal with a modular function $\mu$ which is defined on an orthomodular lattice $\mathcal{L}$. As we said before, modular functions on OMLs are the corresponding of measures on OMSs when Frechét-Nikodym topological methods are used in connection with non-distributive lattice operations.

The method works for the Boolean case as follows. Consider a $\sigma$-complete Boolean algebra $\mathcal{L}$, a sub-algebra $\mathcal{M}$ of $\mathcal{L}$ and a measure $\mu : \mathcal{M} \to \mathcal{G}$ $\sigma$-order continuous and exhaustive. Assume for simplicity that the group $\mathcal{G}$ is metrizable and denote by $\sigma(\mathcal{M})$ the $\sigma$-sub-algebra generated by $\mathcal{M}$.

Starting from the measure $\mu$ one defines a monotone and sub-additive function $\mu^* : \mathcal{L} \to [0, \infty]$ by the positions

$$\tilde{\mu}(a) = \sup\{|\mu(b)| : b \in \mathcal{M},\ b \in [0, a]\} \quad (a \in \mathcal{M})$$

and

$$\mu^*(a) = \inf\{\tilde{\mu}(b) : b \in \mathcal{M}_\sigma,\ b \geq a\} \quad (a \in \mathcal{L})$$

where $\mathcal{M}_\sigma$ is the set of all order limits of increasing sequence in $\mathcal{M}$. Then it can be shown that $\mu^*$ is a $\sigma$-sub-measure extending $\tilde{\mu}$ on $\mathcal{L}$.

To the sub-measure $\mu^*$ is associated in the standard way a Frechét-Nikodym topology $\tau_{\mu^*}$ on $\mathcal{L}$. It turns out that this topology makes $\mu$ uniformly continuous. Then, by completeness of $\mathcal{G}$, $\mu$ can be extended in a unique way to a uniformly continuous function $\bar{\mu}$ defined on the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ with respect to $\tau_{\mu^*}$. Since the lattice operations are continuous with respect to $\tau_{\mu^*}$, $\overline{\mathcal{M}}$ is a sub-algebra and $\bar{\mu}$ is a measure. Moreover $\bar{\mu}$ is $\sigma$-order continuous. Finally, since by exhaustivity $\overline{\mathcal{M}}$ is a $\sigma$-complete lattice, the conclusion that the measure $\mu$ can be extended in a unique way to a measure defined on the $\sigma$-complete sub-algebra generated by $\mathcal{M}$ $\sigma(\mathcal{M})$ preserving the properties of $\mu$.

The previous result can be immediately extended to non-negative real valued modular functions on an orthomodular lattice $\mathcal{L}$ observing that they are in particular exhaustive.

Denote by $\mathcal{M}$ a sub-orthomodular lattice of $\mathcal{L}$, i.e. a sub-lattice closed with respect to orthocomplementation, and by $\sigma(\mathcal{M})$ the $\sigma$-complete lattice generated by $\mathcal{M}$ that turns out to be an orthomodular lattice.

THEOREM 3.1 ([1], Cor. 2.2.5). – *Let $\mu : \mathcal{M} \to [0, \infty[$ be a $\sigma$-order continuous modular function. Then there exists a unique extension of $\mu$ to a $\sigma$-order continuous modular function $\bar{\mu}: \sigma(\mathcal{M}) \to [0, \infty[$ and $\mathcal{M}$ is dense in $(\sigma(\mathcal{M}), \mathcal{U}_{\bar{\mu}})$.*

If $\mathcal{G}$ is metrizable the previous result is still true. Indeed, by a generalized version of the control measure theorem of Bartle-Dunford-Schwartz there exists a modular function $\nu : \mathcal{L} \to [0, 1]$ such that $\mathcal{U}_\mu = \mathcal{U}_\nu$. Therefore, at a first step one can extend the measure $\nu$ by Theorem 2.1 to a measure $\bar{\nu}$ on $\sigma(\mathcal{M})$ finding, subsequently, by means of uniform continuity of $\mu$ with respect to $\mathcal{U}_\nu$, the extension of the original function. When the group $\mathcal{G}$ is not metrizable, embedding $\mathcal{G}$ in a product of Banach spaces $\mathcal{G}_i$, $i \in I$, and considering $\mu$ as a function $(\mu_i)_{i \in I}$ the proof still works. The result can be formulated as follows

THEOREM 3.2 ([1], Theor. 2.2.7). – *Let $\mathcal{G}$ be a locally convex complete Hausdorff linear space and $\mu : \mathcal{M} \to \mathcal{G}$ be an exhaustive $\sigma$-order continuous modular function. Then there exists a unique extension of $\mu$ to a $\sigma$-order continuous exhaustive modular function $\bar{\mu}: \sigma(\mathcal{M}) \to \mathcal{G}$ and $\mathcal{M}$ is dense in $(\sigma(\mathcal{M}), \mathcal{U}_{\bar{\mu}})$.*

The principal difficulty in the application of the previous topological method when $\mathcal{G}$ is an Abelian metrizable topological group is due to the fact that the uniformity associated to $\mu^*$ is not in general a lattice uniformity. Therefore Theorem 2.2 still holds provided that an additional condition is introduced in order to guaranty the continuity of lattice operations. This difficulty seems to be caused again by non-distributivity of lattice operations. In fact, the method substantially works as in the Boolean case for distributive OMSs such as MV-algebras or $\Delta$-l-semigroups where the uniformity associated to a sub-measure is always a lattice uniformity (see [2]).

Another well known problem in the classical measure theory is that of finding a common extension on the algebra $\mathcal{L}$ of sets of two consistent measures $\mu_1$ and $\mu_2$ defined, respectively, on the sub-algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}$ (without loss of generality $\mathcal{L}$ may be the algebra generated by $\mathcal{L}_1$ and $\mathcal{L}_2$).

The measures $\mu_1$ and $\mu_2$ admit a common extension if there exists a measure $\mu$ defined on $\mathcal{L}$ such that its restrictions to $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal to $\mu_1$ and $\mu_2$, respectively. A classical result states that, in the case of real-valued measures, a common extension exists if and only if $\mu_1$ and $\mu_2$ are consistent, that is if and only if they coincide on the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$. If the measures $\mu_1$ and $\mu_2$ are assumed to be bounded, their common extension, generally, need not be bounded.

Along a line of research begun by Lipecki [26], several authors have dealt with the problem of providing necessary and sufficient conditions under which bounded consistent measures have a bounded common extension in the real-valued or in the general vector-valued case (see [3], [11]). Conditions can be given also in the general case of order bounded vector measures with values in Dedekind complete Riesz spaces in order their common extension to be order bounded. The proofs in this case are based on representation of vector measures by linear operators and the Hahn-Banach theorem for operators in order complete Riesz-spaces (see [34]).

Results concerning the problem of common extension for states defined on sub-algebras of OMPs are known in the case of concrete OMPs. A concrete OMP $\mathcal{L}$ is defined as a collection of subsets of a non-empty set $\Omega$ containing the empty set, the complement of any element and any finite disjoint intersection of its elements.

THEOREM 3.3. – ([10], Theor. 3.6.1). – *Let $\mathcal{L}$ be a concrete* OMP, $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be Boolean sub-algebras of* $\mathcal{L}$, $\mu_1$ *and* $\mu_2$ *be states defined on* $\mathcal{L}_1$ *and* $\mathcal{L}_2$, *respectively. Then there exists a state which is a common extension of* $\mu_1$ *and* $\mu_2$ *on* $\mathcal{L}$ *if and only if, for every* $(a_1, a_2) \in \mathcal{L}_1 \times \mathcal{L}_2$

$$\mu_1(a_1) \geqslant \mu_2(a_2) \quad \text{if } a_1 \geqslant a_2$$

$$\mu_1(a_1) \leqslant \mu_2(a_2) \quad \text{if } a_1 \leqslant a_2 .$$

Clearly, the previous result implies, if the sub-algebras in the statement coincide, that every state defined on a sub-algebra of a concrete OMP has at least an extension on the whole poset.

A different technique to obtain an extension is developed in [17] making use, substantially, of the geometrical properties of the state space.

The results are proved for the case of commutative BCK-algebras with relative cancelation property, but, by basic representation theorems present in [16], they still hold if the space $\mathcal{L}$ is a positive Vitali space which is, as lattice, Dedekind $\sigma$-complete. The extension is based on special elements of the set $\mathcal{L}$, called characteristic elements, defined with respect to a quasi strong unit. Roughly speaking, a quasi strong unit is an element in $\mathcal{L}$ that generates, as ideal, the whole space. If such unit exists, then the state space $C$ of $\mathcal{L}$ is a compact convex Hausdorff space and the subspace of its extremal points, which is basically disconnected, can be topologically identified with the set of maximal ideals of $\mathcal{L}$.

This result is due to the fact that maximal ideals of $\mathcal{L}$ are in a one-to-one correspondence with maximal ideals of the Boolean sub-algebra of characteristic elements. It allows the extension of any state defined on the sub-algebra of characteristic elements to the space $\mathcal{L}$. Firstly one extends a two-valued state: its kernel is a maximal ideal in the sub-algebra and corresponds to a maximal ideal in $\mathcal{L}$, then it corresponds to an extremal state which is the desired extension. Now, the extension of any state follows standardly since states are weak limits of convex combinations of extremal states.

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