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Pseudo-valuation rings. II

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Sunto. – Viene data una condizione sufficiente affinché un sopra-anello di un anello di pseudo-valutazione (PVR) sia ancora un PVR. Da ciò segue che se $(R, M)$ è un PVR, allora ogni sopra-anello di $R$ è un PVR se (e soltanto se) $R[u]$ è quasi-locale per ciascun elemento $u$ di $(M: M)$. Vari risultati sono dimostrati per un ideale primo di un anello commutativo arbitrario $R$, avente $Z(R)$ come insieme di zero-divisori. Per esempio, se $P$ è un primo «forte» di $R$ e contiene un elemento non-zero divisor di $R$, allora $(P: P)$ è un sopra-anello di $R$ con l’insieme degli ideali totalmente ordinato e con ideale massimale $P$; inoltre, $(P: P)$ è un PVR il cui ideale massimale è un ideale primo anche in $R$ se e soltanto se $P$ e $Z(R)$ sono entrambi ideali primi «forti» di $R$. Se $(R, M)$ è un PVR, viene dimostrato anche che $Z(R)$ può coincidere con nil $(R)$ oppure con un ideale primo propriamente contenuto tra questi due ideali.

1. – Introduction.

We assume throughout that all rings are commutative with $1 \neq 0$. This paper continues our study of pseudo-valuation rings (as introduced in [6]). We begin by recalling some background material. As in [10], an integral domain $R$, with quotient field $K$, is called a pseudo-valuation domain (PVD) in case each prime ideal $P$ of $R$ is strongly prime, in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. In [6], we generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [6] that a prime ideal $P$ of a ring $R$ is said to be strongly prime (in $R$) if $aP$ and $bR$ are comparable for all $a, b \in R$. A ring $R$ is called a pseudo-valuation ring (PVR) if each prime ideal of $R$ is strongly prime. A PVR is necessarily quasilocal [6, Lemma 1(b)]; a chained ring is a PVR [6, Corollary 4]; an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [5, Proposition 3]); and if $R$ is a PVR whose maximal ideal $M$ contains a non-zerodivisor, then $V := (M: M)$ is a chained ring with maximal ideal $M$ [6, Theorem 8].

The following notation will be used throughout. Let $R$ be a ring. Then $Z(R)$ denotes the set of zerodivisors of $R$, and nil $(R)$ denotes the set of nilpotent elements of $R$. Also, $S := R - Z(R) = \{ x \in R \mid x$ is a non-zerodivisor of $R \}$, $T = R_{S}$ is the total quotient ring of $R$, $R'$ denotes the integral closure of $R$ in $T$, and $U(R)$ denotes the set of units of $R$. As usual, we say that a ring $B$ is an overring
of \( R \) if \( R \subset B \subset T \); if \( I \) is an ideal of \( R \), then \( (I: I) = \{x \in T | xI \subset I\} \) is an overring of \( R \) and \( I^{-1} = \{x \in T | xR \subset R\} \); and \( (R, M) \) denotes that \( R \) is quasilocal with maximal ideal \( M \). Any unexplained material is as in [6], [12].

This paper is organized as follows. Section 2 develops a characterization of the PVRs all of whose overrings are PVRs. Section 3 is devoted to a number of results and examples concerning strongly prime ideals, with, as expected, interplay with the PVR concept. Two typical results in this regard are the following part of Theorem 3.6: if a strongly prime ideal \( P \) contains a non-zerodivisor, then \( (P: P) \) is a chained ring with maximal ideal \( P \); and Corollary 3.13: if \( P \in \text{Spec}(R) \), then \( (P: P) \) is a PVR whose maximal ideal is in \( \text{Spec}(R) \) if and only if \( P \) and \( Z(R) \) are both strongly prime ideals of \( R \). Moreover, Example 3.16(c) shows that if \( (R, M) \) is a PVR, then \( Z(R) \) can be nil \( (R), M, \) or a prime ideal properly contained between these two ideals.

2. – PVRs whose overrings are PVRs.

Our first result is a partial converse to the fact that PVRs are quasilocal.

Theorem 2.1. – Let \( (R, M) \) be a PVR and \( u \in V - R \). Then \( R[u] \) is a PVR if and only if \( R[u] \) is quasilocal.

Proof. – The «only if» assertion is immediate since a PVR is quasilocal [6, Lemma 1(b)].

Conversely, suppose that \( R[u] \) is quasilocal. It suffices by [6, Theorem 7] to show that \( M \) is the unique maximal ideal of \( R[u] \). If \( u \notin U(R[u]) \), then \( u + 1 \in U(R[u]) = U(R[u + 1]) \) since \( R[u] \) is quasilocal, whence \( (u + 1)^{-1} \in R' \) by [12, Theorem 15]. Hence \( (u + 1)^{-1} \notin M \) since \( M \) is a proper ideal of \( R[u] \). Since \( R' \) is a PVR with maximal ideal \( M \) by [6, Theorem 19], and \( (u + 1)^{-1} \in R' - M = U(R') \), we have \( u + 1 \in R' \) and \( u = (u + 1) - 1 \in R' \). On the other hand, if \( u \in U(R[u]) \), then [12, Theorem 15] gives \( u^{-1} \in R' \); as \( u^{-1} \notin M \) (since \( M \) is a proper ideal of \( R[u] \)), we have \( u^{-1} \in R' - M = U(R') \). Thus, in both cases, \( u \in R' \), and so \( R[u] \subset R' \).

Consider \( v \in R[u] - M \). As \( v \in R' - M = U(R') \), \( v^{-1} \in R' \) and so, by [12, Theorem 15], \( v^{-1} \in R[v] \subset R[u] \). In particular, \( v \in U(R[u]) \). Hence \( M \) is the maximal ideal of \( R[u] \).

Corollary 2.2. – If \( (R, M) \) is a PVR, then the following conditions are equivalent:

1. \( R' = V = (M: M) = \{x \in T | xM \subset M\} \);
2. Each overring of \( R \) is a PVR;
(3) Each overring of \( R \) that does not contain an element of the form \( 1/s \) for some \( s \in M \) is a PVR;

(4) For each \( u \in V - R, \ R[u] \) is a PVR;

(5) For each \( u \in V - R, \ R[u] \) is quasilocal;

(6) Each overring of \( R \) is quasilocal.

**Proof.** – (1) \( \Leftrightarrow \) (2) by [6, Theorem 21]; (2) \( \Rightarrow \) (3) trivially; and (3) \( \Rightarrow \) (2) by [6, Lemma 20 and Corollary 4]. Moreover, (2) \( \Rightarrow \) (6) trivially; and (5) \( \Rightarrow \) (4) by Theorem 2.1. It suffices to prove that (4) \( \Rightarrow \) (1). For this, note via the proof of Theorem 2.1 that (4) implies that \((M: M) \subset R’\), while [6, Lemma 17] gives the reverse inclusion.

**Example 2.3.** – (a) Theorem 2.1 does not extend to overrings which are generated by more than one element. In fact, if \((R, M)\) is a PVD and \( A \) is a quasilocal overring of \( R \) which is contained in \((M: M)\), then \( A \) need not be a PVD. For an example, consider \( R = \mathbb{Q} + X\mathbb{Q}(s, t)[[X]] = \mathbb{Q} + M \), where \( s, t \) and \( X \) are indeterminates and \( M = X\mathbb{Q}(s, t)[[X]] \). Observe that \( A = \mathbb{Q}[s, t]_{(s, t)} + M \) is a quasilocal overring of \( R \) which is contained in \((M: M) = \mathbb{Q}(s, t)[[X]]\), although \( A \) is not a PVD.

(b) Not all PVRs satisfy the equivalent conditions in Corollary 2.2. We next illustrate this with an example in which \( R \) is an integrally closed PVD. Let \( t \) and \( X \) be indeterminates and let \( V = \mathbb{Q}(t)[[X]] = \mathbb{Q}(t) + M \), where \( M = XV \). Then \( V \) is a valuation domain, and hence \( R = \mathbb{Q} + M \) is a PVD with maximal ideal \( M \) and \((M: M) = V\). However, \( R \) has an overring, namely \( R[t] = \mathbb{Q}[t] + M \), which is not quasilocal.

**Remark 2.4.** – The equivalence of (5) and (6) in Corollary 2.2 has the following counterpart for arbitrary rings. Let \( R \) be a ring with integral closure \( R’ \), and let \( M \subset R \). Then \((R’, M)\) is quasilocal \( \iff \) for each \( u \in R’ \), \((R[u], M)\) is quasilocal \( \iff \) each integral overring of \( R \) is quasilocal with unique maximal ideal \( M \). For a proof, note first via the incomparability and going-up properties that if \((R’, M)\) is quasilocal, then each integral overring of \( R \) is quasilocal with unique maximal ideal \( M \). On the other hand, suppose that \((R[u], M)\) is quasilocal for each \( u \in R’ \). Then if \( R \) has distinct maximal ideals \( M_1 \) and \( M_2 \), pick \( v \in M_1 - M_2 \) and note, via the going-up property of \( R[v] \subset R’ \), that \( M_1 \cap R[v] \) and \( M_2 \cap R[v] \) are distinct maximal ideals of \( R[v] \), the desired contradiction.

3. – Strongly prime ideals.

We next study some properties of strongly prime ideals. Recall that a strongly prime ideal of \( R \) is comparable under inclusion to each ideal
of \( R \) [6, Lemma 1(a)], and hence, that \( Z(R) \) is a prime ideal if \( R \) is a PVR.

**Lemma 3.1.** – Let \( P \) be a strongly prime ideal of a ring \( R \). Then

(a) \( P \) is comparable to \( Z(R) \).
(b) \( P_S \) is a strongly prime ideal of \( R_S \) for any multiplicative subset \( S' \) of \( R \) disjoint from \( P \).
(c) \( P_P \) is a strongly prime ideal of \( R_P \) and \( R_P \) is a PVR.
(d) If \( P \) contains a non-zerodivisor of \( R \), then \( P_P = P \).
(e) Each prime ideal \( Q \) of \( R \) is strongly prime. Moreover, \( (P: P) \subset (Q: Q) \).

**Proof.** – (a) This is clear since \( Z(R) \) is a union of prime ideals of \( R \) [12, page 3] and a strongly prime ideal is comparable to each (prime) ideal of \( R \) [6, Lemma 1(a)].
(b) This follows immediately from the definitions.
(c) By part (b) above, \( P_P \) is a strongly prime ideal of \( R_P \), and hence \( R_P \) is a PVR [6, Theorem 2].
(d) By part (a) above, \( Z(R) \subset P \), and hence \( R_P \subset R_S \). Let \( p/s \in P_P \) with \( p \in P \) and \( s \in R - P \). Then \( P \subset sR \) by [6, Lemma 1(a)]; so \( p/s \in P_P \cap R = P \). Hence \( P_P = P \).
(e) For the first assertion, just use the proof of [6, Theorem 2]. For the «moreover» statement, we may assume that \( Q \not= P \). Let \( x \in (P: P) \). Then \( xP \subset P \), and hence \( xQ \subset P \subset R \). Then \( (xQ) P = (xP) Q \subset Q \) yields \( xQ \subset Q \) since \( Q \) is prime. Thus \( x \in (Q: Q) \), and hence \( (P: P) \subset (Q: Q) \). □

We first concentrate on the case when \( P \) is a strongly prime ideal which contains a non-zerodivisor of \( R \). In this case, we show that \( R \subset R_P \subset (P: P) \subset P^{-1} \subset T \); and that \( (P: P) = P^{-1} \) if \( P \) is not principal (Theorem 3.6); and in Corollary 3.7(b), we determine when \( R_P = (P: P) \).

**Proposition 3.2.** – Let \( P \) be a strongly prime ideal of a ring \( R \) which contains a non-zerodivisor of \( R \). Then

(a) \( R \subset R_P \subset (P: P) \subset P^{-1} \subset T \).
(b) \( P^{-1} \not= T \).

**Proof** (a) By Lemma 3.1(a), \( Z(R) \subset P \), and hence \( R_P \subset T \). Thus we need only show that \( R_P \subset (P: P) \). This follows from Lemma 3.1(d) since \( R_P \subset (P_P: P_P) = (P: P) \).
(b) Let \( s \in P \) be a non-zerodivisor. If \( P^{-1} = T \), then \( 1/s^2 \in P^{-1} \); and hence \( 1/s = s(1/s^2) \in PP^{-1} \subset R \), a contradiction. Thus \( P^{-1} \not= T \). \( \blacksquare \)

We next give another condition for a prime ideal to be strongly prime. This generalizes [6, Theorem 5].

**Proposition 3.3.** – Let \( P \) be a prime ideal of a ring \( R \). Then

(a) Suppose that \( Z(R) \subset P \). Then \( P \) is strongly prime if and only if for every \( a, b \in R \), either \( bR \subset aR \) or \( aP \subset bP \).

(b) Let \( P \) be a strongly prime ideal. If either \( P \) contains a non-zerodivisor or \( P \) is a maximal ideal of \( R \), then for every \( a, b \in R \), either \( bR \subset aR \) or \( aP \subset bP \).

**Proof.** – (a) Suppose that \( P \) is strongly prime. Let \( a, b \in R \). If \( bR \subset aR \) then \( bR \subset aR \). So we may assume that \( aP \subset bP \). If \( aP \not\subset bP \), then \( ap = bc \) for some \( p \in P \) and \( c \in R - P \). Then \( c \) is a non-zerodivisor since \( Z(R) \subset P \), and \( c | p \) since \( P \subset cR \) by [6, Lemma 1(a)]. Hence \( a | b \), and thus \( bR \subset aR \).

Conversely, suppose that for every \( a, b \in R \), either \( bR \subset aR \) or \( aP \subset bP \). Let \( a, b \in R \). If \( aP \subset bP \), then \( aP \subset bR \). So we may assume that \( bR \subset aR \). Then \( b = ac \) for some \( c \in R \). If \( c \in P \), then \( bR \subset aP \). Suppose that \( c \not\in P \). Then \( c \) is a non-zerodivisor since \( Z(R) \subset P \). Let \( 0 \neq p \in P \); then \( bp = acp \). We claim that \( c | p \). If not, then \( cP \subset pP \) by hypothesis. Hence \( cp = pq \) for some \( q \in P \). Thus \( p(c - q) = 0 \), and hence \( c - q \in Z(R) \subset P \). Thus \( c \in P \), a contradiction. Hence \( c | p \) for each \( p \in P \), and thus \( P \subset cR \). Hence \( aP \subset acR = bR \). Thus \( P \) is strongly prime.

(b) In either case, \( Z(R) \subset P \) by Lemma 3.1(a). Thus part (b) follows from part (a) above. \( \blacksquare \)

Recall from [6] that an ideal \( I \) of a ring \( R \) satisfies property \((*)\) if whenever \( xy \in I \) for some \( x, y \in I \), then either \( x \in I \) or \( y \in I \). It was shown [6, Theorem 14] that if \( (R, M) \) is a PVR, then \( M \) satisfies property \((*)\). The following proposition is a generalization of that fact.

**Proposition 3.4.** – Let \( P \) be a strongly prime ideal of a ring \( R \). If \( P \) contains a non-zerodivisor of \( R \), then \( P \) satisfies property \((*)\).

**Proof.** – By parts (c) and (d) of Lemma 3.1 and Proposition 3.2(a), \( R_P \) is a PVR with maximal ideal \( P_P = P \) and total quotient ring \( T \). Thus \( P \) satisfies property \((*)\) by [6, Theorem 14]. \( \blacksquare \)

For our next result, cf. [10, Proposition 1.2] and [6, Lemma 13].

**Proposition 3.5.** – Let \( P \) be a strongly prime ideal of a ring \( R \) which contains a non-zerodivisor of \( R \). Then
(a) \( T - R \subseteq U(T) \).

(b) If \( x \in T - R \), then \( x^{-1}P \subset P \).

Proof. – (a) Let \( x = a/b \in T - R \), where \( a \in R \) and \( b \in R - Z(R) \). Suppose that \( a \in Z(R) \). Since \( P \) is strongly prime, \( aR \) and \( bP \) are comparable. If \( aR \subset bP \), then \( x \in P \subset R \), a contradiction. Thus \( bP \subset aR \subset Z(R) \), and hence \( b \in Z(R) \) since \( P \) contains a non-zerodivisor, again a contradiction. Thus \( a \notin Z(R) \); so \( x^{-1} = b/a \in T \), and thus \( x \in U(T) \).

(b) We have \( x(x^{-1}P) = P \); so \( x^{-1}P \subset P \) since \( P \) satisfies property \((*)\) by Proposition 3.4.

Recall that a ring \( R \) is a chained ring if its ideals are linearly ordered by inclusion (i.e., for every \( x, y \in R \), either \( x \mid y \) or \( y \mid x \)). Any chained ring is necessarily a PVR [6, Corollary 4]. The following result is motivated by [2, Proposition 4.3] and [4, Proposition 5].

Theorem 3.6. – Let \( P \) be a strongly prime ideal of a ring \( R \) which contains a non-zerodivisor of \( R \). Then \((P : P)\) is a chained ring with maximal ideal \( P \). Moreover, if \( P \) is nonprincipal, then \((P : P) = P^{-1}\); and if \( P \) is principal, then \((P : P) = R\).

Proof. – By parts (c) and (d) of Lemma 3.1, \( R_p \) is a PVR with maximal ideal \( P_p = P \). Since \( P \) contains a non-zerodivisor, \((P : P) = (P_p : P_p)\) is a chained ring with maximal ideal \( P_p = P \) by [6, Theorem 8].

For the «moreover» statement, first suppose that \( P \) is not principal. Let \( x \in P^{-1} - (P : P) \). Then \( x^{-1} \in T \) by Proposition 3.5(a), and hence \( P \subset x^{-1}R \).

Since \( x^{-1}(xP) = P \) and \( x \notin (P : P) \), we have \( x^{-1} \in P \) since \( P \) satisfies property \((*)\) by Proposition 3.4. Thus \( P = x^{-1}R \), a contradiction. Hence \((P : P) = P^{-1}\).

If \( P \) is principal, then \( P = sR \) for some non-zerodivisor \( s \in P \). Thus \((P : P) = (sR : sR) = R\).

Corollary 3.7. – Let \( P \) be a strongly prime ideal of a ring \( R \). Then

(a) If \( P \) contains a non-zerodivisor prime \( p \) of \( R \), then \( P \) is maximal, \( P = pR \), and \( R \) is a chained ring (and thus a PVR).

(b) Suppose that \( P \) contains a non-zerodivisor of \( R \). Then \( R_p = (P : P) \) if and only if \( R_p \) is a chained ring.

(c) Let \( Q \) be a prime ideal of \( R \) properly contained in \( P \). If \( Q \) contains a non-zerodivisor of \( R \), then \( Q \) is strongly prime and \( R_q = (Q : Q) \).

Proof. – (a) Let \( y \in R \). Suppose that \( y \notin pR \). Then \( pP \subset yP \) by Proposition 3.3(b). Hence \( p^2 = ym \) for some \( m \in R \). Since \( p^2 \nmid y \) and \( p \) is a non-zerodivisor prime of \( R \), \( p^2 \mid m \). Hence \( m = p^2k \) for some \( k \in R \). Thus \( p^2 = yp^2k \), and hence
yk = 1. Thus \( y \in U(R) \). Hence \( P \) is maximal, \( P = pR \), and \( R \) is a PVR. Thus \( R = (P: P) \) is a chained ring by Theorem 3.6.

(b) If \( R_P = (P: P) \), then \( R_P \) is a chained ring by Theorem 3.6. Conversely, suppose that \( R_P \) is a chained ring. Then \( R_P \subset (P: P) \) by Proposition 3.2(a). Let \( x \in (P: P) \). We may assume that \( x \notin P \), and hence \( x \) is a unit of \( (P: P) \). Thus either \( x \) or \( x^{-1} \) is in \( R_P \) since \( R_P \) is a chained ring. If \( x^{-1} \in R_P \), then \( x^{-1} \in R_P - P \), and hence \( x = (x^{-1})^{-1} \in R_P \). Thus \( R_P = (P: P) \).

(c) By Lemma 3.1(e) and Theorem 3.6, \( (P: P) \subset (Q: Q) \) are chained rings with maximal ideals \( P \) and \( Q \), respectively. Thus \( R_Q = (R_P)_{Q_P} = (P: P)_{Q_P} \) is a chained ring by \([6, \text{Theorem 12}]\); so \( R_Q = (Q: Q) \) by part (b) above. ■

The next result is motivated by \([2, \text{Proposition 4.6}]\).

**Theorem 3.8.** – The following statements are equivalent for a proper ideal \( I \) of a ring \( R \) which contains a non-zerodivisor of \( R \):

1. \( I \) is a nonprincipal strongly prime ideal of \( R \).
2. \( I^{-1} \) is a ring and for every \( a, b \in R \), the ideals \( aI \) and \( bR \) are comparable.

**Proof.** – (1) \( \Rightarrow \) (2): This is clear by the definition of strongly prime ideal and Theorem 3.6.

(2) \( \Rightarrow \) (1): Let \( s \in I \) be a non-zerodivisor of \( R \). The proof of Proposition 3.2(b) shows that \( I \) is not principal. We need only show that \( I \) is prime. Since \( s \in I \) is a non-zerodivisor and for every \( a, b \in R \), the ideals \( aI \) and \( bR \) are comparable, \( Z(R) \subset I \). Suppose that \( xy \in I \) for some \( x, y \in R - I \). Hence \( I \subset xyR \) and \( I \subset yR \) by hypothesis. Since \( x, y \in R - I \) and \( Z(R) \subset I \), both \( x \) and \( y \) are non-zerodivisors. Thus \( 1/x, 1/y \in I^{-1} \), and hence \( 1/(xy)^2 \in I^{-1} \) since \( I^{-1} \) is a ring. Thus \( 1/xy = xy/(xy)^2 \in II^{-1} \subset R \), a contradiction. Hence \( I \) is prime. ■

We have the following partial converse to Theorem 3.6.

**Theorem 3.9.** – Let \( P \) be a prime ideal of a ring \( R \) such that \( B = (P: P) \) is a PVR with maximal ideal \( M \in \text{Spec}(R) \). Then

1. \( Z(R) \subset M \).
2. \( M, P, \) and \( Z(R) \) are strongly prime ideals of \( R \).

In particular, if \( (P: P) \) is a PVR with maximal ideal \( P \), then \( P \) is a strongly prime ideal of \( R \) and \( Z(R) \subset P \).

**Proof.** – (a) Let \( x \in R - M \). Then \( x \in U(B) \). Thus \( x \notin Z(R) \), so \( Z(R) \subset M \).

(b) Let \( a, b \in R \). Since \( M \) is a strongly prime ideal of \( B \), the ideals \( bB \)
and $aM$ are comparable. If $bB \subseteq aM$, then $bR \subseteq aM$. Thus we may assume that $aM \subseteq bB$. If $aM \not\subseteq bR$, then $am = bd$ for some $m \in M$ and $d \in B - R$. Thus $d \in U(B)$, and hence $b = a(d^{-1}m) \in aM$. Thus $bR \subseteq aM$, and hence $M$ is a strongly prime ideal of $R$. Since $P \subseteq M$, $P$ is also a strongly prime ideal of $R$ by Lemma 3.1(e). Since $Z(R) \subseteq M$ and the prime ideals of $R$ contained in $M$ are strongly prime and linearly ordered, $Z(R)$ is a prime ideal of $R$. Hence $Z(R)$ is a strongly prime ideal of $R$ by Lemma 3.1(e).

The «in particular» statement is immediate. ■

We next consider the case when the strongly prime ideal $P$ does not contain a non-zerodivisor, i.e., when $P \subseteq Z(R)$. For this case, the next result is analogous to Proposition 3.2.

**Proposition 3.10.** – Let $P$ be a strongly prime ideal of a ring $R$ such that $P \subseteq Z(R)$. Then

(a) $P_S = P$.

(b) $(P : P) = T$.

(c) $P = P_S$ is a strongly prime ideal of $R_S = T = (P : P)$.

**Proof.** – Let $s \in S$. Then $P \subseteq sR$ by [6, Lemma 1(a)], and hence $(1/s)P \subseteq R$. Thus $s((1/s)P) \subseteq P$, $s \not\in P$, and $P$ a prime ideal yields $(1/s)P \subseteq P$. Hence $P_S = P$ and $(P : P) = T$. That $P_S$ is strongly prime follows from Lemma 3.1(b). ■

**Theorem 3.11.** – Let $P$ be a prime ideal of a ring $R$ such that $P \subseteq Z(R)$. Then

(a) $T$ is a PVR if and only if $Z(R)_S$ is a strongly prime ideal of $T$.

(b) $(P : P)$ is a PVR with maximal ideal $M \in \text{Spec} (R)$ if and only if $Z(R)$ is a strongly prime ideal of $R$.

(c) If $(P : P)$ is a PVR with maximal ideal $M \in \text{Spec} (R)$, then $(P : P) = T$, $P$ is a strongly prime ideal of $R$, and $M = Z(R)$.

**Proof.** – Let $Q := Z(R)$.

(a) If $T$ is a PVR, then $T = R_S$ is quasilocal, necessarily with maximal ideal $Q_S$. Conversely, if $Q_S$ is a strongly prime ideal of $T$, then $T$ is a PVR by [6, Theorem 2].

(b) If $Q$ is a strongly prime ideal of $R$, then $T = R_S$ is a PVR with maximal ideal $Q = Q_S$ by Lemma 3.1(c) and Proposition 3.10(a). Thus $P \subseteq Q$ is also a strongly prime ideal of $R$ by Lemma 3.1(e); so $(P : P) = T$ by Proposition 3.10(b). The converse follows from Theorem 3.9(b).
(c) Suppose that \((P: P)\) is a PVR with maximal ideal \(M \in \text{Spec}(R)\). Then \(P\) is a strongly prime ideal of \(R\) by Theorem 3.9(b), and hence \((P: P) = T\) by Proposition 3.10(b). By part (a) above, necessarily \(M = Q_S = Q_S \cap R = Q\).

The next two corollaries summarize our earlier results on when \((P: P)\) is a PVR.

**Corollary 3.12.** – Let \(P\) be a prime ideal of a ring \(R\). If \((P: P)\) is a PVR with maximal ideal \(M \in \text{Spec}(R)\), then \(P\) and \(Z(R)\) are strongly prime ideals of \(R\) and either \(M \subseteq P\) or \(M \subseteq Z(R)\).

**Proof.** – By Theorem 3.9(b), \(M, P,\) and \(Z(R)\) are strongly prime ideals of \(R\). Thus either \(Z(R) \subseteq P\) or \(P \subseteq Z(R)\) by Lemma 3.1(b). If \(P \subseteq Z(R)\), then \(M = Z(R)\) by Theorem 3.11(c). If \(Z(R)\) is properly contained in \(P\), then \(M = P\) by Theorem 3.6.

**Corollary 3.13.** – Let \(P\) be a prime ideal of a ring \(R\). Then the following statements are equivalent:

1. \((P: P)\) is a PVR with maximal ideal \(M \in \text{Spec}(R)\);
2. \(P\) and \(Z(R)\) are strongly prime ideals of \(R\).

**Proof.** – (1) \(\Rightarrow\) (2) by Theorem 3.9(b).

(2) \(\Rightarrow\) (1): By Lemma 3.1(a), \(P\) and \(Z(R)\) are comparable. If \(P \subseteq Z(R)\), then we are done by Theorem 3.11(b). If \(Z(R)\) is properly contained in \(P\), then we are done by Theorem 3.6.

**Question 3.14.** – Let \(P\) be a strongly prime ideal of a ring \(R\) such that \(P \subseteq Z(R)\) and \((P: P) (= T)\) is a PVR. Then \(T\) has maximal ideal \(Z(R)_S\) and \(Z(R)\) is a prime ideal of \(R\). Is \(Z(R)\) also a strongly prime ideal of \(R\)?

Any PVD which is not a field gives an example of a PVR \((R, M)\) for which \(\text{nil}(R) = Z(R) \neq M\). In [6, Example 10(b)], we constructed a PVR \((R, M)\) with \(\text{nil}(R) \neq Z(R) = M\). These examples raise the question whether there exists a PVR \((R, M)\) for which \(\text{nil}(R)\) is neither \(Z(R)\) nor \(M\). In Example 3.16(c), we show that such behavior is possible. In the next proposition, we give a necessary and sufficient condition for certain rings \(R\) to have \(\text{nil}(R) = Z(R)\).

**Proposition 3.15.** – Let \(R\) be a ring such that either \(R\) is quasilocal or \(\text{nil}(R)\) is a (minimal) prime ideal of \(R\). Then \(Z(R) = \text{nil}(R)\) if and only if for every \(x \in Z(R)\) there exists an integer \(k \geq 1\) such that \(x^k R = x^{k+1} R\).

**Proof.** – We need only prove the «if» assertion. Suppose that there is an
\[ x \in Z(R) - \text{nil}(R). \] Then \( x^k = x^{k+1}m \) for some \( m \in R \) and some integer \( k \geq 1 \).

Hence \( x^k(1-xm) = 0 \in \text{nil}(R). \) If \( \text{nil}(R) \) is prime, then \( 1-xm \in \text{nil}(R) \) since \( x \notin \text{nil}(R). \) Hence \( xm = 1 -(1-xm) \in U(R), \) and thus \( x \in U(R), \) a contradiction. If \( R \) is quasilocal with maximal ideal \( M, \) then \( x \in M, \) and hence \( 1-xm \in U(R). \) Thus \( x^k = 0, \) so \( x \in \text{nil}(R), \) again a contradiction. Thus \( Z(R) = \text{nil}(R). \)

We end the paper with several examples. In particular, Example 3.16(c) shows that if \( R \) is a PVR with maximal ideal \( M, \) then \( Z(R) \) can be nil \((R), M, \) or a prime ideal properly contained between these two ideals.

**Example 3.16.** (a) ([6, Example 10(a)]) Let \( k \) be a field and \( X \) and \( Y \) indeterminates. Then \( R = k[X, Y]/(X^2, XY, Y^2) = k[x, y] \) is a zero-dimensional PVR with (strongly prime) maximal ideal \( M = Z(R) = (x, y), \) and \( (M: M) = R \) is not a chained ring. Thus the non-zero divisor hypothesis is needed in Theorem 3.6.

(b) Let \( W \) be a valuation domain with maximal ideal \( N. \) For any \( 0 \neq x \in N, R = W/xW \) is a PVR [6, Corollary 3] with maximal ideal \( M = Z(R) = N/xW \) and \( \text{nil}(R) = Q/xW, \) where \( Q = \sqrt{xW} \) is the (unique) prime ideal of \( W \) minimal over \( xW. \) To see that \( Z(R) = N/xW, \) observe that for any \( m \in N - xW, \) then \( x = rm \) for some \( r \notin xW, \) and hence \((r + xW)(m + xW) = 0 \) in \( W/xW \) with \( r + xW \) nonzero. This example generalizes [6, Example 10(b)].

(c) Let \( W \) be a valuation domain with maximal ideal \( N \) and let \( 0 \neq x \in N. \) Then by part (b) above, \( W^* = W/xW \) is a chained ring with maximal ideal \( N^* = N/xW = Z(W^*), \) \( \text{nil}(W^*) = \sqrt{xW}/xW, \) and residue field \( k = W^*/N^* = W/N. \) Let \( \pi: W^* \to k \) be the natural surjection, let \( D \) be a valuation domain with maximal ideal \( P \) and quotient field \( k, \) and let \( R = \pi^{-1}(D). \) Then \( R \) is a chained ring with maximal ideal \( M = \pi^{-1}(P) \supset N^*, \) \( \text{nil}(R) = \text{nil}(W^*), \) and \( Z(R) = Z(W^*) = N^*. \) (Also note that \( R = \mu^{-1}(D)/xW, \) where \( \mu: W \to k \) is the natural surjection; so \( \mu^{-1}(D) \subset W \) is a valuation domain.)

By standard gluing techniques (cf. [9, Corollary 1.5]), Spec \((R) \) is order-isomorphic to the result of gluing Spec \((D) \) "above" Spec \((W^*), \) where \( 0 \) in Spec \((D) \) is identified with \( N^* \) in Spec \((W^*). \) Thus for any integers \( i \) and \( n \) with \( 1 \leq i \leq n, \) there is an \((n-1)\)-dimensional chained ring \((R, M) \) with distinct prime ideals \( \text{nil}(R) = M_1 \subset M_2 \subset \cdots \subset M_n = M \) such that \( Z(R) = M_i. \)

More generally, let \((I, \leq)\) be any set which can be realized as the spectrum of some valuation domain (i.e., by [13, Corollary 3.6], \( I \) is linearly ordered and satisfies properties (K1) and (K2) (cf. [12, pages 6-7])). Let \( m \) be the minimum element of \( I, L \) the maximum element of \( I, \) and \( i \in I \) with \( m \leq i \leq L. \) By the above construction, there is a chained ring (and hence a PVR) \((R, M) \) with Spec \((R) \) order-isomorphic to \( I, \) where \( \text{nil}(R) \leftrightarrow m, Z(R) \leftrightarrow i, \) and \( M \leftrightarrow L. \)
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