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A Stationary Flow of Fresh and Salt Groundwater in a Heterogeneous Coastal Aquifer.

S. CHALLAL - A. LYAGHFOURI

Sunto. – *Si stabilisce l'esistenza e l'unicità di una soluzione monotona per il problema di frontiera libera correlato al flusso stazionare d'acqua dolce e salata intorno ad un acquifero eterogeneo. Si provano la continuità e l'esistenza di un limite asintotico della frontiera libera.*

Introduction.

We study a two phase free boundary problem modeling a stationary flow of fresh and salt water through a heterogeneous, horizontal and unbounded two dimensional coastal aquifer. We recall that this problem was studied in [AD1] for the homogeneous case. Existence of a solution was proved together with the continuity of the free boundary (see also [C]) and some qualitative properties. The uniqueness of the solution was left as an open problem.

After setting the problem, we establish an existence result for a general matrix permeability. When the permeability depends only on the vertical direction, we prove existence of monotone solutions. For this kind of solutions, we give an asymptotic behavior far away on the left and on the right of the aquifer. We also prove the continuity of the free boundary separating the two fluids. Moreover, we prove uniqueness of these solutions. Finally we study the behavior of the free boundary at the left boundary of its definition interval.

The case of a flow governed by a nonlinear Darcy's law is considered in [CL1] and [CCL].

1. – Statement of the problem.

The aquifer is represented by the open set $\Omega = \mathbb{R} \times (-h, 0)$, $(h > 0)$. Fresh water is injected through the segment $[OA]$ $(A = (0, a),\ a > 0)$ with total amount Q_f and with uniform velocity, while salt water is injected far away on the left side of the aquifer over the height h with a total amount Q_s (see Figure 1). The aquifer considered here is heterogeneous with permeability $\mathsf{a}(X) = (\mathsf{a}_{ij}(X))_{1 \leq i, j \leq 2}$, $X = (x, z)$. The flow is governed by the following Dar-

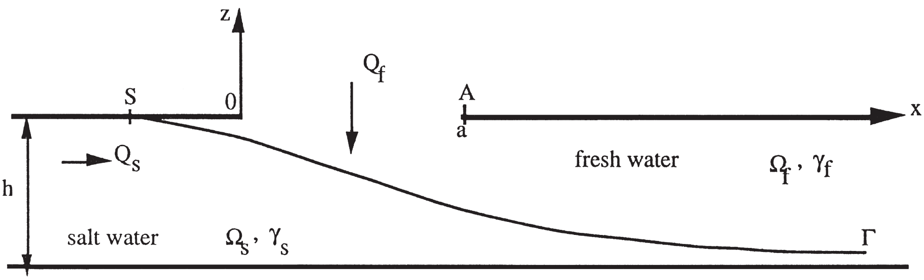

Figure 1

cy law:

(1.1) $$v = -\mathsf{a}(X)(\nabla p + \gamma e_z) \quad \text{in } \Omega$$

where $e_z = (0, 1)$, v is the fluid velocity, p its pressure and γ is the specific weight of the fluid given by

(1.2) $$\gamma = \gamma_f \chi(\Omega_f) + \gamma_s \chi(\Omega_s) \quad \text{with } \gamma_s > \gamma_f > 0\,,$$

$\chi(E)$ denotes the characteristic function of the set E, Ω_f (resp. Ω_s) denotes the subset of Ω occupied by fresh (resp. salt) water.

We assume that the flow is incompressible and the two fluids are unmixed and separated by an interface Γ. Moreover $\partial\Omega \backslash [OA]$ is assumed to be impervious. This leads to:

(1.3) $$\mathrm{div}\,(v) = 0 \qquad \text{in } \Omega$$

(1.4) $$v = \frac{-Q_f}{a} e_z \qquad \text{on } [OA]$$

(1.5) $$v.\nu = 0 \qquad \text{on } \partial\Omega \backslash [\mathrm{OA}]$$

(1.6) $$v_i.\nu = 0 \qquad \text{on } \Gamma \ (i = s, f)$$

v_i denotes the restriction of v to Ω_i $(i = s, f)$ and ν is the outward unit normal to $\partial\Omega$ or Γ.

From (1.3), there exists a stream function ψ such that

(1.7) $$v = \mathrm{Rot}\,\psi = \left(-\frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial x}\right) \qquad \text{in } \Omega\,.$$

Let $\zeta \in \mathcal{D}(\Omega)$, we have by (1.2):

(1.8) $$\int_\Omega (\nabla p + \gamma e_z).\mathrm{Rot}\,\zeta = -\int_\Omega \mathsf{a}^{-1}(X)\,\mathrm{Rot}\,\psi.\mathrm{Rot}\,\zeta$$

and

$$\int_\Omega (\nabla p + \gamma e_z).\mathrm{Rot}\,\zeta = \int_\Omega \gamma e_x.\,\nabla\zeta \quad \text{with } e_x = (1, 0). \tag{1.9}$$

Now for

$$b(X) = \frac{{}^t\mathsf{a}(X)}{\det(\mathsf{a}(X))} \tag{1.10}$$

with $\det(\mathsf{a}(X)) = (\mathsf{a}_{11}.\mathsf{a}_{22} - \mathsf{a}_{12}.\mathsf{a}_{21})(X)$ and ${}^t\mathsf{a}(X) = (\mathsf{a}_{ji}(X))_{1 \leqslant i,\, j \leqslant 2}$, one can easily verify that:

$$\int_\Omega \mathsf{a}^{-1}(X)\,\mathrm{Rot}\,\psi.\mathrm{Rot}\,\zeta = \int_\Omega b(X)\,\nabla\psi.\nabla\zeta \tag{1.11}$$

Using (1.7)-(1.11) and the strong formulation (1.1)-(1.6), we obtain the following weak formulation:

$$(\mathrm{P}) \quad \begin{cases} \text{Find } (\psi, \gamma) \in H^1_{\mathrm{loc}}(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \text{i) } \displaystyle\int_\Omega (b(X)\,\nabla\psi + \gamma e_x).\nabla\zeta = 0 \qquad \forall\zeta \in H^1_0(\Omega) \text{ with compact support in } \overline{\Omega} \\ \text{ii) } \gamma \in H(\psi) \qquad \text{a.e. in } \Omega \\ \text{iii) } -Q_f \leqslant \psi \leqslant Q_s \qquad \text{a.e. in } \Omega \\ \text{iv) } \psi(x, -h) = Q_s, \quad \psi(x, 0) = \phi_0(x) \qquad \forall x \in \mathbb{R} \end{cases}$$

where H is the maximal monotone graph defined by:

$$H(t) = \begin{cases} \gamma_s & \text{if } t > 0 \\ [\gamma_f, \gamma_s] & \text{if } t = 0 \\ \gamma_f & \text{if } t < 0 \end{cases} \tag{1.12}$$

and

$$\phi_0(x) = -Q_f \min\left(\frac{x^+}{a}, 1\right), \qquad x^+ = \max(x, 0). \tag{1.13}$$

The condition (P) iii) expresses that the discharge of the flow through the section $[-h, z]$ lies between 0 and $Q_s + Q_f$ (see [CCL]).

We will assume that b satisfies:

$$b \in [L^\infty(\Omega)]^4, \quad \exists\alpha > 0, \quad \langle b(X)\,\xi, \xi\rangle \geqslant \alpha|\xi|^2 \quad \text{a.e. } X \in \Omega,\ \forall\xi \in \mathbb{R}^2. \tag{1.14}$$

REMARK 1.1. – Note that if $\mathsf{a}(X)$ satisfies (1.14) with $\det(\mathsf{a}(X))>0$ or ${}^t\mathsf{a}(X)=\mathsf{a}(X)$, then $b(X)$ satisfies (1.14).

2. – Existence of a solution.

To prove the existence of a solution, we consider a sequence of approximated problems on bounded subdomains $\Omega_m=(-m, m)\times(-h, 0)$ $(m>a)$ of Ω.

First, we define the function Φ by:

$$\Phi(x, z)=\phi_0(x)+\phi_1(z)(Q_s-\phi_0(x)) \tag{2.1}$$

where ϕ_0 is defined by (1.13) and ϕ_1 defined by

$$\phi_1(z)=\int_0^a\int_z^0\frac{dX}{b_{22}(X)}\bigg/\int_0^a\int_{-h}^0\frac{dX}{b_{22}(X)}\,. \tag{2.2}$$

Next, for $m>a$, we consider the following problem:

$$(\mathrm{P_m})\qquad\begin{cases}\text{Find } (\psi_m, \gamma_m)\in H^1(\Omega_m)\times L^\infty(\Omega_m) \text{ such that:}\\ \text{i) } \displaystyle\int_{\Omega_m}(b(X)\,\nabla\psi_m+\gamma_m e_x).\nabla\zeta=0 \qquad \forall\zeta\in H_0^1(\Omega_m)\\ \text{ii) } \gamma_m\in H(\psi_m) \quad \text{a.e. in } \Omega_m\\ \text{iii) } -Q_f\leqslant\psi_m\leqslant Q_s \quad \text{a.e. in } \Omega_m\\ \text{iv) } \psi_m=\Phi \quad \text{on } \partial\Omega_m.\end{cases}$$

THEOREM 2.1.

i) *There exists a solution* (ψ_m, γ_m) *of problem* $(\mathrm{P_m})$.

ii) *If* $b(X)=b(z)$ *a.e. in* Ω, *then there exists a solution* (ψ_m, γ_m) *such that*

$$\partial_x\psi_m\leqslant 0 \quad \textit{and} \quad \partial_x\gamma_m\leqslant 0 \quad \textit{in } \mathcal{D}'(\Omega_m). \tag{2.3}$$

To prove Theorem 2.1, we consider for $\varepsilon>0$, the following problem:

$$(\mathrm{P_m^\varepsilon})\qquad\begin{cases}\text{Find } \psi_m^\varepsilon\in H^1(\Omega_m) \text{ such that:}\\ \text{i) } \displaystyle\int_{\Omega_m}(b(X)\,\nabla\psi_m^\varepsilon+H_\varepsilon(\psi_m^\varepsilon)\,e_x).\nabla\zeta=0 \qquad \forall\zeta\in H_0^1(\Omega_m)\\ \text{ii) } \psi_m^\varepsilon=\Phi \quad \text{on } \partial\Omega_m\end{cases}$$

where

$$H_\varepsilon(t) = \gamma_f + (\gamma_s - \gamma_f) \min\left(\frac{t^+}{\varepsilon}, 1\right). \tag{2.4}$$

Arguing as in [L], we establish that $(\mathrm{P}^\varepsilon_\mathrm{m})$ admits a unique solution satisfying, up to a subsequence

$$(\psi^\varepsilon_m, H_\varepsilon(\psi^\varepsilon_m)) \rightharpoonup (\psi_m, \gamma_m) \quad \text{in } H^1(\Omega_m) \times L^2(\Omega_m) \tag{2.5}$$

with $\gamma_m \in H(\psi_m)$. Then taking $(\psi^\varepsilon_m - Q_s)^+$ and $(-Q_f - \psi^\varepsilon_m)^+$ as test functions of (P^ε_m), we prove that for all $\varepsilon \in (0, Q_s)$

$$-Q_f \leqslant \psi^\varepsilon_m \leqslant Q_s \quad \text{a.e. in } \Omega_m. \tag{2.6}$$

PROOF OF THEOREM 2.1.

i) Using Rellich's theorem and the continuity of the trace operator, we deduce that (ψ_m, γ_m) is a solution of $(\mathrm{P_m})$.

ii) Since $\partial_x(H_\varepsilon(\psi^\varepsilon_m)) = H'_\varepsilon(\psi^\varepsilon_m)\,\partial_x \psi^\varepsilon_m$ and $H'_\varepsilon(\psi^\varepsilon_m) \geqslant 0$, it suffices to show that $\partial_x \psi^\varepsilon_m \leqslant 0$.

Let $\delta > 0$ and set $\psi^{\varepsilon\delta}_m(X) = \psi^\varepsilon_m(x+\delta, z)$. To compare $\psi^{\varepsilon\delta}_m$ and ψ^ε_m on $\Omega^\delta_m \cap \Omega_m$ where $\Omega^\delta_m = (-m-\delta, m-\delta) \times (-h, 0)$, we need the two following lemmas:

LEMMA 2.2. – *Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^2$. Let $F : \mathcal{O} \to \mathbb{R}^2$ be a Lipschitz continuous function. Let u_1 and u_2 satisfying*:

$$\int_{\mathcal{O}} (b(X)\,\nabla u_i + F(u_i)) \cdot \nabla\zeta = 0 \qquad \forall \zeta \in H^1_0(\mathcal{O}), \quad i = 1, 2\,. \tag{2.7}$$

If $(u_1 - u_2)^+ \in H^1_0(\mathcal{O})$ then $u_1 \leqslant u_2$ a.e. in $\mathcal{O}$.

PROOF. – Let $\eta > 0$ and $f_\eta(t) = (1 - \eta/t^+)^+$. Set $u = (u_1 - u_2)^+$. It is clear that $f_\eta(u) \in H^1_0(\mathcal{O})$. Then we deduce

$$\int_{\mathcal{O}} (b(X)\,\nabla u + (F(u_1) - F(u_2))).\nabla(f_\eta(u)) = 0$$

and

$$\int_{\mathcal{O} \cap [u \geqslant \eta]} \frac{\eta}{u^2} b(X)\,\nabla u . \nabla u = \tag{2.8}$$

$$- \int_{\mathcal{O} \cap [u \geqslant \eta]} \frac{\eta}{u^2} (F(u_1) - F(u_2)).\nabla u \leqslant L\eta \int_{\mathcal{O} \cap [u \geqslant \eta]} \frac{|\nabla u|}{|u|}$$

where L is the Lipschitz constant of F.

Using (1.14) and the Cauchy schwartz inequality, we deduce from (2.8):

$$\int_{\mathcal{O}} \left| \nabla \log \left(1 + \frac{(u-\eta)^+}{\eta} \right) \right|^2 = \int_{\mathcal{O} \cap [u \geq \eta]} \frac{|\nabla u|^2}{u^2} \leq c$$

where c is a constant independent of η.

By Poincaré's inequality, we get

$$\int_{\mathcal{O}} \left| \log \left(1 + \frac{(u-\eta)^+}{\eta} \right) \right|^2 \leq c\,.$$

Letting $\eta \to 0$, we deduce that $u = 0$ a.e. in $\mathcal{O}$. ■

Now, let us define for $z \in [-h, 0]$

$$\text{(2.9)} \qquad \begin{cases} v_{+\infty}(z) = -Q_f + (Q_s + Q_f)\,\phi_1(z)\,, \\ v_{-\infty}(z) = Q_s \phi_1(z)\,. \end{cases}$$

We have

LEMMA 2.3. – *Assume that* $b(X) = b(z)$. *For any* $\varepsilon > 0$, *we have*:

$$\text{(2.10)} \qquad v_{+\infty} \leq \psi_m^\varepsilon \leq v_{-\infty} \quad \text{a.e. in } \Omega_m.$$

PROOF. – First remark that for $\zeta \in H_0^1(\Omega_m)$, we have

$$\int_{\Omega_m} (b(z)\,\nabla v_{+\infty} + H_\varepsilon(v_{+\infty})\,e_x).\nabla\zeta =$$

$$(Q_s + Q_f) \int_{\Omega_m} \phi_1'(z)\, b_{12}(z) \frac{\partial \zeta}{\partial x} + \phi_1'(z)\, b_{22}(z) \frac{\partial \zeta}{\partial z} = 0\,.$$

Moreover $(v_{+\infty} - \psi_m^\varepsilon)^+ = 0$ on $\partial\Omega_m$. Then applying Lemma 2.2 with $b(X) = b(z)$, $F = (H_\varepsilon, 0)$, we get $v_{+\infty} \leq \psi_m^\varepsilon$ a.e. in Ω_m. In the same way we establish that $\psi_m^\varepsilon \leq v_{-\infty}$ a.e. in Ω_m. ■

END OF THE PROOF OF THEOREM 2.1. – Since b does not depend on x, $\psi_m^{\varepsilon\delta}$ satisfies the same equation satisfied by ψ_m^ε in $\Omega_m^\delta \cap \Omega_m$ i.e.

$$\int_{\Omega_m^\delta \cap \Omega_m} (b(z)\,\nabla\psi_m^{\varepsilon\delta} + H_\varepsilon(\psi_m^{\varepsilon\delta})\,e_x).\nabla\zeta = 0 \qquad \forall \zeta \in H_0^1(\Omega_m^\delta \cap \Omega_m)\,.$$

Moreover, we have by Lemma 2.3

$$\psi_m^{\varepsilon\delta}(-m, z) = \psi_m^{\varepsilon}(-m+\delta, z) \leqslant v_{-\infty}(z) = \psi_m^{\varepsilon}(-m, z)$$

$$\psi_m^{\varepsilon\delta}(m-\delta, z) = \psi_m^{\varepsilon}(m, z) = v_{+\infty}(z) \leqslant \psi_m^{\varepsilon}(m-\delta, z)$$

and

$$\psi_m^{\varepsilon\delta}(x, -h) = \psi_m^{\varepsilon}(x+\delta, -h) = Q_s = \psi_m^{\varepsilon}(x, -h)$$

$$\psi_m^{\varepsilon\delta}(x, 0) = \psi_m^{\varepsilon}(x+\delta, 0) = \phi_0(x+\delta) \leqslant \phi_0(x) = \psi_m^{\varepsilon}(x, 0).$$

Thus $(\psi_m^{\varepsilon\delta} - \psi_m^{\varepsilon})^+ \in H_0^1(\Omega_m^\delta \cap \Omega_m)$ and by Lemma 2.2, we get $\psi_m^{\varepsilon\delta} \leqslant \psi_m^{\varepsilon}$ a.e. in $\Omega_m^\delta \cap \Omega_m$ from which we deduce:

$$\partial_x \psi_m^{\varepsilon} \leqslant 0 \quad \text{in } \mathcal{D}'(\Omega_m). \qquad \blacksquare \tag{2.11}$$

Now we can state the main result of this section:

THEOREM 2.4.

i) *There exists a solution* (ψ, γ) *of problem* (P).

ii) *If* $b(X) = b(z)$ *a.e. in* Ω, *then there exists a solution* (ψ, γ) *such that*

$$\partial_x \psi \leqslant 0 \quad \textit{and} \quad \partial_x \gamma \leqslant 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.12}$$

PROOF. – First let $m_0 > a$ and $\eta \in C^\infty(\mathbb{R})$ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $(-m_0, m_0)$, $\eta = 0$ for $|x| \geqslant m_0 + 1$, $|\eta'| \leqslant c$. Then for $m \geqslant 1 + m_0$, $\eta^2(\psi_m - \Phi)$ is a test function for $(\mathrm{P_m})$. So

$$\int_{\Omega_{m_0+1}} \eta^2 b(X) \nabla\psi_m . \nabla\psi_m = -2 \int_{\Omega_{m_0+1}} \eta\psi_m b(X) \nabla\psi_m . \nabla\eta + \int_{\Omega_{m_0+1}} b(X) \nabla\psi_m . \nabla(\eta^2 \Phi) -$$

$$\int_{\Omega_{m_0+1}} \gamma_m \eta^2 \partial_x \psi_m + \int_{\Omega_{m_0+1}} \gamma_m \eta^2 \partial_x \Phi - \int_{\Omega_{m_0+1}} \gamma_m 2\eta\eta'(\psi_m - \Phi).$$

Using (1.14), the Cauchy-Schwartz inequality and the fact that ψ_m, γ, η, η', Φ, $\nabla\Phi$ are uniformly bounded, we deduce that: $|\nabla\psi_m|_{L^2(\Omega_{m_0})} \leqslant c(m_0)$ which leads by $(\mathrm{P_m})$iii) to

$$|\psi_m|_{1, \Omega_{m_0}} \leqslant c(m_0). \tag{2.13}$$

Let us now extend ψ_m by $v_{+\infty}$ (resp. $v_{-\infty}$) for $x \geqslant m$ (resp. $x \leqslant -m$). We also extend γ_m to $\Omega \setminus \Omega_m$ in such a way to have $\gamma_m \in H(\psi_m)$ a.e. in Ω. Then by (2.13) and a diagonal process there exists a subsequence of (ψ_m, γ_m) and $(\psi, \gamma) \in$

$H^1_{\mathrm{loc}}(\Omega) \times L^\infty(\Omega)$ such that

$$(\psi_m, \gamma_m) \rightharpoonup (\psi, \gamma) \quad \text{in } H^1_{\mathrm{loc}}(\Omega) \times L^2(\Omega). \tag{2.14}$$

Now, (2.14) allows us to check that (ψ, γ) is a solution of (P). This proves i). Using Theorem 2.1 ii), we prove ii). ■

Let us now give some properties of the solutions of (P).

3. – Properties and asymptotic behavior of solutions.

PROPOSITION 3.1. – *Let* (ψ, γ) *be a solution of* (P). *We have*:

i) $\psi \in C^{0,\beta}_{\mathrm{loc}}(\overline{\Omega})$ *for some* $\beta \in (0, 1)$;

ii) $[\psi > 0]$ *and* $[\psi < 0]$ *are open sets*;

iii) *If* $b \in C^{k,\sigma}(\Omega)$ *then* $\psi \in C^{k+1,\sigma}([\psi > 0] \cup [\psi < 0])$.

If b *is analytic in* Ω *then* ψ *is analytic in* $[\psi > 0] \cup [\psi < 0]$.

PROOF. – Taking $\xi \in \mathcal{D}(\Omega)$ as a test function for (P), we get

$$\operatorname{div}(b(X)\, \nabla\psi) = -\gamma_x \quad \text{in } \mathcal{D}'(\Omega). \tag{3.1}$$

Then i) is a direct consequence of (3.1) and usual results of regularity (see [GT]). ii) is a consequence of i). Using (P)ii) and (3.1), we get

$$\operatorname{div}(b(X)\, \nabla\psi) = 0 \quad \text{in } \mathcal{D}'([\psi > 0]) \quad (\text{resp. } \mathcal{D}'([\psi < 0])) \tag{3.2}$$

from which we deduce iii) (see [GT]). ■

In the remainder of this paper, we will assume that $b(X) = b(z)$ a.e. in Ω and consider only monotone solutions (ψ, γ) of (P) (i.e. $\partial_x \psi \leqslant 0$, $\partial_x \gamma \leqslant 0$ in $\mathcal{D}'(\Omega)$).

PROPOSITION 3.2. – *Let* (ψ, γ) *be a solution of* (P). *Then we have*:

$$\operatorname{div}(b(z)\, \nabla\psi) \geqslant 0, \qquad \operatorname{div}(b(z)\, \nabla\psi^+) \geqslant 0, \qquad \operatorname{div}(b(z)\, \nabla\psi^-) \geqslant 0 \quad \textit{in } \mathcal{D}'(\Omega)$$

where $\psi^+ = \max(\psi, 0)$ *and* $\psi^- = (-\psi)^+$.

PROOF. – The first inequality is a consequence of (3.1) and (2.12). Now, let $\xi \in \mathcal{D}(\Omega)$, $\xi \geqslant 0$ and $\varepsilon > 0$, then $\min(\psi^+/\varepsilon, \xi)$ is a test function for (P) and we have:

$$\int_\Omega b(z)\, \nabla\psi . \nabla\left(\min\left(\frac{\psi^+}{\varepsilon}, \xi\right)\right) + \gamma \partial_x\left(\min\left(\frac{\psi^+}{\varepsilon}, \xi\right)\right) = 0\,. \tag{3.3}$$

Note that:

$$\int_\Omega \gamma \partial_x \left(\min \left(\frac{\psi^+}{\varepsilon}, \xi \right) \right) = \gamma_s \int_\Omega \partial_x \left(\min \left(\frac{\psi^+}{\varepsilon}, \xi \right) \right) = 0 .$$

Then (3.3) becomes:

$$\int_{\Omega \cap [\psi^+ \geqslant \varepsilon\xi]} b(z)\, \nabla\psi . \nabla\xi =$$

$$-\frac{1}{\varepsilon} \int_{\Omega \cap [\psi^+ < \varepsilon\xi]} b(z)\, \nabla\psi . \nabla\psi^+ = -\frac{1}{\varepsilon} \int_{\Omega \cap [\psi^+ < \varepsilon\xi]} b(z)\, \nabla\psi^+ . \nabla\psi^+ \leqslant 0 .$$

Letting $\varepsilon \to 0$, we get:

$$\int_\Omega b(z)\, \nabla\psi . \nabla\xi \leqslant 0 \qquad \forall \xi \in \mathcal{D}(\Omega), \quad \xi \geqslant 0 .$$

Thus we obtain:

$$\operatorname{div}(b(z)\, \nabla\psi^+) \geqslant 0 .$$

To prove $\operatorname{div}(b(z)\, \nabla\psi^-) \geqslant 0$, we take $\min(\psi^-/\varepsilon, \xi)$ as a test function for (P) and we argue as above. ■

THEOREM 3.3. – *Let* (ψ, γ) *be a solution of* (P). *Then we have*:

i) *For all* $z \in [-h, 0]$,

(3.4) $\quad \psi(x, z) \to v_{+\infty}(z)$ *(resp.* $v_{-\infty}(z)$*)* *as* $x \to +\infty$ *(resp.* $-\infty$*).*

ii) *For a.e.* $(x, z) \in \Omega$, we set $\gamma_R(x, z) = \gamma(x + R, z)$. Then we have:

(3.5) $\quad \gamma_R(x,z) \rightharpoonup \gamma_{+\infty}(z)$ *(resp.* $\gamma_{-\infty}(z)$*)* *as* $R \to +\infty$ *(resp.* $-\infty$*) in* $L^2(\Omega_{0,1})$

where $\gamma_{+\infty} \in H(v_{+\infty})$ *(resp.* $\gamma_{-\infty} \in H(v_{-\infty})$*) and* $\Omega_{m,n} = (m, n) \times (-h, 0)$ *for* $m, n \in \mathbb{R}$.

First, we need the following lemma:

LEMMA 3.4. – *Let* (ψ, γ) *be a solution of* (P). *Then we have*:

(3.6) $$\lim_{R \to +\infty} \int_{\Omega_{R, R+1}} |\nabla(\psi - v_{+\infty})|^2 = 0 \quad \textit{and} \quad \lim_{R \to -\infty} \int_{\Omega_{R, R+1}} |\nabla(\psi - v_{-\infty})|^2 = 0.$$

PROOF. – Let $R > a$. Set $\omega_R(x, z) = \psi(x + R, z)$ and consider $\eta_R \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \eta_R \leqslant 1$, $\eta_R = 1$ in $\Omega_{0,1}$, $\eta_R = 0$ for $|x| \geqslant R/2$ and $|\eta_R'| \leqslant c/R$. We have by (1.14)

$$I^R = \int_\Omega \eta_R^2 \, |\nabla(\omega_R - v_{+\infty})|^2 \leqslant \frac{1}{\alpha}\int_\Omega \eta_R^2 b(z)\, \nabla(\omega_R - v_{+\infty}).\nabla(\omega_R - v_{+\infty}) =$$

$$\frac{1}{\alpha}\int_\Omega b(z)\, \nabla(\omega_R - v_{+\infty}).\nabla(\eta_R^2(\omega_R - v_{+\infty})) -$$

$$\frac{2}{\alpha}\int_\Omega \eta_R(\omega_R - v_{+\infty})\, b(z)\, \nabla(\omega_R - v_{+\infty}).\nabla\eta_R.$$

Since $\gamma_x \leqslant 0$ and $\eta_R^2(\omega_R - v_{+\infty}) \geqslant 0$ then

$$\int_\Omega b(z)\, \nabla(\omega_R - v_{+\infty}).\nabla(\eta_R^2(\omega_R - v_{+\infty})) = -\int_\Omega \gamma\partial_x(\eta_R^2(\omega_R - v_{+\infty})) \leqslant 0\,.$$

So

$$I^R \leqslant -\frac{2}{\alpha}\int_\Omega \eta_R(\omega_R - v_{+\infty})\, b(z)\, \nabla(\omega_R - v_{+\infty}).\nabla\eta_R \leqslant$$

$$c'\int_\Omega \eta_R\, |\nabla(\omega_R - v_{+\infty})|.|\nabla\eta_R| \leqslant \frac{1}{2}\int_\Omega \eta_R^2\, |\nabla(\omega_R - v_{+\infty})|^2 + \frac{c'^2}{2}\int_\Omega |\nabla\eta_R|^2.$$

Then

$$I^R \leqslant \frac{c''}{R}, \qquad 0 \leqslant \int_{\Omega_{0,1}} |\nabla(\omega_R - v_{+\infty})|^2 \leqslant I^R \leqslant \frac{c''}{R}$$

and the first part of (3.6) holds. The second part can be proved similarly. ■

PROOF OF THEOREM 3.3. – Using the fact that ψ is uniformly bounded in Ω and nonincreasing in the x-direction, it admits limits when $x \to \pm\infty$. Moreover using (3.6) and Poincaré's inequality, we get (3.4). Finally, H being a maximal monotone graph, we deduce (3.5). ■

REMARK 3.5.

i) From the monotonicity and the asymptotic behavior of ψ, we deduce that

$$v_{+\infty} \leqslant \psi \leqslant v_{-\infty} \quad \text{in } \Omega\,. \tag{3.7}$$

ii) Note that

$$v_{+\infty}(z) = 0 \;\Leftrightarrow\; \phi_1(z) = \frac{Q_f}{Q_s + Q_f}\,.$$

But since

$$\phi_1'(z) = -\frac{1}{b_{22}(z)} \Big/ \int_{-h}^{0} \frac{ds}{b_{22}(s)} < 0\,,$$

then $\phi_1 : [-h, 0] \to [0, 1]$ is one to one and there exists a unique $h^* \in (0, h)$ such that

$$\phi_1(-h^*) = \frac{Q_f}{Q_s + Q_f}\,.$$

For all $-h < z < -h^*$, $v_{+\infty}(z) > 0$. So the set $\mathbb{R} \times (-h, -h^*)$ is contained in $[\psi > 0]$.

4. – Study of the free boundary.

The free boundary is defined by $\Gamma = \{(x, z) \in \Omega / \psi(x, z) = 0\}$.

Due to the asymptotic behavior of ψ, one can define two functions g_1 and g_2 by:

$$g_1(z) = \sup\{x/\psi(x, z) > 0\} \qquad \text{for } z \in (-h^*, 0)$$

$$g_2(z) = \inf\{x/\psi(x, z) < 0\} \qquad \text{for } z \in (-h^*, 0)\,.$$

Then, we have:

PROPOSITION 4.1.

$$G = \{(x, z) \in \Omega / -h^* < z < 0 \;\; and \;\; g_1(z) \leqslant x \leqslant g_2(z)\} \subset \Gamma \subset G \cup [z = -h^*]\,.$$

PROOF. – It is a consequence of definitions of g_1, g_2 and the monotonicity of ψ. ■

THEOREM 4.2. – *$g_1 = g_2 = g$, $G = [x = g(z)]$ and g is continuous on $(-h^*, 0)$.*

To prove Theorem 4.2, we need two lemmas:

LEMMA 4.3. – *Let $z_0 \in (-h^*, 0)$, $x_0 \in \mathbb{R}$ and $r > 0$.*

Assume that $S = \{(x, z_0)/|x - x_0| \leq r\} \subset \Gamma$, *then we cannot have*

$$\forall (x, z) \in B_r(x_0, z_0) \backslash S\,, \qquad \psi(x, z) \neq 0$$

where $B_r(x_0, z_0)$ *is the open ball of center* (x_0, z_0) *and radius* r.

PROOF. – We can assume that $B_r(x_0, z_0) \subset \mathbb{R} \times (-h^*, 0)$. Suppose that $\psi(x, z) \neq 0$ $\forall (x, z) \in B_r(x_0, z_0) \backslash S$. Then for $\xi \in \mathcal{D}(B_r(x_0, z_0))$, we have by (P) i)-ii)

$$\int_{B_r(x_0, z_0)} b(z)\, \nabla\psi . \nabla\xi = - \int_{B_r(x_0, z_0)} \gamma \partial_x \xi = 0\,. \tag{4.1}$$

For $0 < \delta < r/2$, the function defined by: $\psi_\delta(x, z) = \psi(x - \delta, z)$ satisfies by (4.1): $\operatorname{div}(b(z)\, \nabla\psi_\delta) = 0$ in $B_{r/2}(x_0, z_0)$. Moreover, we have $\psi_\delta \geq \psi$ in $B_{r/2}(x_0, z_0)$ and $\psi = \psi_\delta$ on $S \cap B_{r/2}(x_0, z_0)$. Thus by the strong maximum principle (see [GT]), $\psi = \psi_\delta$ in $B_{r/2}(x_0, z_0)$. Thus $\partial_x \psi = 0$ and $\psi(x, z) = \kappa(z)$ in $B_{r/2}(x_0, z_0)$. This leads by (4.1) to

$$\kappa(z) = \lambda \int_{z_0}^{z} \frac{ds}{b_{22}(s)} \qquad \text{in } B_{r/2}(x_0, z_0) \text{ for } \lambda \in \mathbb{R}\,.$$

We distinguish two cases:

1) If $\lambda > 0$ then $\psi > 0$ in $B_{r/2}^+ = B_{r/2}(x_0, z_0) \cap [z > z_0]$ and $\psi < 0$ in $B_{r/2}^- = B_{r/2}(x_0, z_0) \cap [z < z_0]$. Since ψ is monotone, then $\psi \geq \kappa(z) > 0$ in $D = (-\infty, x_0) \times (z_0, z_0 + r/2)$. So we have

$$\begin{cases} \operatorname{div}(b(z)\, \nabla(\psi - \kappa)) = 0 & \text{in } D \\ \psi - \kappa \geq 0 & \text{in } D \\ \psi - \kappa = 0 & \text{in } B_{r/2}^+ \end{cases}$$

and by the strong maximum principle, $\psi = \kappa$ in D.

Now for $x \to -\infty$, we have by (3.4), $\kappa(z) = v_{-\infty}(z)$ which leads to a contradiction since $\kappa(z_0) = 0$ and

$$v_{-\infty}(z_0) = Q_s \int_{z_0}^{0} \frac{ds}{b_{22}(s)} \Bigg/ \int_{-h}^{0} \frac{ds}{b_{22}(s)} > 0\,.$$

2) If $\lambda < 0$ then $\psi < 0$ in $B_{r/2}^+$ and $\psi > 0$ in $B_{r/2}^-$. So we have $\psi > 0$ in $D' = (-\infty, x_0) \times (z_0 - r/2, z_0)$. We get $\psi = \kappa$ in D' and we obtain a contradiction with the asymptotic behavior of ψ at $-\infty$. ■

LEMMA 4.4. – *Consider* $R = (x_1, x_2) \times (z_1, z_2) \subset \Omega$ *such that on its bound-*

ary we have:

$$\begin{cases} \psi(x, z) \leqslant 0 & \textit{for } z = z_1 \textit{ and } z = z_2 \\ \psi(x, z) \leqslant \delta & \textit{for } x = x_1 \\ \psi(x, z) \leqslant -\delta & \textit{for } x = x_2 \end{cases}$$

for some $\delta > 0$. *Then*:

$$\psi(x, z) < 0 \quad \textit{for} \quad x > \frac{x_1 + x_2}{2}, \quad z_1 < z < z_2 .$$

PROOF. – Let u be the function defined by:

$$\begin{cases} \operatorname{div}(b(z)\, \nabla u) = 0 \quad \text{in} \quad R^- = \left(\dfrac{x_1 + x_2}{2}, x_2\right) \times (z_1, z_2) \\ u = 0 \quad \text{for} \quad \dfrac{x_1 + x_2}{2} \leqslant x \leqslant x_2, \ \ z \in \{z_1, z_2\} \ \text{ and } \ x = \dfrac{x_1 + x_2}{2}, \ \ z_1 \leqslant z \leqslant z_2 \\ u = -\delta \quad \text{for} \quad x = x_2, \ \ z_1 \leqslant z \leqslant z_2 . \end{cases}$$

Note that we have $u \leqslant 0$ on ∂R^- and $u \not\equiv 0$ in R^-, so by the weak and strong maximum principles, we deduce that: $u < 0$ in R^-.

Consider now w defined by:

$$w(x, z) = \begin{cases} u(x, z) & \text{in } R^- \\ -u(x_1 + x_2 - x, z) & \text{in } R^+ = \left(x_1, \dfrac{x_1 + x_2}{2}\right) \times (z_1, z_2) . \end{cases}$$

Let us verify that w satisfies:

$$\begin{cases} \operatorname{div}(b(z)\, \nabla w) = 0 & \text{in } R \\ w \geqslant \psi & \text{on } \partial R . \end{cases}$$

Let $\xi \in \mathcal{D}(R)$. We have:

$$\int_R b(z)(\nabla w) . \nabla \xi = \int_{R^-} b(z)(\nabla u) . \nabla \xi - \int_{R^+} b(z)\big(\nabla(u(x_1 + x_2 - x, z))\big) . \nabla \xi$$

$$= \int_{R^-} b(z)(\nabla u) . \nabla \xi - \int_{R^-} b(z)(\nabla u) . \nabla \xi(x_1 + x_2 - x, z)$$

$$= \int_{R^-} b(z)(\nabla u) . \nabla(\xi - \xi(x_1 + x_2 - x, z)) = 0$$

since $\xi(x, z) - \xi(x_1 + x_2 - x, z) = 0$ on ∂R^-. Then we get by the weak maximum principle $\psi \leqslant u < 0$ in R^- since by Proposition 3.2, we have $\operatorname{div}(b(z)\,\nabla\psi) \geqslant 0$ in R. ■

PROOF OF THEOREM 4.2. – Using Lemma 4.3 and Lemma 4.4, one can adapt the proof given in [AD1] or [CCL]. ■

COROLLARY 4.5. – *Let* (ψ, γ) *be a solution of* (P). *Then we have*

i) $\gamma = \gamma_s \chi([\psi > 0]) + \gamma_f \chi([\psi < 0])$ *a.e. in* Ω

ii) *The sets* $[\psi > 0]$ *and* $[\psi < 0]$ *are connected by arcs.*

PROOF. – i) We have by (P)ii), $\gamma = \gamma_s$ a.e. in $[\psi > 0]$ and $\gamma = \gamma_f$ a.e. in $[\psi < 0]$. Moreover the set $[\psi = 0] = \Gamma$ is of measure zero by Proposition 4.1 and Theorem 4.2. Thus $\gamma = \gamma_s \chi([\psi > 0]) + \gamma_f \chi([\psi < 0])$ a.e. in Ω.

ii) We argue as in [CCL]. ■

5. – Uniqueness of the solution

THEOREM 5.1. – *There exists a unique solution* (ψ, γ) *of* (P).

First, we have

LEMMA 5.2. – *Let* (ψ_1, γ_1), (ψ_2, γ_2) *be two solutions of* (P). *Then we have for* $i = 1, 2$

$$\mathcal{F}_i(\zeta) = \int_\Omega (b(z)\,\nabla(\psi_i - \psi_0) + (\gamma_i - \gamma_0)\, e_x).\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(\mathbb{R}^2) \tag{5.1}$$

where $\psi_0 = \min(\psi_1, \psi_2)$ *and* $\gamma_0 = \min(\gamma_1, \gamma_2)$.

PROOF. – Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geqslant 0$ and $\varepsilon > 0$. Set $\xi = \min(\zeta, (\psi_i - \psi_0)/\varepsilon)$. Using the fact that ξ is a test function for (P) written for (ψ_1, γ_1) and (ψ_2, γ_2), we obtain by subtracting the equations:

$$\int_{[\psi_i - \psi_0] \geqslant \varepsilon\zeta]} b(z)\,\nabla(\psi_i - \psi_0).\nabla\zeta + \int_\Omega (\gamma_i - \gamma_0)\,\zeta_x \leqslant \int_\Omega (\gamma_i - \gamma_0)\left(\zeta - \frac{\psi_i - \psi_0}{\varepsilon}\right)^+_x . \tag{5.2}$$

Setting $g_0 = \min(g_1, g_2)$ and using Corollary 4.5i), we obtain

$$\int_{\Omega} (\gamma_i - \gamma_0)\left(\zeta - \frac{\psi_i - \psi_0}{\varepsilon}\right)_x^+ = \tag{5.3}$$

$$(\gamma_s - \gamma_f)\int_{I_0}\left\{\left(\zeta - \frac{\psi_i - \psi_0}{\varepsilon}\right)^+ (g_i(z), z) - \left(\zeta - \frac{\psi_i - \psi_0}{\varepsilon}\right)^+ (g_0(z), z)\right\}$$

with $I_0 = \{z \in (-h^*, 0)/g_0(z) < g_i(z)\}$.

Taking into account (5.3) and letting $\varepsilon \to 0$ in (5.2), we get:

$$\mathcal{F}_i(\zeta) \leqslant 0 \qquad \forall \zeta \in \mathcal{D}(\mathbb{R}^2), \qquad \zeta \geqslant 0\,. \tag{5.4}$$

Now, we consider $\zeta \in \mathcal{D}(\mathbb{R}^2)$. Let $K = \operatorname{supp} \zeta$ and $M = \sup\limits_K |\zeta|$. It is clear that there exists $R_0 > a$ such that $\forall R \geqslant R_0$, $K \subset (-R, R) \times \mathbb{R}$.

Consider $\zeta_R \colon \mathbb{R} \to \mathbb{R}^+$ defined by:

$$\zeta_R(x) = \begin{cases} 0 & \text{if } |x| \geqslant R+1 \\ M & \text{if } |x| \leqslant R \\ M(-x+R+1) & \text{if } R \leqslant x \leqslant R+1 \\ M(x+R+1) & \text{if } -R-1 \leqslant x \leqslant -R\,. \end{cases}$$

Then we have

$$\forall R \geqslant R_0, \qquad \forall (x, z) \in \mathbb{R}^2, \qquad -\zeta_R(x) \leqslant \zeta(x, z) \leqslant \zeta_R(x)\,. \tag{5.5}$$

Using (5.4)-(5.5), we get

$$\mathcal{F}_i(\zeta_R) \leqslant \mathcal{F}_i(\zeta) \leqslant -\ \mathcal{F}_i(\zeta_R)\,. \tag{5.6}$$

Let us compute $\mathcal{F}_i(\zeta_R)$:

$$\mathcal{F}_i(\zeta_R) = M \int_{\Omega_{-R-1,\,-R}} b(z)\,\nabla(\psi_i - \psi_0)\,.e_x + M \int_{\Omega_{-R-1,\,-R}} (\gamma_i - \gamma_0)$$

$$-M \int_{\Omega_{R,\,R+1}} b(z)\,\nabla(\psi_i - \psi_0)\,.e_x - M \int_{\Omega_{R,\,R+1}} (\gamma_i - \gamma_0)\,.$$

By Lemma 3.4 and Theorem 3.3, we deduce that:

$$\lim_{R \to +\infty} \mathcal{F}_i(\zeta_R) = 0$$

from which we deduce that $\mathcal{F}_i(\zeta) = 0$ and the lemma follows. ■

REMARK 5.3. – Note that (ψ_0, γ_0) is a solution of (P) since by density

(5.1) is still true for $\zeta \in H_0^1(\Omega)$ with compact support. Clearly we have also $\psi_{0x} \leqslant 0$ and $\gamma_{0x} \leqslant 0$ in $\mathcal{D}'(\Omega)$.

PROOF OF THEOREM. – 5.1. – Since for $x \geqslant a$, we have $\psi_i(x, 0) = \psi_0(x, 0) = -Q_f$, there exists by continuity a small ball B centered on a point $(a_1, 0)$ with $a_1 > a$ such that $B \cap \Omega \subset [\psi_i < 0] \cap [\psi_0 < 0]$. Now, let $\zeta \in \mathcal{D}([\psi_i < 0] \cup B)$. Using (5.1) and the fact that $\gamma_i = \gamma_0 = \gamma_f$ a.e. in $[\psi_i < 0]$, we get:

$$\int_{[\psi_i < 0]} b(z)\, \nabla\psi . \nabla\zeta = 0 \tag{5.7}$$

with $\psi = \psi_i - \psi_0$. Now because $\psi = 0$ on $B \cap [z = 0]$, we may extend ψ by 0 into $B \backslash \Omega$ in such a way that $\psi \in H^1_{\mathrm{loc}}([\psi_i < 0] \cup B)$. We also extend $b(z)$ by I_2 into $B \backslash \Omega$. Then we obtain from (5.7):

$$\int_{[\psi_i < 0] \cup B} b(z)\, \nabla\psi . \nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}([\psi_i < 0] \cup B). \tag{5.8}$$

Moreover we have $\psi \geqslant 0$ in the open connected set $\Omega_i = [\psi_i < 0] \cup B$, $\psi = 0$ in $B \backslash [\psi_i < 0]$ and b strictly elliptic, thus we deduce by the strong maximum principle that $\psi = 0$ in Ω_i which leads to $\psi_i = \psi_0$ in $[\psi_i < 0]$ and then $\psi_1 = \psi_2$ in $[\psi_1 < 0] \cap [\psi_2 < 0]$. But we can verify that we have now $[\psi_1 < 0] = [\psi_2 < 0]$. Similarly we prove that $\psi_1 = \psi_2$ in $[\psi_1 > 0] \cap [\psi_2 > 0]$ and $[\psi_1 > 0] = [\psi_2 > 0]$. Finally, we have proved that $\psi_1 = \psi_2$ in Ω and by Corollary 4.5i), we have also $\gamma_1 = \gamma_2$ in Ω. ■

6. – Study of the free boundary near $z = -h^*$.

The goal of this section, is to prove the following theorem which means that Γ does not contain the ray $[z = -h^*]$.

THEOREM 6.1.. – *The set* $S = \{x \in \mathbb{R} / \psi(x, -h^*) = 0\}$ *is empty and* $\Gamma = [x = g(z)]$.

PROOF. – We argue by contradiction. Assume that $S \neq \emptyset$. Since ψ is continuous in Ω and nonincreasing, S is a closed interval. Set $\alpha = \inf S$ and $\beta = \sup S$.

Let $x_0 \in S$, then by monotonicity of ψ, $\psi(x, -h^*) \leqslant \psi(x_0, -h^*) = 0$ $\forall x \geqslant x_0$. Moreover $\psi(x, -h^*) \geqslant v_{+\infty}(-h^*) = 0$, then $\psi(x, -h^*) = 0$ $\forall x \geqslant x_0$. So $\beta = +\infty$.

Now, if $\alpha = -\infty$, then $\psi(x, -h^*) = 0$ $\forall x \in \mathbb{R}$. But this leads to a contradiction with the asymptotic behavior of ψ at $-\infty$. Thus

$$S = \{x \in \mathbb{R},\ \ \psi(x, -h^*) = 0\} = [\alpha, +\infty). \tag{6.1}$$

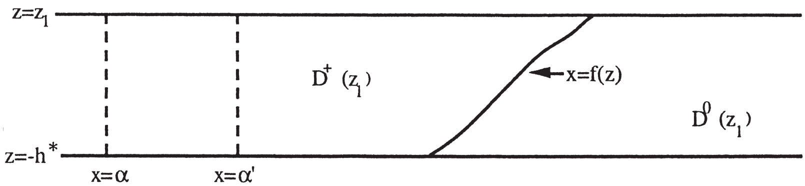

Figure 2

Let $\alpha' = \max(\alpha + 1/2, 1) > \alpha$ and C a constant satisfying

$$C > \frac{\alpha'}{h - h^*} b_{22} - b_{21} \qquad \text{a.e. in } \Omega \tag{6.2}$$

which is possible since $b \in L^\infty(\Omega)$.

Define $f(z)$ by:

$$f(z) = \int_{-h}^{z} \frac{b_{21}(s) + C}{b_{22}(s)} ds \tag{6.3}$$

then we have:

LEMMA 6.2.

i) $f \in W^{1,\infty}(-h, 0)$ *and* $f'(z) > \dfrac{\alpha'}{h - h^*} > 0$

ii) $f(-h^*) > \alpha'$

iii) $(b_{22}(z) f'(z) - b_{21}(z))' = 0$ *in* $\mathcal{D}'(-h, 0)$.

PROOF. – i) By (1.14), we have for $z_1, z_2 \in (-h, 0)$

$$|f(z_1) - f(z_2)| = \left| \int_{z_1}^{z_2} \frac{b_{21}(s) + C}{b_{22}(s)} ds \right| \leq c_1 |z_1 - z_2|$$

and $|f(z)| \leq c_1 |z + h| \leq 2 c_1 h$ for some constant c_1. Then $f \in W^{1,\infty}(-h, 0)$ and by (6.2)-(6.3)

$$f'(z) = \frac{b_{21}(z) + C}{b_{22}(z)} > \frac{\alpha'}{h - h^*} > 0 . \tag{6.4}$$

ii) Using (6.4), we have from (6.3): $f(z) > (z + h)\alpha'/(h - h^*)$. By i) $f \in C^0([-h, 0])$, then $f(-h^*) > \alpha'$.

iii) We have by (6.4), $b_{22}(z) f'(z) - b_{21}(z) = C$ and then iii) holds.

Let $k > 0$. We define the functions v and θ by:

$$(6.5) \qquad \begin{cases} v(x, z) = k(\gamma_s - \gamma_f)(f(z) - x)^+ \\ \theta(x, z) = \gamma_s \chi([x < f(z)]) + \gamma_f \chi([x > f(z)]) \end{cases} \qquad (x, z) \in D(z_1)$$

where $D(z_1) = (\alpha', +\infty) \times (-h^*, z_1)$ with $z_1 \in (-h^*, 0)$. Then we have

LEMMA 6.3. – *There exists $k > 0$ such that*:

$$(6.6) \qquad \int_{D(z_1)} (b(z)\, \nabla v + \theta e_x)\, \nabla \xi \geq 0 \qquad \forall \xi \in \mathcal{D}(D(z_1)), \quad \xi \geq 0 .$$

PROOF. – Set $D^+(z_1) = D(z_1) \cap [x < f(z)]$ and $D^0(z_1) = D(z_1) \cap [x > f(z)]$ (see Figure 2). Let $\xi \in \mathcal{D}(D(z_1))$, $\xi \geq 0$. We have

$$(6.7) \quad \int_{D(z_1)} (b(z) \nabla v + \theta e_x)\, \nabla \xi = \int_{D^+(z_1)} (b(z)\, \nabla v + \gamma_s e_x)\, \nabla \xi + \int_{D^0(z_1)} \gamma_f e_x . \nabla \xi .$$

In $D^+(z_1)$, we have $\nabla v = k(\gamma_s - \gamma_f)(-e_x + f'(z)\, e_z)$ and

$$b(z)\, \nabla v = k(\gamma_s - \gamma_f)((-b_{11} + b_{12} f'(z))\, e_x + (-b_{21} + b_{22} f'(z))\, e_z) =$$

$$k(\gamma_s - \gamma_f)((-b_{11} + b_{12} f'(z))\, e_x + C e_z) .$$

Then

$$(6.8) \qquad \operatorname{div}(b(z)\, \nabla v) = 0 \quad \text{in } D^+(z_1)$$

since $(-b_{11} + b_{12} f'(z))$ does not depend on x. Using (6.7) and (6.8), we get by applying the Green formula

$$\int_{D(z_1)} (b(z)\, \nabla v + \theta e_x)\, \nabla \xi = \int_{[x = f(z)]} (b(z)\, \nabla v + \gamma_s e_x) . \nu \xi + \int_{[x = f(z)]} \gamma_f e_x . (-\nu)\, \xi =$$

$$\int_{[x = f(z)]} ([k(\gamma_s - \gamma_f)(-b_{11} + b_{12} f'(z)) + (\gamma_s - \gamma_f)]\, \nu_x + k(\gamma_s - \gamma_f)\, C \nu_z)\, \xi$$

where ν denotes the unit normal vector to $(\partial D^+(z_1) \cap [x = f(z)])$ pointing into $D^0(z_1)$. ν is given explicitely by

$$\nu = \nu_x e_x + \nu_z e_z = \frac{1}{\sqrt{1 + f'^2(z)}}\, (e_x - f'(z)\, e_z) .$$

Then we have

$$\int_{D(z_1)} (b(z)\,\nabla v + \theta e_x)\,\nabla \xi = (\gamma_s - \gamma_f) \int_{[x=f(z)]} \frac{u_k(z)}{\sqrt{1+f'^2(z)}}\,\xi$$

with $u_k(z) = 1 + k(-b_{11} + b_{12} f'(z) - Cf'(z))$.

Note that, since $b_{11}, b_{12}, f' \in L^\infty(-h, 0)$, we have $|u_k(z) - 1| \leqslant kc_1$ for some constant c_1. Then $u_k(z) \geqslant 1 - kc_1$. If we choose k such that $1 - kc_1 > 0$ i.e. $0 < k < 1/c_1$ then (6.6) holds. ■

Now, we will compare (ψ^+, γ) with (v, θ).

LEMMA 6.4. – *Let* (ψ, γ) *be a solution of* (P) *and* (v, θ) *defined by* (6.5). *Then there exists* $z_0 \in (-h^*, 0)$ *such that*

$$\int_{D(z_0)} (b(z)\,\nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\,\nabla\zeta = 0 \qquad \forall \zeta \in \mathcal{D}(\mathbb{R}^2), \tag{6.9}$$

where $v_0 = \min(\psi^+, v)$, $\theta_0 = \min(\gamma, \theta)$.

PROOF. – The proof is done in several steps.

1st step.

We have $\psi(\alpha', -h^*) = 0$ then by continuity of ψ, there exists $\delta > 0$ such that

$$\psi(\alpha', z) \leqslant k(\gamma_s - \gamma_f)(f(-h^*) - \alpha') \qquad \forall z \in (-h^*, -h^* + \delta).$$

Moreover $\exists z_0 \in (-h^*, -h^* + \delta)$ such that $\psi(\alpha', z_0) = 0$. If not, we distinguish two cases:

i) $\psi(\alpha', z) > 0$ $\forall z \in (-h^*, -h^* + \delta)$ leads to $\psi(x, z) \geqslant \psi(\alpha', z) > 0$ $\forall x \in (\alpha, \alpha')$ which contradicts Lemma 4.3 since we have $\psi(x, z) > 0$ in $(\alpha, \alpha') \times (-h, -h^*)$ and $\psi(x, -h^*) = 0$ for $x \in (\alpha, \alpha')$.

ii) $\psi(\alpha', z) < 0$ $\forall z \in (-h^*, -h^* + \delta)$ leads to $\psi(x, z) \leqslant \psi(\alpha', z) < 0$ $\forall x \geqslant \alpha'$ which leads again to a contradiction by Lemma 4.3.

Set $D = D(z_0)$. Then since $f'(z) > 0$

$$\psi^+(\alpha', z) \leqslant k(\gamma_s - \gamma_f)(f(-h^*) - \alpha') \leqslant$$

$$k(\gamma_s - \gamma_f)(f(z) - \alpha') = v(\alpha', z) \qquad \forall z \in (-h^*, z_0)$$

$$\psi^+(x, -h^*) = 0 \leqslant v(x, -h^*) \qquad \forall x \geqslant \alpha'$$

$$\psi^+(x, z_0) = 0 \leqslant v(x, z_0) \qquad \forall x \geqslant \alpha'$$

then

$$\psi^{+} \leqslant v \quad \text{ on } \partial D . \tag{6.10}$$

2nd step.

We have $\forall \zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geqslant 0$

$$\int_D (b(z)\, \nabla(\psi^{+} - v_0) + (\gamma - \theta_0)\, e_x) \nabla\zeta \leqslant \int_I (\gamma_s - \gamma_f)\, \zeta(g(z), z)\, dz\, , \tag{6.11}$$

where $I = \{z \in (-h^*, z_0) / g(z) > f(z)\}$.

Indeed, let $\varepsilon > 0$, $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geqslant 0$. Set

$$\xi = \min\left(\frac{(\psi^{+} - v_0)}{\varepsilon}, \zeta\right) = \min\left(\frac{(\psi^{+} - v)^{+}}{\varepsilon}, \zeta\right).$$

We have $\xi \in H^1(D)$, $\xi \geqslant 0$ and $\xi = 0$ for large x. Moreover $\xi = 0$ on ∂D by (6.10). By Proposition 3.2, we have $\operatorname{div}(b(z)\, \nabla\psi^{-}) \geqslant 0$ in $\mathcal{D}'(\Omega)$, then $\operatorname{div}(b(z)\, \nabla\psi^{+}) + \partial_x \gamma \geqslant 0$. So we have:

$$\int_D (b(z)\, \nabla\psi^{+} + \gamma e_x)\, \nabla\xi \leqslant 0\, .$$

By (6.6), we have:

$$-\int_D (b(z)\, \nabla v + \theta e_x)\, \nabla\xi \leqslant 0\, .$$

Adding these inequalities, we get:

$$\int_D (b(z)\, \nabla(\psi^{+} - v) + (\gamma - \theta)\, e_x)\, \nabla\xi \leqslant 0$$

which can be written:

$$\int_D (b(z)\, \nabla(\psi^{+} - v_0) + (\gamma - \theta_0)\, e_x) \nabla\xi \leqslant 0\, . \tag{6.12}$$

Since $\xi = \zeta - (\zeta - (\psi^{+} - v_0)/\varepsilon)^{+}$, we have:

$$\int_{D \cap [\psi^{+} - v_0 \geqslant \varepsilon\zeta]} b(z)\, \nabla(\psi^{+} - v_0)\, \nabla\zeta + \int_D (\gamma - \theta_0)\, e_x . \nabla\zeta \leqslant$$

$$-\frac{1}{\varepsilon} \int_{D \cap [\psi^{+} - v_0 < \varepsilon\zeta]} b(z)\, \nabla(\psi^{+} - v_0)\, \nabla(\psi^{+} - v_0) + \int_D (\gamma - \theta_0) \left(\zeta - \frac{(\psi^{+} - v_0)}{\varepsilon}\right)^{+}_{x} .$$

By (1.14), we get:

$$\int_{D \cap [\psi^+ - v_0 \geq \varepsilon\zeta]} b(z)\, \nabla(\psi^+ - v_0)\, \nabla\zeta + \int_D (\gamma - \theta_0)\, e_x . \nabla\zeta \leq \int_D (\gamma - \theta_0) \left(\zeta - \frac{(\psi^+ - v_0)}{\varepsilon} \right)_x^+ . \tag{6.13}$$

Note that we have:

$$\int_D (\gamma - \theta_0) \left(\zeta - \frac{(\psi^+ - v_0)}{\varepsilon} \right)_x^+ = \int_{D \cap [v_0 > 0]} (\gamma - \theta_0) \left(\zeta - \frac{(\psi^+ - v_0)}{\varepsilon} \right)_x^+ + \int_{D \cap [v_0 = 0]} (\gamma - \theta_0) \left(\zeta - \frac{\psi^+}{\varepsilon} \right)_x^+ .$$

Since $D \cap [v_0 > 0] = D \cap ([\psi^+ > 0] \cap [v > 0])$, we have $\gamma = \theta = \theta_0 = \gamma_s$ in this set and then

$$\int_{D \cap [v_0 > 0]} (\gamma - \theta_0) \left(\zeta - \frac{(\psi^+ - v_0)}{\varepsilon} \right)_x^+ = 0$$

For the other integral, we have:

$$\begin{aligned}
\int_{D \cap [v_0 = 0]} (\gamma - \theta_0) \left(\zeta - \frac{\psi^+}{\varepsilon} \right)_x^+
&= \int_{D \cap [\psi^+ = v_0 = 0]} (\gamma_f - \gamma_f)\, \zeta_x + \int_{D \cap [\psi^+ > 0,\, v_0 = 0]} (\gamma_s - \gamma_f) \left(\zeta - \frac{\psi^+}{\varepsilon} \right)_x^+ \\
&= (\gamma_s - \gamma_f) \int_{D \cap [f(z) < x < g(z)]} \left(\zeta - \frac{\psi^+}{\varepsilon} \right)_x^+ \\
&= (\gamma_s - \gamma_f) \int_I \int_{f(z)}^{g(z)} \left(\zeta - \frac{\psi^+}{\varepsilon} \right)_x^+ \\
&= (\gamma_s - \gamma_f) \int_I \left(\zeta - \frac{\psi^+}{\varepsilon} \right)^+ (g(z), z) - \left(\zeta - \frac{\psi^+}{\varepsilon} \right)^+ (f(z), z)\, dz \\
&\leq (\gamma_s - \gamma_f) \int_I \zeta(g(z), z)\, dz \qquad \text{since } \psi^+(g(z), z) = 0 .
\end{aligned}$$

Thus:

$$\int_D (\gamma - \theta_0)\left(\zeta - \frac{(\psi^+ - v_0)}{\varepsilon}\right)_x^+ \leqslant (\gamma_s - \gamma_f)\int_I \zeta(g(z), z)\, dz\,. \tag{6.14}$$

Using (6.14) and letting $\varepsilon \to 0$ in (6.13), we get (6.11).

3rd step.

We have $\forall \zeta \in \mathcal{D}(\mathbb{R}^2), \quad \zeta \geqslant 0$

$$\int_D (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla\zeta \leqslant 0\,. \tag{6.15}$$

Indeed, let $\delta > 0$, $\zeta \in \mathcal{D}(\mathbb{R}^2)$, $\zeta \geqslant 0$. Set $A_0 = [v_0 > 0]$ and define $\alpha_\delta(X) = (1 - d(X, A_0)/\delta)^+$. Note that

$$A_0 = [\psi^+ > 0] \cap [v > 0] = [x < g(z)] \cap [x < f(z)] = [x < \min(f(z), g(z))] \subset [x < f(z)]$$

and $\alpha_\delta = 1$ in $\overline{A}_0$. Then we write:

$$\begin{aligned}\int_D (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla\zeta &= \int_D (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla(\alpha_\delta \zeta) \\ &\quad + \int_D (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla((1 - \alpha_\delta)\, \zeta) \\ &= I_1^\delta + I_2^\delta.\end{aligned}$$

First by the previous step we have:

$$I_1^\delta \leqslant (\gamma_s - \gamma_f)\int_I (\alpha_\delta \zeta)(g(z), z)\, dz$$

Since $\{(g(z), z)/z \in I\} \subset \mathbb{R}^2 \setminus \overline{A}_0$, we get:

$$\overline{\lim}_{\delta \to 0} I_1^\delta \leqslant 0\,. \tag{6.16}$$

Next we write:

$$\begin{aligned} I_2^\delta = &\int_{D \cap [g(z) > \alpha']} (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla((1 - \alpha_\delta)\, \zeta) \\ &+ \int_{D \cap [g(z) \leqslant \alpha']} (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla((1 - \alpha_\delta)\, \zeta)\,.\end{aligned}$$

We remark that for all $(x, z) \in D \cap [g(z) \leqslant \alpha']$, we have $x > g(z)$ then $\psi^+(x, z) = 0$. Moreover, since $v \geqslant 0$, we have $v_0 = \min(\psi^+, v) = 0$ in $D \cap$

$[g(z) \leqslant \alpha']$. Then $\gamma = \theta_0 = \gamma_f$ in $D \cap [g(z) \leqslant \alpha']$ and

$$\int_{D \cap [g(z) \leqslant \alpha']} (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla((1 - \alpha_\delta)\, \zeta) = 0\,.$$

So

$$I_2^\delta = \int_{D \cap [g(z) > \alpha']} (b(z)\, \nabla(\psi^+ - v_0) + (\gamma - \theta_0)\, e_x)\, \nabla((1 - \alpha_\delta)\, \zeta)$$

$$= \int_{D \cap [g(z) > \alpha']} (b(z)\, \nabla\psi^+ + \gamma e_x)\, \nabla((1-\alpha_\delta)\, \zeta) - \int_{D \cap [g(z) > \alpha']} (b(z)\, \nabla v_0 + \theta_0 e_x)\, \nabla((1-\alpha_\delta)\, \zeta)$$

$$= I_{2,1}^\delta - I_{2,2}^\delta\,.$$

Let us consider the set

$$J = \{ z \in (-h^*, z_0) / g(z) > \alpha' \}\,.$$

If $J = \emptyset$ then $g(z) \leqslant \alpha'$ $\forall z \in (-h^*, z_0)$ and then $\psi^+(\alpha', z) = 0$ $\forall z \in [-h^*, z_0]$. By monotonicity of ψ, we have $\psi < 0$ in D which leads to a contradiction with Lemma 4.3 since we have $\psi > 0$ in $\Omega \cap [z < -h^*]$ and $\psi = 0$ on $(\alpha', +\infty) \times \{-h^*\}$.

Assume now that $J \neq \emptyset$. For any $z \in J$ we define

$$m(z) = \inf \{ s \in (-h^*, z_0) / \forall t \in [s, z], \quad t \in J \}$$

$$M(z) = \sup \{ s \in (-h^*, z_0) / \forall t \in [z, s], \quad t \in J \}\,.$$

Since g is continuous, we have $\forall z \in J$, $m(z), M(z) \notin J$. Set

$$\mathfrak{J} = \{ [m(z), M(z)] \subset [-h^*, z_0] / z \in J \}\,.$$

Let us define $\varphi : \mathfrak{J} \to \boldsymbol{Q}$ by $\varphi([m(z), M(z)]) = r \in \boldsymbol{Q} \cap [m(z), M(z)]$ (r is chosen arbitrarily, $\boldsymbol{Q}$ is the set of rational numbers). φ is one to one from $\mathfrak{J}$ to $\varphi(\mathfrak{J}) \subset \boldsymbol{Q}$. Indeed, let $[m(z), M(z)]$ and $[m(z'), M(z')] \in \mathfrak{J}$ such that $\varphi([m(z), M(z)]) = \varphi([m(z'), M(z')])$, then $[m(z), M(z)] \cap [m(z'), M(z')] \neq \emptyset$. This leads by definition of $m(z)$ and $M(z)$ to $[m(z), M(z)] = [m(z'), M(z')]$.

Thus, we have

$$\mathfrak{J} = \{ [m(z_n), M(z_n)] \subset [-h^*, z_0] / z_n \in J, \quad n \in \mathcal{N} \} \qquad \text{with } \mathcal{N} \subset \mathbb{N}\,.$$

Now we can write $I_{2,1}^\delta$ as follows:

$$I_{2,1}^\delta = \int_{\cup_{n \in \mathcal{N}} (\alpha', +\infty) \times (m(z_n), M(z_n))} (b(z)\, \nabla\psi^+ + \gamma e_x)\, \nabla((1 - \alpha_\delta)\, \zeta)$$

$$= \sum_{n \in \mathcal{N}} \int_{(\alpha', +\infty) \times (m(z_n), M(z_n))} (b(z) \nabla \psi^+ + \gamma e_x) \nabla((1-\alpha_\delta)\zeta) .$$

We have $\forall z \in (m(z_n), M(z_n))$, $(\alpha', z) \in \overline{A}_0$. Indeed, we have for $z \in (m(z_n), M(z_n))$, $g(z) > \alpha'$, then $\psi^+(\alpha', z) > 0$. By continuity, $\exists r > 0$ such that $\psi^+ > 0$ in $B_r(\alpha', z)$. We also have $v > 0$ in $B_r(\alpha', z) \cap D$, then $v_0 > 0$ in $B_r(\alpha', z) \cap D$. Thus $B_r(\alpha', z) \cap D \subset A_0$.

We deduce that $(1-\alpha_\delta)(\alpha', z) = 0$. Moreover $\psi^+(x, m(z_n)) = \psi^+(x, M(z_n)) = 0$ $\forall x > \alpha'$. Using Lemma 6.5 below, we get

$$\int_{(\alpha', +\infty) \times (m(z_n), M(z_n))} (b(z) \nabla \psi^+ + \gamma e_x) \nabla((1-\alpha_\delta)\zeta) \leqslant 0 \qquad \forall n \in \mathcal{N}$$

and then

$$I_{2,1}^\delta \leqslant 0 . \tag{6.17}$$

Now, we have

$$I_{2,2}^\delta = \int_{(D \setminus \overline{A}_0) \cap [g(z) > \alpha']} \gamma_f((1-\alpha_\delta)\zeta)_x$$

$$= \int_J \int_{\min(f(z), g(z))}^{+\infty} \gamma_f((1-\alpha_\delta)\zeta)_x$$

$$= -\int_J \gamma_f((1-\alpha_\delta)\zeta)\big(\min(f(z), g(z)), z\big)$$

$$= 0 \qquad \text{since } \big(\min(f(z), g(z)), z\big) \in \overline{A}_0 \text{ when } z \in J .$$

Thus $I_{2,2}^\delta = 0$ and by (6.16)-(6.17) we get (6.15).

4th step.

Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$. Set $K = \operatorname{supp} \zeta$ and $M = \sup_K |\zeta|$. Then there exists $R_0 > a$ such that $\forall R \geqslant R_0$, $K \subset (-R, R) \times \mathbb{R}$. Define ζ_R as in the proof of Lemma 5.2, then we have:

$$-\zeta_R \leqslant \zeta \leqslant \zeta_R \qquad \forall R \geqslant R_0 .$$

Using (6.15) for $\zeta_R - \zeta$ and $\zeta_R + \zeta$ respectively, we get:

$$T(\zeta_R) \leqslant T(\zeta) \leqslant -T(\zeta_R)$$

with $T(\zeta) = \int_D (b(z) \nabla(\psi^+ - v_0) + (\gamma - \theta_0) e_x) \nabla \zeta$.

Moreover, we have for large R:

$$T(\zeta_R) = -M \int_{D_{R,R+1}} b(z)\,\nabla(\psi^+ - v_0).e_x - M \int_{D_{R,R+1}} (\gamma - \theta_0)$$

$$= -M \int_{D_{R,R+1}} b(z)\,\nabla\psi^+ .e_x - M \int_{D_{R,R+1}} (\gamma - \gamma_f)$$

$$= -M \int_{D_{0,1}} b(z)\,\nabla\psi_R^+ .e_x - M \int_{D_{0,1}} (\gamma_R - \gamma_f)$$

with $D_{m,n} = \Omega_{m,n} \cap D$. By (3.5), we have:

$$\lim_{R \to +\infty} \int_{D_{0,1}} (\gamma_R - \gamma_f) = 0\,.$$

Moreover by (3.6), we deduce up to a subsequence of R, still denoted by R, that:

$$\nabla\psi_R^+ \rightharpoonup \nabla v_{+\infty}^+ = 0 \qquad \text{in } L^2(D_{0,1})\,.$$

Thus:

$$\lim_{R \to +\infty} \int_{D_{0,1}} b(z)\,\nabla\psi_R^+ .\, e_x = 0\,.$$

This completes the proof of Lemma 6.4. ■

LEMMA 6.5. – *If* $D = (x_0, +\infty) \times (z_1, z_2) \subset \Omega$, $(z_1 < z_2)$ *and*

$$\psi(x, z_i) \leqslant 0 \qquad i = 1, 2 \ \ \forall x \geqslant x_0,$$

then we have

$$(6.18) \qquad \int_D (b(z)\,\nabla\psi^+ + \gamma e_x)\,\nabla\xi \leqslant 0 \qquad \forall \xi \in H^1(D), \quad \xi \geqslant 0$$

$$\xi(x_0, z) = 0 \ \ a.e. \ \ z \in (z_1, z_2) \ \ and \ \ \xi = 0 \ \ for\ large\ \ x\,.$$

PROOF. – Using Proposition 3.2 we deduce that

$$(6.19) \qquad \int_D (b(z)\,\nabla\psi^+ + \gamma e_x)\,\nabla\xi \leqslant 0 \quad \forall \xi \in H_0^1(D), \quad \xi \geqslant 0 \ \text{ with bounded support}\,.$$

Now, let $\xi \in C^\infty(\overline{\Omega})$ such that $\xi \geqslant 0$, $\xi(x_0, z) = 0$ and $\xi = 0$ for large x. Let $\delta > 0$ and define $d_\delta(z) = \min((z - z_1)^+/\delta, 1).\min((z_2 - z)^+/\delta, 1)$. Since $d_\delta(z_1) = d_\delta(z_2) = 0$, we have $d_\delta\,\xi \in H_0^1(D)$ with compact support in $\overline{D}$, so it is a test function for (6.19). Then

$$\int_D (b(z)\,\nabla\psi^+ + \gamma e_x)\,\nabla(d_\delta\,\xi) \leqslant 0\,.$$

But, since d_δ does not depend on x, we obtain:

$$\int_D b(z)\,\nabla\psi^+\,\nabla(d_\delta\,\xi) + d_\delta\,\gamma e_x\,.\nabla\xi \leqslant 0\,. \tag{6.20}$$

Moreover we have

$$\int_D b(z)\,\nabla\psi^+\,\nabla((1-d_\delta)\,\xi) \leqslant 0\,. \tag{6.21}$$

Indeed, set $\zeta_0 = (1-d_\delta)\xi$ and for $\varepsilon > 0$, let $\zeta = \min(\psi^+/\varepsilon,\ \zeta_0)$. We have $\zeta = 0$ on ∂D and $\zeta \geqslant 0$, then since $\operatorname{div}(b(z)\,\nabla\psi^+) \geqslant 0$, we get

$$\int_D b(z)\,\nabla\psi^+\,\nabla\zeta \leqslant 0$$

which can be written

$$\int_{D\cap[\psi^+ \geqslant \varepsilon\zeta_0]} b(z)\,\nabla\psi^+\,.\nabla\zeta_0 + \frac{1}{\varepsilon}\int_{D\cap[\psi^+ < \varepsilon\zeta_0]} b(z)\,\nabla\psi^+\,.\nabla\psi^+ \leqslant 0$$

then by (1.14),

$$\int_{D\cap[\psi^+ \geqslant \varepsilon\zeta_0]} b(z)\,\nabla\psi^+\,.\nabla\zeta_0 \leqslant 0\,.$$

Letting $\varepsilon \to 0$, we get (6.21).

Now, adding (6.20) and (6.21), we get

$$\int_D b(z)\,\nabla\psi^+\,.\nabla\xi + d_\delta\,\gamma e_x\,.\nabla\xi \leqslant 0\,. \tag{6.22}$$

(6.18) holds by letting $\delta \to 0$ in (6.22). ■

END OF PROOF OF THEOREM 6.1. – Let $z_* \in (-h^*, z_0)$.

1) If $\psi^+(\alpha', z_*) = 0$ then by monotonicity $\psi^+(x, z_*) = 0\ \forall x \geqslant \alpha'$.

2) If $\psi^+(\alpha', z_*) > 0$ then by continuity of ψ, there exists a small ball $B_r(\alpha', z_*)$ such that $\psi^+ > 0$ in $B_r(\alpha', z_*)$. Let us denote by C_* the connected component of $D(z_0) \cap [\psi^+ > 0]$ which contains $B_r(\alpha', z_*) \cap D(z_0)$. Let $\zeta \in \mathcal{D}(\mathbb{R}^2)$ with $\operatorname{supp}\zeta \subset \Delta = \{(x, z)/(\alpha+\alpha')/2 < x < \alpha',\ |z - z_*| < r\} \cup (C_* \cap$

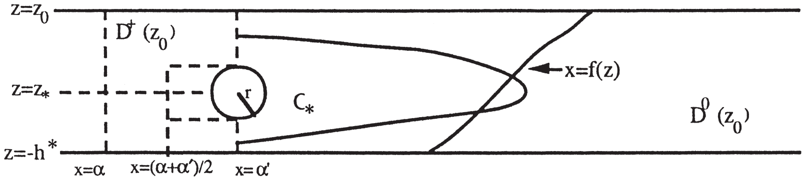

Figure 3

$D^+(z_0)$) (see Figure 3). Applying Lemma 6.4, we obtain:

$$\int_{C_* \cap D^+(z_0)} b(z)\, \nabla(\psi^+ - v)^+\, \nabla\zeta = 0 \tag{6.23}$$

since we integrate on $[\psi^+ > 0] \cap [v > 0]$, in which we have $\gamma = \theta = \gamma_s$.

Let us define a function w by:

$$w = \begin{cases} (\psi^+ - v)^+ & \text{in } C_* \cap D^+(z_0) \\ 0 & \text{in } \varDelta \setminus (C_* \cap D^+(z_0)) \end{cases},$$

then since $\psi^+(\alpha', z) \leq v(\alpha', z)\ \forall z \in (-h^*, z_0)$, it is clear that $w \in H^1(\varDelta)$ and from (6.23), we get:

$$\int_{\varDelta} b(z)\, \nabla w\, \nabla \zeta = 0 \qquad \forall \zeta \in \mathcal{D}(\varDelta). \tag{6.24}$$

Now, since $w \geq 0$ in $\varDelta$, $w = 0$ in $\varDelta \setminus (C_* \cap D^+(z_0))$, we deduce from (6.24) and the strong maximum principle that $w = 0$ in $\varDelta$ which leads to

$$\psi^+ \leq v \quad \text{in } \varDelta. \tag{6.25}$$

1) If $(f(z_*), z_*) \in C_*$, then $(f(z_*), z_*) \in \partial(C_* \cap D^+(z_0))$ and by (6.25) $\psi^+(f(z_*), z_*) \leq v(f(z_*), z_*) = 0$. So $\psi^+(f(z_*), z_*) = 0$ and then $\psi^+(x, z_*) = 0\ \forall x \geq f(z_*)$.

2) If $(f(z_*), z_*) \notin C_*$. Assume that $\psi^+(f(z_*), z_*) > 0$ then $\psi^+(x, z_*) > 0\ \ \forall x \leq f(z_*)$. Since $C_* \cap \big((\alpha', f(z_*)] \times \{z_*\}\big) \neq \emptyset$, we get $(\alpha', f(z_*)] \times \{z_*\} \subset C_*$ and we have a contradiction. Thus $\psi^+(f(z_*), z_*) = 0$ and $\psi^+(x, z_*) = 0\ \forall x \geq f(z_*)$.

Hence, we have proved that

$$\psi^+(x, z) = 0 \qquad \forall x \geq f(z),\ (x, z) \in D(z_0).$$

This means that $\psi(x, z) \leq 0$ in $D^0(z_0)$. We then have:

$$\operatorname{div}(b(z)\, \nabla\psi) = 0 \quad \text{in } \mathcal{D}'(D^0(z_0)).$$

By the asymptotic behaviour of ψ at $+\infty$ and the strong maximum principle, we deduce that:

$$\psi < 0 \quad \text{in } D^0(z_0).$$

Since we have $\psi > 0$ in $[z < -h^*] \cap \Omega$ and $\psi(x, -h^*) = 0$ for $x \geqslant \alpha'$, we get a contradiction with Lemma 4.3.

We conclude that $\Gamma \cap [z = -h^*] = \emptyset$ and $\Gamma = [x = g(z)]$. ■

COROLLARY 6.6. – *We have*:

$$\lim_{z \to -h^*, z > -h^*} g(z) = +\infty .$$

PROOF. – Let $A > 0$. By Theorem 6.1, we have $\psi(A, -h^*) > 0$. Since ψ is continuous, there exists $\delta > 0$ such that:

$$\forall z \in (-h^*, -h^* + \delta) \qquad \psi(A, z) > 0$$

this leads to:

$$\forall z \in (-h^*, -h^* + \delta) \qquad g(z) > A$$

which means that:

$$\lim_{z \to -h^*, z > -h^*} g(z) = +\infty . \qquad ■$$

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