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Partial Discretization of Topologies.

M. Bonanzinga - F. Cammaroto - M. V. Matveev

Sunto. – *In questo lavoro daremo una construzione che aumenta il numero di sottospazi chiusi e discreti dello spazio e daremo alcune applicazioni di tale construzione.*

1. – Introduction.

Many interesting examples of topological spaces were constructed by means of refining the topology of some well-known space. For example, in the construction of the Michael line (see [10], 5.1.32) one declares all irrational points of the real line isolated and thus obtains a stronger topology in which there are more open discrete subsets than in the original topology of $\mathbb{R}$.

In this paper we follow the opposite approach: we encrease the number of *closed* discrete subsets. The new topology add to the old one new closed discrete subspaces; for this reason we will speak about the «discretization» of the original topology. This approach is not new; let us recall two well-known examples. The first one is the simplest example of a Hausdorff, nonregular space, the second one provides the simplest construction of a Hausdorff L-space.

Example 1.1 ([10], 1.5.7). – The «usual» Hausdorff, nonregular space.

Consider the unit interval $[0, 1]$ with the topology inherated from the usual topology of the real line. Refine this topology on $[0, 1]$ declaring that basic neighbourhoods at points different from 0 are the same that in the original topology, while a basic neighbourhood of the point 0 takes the following form: $[0, \varepsilon)\setminus\{1/n : n = 1, 2, \ldots\}$, where $\varepsilon > 0$. The new space is a Hausdorff nonregular space. Note that with respect to the original topology, a new closed discrete subspace has been added.

Example 1.2 [22]. – A Hausdorff L-space.

Consider the space $X = \mathbb{R}$ with the topology generated by the base consisting of all sets of the form $(a, b)\setminus A$, where A is arbitrary countable set. This space X is hereditarily Lindelöf but not separable. That it is not separable follows directly from the definition, that it is (hereditarily) Lindelöf either can be easily seen directly, or it follows from Corollary 3.7 below.

Now we give the general definition.

DEFINITION 1.3. – *Let* $(X, \mathcal{T})$ *be a topological space,* $\mathcal{B}$ *be a base for* X, $Y \subset X$ *and* τ *be a cardinal. Denote* $\mathcal{B}_{Y\tau} = \{U\setminus A\colon U \in \mathcal{B}, A \subset Y, |A| < \tau\}$; *this family is a base for a new topology* $\mathcal{T}_{Y\tau}$ *on* X *that we will call the* partial τ-discretization *of* $\mathcal{T}$. *For* $Y = X$ *we write just* $\mathcal{T}_\tau$ *and we will say that* $\mathcal{T}_\tau$ *is the* τ-discretization *of* $\mathcal{T}$.

Further we will often call $\mathcal{T}$ and $\mathcal{T}_{Y\tau}$ (resp., $\mathcal{T}_\tau$) the *old* and the *new* topology, respectively. First of all, we note that the topology $\mathcal{T}_{Y\tau}$ does not depend on the base $\mathcal{B}$. In this sense the choise of $\mathcal{B}$ is not important, and we will often address to the elements of $\mathcal{B}_{Y\tau}$ at to *basic open sets* without indicating which particular base $\mathcal{B}$ is considered.

In the topology $\mathcal{T}_{Y\tau}$, all the sets $A \subset Y$ with $|A| < \tau$ are closed in X and discrete. In Example 1.1 we have that $Y = \{1/n\colon n = 1, 2, \ldots\}$ and $\tau = \omega_1$; in Example 1.2, $Y = X$ and $\tau = \omega_1$.

Also note that if $\tau > |X|$, then $(X, \mathcal{T}_\tau)$ is a discrete space. To avoid this trivial case, henceforward we usually assume that $\tau \leqslant \Delta(X)$, where

$$\Delta(X) = \min\{|U|\colon U \text{ is a nonempty open set in } X\}$$

is usually called the *dispersion character of* X. Note that for every infinite cardinal τ we have that $\Delta(X, \mathcal{T}) \geqslant \tau$ iff $\Delta(X, \mathcal{T}_\tau) \geqslant \tau$; for this reason we simply write $\Delta(X) \geqslant \tau$ without indicating which topology, $\mathcal{T}$ or $\mathcal{T}_\tau$, is considered.

Being refinements of the original topology, partial τ-discretization and τ-discretization of course preserve the Hausdorff axiom of separation. However, the regularity of a space typically is not preserved; from a regular or even a normal space $(X, \mathcal{T})$ we «nearly always» obtain a nonregular space $(X, \mathcal{T}_{Y\tau})$, where $Y \subset X$, or $(X, \mathcal{T}_\tau)$; why this happens can be easily seen from Examples 1.1 and 1.2. So, partial τ-discretization and τ-discretization are ways to obtain Hausdorff examples in the cases when constructing regular ones is impossible or difficult.

The paper is organized in the following way: we start with some basic facts about τ-discretizations, then we consider the behavior of various topological properties under the operation of τ-discretization, further we find out how the operation of τ-discretization interlaps with other operations on topological spaces, and we finish with a selection of applications.

2. – How much different is the new topology from the old one?

It turns out that it is different, but not very much different. This gives hope that preserving some properties of the old topology and destroying other properties one can obtain spaces with interesting sets of properties.

First, let us compare the closed sets in the old and in the new topology. Of course, in general, new closed sets do appear after a (partial) τ-discretization, but the following result shows that (under some assumptions) *regular closed (and hence regular open) sets in the old and in the new topologies are the same.*

PROPOSITION 2.1. – *Let τ be a cardinal, $(X, \mathcal{T})$ be a space and V be an element of $\mathcal{T}_\tau$. We have that*

(*A*) *if $\Delta(X) \geqslant \tau$, then $\mathrm{cl}_{\mathcal{T}_\tau} V = \mathrm{cl}_{\mathcal{T}} V = \mathrm{cl}_{\mathcal{T}} U$, for some $U \in \mathcal{T}$;*

(*B*) *if Y is a subset of X such that $X \setminus Y$ is dense in $(X, \mathcal{T})$, then $\mathrm{cl}_{\mathcal{T}_{Y\tau}} V = \mathrm{cl}_{\mathcal{T}} V = \mathrm{cl}_{\mathcal{T}} U$, for some $U \in \mathcal{T}$.*

PROOF. – (*A*): For every $y \in V$, fix $U_y \in \mathcal{T}$ and $A_y \subset X$, $|A_y| < \tau$ so that $U_y \setminus A_y \subset V$. Put $U = \cup\{U_y : y \in V\}$. Then $U \in \mathcal{T}$ and $U \supset V$. It is clear that $\mathrm{cl}_{\mathcal{T}_\tau} V \subseteq \mathrm{cl}_{\mathcal{T}} V \subseteq \mathrm{cl}_{\mathcal{T}} U$. It remains to check that $\mathrm{cl}_{\mathcal{T}} U \subseteq \mathrm{cl}_{\mathcal{T}_\tau} V$. Let $x \in \mathrm{cl}_{\mathcal{T}} U$ and let W be a basic neighbourhood of x in $\mathcal{T}_\tau$, i.e. $x \in W = O_W \setminus A_W$ where $O_W \in \mathcal{T}$ and $|A_W| < \tau$. Since $x \in \mathrm{cl}_{\mathcal{T}} U$, there is $z \in U \cap O_W$. By the definition of U, $z \in U_y \cap O_W$ for some $y \in V$. So, $U_y \cap O_W$ is a nonempty open (in the topology $\mathcal{T}$) set. Since $\Delta(X) \geqslant \tau$ and $|A_y \cup A_W| < \tau$, there is $t \in (U_y \cap O_W) \setminus (A_y \cup A_W) \subset (U_y \setminus A_y) \cap (O_W \setminus A_W) \subset V \cap W$. So, every neighbourhood of x in $\mathcal{T}_\tau$ intersects V, i.e. $x \in \mathrm{cl}_{\mathcal{T}_\tau} V$.

(*B*): The proof of this fact repeats the proof of case (*A*) almost wordwise. Now the hypothesis that $X \setminus Y$ is dense in $(X, \mathcal{T})$ is used instead of the condition that X has dispersion character greater or equal to τ and the set U is defined as follows: for every $x \in V$, we can fix the sets $U_x \in \mathcal{T}$ and $A_x \subset Y$, with $|A_x| < \tau$, such that $x \in U_x \setminus A_x \subset V$; then $U = \cup\{U_x : x \in V\}$. ∎

Recall that a space is *almost regular* if every point can be separated by open sets from every regular closed set not containing this point [26]; a space is *semiregular* if regular open sets form a base [10]; a T_1 space is regular if and only if it is both almost regular and semiregular [26]. Since (under our usual assumptions) regular closed sets in the old topology remain regular closed in the new topology, we have the following proposition:

PROPOSITION 2.2. – *Let τ be a cardinal, $(X, \mathcal{T})$ be a space and V be an element of $\mathcal{T}_\tau$. We have that*

(*A*) *if* $\Delta(X) \geqslant \tau$ *and* $(X, \mathcal{T})$ *is almost regular, then* $(X, \mathcal{T}_\tau)$ *is almost regular*;

(*B*) *if* Y *is a subset of* X *such that* $X \backslash Y$ *is dense in* $(X, \mathcal{T})$ *and* $(X, \mathcal{T})$ *is almost regular, then* $(X, \mathcal{T}_{Y\tau})$ *is almost regular.*

So we see that even though in general regularity is not preserved by τ-discretization and partial τ-discretization, a «part» of regularity, namely almost regularity, is preserved by these operations. In fact, regularity is destroyed by means of the loss of semiregularity: the regular open sets are (again under our usual assumptions) the same in the old and in the new topology, and since they formed a base for the old topology, they can not form a base for the new topology, because the new topology is, tipically, finer. In other words, in the case of the τ-discretization we have the following simple proposition (say that a space is τ-discrete if all its subsets of cardinality less than τ are closed and discrete):

PROPOSITION 2.3. – *Let* $\Delta(X) \geqslant \tau$. *Then either* $(X, \mathcal{T})$ *is* τ*-discrete* (*i.e.* $\mathcal{T} = \mathcal{T}_\tau$), *or* $(X, \mathcal{T}_\tau)$ *is not semiregular.*

We conclude the consideration of regular open sets and almost regularity with the example demonstrating that the assumptions «$\Delta(X) \geqslant \tau$» and «$X \backslash Y$ is dense in $(X, \mathcal{T})$» in Proposition 2.2 are essential.

EXAMPLE 2.4. – An almost regular space $(X, \mathcal{T})$ such that the ω_1-discretization $(X, \mathcal{T}_{\omega_1})$ is not almost regular.

By transfinite induction, it is easy to construct three disjoint dense subspaces $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ of the real line $\mathbb{R}$ with the usual topology $\mathcal{S}$ such that $\mathbb{R} = \mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \mathbb{Q}_3$ and $\Delta(\mathbb{Q}_i) = \boldsymbol{c}$, for $i = 1, 2, 3$. Since $(\mathbb{R}, \mathcal{S})$ is hereditarily separable, there exist countable and dense sets C_1, C_2, C_3 such that $C_i \subset \mathbb{Q}_i$, for $i = 1, 2, 3$. Define $S = C_1 \cup C_2 \cup C_3$ with the topology inhereted from $(\mathbb{R}, \mathcal{S})$ and let $\{a_n : n \in \omega\} \subset \mathbb{R}$ be a sequence of pairwise distinct points converging to 0 in the topology $\mathcal{S}$. Also consider $\mathbb{R}$ with the topology $\mathcal{S}^*$ generated by $\mathcal{S} \cup \{\mathbb{Q}_1\} \cup \{\mathbb{Q}_2\}$. Consider the set $X = (\mathbb{R} \times \{0\}) \cup (S \times \{a_n : n \in \omega\})$ with the following topology $\mathcal{T}$: $Y = S \times \{a_n : n \in \omega\}$ with the product topology $\mathcal{R}$ is open in X; a basic neighbourhood of the point $(q_1, 0) \in \mathbb{Q}_1 \times \{0\}$ takes the form $(((q_1 - \delta, q_1 + \delta) \cap \mathbb{Q}_1) \times \{0\}) \cup (((q_1 - \delta, q_1 + \delta) \cap S) \times \{a_m : m > n\})$, where $\delta > 0$ and $n \in \omega$; a basic neighbourhood of the point $(q_2, 0) \in \mathbb{Q}_2 \times \{0\}$ takes the form $(((q_2 - \delta, q_2 + \delta) \cap \mathbb{Q}_2) \times \{0\}) \cup (((q_2 - \delta, q_2 + \delta) \cap S) \times \{a_m : m > n\})$, where $\delta > 0$ and $n \in \omega$; a basic neighbourhood of the point $(q_3, 0) \in \mathbb{Q}_3 \times \{0\}$ takes the form $((q_3 - \delta, q_3 + \delta) \times \{0\}) \cup (((q_3 - \delta, q_3 + \delta) \cap S) \times \{a_m : m > n\})$, where $\delta > 0$ and $n \in \omega$. Note that $\Delta(X) = \omega$.

Now we prove that $(X, \mathcal{T})$ is almost regular. Since Y is an open subset of

$(X, \mathcal{T})$ and a dense subspace of the almost regular space $\mathbb{R} \times \{a_n : n \in \omega\}$, we have that $(X, \mathcal{T})$ is almost regular at all points of the form (x, a_n). Then we have to prove that X is almost regular at the point $(x, 0)$. Let $U \subset X$ be a regular open subset such that $(x, 0) \in U$; we want to find an open set $T \subset X$ such that $(x, 0) \in T \subseteq \mathrm{cl}_X(T) \subseteq U$. Assume that $(x, 0) \in \mathbb{Q}_3$. Then there exists $\delta > 0$ and $n \in \omega$ such that $(x, 0) \in ((x-\delta, x+\delta) \times \{0\}) \cup (((x-\delta, x+\delta) \cap S) \times \{a_m : m > n\}) \subseteq U$.

Define $T = \left(\left(x - \dfrac{\delta}{2},\ x + \dfrac{\delta}{2}\right) \times \{0\}\right) \cup \left(\left(\left(x - \dfrac{\delta}{2},\ x + \dfrac{\delta}{2}\right) \cap S\right) \times \{a_m : m > n\}\right)$. Then $(x, 0) \in T \subseteq \mathrm{cl}_X(T) = \left(\left[x - \dfrac{\delta}{2}, x + \dfrac{\delta}{2}\right] \times \{0\}\right) \cup \left(\left(\left[x - \dfrac{\delta}{2}, x + \dfrac{\delta}{2}\right] \cap S\right) \times \{a_m : m \geqslant n\}\right) \subset U$.

Now assume that $x \in \mathbb{Q}_1$; for $x \in \mathbb{Q}_2$ the proof is similar. There exist $\delta > 0$ and $n \in \omega$ such that $(x, 0) \in (((x-\delta, x+\delta) \cap \mathbb{Q}_1) \times \{0\}) \cup (((x-\delta, x+\delta) \cap S) \times \{a_m : m > n\}) \subseteq U$. Define $A = (((x-\delta, x+\delta) \cap \mathbb{Q}_1) \times \{0\}) \cup (((x-\delta, x+\delta) \cap S) \times \{a_m : m > n\})$. We have $\mathrm{cl}_X(A) = \mathrm{cl}_X(\mathrm{cl}_Y(A \cap Y)) = \mathrm{cl}_X(([x-\delta, x+\delta] \cap S) \times \{a_m : m \geqslant n\}) = ([x-\delta, x+\delta] \times \{0\}) \cup (([x-\delta, x+\delta] \cap S) \times \{a_m : m \geqslant n\})$. $\mathrm{Int}_X(\mathrm{cl}_X(A)) = ((x-\delta, x+\delta) \times \{0\}) \cup (((x-\delta, x+\delta) \cap S) \times \{a_m : m > n\}) \subseteq U$. Define the open set $T = ((x-\delta/2, x+\delta/2) \times \{0\}) \cup (((x-\delta/2, x+\delta/2) \cap S) \times \{a_m : m > n\})$. Then $(x, 0) \in T \subseteq \mathrm{cl}_X(T) = ([x-\delta/2, x+\delta/2] \times \{0\}) \cup (([x-\delta/2, x+\delta/2] \cap S) \times \{a_m : m \geqslant n\}) \subseteq U$.

Now consider the space $(X, \mathcal{T}_{\omega_1})$, and we show that this space is not almost regular. $\mathbb{R} \times \{0\}$ is clopen in $(X, \mathcal{T}_{\omega_1})$; indeed it was closed in $(X, \mathcal{T})$ and $X \setminus (\mathbb{R} \times \{0\})$ is countable. Since almost regularity is preserved by clopen subspaces, it remains to show that $\mathbb{R} \times \{0\}$ is not an almost regular subspace of $(X, \mathcal{T}_{\omega_1})$. Let $a, b \in \mathbb{Q}_3$, with $a < b$. Put $H = ([a, b] \setminus \mathbb{Q}_2) \times \{0\}$. Then H is a regular closed subset of $(\mathbb{R} \times \{0\}, \mathcal{T}_{\omega_1}|_{\mathbb{R} \times \{0\}})$. Then consider a point $x \in (a, b) \cap \mathbb{Q}_2$ and define $p = (x, 0)$. We have that p can not be separated by open sets from H. Indeed, since $\Delta(\mathbb{Q}_2) = \boldsymbol{c}$, the interesection of every neighbourhood of p with every neighbourhood of H in $(\mathbb{R} \times \{0\}, \mathcal{T}|_{\mathbb{R} \times \{0\}})$ has cardinality greater or equal to $\boldsymbol{c}$. Then the interesection of every neighbourhood of p with every neighbourhood of H in $(\mathbb{R} \times \{0\}, \mathcal{T}_{\omega_1}|_{\mathbb{R} \times \{0\}})$ is greater or equal to $\boldsymbol{c}$ and then nonempty.

Note that $X \setminus Y$ is closed in $(X, \mathcal{T})$ and $(X, \mathcal{T}_{Y\omega_1})$ is not almost regular. To prove this last fact it is enough to note that $X \setminus Y$ is a clopen subspace of $(X, \mathcal{T}_{Y\omega_1})$ which is not almost regular.

To finish the discussion of the axioms of separation, we note that $(X, \mathcal{T}_\tau)$ is a T_1 space for every topology $\mathcal{T}$ and every cardinal $\tau \geqslant 2$.

Now we will show that the old and the new topologies are similar yet from one more viewpoint: we will prove that, under our usual assumptions, the spaces $(X, \mathcal{T})$ and $(X, \mathcal{T}_\tau)$, respectively $(X, \mathcal{T})$ and $(X, \mathcal{T}_{Y\tau})$, have the same set of real valued continuous functions. Of course, the functions which were continuous in the old topology remain continuous in the new one. It is necessary to check that new continuous functions do not appear.

PROPOSITION 2.5. – *Let $(X, \mathcal{T})$ be a space and τ be a cardinal. We have that*

(*A*) *if $\Delta(X) \geqslant \tau$ and $f: (X, \mathcal{T}_\tau) \to \mathbb{R}$ is a continuous function, then f is a continuos function with respect to the old topology $\mathcal{T}$;*

(*B*) *if Y is a subset of X such that $X \setminus Y$ is dense in $(X, \mathcal{T})$ and $f: (X, \mathcal{T}_{Y\tau}) \to \mathbb{R}$ is a continuous function, then f is a continuos function with respect to the old topology $\mathcal{T}$.*

PROOF. – (*A*): Let $f: (X, \mathcal{T}_\tau) \to \mathbb{R}$ be a continuous function and $\{a_\sigma : \sigma \in \Sigma\}$ be a sequence converging to the point $a \in X$ in the sense of the old topology. We have to prove that $\lim_{\sigma \in \Sigma} f(a_\sigma) = f(a)$. By contradiction, assume that there exists $\varepsilon > 0$ and a cofinal subnet A of $\{a_\sigma : \sigma \in \Sigma\}$ such that $|f(a_\sigma) - f(a)| > \varepsilon$, for every $a_\sigma \in A$. Let W_a be a neighbourhood of a in $\mathcal{T}_\tau$. Now we show that there exists $a_\sigma \in A$ such that for every neighbourhood V_{a_σ} of it in $\mathcal{T}_\tau$ we have that $|W_a \cap V_{a_\sigma}| \geqslant \tau$. Consider the set $W_a = O_a \setminus H_a$, where $O_a \in \mathcal{T}$ and $H_a \subset X$, $|H_a| < \tau$. Then there exists $a_\sigma \in A$ such that $a_\sigma \in O_a$. Let V_{a_σ} be a neighbourhood of a_σ in the new topology; we assume that $V_{a_\sigma} = U_{a_\sigma} \setminus H_{a_\sigma}$ where $U_{a_\sigma} \in \mathcal{T}$ and $H_{a_\sigma} \subset X$, with $|H_{a_\sigma}| < \tau$. Then $W_a \cap V_{a_\sigma} = (O_a \setminus H_a) \cap (U_{a_\sigma} \setminus H_{a_\sigma}) = (O_a \cap U_{a_\sigma}) \setminus (H_a \cup H_{a_\sigma})$. Since $\Delta(X) \geqslant \tau$ and O_a and U_{a_σ} belong to $\mathcal{T}$, we have that $O_a \cap U_{a_\sigma}$ is a nonempty open set of $(X, \mathcal{T})$ and then it has cardinality $\geqslant \tau$; then, as both H_a and H_{a_σ} have cardinality $< \tau$, we conclude that $|W_a \cap V_{a_\sigma}| \geqslant \tau$. Since $f: (X, \mathcal{T}_\tau) \to \mathbb{R}$ is a continuous function, there exists a neighbourhood W_{a_σ} of a_σ in $(X, \mathcal{T}_\tau)$ such that $|f(y) - f(a_\sigma)| < \varepsilon/2$, for every $y \in W_{a_\sigma}$. Then there exists a point $z \in W_a \cap W_{a_\sigma}$ such that $|f(z) - f(a)| > \varepsilon/2$. By arbitrarity of W_a, we obtain a contradiction.

(*B*): The proof of this fact repeats proof for the case (*A*) almost wordwise. Now the hypothesis that $X \setminus Y$ is dense in $(X, \mathcal{T})$ is used instead of the condition that X has dispersion character greater or equal to τ to prove that for every neighbourhood W_a of a in $\mathcal{T}_{Y\tau}$ there exists a point $a_\sigma \in A$ such that for every neighbourhood V_{a_σ} of a_σ in $\mathcal{T}_{Y\tau}$ one has $W_a \cap V_{a_\sigma} \neq \emptyset$. ■

Below $C_p(X, \mathcal{T})$, $C_{co}(X, \mathcal{T})$ and $C_u(X, \mathcal{T})$ denote the space of all real continuous functions on $(X, \mathcal{T})$ with the topology of pointwise convergence, the compact-open topology and the topology of uniform convergence, respectively.

COROLLARY 2.6. – *Let* $(X, \mathcal{T})$ *be a space and* τ *a cardinal. We have that*

(*A*) *if* $\Delta(X) \geqslant \tau$, *then* $C_p((X, \mathcal{T}_\tau)) = C_p((X, \mathcal{T}))$; $C_{\mathrm{co}}((X, \mathcal{T}_\tau)) \approx C_p((X, \mathcal{T}))$ *and* $C_u((X, \mathcal{T}_\tau)) = C_u((X, \mathcal{T}))$.

(*B*) *if* Y *is a subset of* X *such that* $X \backslash Y$ *is dense in* $(X, \mathcal{T})$, *then* $C_p((X, \mathcal{T}_{Y\tau})) = C_p((X, \mathcal{T}))$ *and* $C_u((X, \mathcal{T}_{Y\tau})) = C_u((X, \mathcal{T}))$.

PROOF. – The proofs are obvious in the cases of the topology of pointwise convergence and of the topology of uniform convergence. In the case of compact-open topology it is enough to note that all compact sets in the τ-discretization are finite. ■

3. – Partial τ-discretization (τ-discretization) and...

3.1. – ...weak compactness-type properties.

Note that, if $\tau > \omega$, then all countable subsets of $(X, \mathcal{T}_\tau)$ are closed and discrete and hence $(X, \mathcal{T}_\tau)$ is *not countably compact*; further, if $\tau > \omega$ and $|Y| \geqslant \omega$, then all countable subsets of $(X, \mathcal{T}_{Y\tau})$ contained in Y are closed and discrete and hence $(X, \mathcal{T}_{Y\tau})$ is *not countably compact*. Then it is natural to ask which compactness-type properties weaker than countable compactness, such as pseudocompactness, feeble compactness, and H-closedness, are preserved by operations $\mathcal{T} \to \mathcal{T}_\tau$ and $\mathcal{T} \to \mathcal{T}_{Y\tau}$. Recall that a space X is pseudocompact provided all continuous real-valued functions are bounded; a space X is feebly compact (see, for example [25]) provided every locally finite family of nonempty open sets in X is finite. Feeble compactness, being, in general, a stronger property, is equivalent to pseudocompactness for Tychonoff spaces (see [10]). Further, recall that a Hausdorff space X is *H-closed* [20] iff for every open cover of X there is a finite subfamily whose union is dense in X.

That (under our usual assumptions) pseudocompactness is preserved by τ-discretizations and partial τ-discretizations is a direct corollary of Proposition 2.5:

PROPOSITION 3.1. – *Let* $(X, \mathcal{T})$ *be a space and* τ *be a cardinal. We have that*

(*A*) *if* $\Delta(X) \geqslant \tau$ *and* $(X, \mathcal{T})$ *is pseudocompact, then* $(X, \mathcal{T}_\tau)$ *is pseudocompact*;

(*B*) *if* Y *is a subset of* X *such that* $X \backslash Y$ *is dense in* $(X, \mathcal{T})$ *and* $(X, \mathcal{T})$ *is pseudocompact, then* $(X, \mathcal{T}_{Y\tau})$ *is pseudocompact.*

However, since $(X, \mathcal{T}_\tau)$ and $(X, \mathcal{T}_{Y\tau})$ usually are not regular, for non-Tychonoff spaces it is more natural to speak about feeble compactness than about pseudocompactness.

PROPOSITION 3.2. – *Let* $(X, \mathcal{T})$ *be a space and* τ *a cardinal. We have that*

(*A*) *if* $\Delta(X) \geqslant \tau$ *and* $(X, \mathcal{T})$ *is feebly compact, then* $(X, \mathcal{T}_\tau)$ *is feebly compact*;

(*B*) *if* Y *is a subset of* X *such that* $X \backslash Y$ *is dense in* $(X, \mathcal{T})$ *and* $(X, \mathcal{T})$ *is feebly compact, then* $(X, \mathcal{T}_{Y\tau})$ *is feebly compact.*

PROOF. – (*A*): Let $\xi = \{V_n : n \in \omega\}$ be a sequence of nonempty open sets in $(X, \mathcal{T}_\tau)$. We assume that all the sets V_n are elements of $\mathcal{B}_\tau$, i.e. there are $U_n \in \mathcal{T}$ and $A_n \subset X$, $|A_n| < \tau$ such that $V_n = U_n \backslash A_n$. Then the sequence $\eta = \{U_n : n \in \omega\}$ consists of nonempty open sets in $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is feebly compact, there exists a point $p \in X$ each neighbourhood of which in $(X, \mathcal{T})$ intersects infinitely many elements of η. We claim that every neighbourhood O of p in $(X, \mathcal{T}_\tau)$ intersects infinitely many elements of ξ. Without loss of generality we assume that $O \in \mathcal{B}_\tau$ is a basic neighbourhood of p in $(X, \mathcal{T}_\tau)$, i.e. $O = W_O \backslash B_O$ for some $W_O \in \mathcal{T}$ and some $B_O \subset X$, $|B_O| < \tau$. We know that W_O intersects infinitely many elements of η. Let U_{n^*} be one of them. We claim that O intersects corresponding $V_{n^*} = U_{n^*} \backslash A_{n^*}$; this will finish the proof. We have that $V_{n^*} \cap O = (U_{n^*} \backslash A_{n^*}) \cap (W_O \backslash B_O) = (U_{n^*} \cap W_O) \backslash (A_{n^*} \cup B_O)$. Since $\Delta(X) \geqslant \tau$ and $W_O \cap U_{n^*}$ is a nonempty open set of $(X, \mathcal{T})$ we have that $|W_O \cap U_{n^*}| \geqslant \tau$; further, as both the sets A_{n^*} and B_0 have cardinality $< \tau$, we have that $|A_{n^*} \cup B_O| < \tau$. Then we conclude that $V_{n^*} \cap O \neq \emptyset$.

(*B*): The proof of this fact repeats the proof of case (*A*) almost wordwise. Now the hypothesis that $X \backslash Y$ is dense in $(X, \mathcal{T})$ is used instead of the condition that X has dispersion character greater or equal to τ to prove that every neighbourhood O of p in $(X, \mathcal{T}_{Y\tau})$ intersects every $V_{n^*} = U_{n^*} \backslash A_{n^*}$, where in this case $A_{n^*} \subset Y$.

PROPOSITION 3.3. – *Let* $(X, \mathcal{T})$ *be a space and* τ *a cardinal. We have that*

(*A*) *if* $\Delta(X) \geqslant \tau$ *and* $(X, \mathcal{T})$ *is H-closed, then* $(X, \mathcal{T}_\tau)$ *is H-closed*;

(*B*) *if* Y *is a subset of* X *such that* $X \backslash Y$ *is dense in* $(X, \mathcal{T})$ *and* $(X, \mathcal{T})$ *is H-closed, then* $(X, \mathcal{T}_{Y\tau})$ *is H-closed.*

PROOF. – (*A*): Let $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $(X, \mathcal{T}_\tau)$. For every $\lambda \in \Lambda$ assume $V_\lambda = U_\lambda \backslash A_\lambda$, where $U_\lambda \in \mathcal{T}$ and $A_\lambda \subset X$, $|A_\lambda| < \tau$. The family $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of the H-closed space $(X, \mathcal{T})$; then there exists a finite subset

$\Lambda_0 \subset \Lambda$ such that $\mathrm{cl}_{\mathcal{T}}(\cup\{U_\lambda : \lambda \in \Lambda_0\}) = X$. By Proposition 2.1 ($A$), we have that $\mathrm{cl}_{\mathcal{T}}(\cup\{U_\lambda : \lambda \in \Lambda_0\}) = \mathrm{cl}_{\mathcal{T}_\tau}(\cup\{V_\lambda : \lambda \in \Lambda_0\})$ and hence $(X, \mathcal{T}_\tau)$ is H-closed.

(B): The proof of this fact repeats the proof of case (A) almost wordwise, using case (B) of Proposition 2.1 instead of case (A).

3.2. – ...Lindelöf-type properties.

Recall that the Lindelöf number $l(X)$ of a space X is the smallest cardinal τ such that every open cover of X has a subcover of cardinality $\leqslant \tau$. It is convenient to consider also the following cardinal function:

DEFINITION 3.4. – *Let X be a topological space.*

$l^*(X) = \min\{\tau$: *every open cover of X has a subcover of cardinality* $< \tau\}$.

Of course $(l(X))^+ \geqslant l^*(X) \geqslant l(X)$; further, $l^*(X) = l(X)^+$, if $l(X)$ is a non-limit cardinal, and $l^*(X)$ can equal either $l(X)$ or $l(X)^+$ if $l(X)$ is a limit cardinal.

PROPOSITION 3.5. – *Let τ be a regular cardinal. If $l^*((X, \mathcal{T})) \leqslant \tau$, then $l^*((X, \mathcal{T}_{Y\tau})) \leqslant \tau$ for every $Y \subset X$.*

PROOF. – Let $\mathcal{V}$ be an open cover of $(X, \mathcal{T}_{Y\tau})$. Without loss of generality we can assume that $\mathcal{V} \subset \mathcal{B}_\tau$. For each $V \in \mathcal{V}$ we fix the sets $U_V \in \mathcal{T}$ and $A_V \subset Y$, $|A_V| < \tau$ such that $V = U_V \backslash A_V$. Then $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$ is an open cover of $(X, \mathcal{T})$. Since $l^*((X, \mathcal{T})) \leqslant \tau$, $\mathcal{U}$ contains a subcover $\mathcal{U}_0$ of cardinality $< \tau$. Denote $A = \cup\{A_V : U_V \in \mathcal{U}_0\}$. Since $|A_V| < \tau$ for every $V \in \mathcal{V}$, $|\mathcal{U}_0| < \tau$ and τ is a regular cardinal, we have that $|A| < \tau$. Denote $\mathcal{V}_0 = \{V \in \mathcal{V} : U_V \in \mathcal{U}_0\}$; then $|\mathcal{V}_0| < \tau$. Put $H = X \backslash \cup \mathcal{V}_0$. Then we have $H \subset A$ and then $|H| < \tau$. For each $x \in H$ we choose $W_x \in \mathcal{V}$ so that $x \in W_x$. Then $\mathcal{V}_0 \cup \{W_x : x \in H\}$ is a subfamily of $\mathcal{V}$ of cardinality $< \tau$ that covers X. So $l^*((X, \mathcal{T}_\tau)) \leqslant \tau$. ■

COROLLARY 3.6. – *If $(X, \mathcal{T})$ is Lindelöf, then $(X, \mathcal{T}_{Y\omega_1})$ is Lindelöf for every subset $Y \subset X$.*

COROLLARY 3.7. – *If $(X, \mathcal{T})$ is Lindelöf, then $(X, \mathcal{T}_{\omega_1})$ is Lindelöf.*

Motivated by previous results, it is natural to ask which Lindelöf-type properties weaker than Lindelöfness – say, near-Lindelöfness, almost-Lindelöfness and weak-Lindelöfness, are preserved by operations $\mathcal{T} \to \mathcal{T}_\tau$ and $\mathcal{T} \to \mathcal{T}_{Y\tau}$. Recall the definitions: a space X is said to be *nearly-Lindelöf* [1] provided every cover of X by regular open sets admits a countable subcover; X is said to be *almost-Lindelöf* [28] provided for every open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ there exists a countable subset Λ_0 of Λ such that $X = \cup\{\mathrm{cl}\, U_\lambda : \lambda \in \Lambda_0\}$; X is said to be *weakly-Lindelöf* [9] provided for every open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ there exists a countable subset Λ_0 of Λ such that $X = \mathrm{cl}(\cup\{U_\lambda : \lambda \in \Lambda_0\})$. The following implications are true (see [7]): nearly-Lindelöf $\Rightarrow$ almost-Lindelöf $\Rightarrow$ weakly-Lindelöf.

Under our usual assumptions, near-Lindelöfness, almost-Lindelöfness and weak-Lindelöfness are preserved by operations $\mathcal{T} \to \mathcal{T}_\tau$ and $\mathcal{T} \to \mathcal{T}_{Y\tau}$ as the following propositions shows.

PROPOSITION 3.8. – *Let $(X, \mathcal{T})$ be a space and τ a cardinal. We have that*

(*A*) *if $\Delta(X) \geqslant \tau$ and $(X, \mathcal{T})$ is nearly-Lindelöf (almost-Lindelöf, weakly Lindelöf), then $(X, \mathcal{T}_\tau)$ is nearly-Lindelöf (almost-Lindelöf, weakly Lindelöf);*

(*B*) *if Y is a subset of X such that $X \setminus Y$ is dense in $(X, \mathcal{T})$ and $(X, \mathcal{T})$ is nearly-Lindelöf (almost-Lindelöf, weakly Lindelöf), then $(X, \mathcal{T}_{Y\tau})$ is nearly-Lindelöf (almost-Lindelöf, weakly Lindelöf).*

PROOF. – In case of near-Lindelöfness, the proof follows by definition because $(X, \mathcal{T})$ and $(X, \mathcal{T}_\tau)$ (respectively, $(X, \mathcal{T})$ and $(X, \mathcal{T}_{Y\tau})$) have the same regular open sets. In cases of almost- and weak Lindelöfness, the proofs are similar to the proof of Proposition 3.3. ■

So we see that, in some sense, near-, almost- and weak-Lindelöfness are better preserved by partial τ-discretization and by τ-discretization than Lindelöfness. Indeed, Lindelöfness is preserved only by partial ω_1-discretization: if we take a space $(X, \mathcal{T})$ such that $|X| > \omega$ and $\tau > \omega_1$, then after τ-discretization (and hence after partial τ-discretization) we obtain a space containing uncountable, closed and discrete subspaces which is impossible for Lindelöf spaces. On the other hand, Proposition 3.8 confirms that near-, almost- and weak-Lindelöfness are preserved by arbitrary τ-discretizations and partial τ-discretizations, the only condition is the restriction on the dispersion character and on the complement of the subspace Y, respectively. By means of a discretization, from a Lindelöf space one can easily construct a nearly-Lindelöf space with arbitrarily big extent. For example, consider the space $(X, \mathcal{T})$, where $X = D^\tau$ and $\mathcal{T}$ is the usual Tychonoff product topology on X. Let Y be a closed, nowhere dense subspace of $(X, \mathcal{T})$ of cardinality 2^τ. By Proposition 3.8 we have that the space $(X, \mathcal{T}_{Y(2^\tau)^+})$ is nearly-Lindelöf; while this space has the extent equal to 2^τ.

However, in another sense, Lindelöfness behaves better: in Proposition 3.5 we did not use the assumption about the dispersion character, while in Propositions 3.8 this assumption was essential as the following example demonstrates.

EXAMPLE 3.9. – A nearly-Lindelöf space the ω_1-discretization of which is not weakly-Lindelöf.

Let $\mathcal{T}$ be the usual topology on the real line $\mathbb{R}$ and $\mathbb{P}$ the set of irrationals. By Proposition 3.8, the space $(\mathbb{R}, \mathcal{T}_{\mathbb{P}_{\boldsymbol{c}^+}})$ is nearly-Lindelöf. The space $(\mathbb{R}, (\mathcal{T}_{\mathbb{P}_{\boldsymbol{c}^+}})_{\omega_1})$ is discrete and hence not weakly-Lindelöf.

3.3. – ...inverse compactness.

Recall that a space X is said to be *inversely compact* [14] provided for every open cover $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ of X there exist a finite subset B of A and a family $\mathcal{V} = \{V_\alpha \colon \alpha \in B\}$ such that for every $\alpha \in B$, either $V_\alpha = U_\alpha$ or $V_\alpha = X \setminus U_\alpha$ and $\mathcal{V}$ covers X.

The space of the Example 5.1 in [14] was the ω-modification of the space $(X, \mathcal{T})$, where $X = \omega_1$ and $\mathcal{T} = \{\emptyset\} \cup \{[0, \alpha) \colon \alpha \leqslant \omega_1\}$. It is trivial that $(X, \mathcal{T})$ is inversely compact and non-compact. It is proved in [14] that $(X, \mathcal{T}_\omega)$ is also inversely compact. Then it is natural to ask the following question

QUESTION 3.10. – *Let $(X, \mathcal{T})$ be an inversely compact space. Is $(X, \mathcal{T}_\omega)$ inversely compact?*

3.4. – ...other cardinal functions.

Let f be a cardinal function. It is easy to prove that if f is the cellularity or the density, then

$$f((X, \mathcal{T}_\tau)) \neq \max\{f((X, \mathcal{T})), \tau\}.$$

Indeed, consider the usual Isbell Mrówka Ψ-space $(X, \mathcal{T})$ where $X = \omega \cup \mathcal{R}$, $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \boldsymbol{c}$. We have that $d((X, \mathcal{T})) = c((X, \mathcal{T})) = \omega$; however, the space $(X, \mathcal{T}_{\omega_1})$ is discrete and so $d((X, \mathcal{T})) = c((X, \mathcal{T})) = |X| = \boldsymbol{c}$.

However, under our usual assumptions cellularity is preserved by (partial) τ-discretization:

PROPOSITION 3.11. – *Let $(X, \mathcal{T})$ be a space, Y be a subspace of X and τ be a cardinal.*

(A) If $\Delta(X) \geqslant \tau$, then $c((X, \mathcal{T}_\tau)) = c((X, \mathcal{T}))$.

(B) If $X \setminus Y$ is dense in $(X, \mathcal{T})$, then $c((X, \mathcal{T}_{Y_\tau})) = c((X, \mathcal{T}))$.

PROOF. – (A): We only have to prove that $c((X, \mathcal{T}_\tau)) \leqslant c((X, \mathcal{T}))$. Put $c((X, \mathcal{T})) = \lambda$. By contraposition, suppose there exists a family $\{A_\alpha\}_{\alpha \in \kappa}$, with $\kappa > \lambda$, of pairwise disjoint open sets in $(X, \mathcal{T}_\tau)$. We can assume that for every $\alpha \in \kappa$, $A_\alpha = U_\alpha \setminus C_\alpha$, where $U_\alpha \in \mathcal{T}$ and $C_\alpha \subset X$, with $|C_\alpha| < \tau$. Now we prove that for every α, $\alpha' \in \kappa$ such that $\alpha \neq \alpha'$, we have that $U_\alpha \cap U_{\alpha'} = \emptyset$; this will imply that $c((X, \mathcal{T})) > \lambda$, a contradiction. Suppose, by contradiction, that there exists α, $\alpha' \in \kappa$ such that $U_\alpha \cap U_{\alpha'} \neq \emptyset$. Since $U_\alpha \cap U_{\alpha'} \in \mathcal{T}$ and $\Delta(X) \geqslant \tau$, we have that $|U_\alpha \cap U_{\alpha'}| \geqslant \tau$. So, as C_α and $C_{\alpha'}$ have cardinality less than τ, we have that $|(U_\alpha \cap U_{\alpha'}) \setminus C_\alpha| \geqslant \tau$ and $|(U_\alpha \cap U_{\alpha'}) \setminus C_{\alpha'}| \geqslant \tau$; then $A_\alpha \cap A_{\alpha'} \neq \emptyset$, a contradiction.

(B): Since cellularity is hereditary with respect to dense subspaces and $X \setminus Y$ is dense in both topologies $\mathcal{T}$ and $\mathcal{T}_{Y\tau}$, we have the following equalities: $c((X, \mathcal{T})) = c((X \setminus Y, \mathcal{T}|_{X\setminus Y})) = c((X\setminus Y, \mathcal{T}_{Y\tau}|_{X\setminus Y})) = c((X, \mathcal{T}_{Y\tau}))$. ■

Also we have the following

PROPOSITION 3.12. – *Let* $(X, \mathcal{T})$ *be a topological space and* Y *be a subset of* X. *If every point of* $\mathrm{cl}_{\mathcal{T}} Y$ *is a complete accumulation point of some set of cardinality* τ *in* $(X, \mathcal{T})$, *then* $d((X, \mathcal{T}_{Y\tau})) \leqslant \max\{d((X, \mathcal{T})), \tau\}$.

PROOF. – Let A be a dense subset of $(X, \mathcal{T})$ such that $|A| = d((X, \mathcal{T}))$. For each $x \in A \cap \mathrm{cl}\, Y$, fix a set $B_x \subset X$ such that $|B_x| = \tau$ and x is a complete accumulation point of B_x in $(X, \mathcal{T})$; for each $x \in A \cap (X \setminus \mathrm{cl}_{\mathcal{T}} Y)$, put $B_x = \{x\}$. Consider the set $B = \cup\{B_x \colon x \in A\}$. Then $|B| \leqslant \tau |A|$. B is dense in $(X, \mathcal{T}_{Y\tau})$. Indeed, let $V \in \mathcal{B}_{Y\tau}$ be a nonempty set. Then there exist $U_V \in \mathcal{T}$ and $A_V \subset Y$ such that $|A_V| < \tau$ and $V = U_V \setminus A_V$. Since A is dense in $(X, \mathcal{T})$, there exists a point $x \in U_V \cap A$. Then, if $x \in U_V \cap (A \cap \mathrm{cl}_{\mathcal{T}} Y)$, we have that x is a complete accumulation point of the set B_x in $(X, \mathcal{T})$ and then $|U_V \cap B_x| = |B_x| = \tau$; so, since $|U_V \setminus V| < \tau$, there exists a point $z \in (U_V \cap B_x) \setminus (U_V \setminus V) = V \cap B_x \subset V \cap B$, that is $V \cap B \neq \emptyset$. If $x \in U_V \cap (A \cap (X \setminus \mathrm{cl}_{\mathcal{T}} Y))$, we have that $x \in B \cap U_V$ and, since $x \in X \setminus \mathrm{cl}_{\mathcal{T}} Y$, we have that $x \notin A_V$. So $x \in B \cap V$, that is $B \cap V \neq \emptyset$. ■

In particular

COROLLARY 3.13. – *Let* $(X, \mathcal{T})$ *be a topological space. If every point of* $(X, \mathcal{T})$ *is a complete accumulation point of some set of cardinality* τ, *then* $d((X, \mathcal{T}_\tau)) = \max\{d((X, \mathcal{T})), \tau\}$.

COROLLARY 3.14. – *Let* $(X, \mathcal{T})$ *be a topological space and* Y *a subset of* X. *If* $\Delta(X) \geqslant \tau$ *and* $\chi(x, (X, \mathcal{T})) \leqslant \tau$, *for every* $x \in \mathrm{cl}_{\mathcal{T}} Y$, *then* $d((X, \mathcal{T}_{Y\tau})) \leqslant \max\{d((X, \mathcal{T})), \tau\}$.

PROOF. – It is enough to check that every point $x \in \mathrm{cl}_{\mathcal{T}} Y$ is a complete accumulation point of some set of cardinality τ in $(X, \mathcal{T})$. Fix a local base $\mathcal{B}_x$ of $(X, \mathcal{T})$ at x such that $|\mathcal{B}_x| \leqslant \tau$. For each $U \in \mathcal{B}_x$, fix a subset $A_U \subset U$ such that $|A_U| = \tau$. Put $A = \cup\{A_U: U \in \mathcal{B}_x\}$. Then $|A| = \tau$ and x is a complete accumulation point of A in $(X, \mathcal{T})$.

COROLLARY 3.15. – *Let* $(X, \mathcal{T})$ *be a topological space. If* $\Delta(X) \geqslant \tau$ *and* $\chi((X, \mathcal{T})) \leqslant \tau$, *then* $d((X, \mathcal{T}_\tau)) = \max\{d((X, \mathcal{T})), \tau\}$.

4. – τ-discretization and other operations.

In this section we consider how the operation of the (partial) τ-discretization commutates with other operations on topological spaces.

4.1. – Subspaces.

Let Y and Z be subspaces of a space X, and let $\mathcal{T}$ be a topology on X. Then

$$(Z, (\mathcal{T}|_Z)_\tau) = (Z, \mathcal{T}_\tau|_Z),$$

and more general,

$$(Z, (\mathcal{T}|_Z)_{Y \cap Z\tau}) = (Z, \mathcal{T}_{Y\tau}|_Z).$$

This means that the result of the two consequent operations, (partial) τ-discretization and taking a subspace, does not depend on the order in which we do this operations. With operations other than taking a subspace, in general, it is not so.

4.2. – Products.

Let X and Y be two sets, $\mathcal{T}$ and $\mathcal{S}$ topologies on X and Y, respectively, and let $\mathcal{T} \otimes \mathcal{S}$ denote the Tychonoff product topology. Is it true that

$$(X \times Y, (\mathcal{T} \otimes \mathcal{S})_\tau) \approx (X, \mathcal{T}_\tau) \times (Y, \mathcal{S}_\tau)\ ?$$

Typically, it is not. The reason can be seen from the following simple example: let $X = Y = \mathbb{R}$, $\mathcal{T} = \mathcal{S}$ is the usual topology of the real line, and $\tau = \omega_1$. Then the set $(X \setminus \mathbb{Q}) \times Y$ is open in the space $(X, \mathcal{T}_\tau) \times (Y, \mathcal{S}_\tau)$, but it is not open in the space $(X \times Y, (\mathcal{T} \otimes \mathcal{S})_\tau)$.

4.3. – Hyperspace.

Now it is reasonable to ask the question similar to the one we have just answered for products about the hyperspace. Which topology is stronger: the Vietoris topology on the hyperspace of the space to which the τ-discretization has been applied, or the τ-discretization of the Vietoris topology? However, stated like this, the question is formally non correct because after the τ-discretization we, in general, obtain new closed sets, so the two topologies are defined on two different groundsets (recall that the Vietoris topology is defined on the set of all closed subsets of the space). However, if the consideration is restricted to the set of all *finite* subsets (let us assume that the space is T_1, so that finite subsets are closed), then the question becomes reasonable.

Let $(X, \mathcal{T})$ be a space, $J(X)$ denote the set of all nonempty finite subsets of X, and $\mathcal{E}(\mathcal{T})$ be the Vietoris topology on $J(X)$, i.e. the topology generated by the base consisting of all sets of the following form (as usual we use «F» or «(F)» to denote a subset of X or a point of the corresponding set $J(X)$, respectively)

$$\langle U_1, \dots, U_n \rangle = \{(F) \in J(X) \colon F \subset U_1 \cup \dots \cup U_n, \quad F \cap U_1 \neq \emptyset, \dots, F \cap U_n \neq \emptyset\}$$

where $U_1, \dots U_n \in \mathcal{T}$. Similary, $\mathcal{E}(\mathcal{T}_\tau)$ also is the Vietoris topology, but now $U_1, \dots, U_n \in \mathcal{T}_\tau$.

PROPOSITION 4.1. – $(\mathcal{E}(\mathcal{T}))_\tau \subset \mathcal{E}(\mathcal{T}_\tau)$ *whenever* $\mathcal{T}$ *is a Hausdorff topology.*

PROOF. – It is enough to show that every basic open set in the topology $(\mathcal{E}(\mathcal{T}))_\tau$ is open in the topology $\mathcal{E}(\mathcal{T}_\tau)$. Let $\mathcal{U} = \langle U_1, \dots U_n \rangle \setminus A \in (\mathcal{E}(\mathcal{T}))_\tau$ and let $(F) \in \mathcal{U}$. We have to find a basic $\mathcal{E}(\mathcal{T}_\tau)$-neighbourhood, $\mathcal{V}$, of (F) contained in $\mathcal{U}$. We have $F = \{x_1, \dots, x_m\}$ for some m. There are pairwise disjoint neighbourhoods $\mathcal{T} \ni V_i \ni x_i$ for $i = 1, \dots m$ such that each V_i is contained in some $U_{j(i)}$. Put $B = \cup\{H \setminus F \colon (H) \in A\}$, then $|B| < \tau$. Further, put $\mathcal{V} = \langle V_1 \setminus B, \dots, V_m \setminus B \rangle$. $\mathcal{V}$ is a basic open set of $\mathcal{E}(\mathcal{T}_\tau)$ and $(F) \in \mathcal{V}$ since $x_1 \in V_1 \setminus B, \dots, x_m \in V_m \setminus B$. Further, $\mathcal{V} \subset \langle U_1, \dots, U_n \rangle$ since each V_i is in some $U_{j(i)}$ and for each U_j there is a point $x_{i(j)} \in V_{i(j)}$ contained in U_j. It remains to check that $\mathcal{V} \cap A = \emptyset$. Let $(H) \in A$. Since $(F) \in \mathcal{U}$, we have $F \neq H$, and so there are only two possibilities:

Case 1. $H \setminus F \neq \emptyset$. Then $H \cap B \neq \emptyset$, and hence $(H) \notin \mathcal{V}$.

Case 2. H is a proper subset of F. Then $(H) \notin \mathcal{V}$ since $\mathcal{V}$ consists of the points corresponding to $\geqslant m$-point subsets of X. ∎

EXAMPLE 4.2. – The two topologies are different.

Let $\tau = \omega_1$, $X = \mathbb{R}$ and let $\mathcal{T}$ be the usual topology of $\mathbb{R}$. Then the set of the irrationals, $\mathbb{P}$, is open in the topology $\mathcal{T}_\tau$, so $\langle \mathbb{P} \rangle \in \mathcal{E}(\mathcal{T}_\tau)$. Note that $\langle \mathbb{P} \rangle = \{(F) \colon \emptyset \neq F \subset \mathbb{P}\}$. Now we show that $\langle \mathbb{P} \rangle \notin (\mathcal{E}(\mathcal{T}))_\tau$. Pick $x \in \mathbb{P}$, then $(\{x\}) \in \langle \mathbb{P} \rangle$. We check that no $(\mathcal{E}(\mathcal{T}))_\tau$-neighbourhood of $(\{x\})$ is contined in $\mathbb{P}$. It is enough to consider a basic neighbourhood of the form $\langle U \rangle \setminus A$ where $x \in U \in \mathcal{T}$ and $|A| < \tau$. Note that $|\langle U \rangle \setminus \langle \mathbb{P} \rangle| = \boldsymbol{c}$ since U contains

$\boldsymbol{c}$ many points (K) with $K \cap \mathbb{Q} \neq \emptyset$. So $\langle U \rangle \backslash \langle \mathbb{P} \rangle \not\subset A$ and thus $\langle U \rangle \backslash A \not\subset \langle \mathbb{P} \rangle$.

By a slight modification of this argument, one can show even some more: $\mathrm{Int}_{(\mathcal{E}(\mathcal{T}))_\tau}(\langle \mathbb{P} \rangle) = \emptyset$.

5. – Some applications.

In this section we observe that several constructions known in the literature are in fact special cases of τ-discretizations. Also we give some new applications.

5.1. – The extent of Hausdorff separable spaces.

By means of the τ-discretization, one can show that the extent of a Hausdorff, separable space can be as big as 2^c (clearly it can not be bigger, because the cardinality can not be bigger). Let $\mathcal{T}$ be the usual Tychonoff product topology of $X = D^c$ and Y a dense, countable subspace of D^c. Then $(X, \mathcal{T}_{X \backslash Y (2^c)^+})$ is a Hausdorff, separable space (Y remains dense), and $X \backslash Y$ is a closed discrete subspace of cardinality 2^c.

The same result can be obtained in another way: let $\mathcal{T}$ be the usual topology of $\beta\omega$, then $\kappa\omega = (\beta\omega, \mathcal{T}_{\beta\omega \backslash \omega (2^c)^+})$ is nothing else but the Katetov extension of ω (see [10], 3.12.6), the set $\beta\omega \backslash \omega$ has cardinality 2^c and is closed in $\kappa\omega$ and discrete.

5.2. – A pseudocompact (a)-space which is not countably compact.

A space X is absolutely countably compact [15] — briefly *acc* — if for every open cover $\mathcal{U}$ of X and every dense subspace Y of X there exists a finite subset $A \subset Y$ such that $St(A, \mathcal{U}) = X$. Recall (see [8]) that a Hausdorff space is countably compact iff it is *starcompact* i.e. for every open cover $\mathcal{U}$ of X, there exists a finite subset $A \subset X$ such that $St(A, \mathcal{U}) = X$. Clearly, a T_1 space X is *acc* iff it is countably compact and satisfies the following property (a) ([18], see also [23])

(a) for every open cover $\mathcal{U}$ of X and every dense subspace $Y \subset X$ there exists a closed in X and discrete subset $A \subset Y$ such that $St(A, \mathcal{U}) = X$.

A space with property (a) is said to be an (a) space.

Note that a property which from the first glance is just a little bit weaker than (a) — for every open cover $\mathcal{U}$ of X there exists a discrete subset $A \subset X$ such that $St(A, \mathcal{U}) = X$ — is, in fact, a property of almost every topological space! (see [18]).

Since (a) together with countable compactness gives a property considerably stronger than countable compactness (see [15], [16], [19], [27], [3], [4]) it is natural to ask what is (a) taken together with compactness-type properties weaker than countable compactness – say, pseudocompactness or feebly compactness for non-Tychonoff spaces.

QUESTION 5.1. – *When is a pseudocompact,* (*a*) *space countably compact* (*and hence acc*)?

First of all: are there pseudocompact (feebly compact) non-countably compact (*a*) spaces at all? The answer is «yes» as demonstrates the following proposition and the example below (a space X is *absolutely star-Lindelöf* — briefly *a-star-Lindelöf* — [5] if for every open cover $\mathcal{U}$ and every dense subspace Y there is a countable subset $A \subset Y$ such that $St(A, \mathcal{U}) = X$).

PROPOSITION 5.2. – *A space X is a counterexample* (*i.e. pseudocompact or feebly compact,* (*a*) *and not countably compact*) *if it has the following properties*:

(1) *X is pseudocompact* (*or feebly compact*),

(2) *every countable subset of X is closed and discrete, and*

(3) *X is absolutely-star-Lindelöf.*

PROOF. – Let X be a space with properties (1), (2), (3). By (2) it is not countably compact, and it remains to check that it is an (*a*) space. Let $\mathcal{U}$ be an open cover of X and Y be a dense subspace of X. By (3) there exists a countable subset $A \subset Y$ such that $St(A, \mathcal{U}) = X$. By (2) A is closed and discrete in X. ■

A Hausdorff counterexample can be constructed quite easy by means of the τ-discretization:

EXAMPLE 5.3. – A Hausdorff, feebly compact (*a*) space which is not countably compact.

We take the unite interval I with its usual topology $\mathcal{T}$ and consider the space $(I, \mathcal{T}_{\omega_1})$. We have to verify that $(I, \mathcal{T}_{\omega_1})$ has the properties (1), (2), (3). Property (2) holds by the definition of topology $\mathcal{T}_{\omega_1}$.

Since $(I, \mathcal{T})$ is Lindelöf, by Corollary 3.7 we have that $(I, \mathcal{T}_{\omega_1})$ is Lindelöf; then $(I, \mathcal{T}_{\omega_1})$ is a-star-Lindelof and (3) is checked.

To verify property (1) we note that $(I, \mathcal{T})$ is a feebly compact space with dispersion character ω_1. Then, by Proposition 3.2 $(I, \mathcal{T}_{\omega_1})$ is feebly compact. ■

The next proposition is a partial answer to Question 5.1. Recall that a subspace $Y \subset X$ is relatively countably compact in X if every infinite subset of Y has a cluster point in X.

PROPOSITION 5.4. – *If a Hausdorff* (*a*) *space X has a dense, relatively countably compact subspace, then X is countably compact* (*and hence acc*).

PROOF. – Let Y be a dense relatively countably compact subspace of X and let $\mathcal{U}$ be an open cover of X. By (a), there exists a closed in X, discrete subspace $A \subset Y$ such that $St(A, \mathcal{U}) = X$. Since Y is relatively countably compact in X, the set A is finite and thus X is starcompact. Since X is Hausdorff, it is countably compact. ■

Every space having a dense, relatively countably compact subspace is feebly compact, and it was an open problem whether every feebly compact space has such a subspace until a counterexample was constructed in [2]. Later, there appeared a series of examples [11], [24], [17], [21] of pseudocompact Tychonoff spaces with property (2) (and different sets of other nice properties including property (2) from Proposition 5.2). May be, a Tychonoff counterexample can be found among these spaces.

5.3. – A centered-Lindelöf space which is not star-Lindelöf.

A space is centered-Lindelöf provided every open cover has a σ-centered subcover. It is easy to see that every star-Lindelöf space is centered-Lindelöf. The converse implication is not true [5], [6]. One of the ways to construct a counterexample is to apply partial $\boldsymbol{c}$-discretization to a «fat Ψ» space obtained by «swelling» the isolated points of the Isbell-Mrowka Ψ-space into the segments of the real line [5].

5.4. – A space without a dense zero-dimensional subspace

The construction of a Tychonoff space without a dense zero-dimensional subspace has proved to be quite a difficult problem [12]. However, a Hausdorff example can be obtained quite easy by means of τ-discretizations [13]: one can just take the $\boldsymbol{c}$-discretization of the usual real line.

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