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J. C. Rosales, P. A. García-Sánchez

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Reduced commutative monoids with two Archimedean components.

J. C. Rosales - P. A. García-Sánchez (*)

Sunto. – *Si studiano i monoidi commutativi ridotti con due componenti archimedee e si forniscono dei teoremi di strutture. Si presta particolare attenzione a quei monoidi che sono finitamente generati, e si danno degli algoritmi che permettono di ottenere informazioni a partire da un delle loro presentazioni.*

1. – Introduction.

Every Archimedean commutative monoid is an Abelian group. We study commutative monoids all of whose elements except the identity are Archimedean. We call this kind of monoids quasi-Archimedean monoids and pay special attention to those which are finitely generated.

The contents of this paper are organized as follows. In the preliminary section we recall the concepts and results required to develop the statements presented later. In Section 3 we give a characterization of the presentations of finitely generated quasi-Archimedean monoids. This characterization yields an algorithmic method for deciding whether a monoid given by a presentation is quasi-Archimedean. Section 4 is devoted to providing a procedure for deciding whether a quasi-Archimedean monoid given by a presentation is cancellative. In Section 5 we present a structure theorem for quasi-Archimedean separative monoids, which states that they are identity extensions of Abelian groups or Tamura's $\mathcal{N}$-semigroups. Applying this theorem to the finitely generated case we obtain algorithms for determining whether a finitely generated commutative semigroup is a $\mathcal{N}$-semigroup or an Abelian group, and for deciding whether a finitely generated commutative monoid given by a presentation is a quasi-Archimedean separative monoid.

2. – Preliminaries.

All semigroups and monoids considered here are commutative, whence we will suppress the adjective commutative every time we refer to a semigroup or

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to a monoid. The binary operation defined over any semigroup is denoted by $+$ and if it has an identity element, then we denote it by 0.

Given an element $a \in \mathbb{N}^p$, the *Apéry set* of a in $\mathbb{N}^p$ is the set

$$\mathrm{ap}(\mathbb{N}^p, a) = \{ b \in \mathbb{N}^p \mid b - a \notin \mathbb{N}^p \}.$$

For $a_1, \ldots, a_k \in \mathbb{N}^p$, we denote the set $\bigcap_{i=1}^{k} \mathrm{ap}(\mathbb{N}^p, a_i)$ by $\mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_k)$.

Let σ be a congruence on $\mathbb{N}^p$. The *σ-class* of $a \in \mathbb{N}^p$ is the set

$$[a] = \{ b \in \mathbb{N}^p \mid (a, b) \in \sigma \}.$$

Let $\leqslant$ a linear admissible order on $\mathbb{N}^p$ (an order such that for every $a, b \in \mathbb{N}^p$ either $a \leqslant b$ or $b \leqslant a$, and so that if $a \leqslant b$, then $a + c \leqslant b + c$ for all $a, b, c \in \mathbb{N}^p$ and $0 \leqslant a$ for all $a \in \mathbb{N}^p$). Since $[a]$ is a nonempty subset of $\mathbb{N}^p$, the set of minimal elements of $[a]$ is finite (Dickson's lemma), which implies that the minimum of $[a]$ with respect to $\leqslant$ exists. The *function minimum* associated to σ with respect to $\leqslant$ is defined by

$$\mu : \mathbb{N}^p \to \mathbb{N}^p, \qquad \mu(x) = min_{\leqslant}([x]).$$

Let ϱ be a subset of $\mathbb{N}^p \times \mathbb{N}^p$. The *congruence generated* by ϱ, denoted by $\langle \varrho \rangle$, is the least (with respect to inclusion) congruence on $\mathbb{N}^p$ containing ϱ. The next result shows that $\langle \varrho \rangle$ always exists (see [1, 5]).

PROPOSITION 1. – *Let ϱ be a subset of $\mathbb{N}^p \times \mathbb{N}^p$ and*

$$\varrho^{-1} = \{ (a, b) \mid (b, a) \in \varrho \}, \Delta(\mathbb{N}^p) = \{ (a, a) \mid a \in \mathbb{N}^p \}.$$

Define

$$\varrho_0 = \varrho \cup \varrho^{-1} \cup \Delta(\mathbb{N}^p), \qquad \varrho_1 = \{ (v + u, w + u) \mid (v, w) \in \varrho_0, u \in \mathbb{N}^p \}.$$

Then $\langle \varrho \rangle$ is the set of pairs $(v, w) \in \mathbb{N}^p \times \mathbb{N}^p$ such that there exist $k \in \mathbb{N}$ and $v_0, \ldots, v_k \in \mathbb{N}^p$ with $v_0 = v$, $v_k = w$ and $(v_i, v_{i+1}) \in \varrho_1$ for all $0 \leqslant i \leqslant k - 1$.

If $\sigma = \langle \varrho \rangle$, we say that ϱ is a system of generators of σ.

A subset $\varrho = \{ (a_1, b_1), \ldots, (a_r, b_r) \}$ of $\mathbb{N}^p \times \mathbb{N}^p$ is reduced with respect to $\leqslant$ if

1. $b_i < a_i$,
2. $a_i \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$,
3) $b_i \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_r)$,

for every $1 \leqslant i \leqslant r$. In this case we can define the map $NF_\varrho : \mathbb{N}^p \to \mathbb{N}^p$ recurrently by

1. if $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_r)$, then $NF_\varrho(x) = x$,

2. if $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_i)$ and $x \notin \mathrm{ap}(\mathbb{N}^p, a_{i+1})$, then $NF_\varrho(x) = NF_\varrho(x - a_{i+1} + b_{i+1})$.

A finite reduced subset ϱ of $\mathbb{N}^p \times \mathbb{N}^p$ is a *canonical system of generators* of σ with respect to a linear admissible order $\preccurlyeq$ if

1. $\langle \varrho \rangle = \sigma$,
2. for all $x \in \mathbb{N}^p$, we have that $\mathrm{NF}_\varrho(\mathrm{x}) = \mu(\mathrm{x})$.

From every system of generators of a congruence σ we can obtain, applying the Knuth-Bendix algorithm, a canonical system of generators of σ with respect to a given linear admissible order. Furthermore, if $\varrho = \{(a_1, b_1), \ldots, (a_t, b_t)\}$ is a canonical system of generators of σ, then $\mathrm{Im}(\mu) = \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$ and $(a, b) \in \sigma$ if and only if $NF_\varrho(a) = NF_\varrho(b)$(see [6] for details).

Let A be a monoid generated by $\{n_1, \ldots, n_p\}$. Define the map

$$\varphi: \mathbb{N}^p \to A\,, \qquad \varphi(x_1, \ldots, x_p) = \sum_{i=1}^{p} x_i n_i .$$

Then A is isomorphic to $\mathbb{N}^p/\sigma$, where σ is the kernel congruence of A. Rédei shows in [5] that every congruence of $\mathbb{N}^p$ is finitely generated. Thus there exists a finite subset ϱ of σ for which $\langle \varrho \rangle = \sigma$. We will refer to ϱ as a *presentation* of A.

The monoid A is *cancellative* if for all $a, b, c \in A$, $a + c = b + c$ implies $a = b$. Let σ be a congruence on $\mathbb{N}^p$. Define

$$M_\sigma = \{a - b \mid (a, b) \in \sigma\} \subseteq \mathbb{Z}^p,$$

where $a - b$ denotes the subtraction in $\mathbb{Z}^p$ performed componentwise. Since σ is a congruence, it follows easily that M_σ is a subgroup of $\mathbb{Z}^p$. Conversely, given a subgroup H of $\mathbb{Z}^p$, define the binary relation

$$\sim_H = \{(a, b) \in \mathbb{N}^p \times \mathbb{N}^p \mid a - b \in H\}.$$

Clearly, $\sim_H$ is a congruence on $\mathbb{N}^p$. In [5] the following two results, showing the relationship between σ, $\sim_{M_\sigma}$ and the property of being cancellative, are proved.

LEMMA 2. – *Let σ be a congruence on $\mathbb{N}^p$.*

1. $\sigma \subseteq \sim_{M_\sigma}$.
2. *For every $(a, b) \in \sim_{M_\sigma}$, there exists $c \in \mathbb{N}^p$ such that $(a+c, b+c) \in \sigma$.*

PROPOSITION 3. – *Let σ be a congruence on $\mathbb{N}^p$. The monoid $\mathbb{N}^p/\sigma$ is cancellative if and only if $\sigma = \sim_{M_\sigma}$.*

A monoid A is *reduced* if its only unit is the identity element, that is, if $a, b \in A$ and $a + b = 0$, then $a = b = 0$. The following characterization of reduced finitely generated monoids in terms of its presentation appears in [7].

Let σ be a congruence on $\mathbb{N}^p$ and ϱ be a canonical system of generators of σ. Let e_i be the element of $\mathbb{N}^p$ all of whose coordinates are equal to 0 except the i-th which is equal to 1. Note that if $(e_i, 0) \in \sigma$, then $(e_i, 0)$ must be in ϱ (this does not hold for systems of generators in general, ϱ must be a canonical system of generators). Observe also that if this is the case, then the i-th coordinate of all a_j (different from e_i) and b_j must be zero (since ϱ is reduced). We can construct from ϱ a subset ϱ' of $\mathbb{N}^{p-1} \times \mathbb{N}^{p-1}$ by eliminating from ϱ the element $(e_i, 0)$ and suppressing the i-th coordinate of the rest of the elements in ϱ. Define σ' by

$$\sigma' = \{((x_1, \ldots, x_{k-1}), (y_1, \ldots, y_{k-1})) \in \mathbb{N}^{p-1} \times \mathbb{N}^{p-1} \mid ((x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{k-1}), (y_1, \ldots, y_{i-1}, 0, y_i, \ldots, y_{k-1})) \in \sigma\}.$$

It is not difficult to prove the next result.

PROPOSITION 4. – *The set ϱ' is a reduced canonical system of generators of σ' and $\mathbb{N}^p/\sigma$ is isomorphic to $\mathbb{N}^{p-1}/\sigma'$.*

This result enables us to eliminate the elements of the form $(e_i, 0)$ in ϱ. In the sequel we assume that ϱ does not contain any such elements. Under these conditions it is straightforward to prove the following statement.

PROPOSITION 5. – *Let σ be a congruence on $\mathbb{N}^p$ and $\varrho = \{(a_1, b_1), \ldots, (a_s, b_s)\}$ be a canonical system of generators of σ. The following conditions are equivalent*:

1. *$\mathbb{N}^p/\sigma$ is reduced,*
2. *for all $i \in \{1, \ldots, s\}$, $b_i \neq 0$.*

Thus once we know a canonical system of generators of σ, we can decide whether $\mathbb{N}^p/\sigma$ is reduced.

3. – Quasi-Archimedean monoids.

Let S be a semigroup. An element $x \in S$ is *Archimedean* if for every $y \in S$, there exist $z \in S$ and $k \in \mathbb{N} \backslash \{0\}$ such that $kx = y + z$. A semigroup is *Archimedean* if all its elements are Archimedean. We say that a monoid is Archimedean if it is Archimedean as semigroup. If A is an Archimedean monoid, then its identity element 0 is Archimedean, which means that for every $y \in A$, there exist $z \in S$ and $k \in \mathbb{N} \backslash \{0\}$ such that $k0 = 0 = y + z$. Thus if A is an Archimedean monoid, then it is a group. The converse is straightforward to prove, whence we have the following result.

PROPOSITION 6. – *A is an Archimedean monoid if and only if A is a group.*

Thus Archimedean monoids have familiar structure. A monoid A is *quasi-Archimedean* if every nonzero element of A is Archimedean and the zero element is not Archimedean. Next we see that a quasi-Archimedean monoid has no units (except of course the identity element), in contrast with Archimedean monoids for which every element is a unit.

PROPOSITION 7. – *Every quasi-Archimedean monoid is reduced.*

PROOF. – Let A be a quasi-Archimedean monoid. Assume that there exists $a, b \in A \setminus \{0\}$ such that $a + b = 0$. Take $c \in A$. Since a is an Archimedean element of A, then there exists $d \in A$ and $k \in \mathbb{N} \setminus \{0\}$ such that $ka = c + d$. Hence $k0 = ka + kb = c + (d + kb)$. Thus 0 is an Archimedean element of A, contradicting the fact that A is a quasi-Archimedean semigroup. ∎

Let S be a semigroup. Define on S the following binary relation: $a \mathcal{N} b$ if there exits $c, d \in S$ and $k, l \in \mathbb{N} \setminus \{0\}$ such that $ka = b + c$ and $lb = a + d$. This binary relation is an equivalence relation. An *Archimedean component* of S is an element of $S/\mathcal{N}$. If a and b are in the Archimedean component C, then $1(a + b) = a + b$ and $(k+1)a = a + ka = (a + b) + c$, which means that $a + b$ belongs to C as well. Hence we have the following result.

PROPOSITION 8. – *Every Archimedean component of a semigroup is one of its subsemigroups.*

With this notation it is straightforward to prove the following statement.

PROPOSITION 9. – *The monoid A is quasi-Archimedean if and only if its Archimedean components are $\{0\}$ and $A \setminus \{0\}$.*

Inspired in this new restatement of the condition of being quasi-Archimedean, we obtain the following characterization.

THEOREM 10. – *Let σ be a congruence on $\mathbb{N}^p$ such that $(e_i, 0) \notin \sigma$ for all $1 \leq i \leq p$. Let $\varrho = \{(a_1, b_1), \ldots, (a_t, b_t)\}$ be a canonical system of generators of σ. Then $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid if and only if ϱ fulfills the following conditions*:

(i) $b_i \neq 0$ *for all* $i \in \{1, \ldots, t\}$,

(ii) *for every nontrivial proper subset X of $\{1, \ldots, p\}$ there exists $(\alpha, \beta) \in \varrho \cup \varrho^{-1}$ such that* $\operatorname{supp}(\alpha) \subseteq X$ *and* $\operatorname{supp}(\beta) \nsubseteq X$.

(*supp* $(x_1, \ldots, x_p) = \{i \in \{1, \ldots, p\} \mid x_i \neq 0\}$.)

PROOF. – *Necessity.* By Proposition 7, $\mathbb{N}^p/\sigma$ is reduced. Condition (i) follows from Proposition 5.

For proving Condition (ii), let $X = \{i_1, \ldots, i_k\}$ be a nontrivial proper subset of $\{1, \ldots, p\}$. Take $x = e_{i_1} + \ldots + e_{i_k}$. By Proposition 9, the Archimedean components of $\mathbb{N}^p/\sigma$ are $\{[0]\}$ and $(\mathbb{N}^p/\sigma)\setminus\{[0]\}$. Since $[x] \neq [0]$ and $[e_1 + \ldots + e_p] \neq [0]$ (this is deduced from Condition (i) if $k \geq 2$ and if $k = 1$, from the assumption $(e_i, 0) \notin \sigma$ for all $i \in \{1, \ldots, p\}$), both $[x]$ and $[e_1 + \ldots + e_p]$ are in the same Archimedean component of $\mathbb{N}^p/\sigma$. Thus there exist $[y] \in \mathbb{N}^p/\sigma$ and $k \in \mathbb{N}\setminus\{0\}$ for which

$$k[x] = [kx] = [e_1 + \ldots + e_p] + [y] = [e_1 + \ldots + e_p + y].$$

By Proposition 1, there exists $v_0, \ldots, v_l \in \mathbb{N}^p$ such that $v_0 = kx$, $v_l = e_1 + \ldots + e_p + y$ and $(v_i, v_{i+1}) \in \varrho_1$ for all $i \in \{0, \ldots, l-1\}$. Since

$$\operatorname{supp}(kx) = X \neq \operatorname{supp}(e_1 + \ldots + e_p + y) = \{1, \ldots, p\},$$

we have that there exists $i \in \{0, \ldots, l-1\}$ for which $\operatorname{supp}(v_i) \subseteq X$ and $\operatorname{supp}(v_{i+1}) \nsubseteq X$. By the definition of ϱ_1, there exist $(\alpha, \beta) \in \varrho \cup \varrho^{-1} \cup \Delta(\mathbb{N}^p)$ and $d \in \mathbb{N}^p$ such that $(v_i, v_{i+1}) = (\alpha, \beta) + (d, d)$. It follows that $(\alpha, \beta) \notin \Delta(\mathbb{N}^p)$, $\operatorname{supp}(\alpha) \subseteq X$ and $\operatorname{supp}(\beta) \nsubseteq X$.

Sufficiency. We prove that the Archimedean components of $\mathbb{N}^p/\sigma$ are $\{[0]\}$ and $(\mathbb{N}^p/\sigma)\setminus\{[0]\}$, which by Proposition 9 implies that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid.

- Assume that $[x] \neq [0]$ is in the same Archimedean component containing $[0]$. Hence there exist $[y] \in \mathbb{N}^p/\sigma$ and $k \in \mathbb{N}\setminus\{0\}$ such that $k[0] = [x] + [y]$. However $k[0] = [k0] = [0]$, which leads to $[0] = [x] + [y]$ contradicting the fact that $\mathbb{N}^p/\sigma$ is reduced (by Condition (i) and Proposition 5). Thus $\{[0]\}$ is an Archimedean component of $\mathbb{N}^p/\sigma$.

- We prove next that $[e_1], \ldots, [e_p], [e_1 + \ldots + e_p]$ are in the same Archimedean component C of $\mathbb{N}^p/\sigma$. By Proposition 8, we have that C is a subsemigroup of $\mathbb{N}^p/\sigma$. Thus proving that $[e_1], \ldots, [e_p]$ are in C we obtain that $C = (\mathbb{N}^p/\sigma)\setminus\{[0]\}$. We show that $[e_1]$ and $[e_1 + \ldots + e_p]$ are in the same Archimedean component (for $[e_i]$, $i \neq 1$, the proof is similar). Take $X_1 = \{1\}$. By Condition (ii) there exists (α_1, β_1) such that $\operatorname{supp}(\alpha_1) \subseteq \{1\}$ and $\operatorname{supp}(\beta_1) \nsubseteq \{1\}$. By Condition (i), it follows that $\alpha_1 = k_1 e_1$ for some $k_1 \in \mathbb{N}\setminus\{0\}$. Take $i_1 \in \operatorname{supp}(\beta_1)\setminus\{1\}$. Then $\beta_1 - e_{i_1} \in \mathbb{N}^p$ and $k_1[e_1] = [\alpha_1] = [\beta_1 - e_{i_1}] + [e_{i_1}]$. Thus

  – $1([e_1 + e_{i_1}]) = [e_1] + [e_{i_1}]$,

  – $(k_1 + 1)[e_1] = [e_1] + [\beta_1 - e_{i_1}] + [e_{i_1}] = [e_1 + e_{i_1}] + [\beta_1 - e_{i_1}]$, which implies that $[e_1]$ and $[e_1 + e_{i_1}]$ are in the same Archimedean component of $\mathbb{N}^p/\sigma$. Set $X_2 = \{1, i_1\}$. Using once more Condition (ii), we obtain that there exists (α_2, β_2) such that $\mathrm{supp}(\alpha_2) \subseteq \{1, i_1\}$ and $\mathrm{supp}(\beta_2) \nsubseteq \{1, i_1\}$. It follows that there is a positive integer k_2 such that $k_2(e_1 + e_{i_1}) = (\alpha_2 + \gamma_2)$ for some $\gamma_2 \in \mathbb{N}^p$. Choose $e_{i_2} \in \mathrm{supp}(\beta_2) \setminus \{1, i_1\}$. Then

– $1[e_1 + e_{i_1} + e_{i_2}] = [e_1 + e_{i_1}] + [e_{i_2}]$,

– $(k_2 + 1)[e_1 + e_{i_1}] = [e_1 + e_{i_1}] + [\alpha_2] + [\gamma_2] = [e_1 + e_{i_1}] + [\beta_2 - e_{i_2}] + [e_{i_2}] + [\gamma_2] = [e_1 + e_{i_1} + e_{i_2}] + [\beta_2 - e_{i_2} + \gamma_2]$,

which implies that $[e_1 + e_{i_1}]$ and $[e_1 + e_{i_1} + e_{i_2}]$ are in the same Archimedean component of $\mathbb{N}^p/\sigma$ (the one containing $[e_1]$). Repeating this procedure several times we obtain that $[e_1]$ and $[e_1 + \ldots + e_p]$ are in the same Archimedean component. ■

4. – Quasi-Archimedean cancellative monoids.

Let σ be a congruence on $\mathbb{N}^p$ such that $(e_i, 0) \notin \sigma$ for all $i \in \{1, \ldots, p\}$. Assume that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid and that ϱ is a canonical system of generators of σ with respect to the linear admissible order $\preccurlyeq$. Under this assumption, ϱ fulfills Conditions (i) and (ii) of Theorem 10. We focus now our attention on describing a procedure that enables us to determine from ϱ whether $\mathbb{N}^p/\sigma$ is cancellative. Recall that $\mathbb{N}^p/\sigma$ is cancellative if and only if $\sigma = \sim_{M_\sigma}$ (in [9] an algorithm for deciding whether a finitely generated monoid is cancellative is presented; here we give an alternative method for the quasi-Archimedean case).

LEMMA 11. – *Let σ be a congruence on $\mathbb{N}^p$ such that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid. If $(a, b) \in \sim_{M_\sigma}$ and $(a, b) \notin \sigma$, then there exists $x_{(a,b)} \in \mathbb{N}^p$ such that $(a + x, b + x) \notin \sigma$ implies that $x < x_{(a,b)}$ (with respect to the usual order defined on $\mathbb{N}^p$).*

PROOF. – Since $(a, b) \in \sim_{M_\sigma}$, Lemma 2 asserts that there exists $c \in \mathbb{N}^p$ such that $(a + c, b + c) \in \sigma$ (this implies that $[c] \neq [0]$, since $(a, b) \notin \sigma$). By hypothesis $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid and thus for every $i \in \{1, \ldots, p\}$ there exists $k_i \in \mathbb{N} \setminus \{0\}$ and $z_i \in \mathbb{N}^p$ such that $k_i[e_i] = [c] + [z_i]$. Take $x_{(a,b)} = (k_1, \ldots, k_p)$. If $x \nless x_{(a,b)}$, then there exists $i \in \{1, \ldots, p\}$ for which the i-th coordinate of x is greater than or equal to k_i, whence $x = k_i e_i + d$ for some $d \in \mathbb{N}^p$.

It follows that

$$[a+x]=[a]+[x]=[a]+[k_ie_i]+[d]=[a]+[c]+[z_i]+[d]=[a+c]+[z_i]+[d]=$$

$$=[b+c]+[z_i]+[d]=[b]+[c]+[z_i]+[d]=[b]+[k_i e_i]+[d]=[b]+[x],$$

which means that $(a+x,\, b+x)\in\sigma$. ■

With this result we are able to sharpen for the quasi-Archimedean case, the characterization of cancellative monoids given in Proposition 2.

PROPOSITION 12. – *Let σ be a congruence on $\mathbb{N}^p$ such that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid. The monoid $\mathbb{N}^p/\sigma$ is cancellative if and only if $(a+e_i,\, b+e_i)\in\sigma$ for all $i\in\{1, \dots, p\}$, implies that $(a,\, b)\in\sigma$.*

PROOF. – *Necessity.* Trivial.

Sufficiency. Assume that $\mathbb{N}^p/\sigma$ is not cancellative. Then there exist $(a,\, b)\in\mathbb{N}^p\times\mathbb{N}^p$ and $c\in\mathbb{N}^p$ for which $(a+c,\, b+c)\in\sigma$ and $(a,\, b)\notin\sigma$. By Lemma 11, there exists $x_{(a,\, b)}\in\mathbb{N}^p$ fulfilling that if $(a+x,\, b+x)\notin\sigma$ for some $x\in\mathbb{N}^p$, then $x<x_{(a,\, b)}$. Since the set of elements in $\mathbb{N}^p$ less than or equal to $x_{(a,\, b)}$ is finite, we have that for a fixed linear admissible order $\preccurlyeq$ there exists the element

$$d=\max_{\preccurlyeq}\{x\in\mathbb{N}^p \mid (a+x,\, b+x)\notin\sigma\}.$$

(Note that this set is not empty because $(a,\, b)\notin\sigma$.) Then $(a+d,\, b+d)\notin\sigma$ and by the maximality of d, we have that $(a+d+e_i,\, b+d+e_i)\in\sigma$ for all $i\in\{1, \dots, p\}$ which contradicts the hypothesis. ■

From Proposition 12 it is derived that if $\mathbb{N}^p/\sigma$ is a non-cancellative quasi-Archimedean monoid, then there exists $(a,\, b)\in\mathbb{N}^p\times\mathbb{N}^p$ such that $(a+e_i,\, b+e_i)\in\sigma$ for all $i\in\{1, \dots, p\}$ and $(a,\, b)\notin\sigma$. Let ϱ be a canonical system of generators of σ with respect to the linear admissible order $\preccurlyeq$ (we assume as usual that $(e_i,\, 0)$ does not belong to σ). Since $(a,\, b)\notin\sigma$, we have that $(\mu(a),\, \mu(b))\notin\sigma$ (where μ is the function minimum associated to σ with respect to $\preccurlyeq$). Furthermore $(\mu(a)+e_i,\, \mu(b)+e_i)\in\sigma$ for all $i\in\{1, \dots, p\}$, since $(a+e_i,\, b+e_i)\in\sigma$. Because of $(\mu(a),\, \mu(b))\notin\sigma$, we have that $\mu(a)\neq\mu(b)$. We can assume without loss of generality that $\mu(b)\prec\mu(a)$, whence $\mu(b)+e_i\prec\mu(a)+e_i$. It follows that $\mu(a)+e_i\neq\mu(a+e_i)$, that is $\mu(a)+e_i\notin\mathrm{Im}\,(\mu)$ (observe also that trivially $\mu(a)\in\mathrm{Im}\,(\mu)$). As a consequence of this remark we obtain the following result.

PROPOSITION 13. – *Let σ be a congruence on $\mathbb{N}^p$ such that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid and $(e_i,\, 0)\notin\sigma$ for all $i\in\{1, \dots, p\}$. Let $\varrho=\{(a_1,\, b_1), \dots, (a_t,\, b_t)\}$ be a canonical system of generators of σ with respect to the linear admissible order $\preccurlyeq$. Then $\mathbb{N}^p/\sigma$ is not cancellative if and only if there exists an element $(x,\, y)\in\mathbb{N}^p\times\mathbb{N}^p$ such that*

(i) $(x, y) \notin \sigma$,

(ii) $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$,

(iii) $x + e_j \notin \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$,

(iv) $(x + e_i, y + e_i) \in \sigma$, *for all* $i \in \{1, \ldots, p\}$,

(v) $y \prec x$.

We see next that there are finitely many elements fulfilling Conditions (ii) and (iii) of Proposition 13. Let $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_p) \in \mathbb{N}^p$, define

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_p, y_p\}).$$

We denote the i-th coordinate, $1 \leqslant i \leqslant p$, of $x \in \mathbb{N}^p$ by $(x)_i$.

PROPOSITION 14. – *Let σ be a congruence on $\mathbb{N}^p$ such that $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid and $(e_i, 0) \notin \sigma$ for all $i \in \{1, \ldots, p\}$. Let $\varrho = \{(a_1, b_1), \ldots, (a_t, b_t)\}$ be a canonical system of generators of σ with respect to the linear admissible order $\leqslant$. If x is an element in $\mathbb{N}^p$ such that $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$ and $x + e_i \notin \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$ for all $i \in \{1, \ldots, p\}$, then there exists $\{a_{i_1}, \ldots, a_{i_p}\} \subseteq \{a_1, \ldots, a_t\}$ fulfilling the following conditions*:

(i) $\#\{a_{i_1}, \ldots, a_{i_p}\} = p$ (*$\#A$ denotes the cardinality of A*),

(ii) *for every $j, k \in \{1, \ldots, p\}$, we have that $j \neq k$, $(a_{i_j})_j > (a_{i_k})_j$*,

(iii) $x = ((a_{i_1})_1 - 1, \ldots, (a_{i_p})_p - 1)$.

PROOF. – Since $x + e_j \notin \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$, there exists $i_j \in \{1, \ldots, t\}$ and $d_j \in \mathbb{N}^p$ such that $x + e_j = a_{i_j} + d_j$. From the fact that $x \in \mathrm{ap}(\mathbb{N}^p, a_{i_j})$, it can be easily deduced that $a_{i_j} - e_j \in \mathbb{N}^p$. It follows that

$$x = (a_{i_1} - e_1) + d_1 = \ldots = (a_{i_p} - e_p) + d_p,$$

whence $x = ((a_{i_1} - e_1) \vee \ldots \vee (a_{i_p} - e_p)) + y$ for some $y \in \mathbb{N}^p$. We show that $y = 0$. Assume that this is not the case, that is $y \neq 0$. Then there exists $j \in \{1, \ldots, p\}$ such that $y - e_j \in \mathbb{N}^p$. Hence $y = (y - e_j) + e_j$, which implies that

$$x = ((a_{i_1} - e_1) \vee \ldots \vee a_{i_j} \vee \ldots \vee (a_{i_p} - e_p)) + (y - e_j)$$

and this leads to $x - a_{i_j} \in \mathbb{N}^p$, contradicting the fact that $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$. Therefore

$$x = ((a_{i_1} - e_1) \vee \ldots \vee (a_{i_p} - e_p)). \tag{1}$$

(i) If $a_{i_j} = a_{i_k}$ and $j \neq k$, then $x \geqslant (a_{i_j} - e_j) \vee (a_{i_k} - e_k) = (a_{i_j} - e_j) \vee (a_{i_j} - e_k) = a_{i_j}$, which contradicts the fact that $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t)$. Therefore $\#\{a_{i_1}, \ldots, a_{i_p}\} = p$.

(ii) From (1) it is deduced that $(x)_k \geqslant (a_{i_j})_k$ for $k \neq j$ and that $(x)_j \geqslant (a_{i_j})_j - 1$. If $(a_{i_j})_j \leqslant (a_{i_k})_j$ for some $k \neq j$, then

$$(a_{i_j})_j - 1 < (a_{i_k})_j \leqslant \max\{(a_{i_1})_j, \ldots, (a_{i_j})_j - 1, \ldots, (a_{i_p})_j\} = (x)_j,$$

which implies $x \geqslant a_{i_j}$, contradicting $x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_p)$.

(iii) The fact that $x = ((a_{i_1})_1 - 1, \ldots, (a_{i_p})_p - 1)$ follows easily from (ii) and (1). ∎

With all this information we are ready to present the algorithm for deciding whether a finitely presented quasi-Archimedean monoid is cancellative.

ALGORITHM 15. – The input of the algorithm is a (finite) presentation $\varrho \subseteq \mathbb{N}^p \times \mathbb{N}^p$ of a quasi-Archimedean monoid A such that $(e_i, 0) \notin \langle \varrho \rangle$. The output is true if A is cancellative and false otherwise.

1. Compute a canonical system of generators $\kappa = \{(a_1, b_1), \ldots, (a_t, b_t)\}$ of $\langle \varrho \rangle$ with respect to a linear admissible order $\preccurlyeq$ (for instance take $\preccurlyeq$ to be the total degree order on $\mathbb{N}^p$).
2. Compute the set

$$X = \{x \in \mathbb{N}^p \mid x \in \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t), x + e_i \notin \mathrm{ap}(\mathbb{N}^p, a_1, \ldots, a_t) \quad \text{for all } 1 \leqslant i \leqslant p\}.$$

3. For every element $x \in X$ construct $A_x = \{z \in [x + e_1]_\sigma \mid z \preccurlyeq x + e_1\}$.
4. For every $y + e_1 \in A_x$, check whether $(x, y) \in \sigma$ or $(x, y) \notin \sigma$; if $(x, y) \notin \sigma$, return false.
5. Return true.

5. – Quasi-Archimedean separative monoids.

A monoid A is *separative* if for all $x, y \in A$, the fact that $2x = x + y = 2y$ implies that $x = y$. The following characterization of the property of being separative in terms of the Archimedean components appears in [4].

PROPOSITION 16. – *A monoid is separative if and only if its Archimedean components are cancellative semigroups.*

An element x in a semigroup S is *idempotent* if $2x = x$. An Archimedean cancellative semigroup without idempotents is called a $\mathcal{N}$-*semigroup*. This kind of semigroup has been widely studied and was introduced by Tamura in [10].

Let S be a semigroup and x be an element not belonging to S. We extend the addition on S to $S \cup \{x\}$ in the following way: $x + s = s = s + x$ for all $s \in$

$S \cup \{x\}$. The resulting semigroup $S \cup \{x\}$ is an *identity extension* of S. It is clear that:

- identity extensions of a semigroup are reduced monoids,
- identity extensions of isomorphic semigroups are isomorphic.

We obtain the following restatement of the condition of being quasi-Archimedean.

PROPOSITION 17. – *A is a quasi-Archimedean monoid if and only if A is an identity extension of an Archimedean semigroup.*

PROOF. – *Necessity.* By Proposition 9 and 8, we have that $A\backslash\{0\}$ is an Archimedean semigroup. Clearly, A is an identity extension of $A\backslash\{0\}$.

Sufficiency. Assume that $A = S \cup \{0\}$ is an identity extension of the Archimedean semigroup S. Clearly, the Archimedean components of A are $\{0\}$ and $S = A\backslash\{0\}$, which by Proposition 9 implies that A is quasi-Archimedean. ∎

For quasi-Archimedean cancellative monoids, we can improve this characterization.

PROPOSITION 18. – *A is a quasi-Archimedean cancellative monoid if and only if A is an identity extension of a $\mathcal{N}$-semigroup.*

PROOF. – *Necessity.* As in Proposition 17, we have that $A\backslash\{0\}$ is an Archimedean semigroup. Since A is cancellative, $A\backslash\{0\}$ is also cancellative. If $2x = x$ for $x \in A$, then $x + x = x$, which by cancellativity implies that $x = 0$. Thus $A\backslash\{0\}$ has no idempotents. Therefore $A\backslash\{0\}$ is a $\mathcal{N}$-semigroup and clearly A is an identity extension of $A\backslash\{0\}$.

Sufficiency. Assume that $A = S \cup \{0\}$ is an identity extension of the $\mathcal{N}$-semigroup S. By Proposition 5, we already know that A is a quasi-Archimedean semigroup. Take $a, b, c \in A$ such that $a + c = b + c$. If $a, b \in S$, then $a = b$, since S is cancellative (it does not matter whether c belongs to S or $c = 0$). If $a \notin S$ or $b \notin S$, then we have an equality of the form $x + y = x$, with $x, y \in S$. By induction on n, it can be deduced that $x + ny = x$ for all $n \in \mathbb{N}$. The semigroup S is Archimedean and for this reason there exists $k \in \mathbb{N}\backslash\{0\}$ and $s \in S$ such that $ky = x + s$. Hence $2(x + s) = x + ky + s = x + s$, which contradicts the fact that S has no idempotent elements. ∎

For the non-cancellative separative case, we obtain a similar characterization.

PROPOSITION 19. – *A is a quasi-Archimedean separative non-cancellative monoid if and only if A is an identity extension of a group.*

PROOF. – *Necessity.* It suffices to show that $A\backslash\{0\}$ is a group. By Propositions 8, 9 and 16, $A\backslash\{0\}$ is an Archimedean cancellative semigroup. Since A is not cancellative, it can be deduced that there exists $x, y \in A\backslash\{0\}$ such that $x + y = x$. As in the proof of Proposition 18, from this fact we can deduce $A\backslash\{0\}$ has an idempotent, say t. Take $a \in A\backslash\{0\}$. Since $2t = t$, we have that $a + t = a + 2t = a + t + t$, which implies that $a = a + t$. Thus t is the identity element of $A\backslash\{0\}$. In addition, since $A\backslash\{0\}$ is an Archimedean semigroup, we have that there exists $k \in \mathbb{N}\backslash\{0\}$ and $b \in S$ such that $kt = a + b$. Since $kt = t$, we conclude that $t = a + b$. Therefore $A\backslash\{0\}$ is a group.

Sufficiency. Assume that $A = G \cup \{x\}$ is an identity extension of the group G. Every group is an Archimedean semigroup, whence from Proposition 5 we obtain that A is a quasi-Archimedean monoid. Since G and $\{x\}$ are the Archimedean components of A and they are cancellative, Proposition 5 asserts that A is separative. Finally $0 + 0 = 0 = 0 + x$, but $x \neq 0$, which means that A is not cancellative. ∎

In the proof of Proposition 19 we have shown that every Archimedean cancellative semigroup with an idempotent is a group, which is a well known fact (see for instance [3]). As a consequence of these two results we have the following statement.

THEOREM 20. – *A is a quasi-Archimedean separative monoid if and only if A is an identity extension of a group or a $\mathcal{N}$-semigroup.*

EXAMPLE 21. – The monoid $(\mathbb{Z}_2, \vee)$ with

$\vee$	0	1
0	0	1
1	1	1

is an identity extension of the trivial group $\{1\}$.

The monoid $\mathbb{N}$ is an identity extension of the $\mathcal{N}$-semigroup $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. ∎

Let $\mathbb{N}^{p*} = \mathbb{N}^p\backslash\{0\}$. Every finitely generated semigroup S is isomorphic to a quotient $\mathbb{N}^{p*}/R$ of $\mathbb{N}^{p*}$ by a congruence R on $\mathbb{N}^{p*}$. Note that $\sigma = R \cup \{(0, 0)\}$ is a congruence on $\mathbb{N}^p$ and that $\mathbb{N}^p/\sigma$ is the identity extension of $\mathbb{N}^{p*}/R$. Using the results presented in this section, we can apply the methods achieved in the preceding sections for obtaining some information about a finitely generated semigroup.

- By Proposition 17, the semigroup $\mathbb{N}^{p*}/R$ is Archimedean if and only if $\mathbb{N}^p/\sigma$ is a quasi-Archimedean monoid. Theorem 10 provides us with a method for deciding, from a system of generators of R, whether $\mathbb{N}^{p*}/R$ is Archimedean.
- Proposition 18 ensures that $\mathbb{N}^{p*}/R$ is a $\mathcal{N}$-semigroup if and only if $\mathbb{N}^p/\sigma$ is a quasi-Archimedean cancellative monoid. Thus Algorithm 15 enables us to check whether $\mathbb{N}^{p*}/R$ is a $\mathcal{N}$-semigroup from a system of generators of R.

In [8] the authors present a procedure for deciding whether a finitely generated monoid is separative. The method presented there is based on the computation of the Archimedean components of the given monoids and then on a decision algorithm for determining whether the quotient of an ideal by a congruence is cancellative (that is, the procedure uses the idea that a monoid is separative if and only if its Archimedean components are cancellative). Here we introduce an alternative way for determining whether a finitely generated quasi-Archimedean monoid is separative once we know one of its presentations. Assume that $\varrho \subseteq \mathbb{N}^p \times \mathbb{N}^p$ is a presentation of the quasi-Archimedean monoid A, that is A is isomorphic to $\mathbb{N}^p/\sigma$, where σ is the congruence generated by ϱ (as usual we assume that $(e_i, 0) \notin \sigma$ for all $1 \leqslant i \leqslant p$).

1. We can use the results presented in [6] for computing a canonical system of generators κ of σ with respect to a fixed linear admissible order $\preccurlyeq$ on $\mathbb{N}^p$. The set κ must fulfill Conditions (i) and (ii) of Theorem 10.

2. Algorithm 15 enables us to decide whether A is cancellative. If this is the case, then A is separative.

3. If A is not cancellative, the we proceed as follows. In the case A is separative, from the proof of Proposition 19, it is deduced that $\mathbb{N}^p/\sigma$ must be an identity extension of its Archimedean component $(\mathbb{N}^p/\sigma)\backslash\{[0]\}$, which is $\mathbb{N}^{p*}/R$, with $R = \sigma\backslash\{(0, 0)\}$. Thus if A is separative, $\mathbb{N}^{p*}/R$ must be a group. By Propositions 8 and 9, $\mathbb{N}^{p*}/R$ is an Archimedean semigroup. By Proposition 6, $\mathbb{N}^{p*}/R$ is a group if and only if it is a monoid, that is it has an identity element. Thus the problem of deciding whether A is separative reduces to determining whether A has a unit. Let μ be the function minimum associated to κ. Then $[u]$ is the identity element of $\mathbb{N}^{p*}/R$ if and only if $(u + e_i, e_i) \in \sigma$ for all $i \in \{1, \dots, p\}$. We can assume that $u = \mu(u)$ (we take it to be the minimum of its σ-class). Note that $e_i \prec u + e_i$ ($0 \neq u$, since $u \in \mathbb{N}^{p*}$), which means that $u + e_i \notin \operatorname{Im}(\mu)$ for all $i \in \{1, \dots, p\}$. By Proposition 14 there is a finite number of elements u fulfilling these conditions and we know how to compute them. Thus we only have to check whether one of them fulfills that $(u + e_i, e_i) \in \sigma$ for all $i \in \{1, \dots, p\}$.

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J. C. Rosales and P. A. García-Sánchez: Departamento de Álgebra, Universidad de Granada
E-18071 Granada, Spain. E-mail: jrosales@ugr.es, pedro@ugr.es

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