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Roberto Peirone

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Convergence and Uniqueness Problems for Dirichlet Forms on Fractals (*)

ROBERTO PEIRONE

Sunto. – *M_1 è un particolare operatore di minimizzazione per forme di Dirichlet definite su un sottoinsieme finito di un frattale K che è, in un certo senso, una sorta di frontiera di K. Viene talvolta chiamato mappa di rinormalizzazione ed è stato usato per definire su K un analogo del funzionale $u \mapsto \int |\text{grad}\, u|^2$ e un moto Browniano. In questo lavoro si provano alcuni risultati sull'unicità dell'autoforma (rispetto a M_1), e sulla convergenza dell'iterata di M_1 rinormalizzata. Questi risultati sono collegati con l'unicità del moto Browniano e con l'omogeneizzazione sui frattali.*

1. – Introduction.

In the last two decades, fractals have been extensively studied, as they appear to have a good likeness to many physical objects. A problem which has been investigated by many authors is that of the construction on a fractal of Dirichlet forms, i.e., functionals analogous to the integral functional $u \mapsto \int |\text{grad} u|^2$ on an open set in $\mathbb{R}^\nu$. Information on the general theory of Dirichlet forms can be found in [3]. The construction of Dirichlet forms can be also seen as a starting point to construct a «Laplacian» and a «Brownian motion» on the given fractal. Note that since, usually, a fractal has no interior, it is impossible to define the gradient (in the usual sense) on it. So, the usual way of constructing a Dirichlet form is based on a finite-difference scheme that I will now illustrate.

In particular, I treat this construction for the nested fractals, a class of highly symmetric fractals introduced by T. Lindstrøm in [10]. In order to make this notion clear I will now describe two typical examples of nested fractals, the Gasket and the Vicsek set (in $\mathbb{R}^2$). Gasket and Vicsek set in $\mathbb{R}^\nu$ for $\nu > 2$ can be defined similarly, and are sometimes studied (see for example [7] for the Gasket), but, as noted in [1], the Vicsek set is nested only for $\nu = 2$. Another usual example of nested fractal is the snowflake described in [10]. In section 2 I recall the exact definition of nested fractal. To construct the Gasket, start with an equilateral triangle T, whose vertices are denoted by P_1, P_2, P_3, and

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consider the three similitudes ψ_i, $i = 1, 2, 3$, in $\mathbb{R}^2$, that are contractions with factor $\frac{1}{2}$ and have P_i as fixed points, in formula $\psi_i(x) = P_i + \frac{1}{2}(x - P_i)$. Then the (Sierpinski) Gasket is the set K defined by

$$K = \bigcap_{n=0}^{\infty} K_n$$

where

$$K_0 = T\,, \qquad K_{n+1} = \bigcup_{i=1}^{3} \psi_i(K_n)\,.$$

The Vicsek set can be constructed analogously, by putting K_0 to be a square, P_1, P_2, P_3, P_4 its vertices, and P_5 its centre, and $\psi_i(x) = P_i + \frac{1}{3}(x - P_i)$, and $K_{n+1} = \bigcup_{i=1}^{4} \psi_i(K_n)$. More generally, every nested fractal is constructed starting from a finite set $\{\psi_1, \dots, \psi_k\}$ of contracting similitudes in $\mathbb{R}^\nu$. Here, a special role is played by the so-called essential fixed points of the similitudes (see section 2), which are the vertices of the triangle for the Gasket and of the square for the Vicsek set.

Given any nested fractal, I denote by $V^{(0)}$ the set of all essential fixed points, by $V_{i_1, \dots, i_n}$ the set $\psi_{i_1, \dots, i_n}(V^{(0)})$, where $\psi_{i_1, \dots, i_n}$ is an abbreviation for $\psi_{i_1} \circ \dots \circ \psi_{i_n}$, and put $V^{(n)} = \bigcup_{i_1, \dots, i_n = 1}^{k} V_{i_1, \dots, i_n}$. The sets $V_{i_1, \dots, i_n}$ are called n-cells and are, in some sense, small copies of $V^{(0)}$. To construct a Dirichlet form on the nested fractal K one has first to consider a Dirichlet form E on $V^{(0)}$, that is a functional $E \colon \mathbb{R}^{V^{(0)}} \to \mathbb{R}$ so that, for some $c_{j_1, j_2}(E)(= c_{j_1, j_2}) \geqslant 0$ $(j_1 \neq j_2)$ with $c_{j_1, j_2} = c_{j_2, j_1}$, we have

$$E(u) = \sum_{j_1 \neq j_2} c_{j_1, j_2}(u(P_{j_1}) - u(P_{j_2}))^2$$

for all $u \colon V^{(0)} \to \mathbb{R}$. It is easily seen that the coefficients c_{j_1, j_2} of E are unique. As a notation for the following, let $\mathcal{D}$ be the set of all functionals E defined as above, and let $\widetilde{\mathcal{D}}$ be the set of those $E \in \mathcal{D}$ that are irreducible, in the sense that $E(u) = 0$ if and only if u is constant. Once $E \in \mathcal{D}$ is given, a Dirichlet form $S_n(E)$ on $V^{(n)}$ and a Dirichlet form $M_n(E)$ on $V^{(0)}$ can be defined in the following way.

$$S_0(E) = E$$

$$S_n(E)(v) = \sum_{i_1, \dots, i_n = 1}^{k} E(v \circ \psi_{i_1, \dots, i_n}) \qquad \text{for } v \in \mathbb{R}^{V^{(n)}}, \qquad n \geqslant 1\,,$$

$$M_n(E)(u) = \inf\{S_n(E)(v) \colon v \in \mathcal{L}(n, u)\} \qquad \forall u \in \mathbb{R}^{V^{(0)}}$$

where $\mathcal{L}(n, u) = \{v \in \mathbb{R}^{V^{(n)}} : v = u \text{ on } V^{(0)}\}$. Thus $S_n(E)$ can be seen as the sum of all «copies» of E on the n-cells, and $M_n(E)$ is, according to the usual terminology, the restriction of $S_n(E)$ to $V^{(0)}$. Note that M_n can be expressed in terms of M_1, namely $M_n = (M_1)^n$. Now if $E \in \widetilde{\mathcal{D}}$ is an eigenform in the sense that there exists $\varrho > 0$ such that $M_1(E) = \varrho E$, a Dirichlet form S on the fractal can be constructed by $S(v) = \lim_{n \to \infty} S_n(E)(v)/\varrho^n$ (see for example [6]).

Problems concerning the operator M_n were discussed in [10] by a probabilistic formulation; in fact, an eigenform, or more precisely the set of all positive multiples of an eigenform, corresponds to a Brownian motion (from another point of view the eigenforms are related to the possibility of defining «harmonic functions» on the fractal, see [6]). The equivalence of the two formulations was pointed out by M. Barlow in [1]. In [10] it was proved that on every nested fractal an eigenform $E \in \widetilde{\mathcal{D}}$ does exist, and in fact $E \in \widetilde{\mathcal{D}}_G$ where $\widetilde{\mathcal{D}}_G$ denotes the set of those $E \in \widetilde{\mathcal{D}}$ that are «distance invariant». A question discussed in [10] concerns the uniqueness (up to a multiplicative constant) of the eigenform in $\widetilde{\mathcal{D}}_G$, which, of course, corresponds to the uniqueness of the Brownian motion.

After some partial results, the uniqueness was proved by C. Sabot in [17], Théorème V.1 and Théorème V.2, where in fact a rather general criterion for existence and uniqueness for eigenforms in a more general combinatorial setting was given. Another proof of the uniqueness was given by V. Metz in [14] (Theorem 4.2), where a stability result (Corollary 5.2) was also proved, in the sense that $\widetilde{M}_n(E)(= M_n(E)/\varrho^n)$ converges to a multiple of the given eigenform if $E \in \widetilde{\mathcal{D}}_G$. As shown by V. Metz in [11] for the Vicsek set, in general the uniqueness result does not hold for eigenforms in $\widetilde{\mathcal{D}}$. The uniqueness proofs of Sabot and Metz are both based on the behaviour of M_1 «near the boundary» of $\widetilde{\mathcal{D}}_G$.

In this paper I propose a different approach to handle the uniqueness and the stability. With regard to the uniqueness, this approach has the advantage that it leads to a shorter proof of uniqueness on nested fractals, and to some results that appear to be not easily obtainable by the approaches of Sabot and Metz. With regard to the stability, using this approach I prove that $\widetilde{M}_n(E)$ converges (in which case I say that E is homogenizable because of the relationship between this property and the homogenization result given by S.Kozlov in [8]), in a context which is much more general than that of nested fractals, possibly without uniqueness.

I will now illustrate my approach in a simple example. Namely, I sketch out how it works to obtain the following statement: In the Sierpinski Gasket, for every $E \in \widetilde{\mathcal{D}}$, $\widetilde{M}_n(E) \xrightarrow[n \to \infty]{} \alpha \overline{E}$ for some $\alpha > 0$. Here, $\overline{E}$ denotes that form in $\widetilde{\mathcal{D}}$ whose coefficients are all equal to 1. In this way, we obtain both uniqueness and stability. Roughly speaking, I prove that $\widetilde{M}_n(E)$ approaches $\overline{E}$ (with respect to Hilbert's projective metric, see Def. 3.1), noting that, if this is not the case then every maximum (resp. minimum) point u of $\widetilde{M}_n(E)/\widetilde{M}_n(\overline{E})$ produces, mapping it by certain positive linear operators, many maximum (resp. minimum) points of $E/\overline{E}$, namely the «restrictions» to the n-cells of the harmonic continuation of u on $V^{(n)}$. Then, using a Perron-Frobenius argument, we see that the existence of «so many» minimum and maximum points of $E/\overline{E}$ implies that some «eigenvector» is at the same time a maximum and a minimum point of $E/\overline{E}$. Hence E is a multiple of $\overline{E}$, and so we conclude immediately.

Note, however, that this argument implies uniqueness, thus it is not applicable in the general case. For this and other reasons, in the actual proofs presented in this paper, I use a modification of it. In sections 2 and 3 I deal with nested fractals, namely in section 2 I present my uniqueness proof and in section 3 my stability proof. It is noteworthy that the arguments of sections 2 and 3 use only the combinatorial structure and not the geometry of the fractal. Accordingly, in section 4 I introduce the notion of combinatorial fractal structure, and then, in this setting, I generalize the stability result of section 3 (Theorem 4.22). This approach of defining combinatorial fractal structures is based on that of [15]. However it is essentially the same approach as that of [4] (cf. also [6], Appendix A), and is similar to that of [17]. The structures defined here can be seen as p.c.f. self-similar sets in the sense of [6] and include the structures of the nested fractals. Theorem 4.22 can be schematized as follows: *Existence implies Stability.* More precisely, if there exists an eigenform in $\widetilde{\mathcal{D}}$, then every $E \in \widetilde{\mathcal{D}}$ is homogenizable. Instead, the argument of [14] shows that existence and uniqueness, with some additional assumptions, imply stability. Hence, for example, unlike the argument of Theorem 4.22, it does not work for $E \in \widetilde{\mathcal{D}}$ even in nested fractals. Note that Theorem 4.22 is in some sense sharp, for, if $E \in \widetilde{\mathcal{D}}$ is homogenizable it easily follows that $\lim_{n\to\infty} \widetilde{M}_n(E)$ is an eigenform. Note also that the same proof shows that the same result holds if, in the definition of S_n, the n-cells are «weighted», as considered for example in [2] and [17], i.e.,

$$S_n(E)(v) = \sum_{i_1, \ldots, i_n = 1}^{k} r_{i_1} \cdots r_{i_n} E(v \circ \psi_{i_1, \ldots, i_n})$$

where r_i are given positive numbers. The existence, uniqueness and stability problems discussed in this paper, can be seen as typical problems for nonexpansive maps with respect to Hilbert's projective metric (see [16]).

In section 5 I return to the uniqueness problem but in this case for combinatorial structures, and I obtain a generalization of the result for nested fractals. I also show some cases in which the uniqueness in $\widetilde{\mathcal{D}}$ (not only in $\widetilde{\mathcal{D}}_G$) takes place. In particular this uniqueness result holds for strongly symmetric fractal structures (for example for the Gasket in $\mathbb{R}^\nu$) (Corollary 5.7). The statement of Corollary 5.7, as well as the homogenizability for strongly symmetric structures, has been stated without proof in [15] by S. Mortola and the present author. For the Gasket (considering only forms having two coefficients equal) the same result has been previously obtained by Kozlov in [8]. He used the homogenizability to prove the Γ-convergence, with respect to a suitable topology, of the functionals $S_n(E)/\varrho^n$, defined for functions on all of the fractal, even if E is not an eigenform. So, in some sense, a Dirichlet form can be defined on the fractal starting from any $E \in \widetilde{\mathcal{D}}$. The uniqueness in $\widetilde{\mathcal{D}}$ and homogenizability for the Gasket were also proved by other authors (see [12], Example 8.8, and references therein), and also the homogenizability in $\widetilde{\mathcal{D}}$ for the Vicsek set (see [11]).

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2. – Uniqueness of eigenforms on nested fractals.

In this section first I recall the definition and the standard properties of nested fractals, then, namely from Lemma 2.4 on, I present my uniqueness proof for the eigenform in $\widetilde{\mathcal{D}}_G$. In the following I do not prove these standard properties (apart from Prop. 2.3 which is less trivial) because they are either explicitly formulated in other papers (see in particular [10] and [14]), or easily proved. Regarding the strong minimum principle (Prop. 2.1), a precise proof of it does not appear to be explicit in other papers. It is used in [14], deduced from Lemma IV.10 and Prop. IV.11 of [10]. However, it seems to me that this deduction is only correct in some cases like those considered in [10], i.e., when $c_{j_1, j_2} > 0$ for P_{j_1}, P_{j_2} such that $\|P_{j_1} - P_{j_2}\|$ is the minimum among the distance between different points of $V^{(0)}$. This is the case of the eigenform E given by the Theorem of Lindstrøm, thus it is not difficult to see that this is sufficient for my proof of uniqueness (it suffices to put $E_1 = E$ in Theorem 2.6). I prefer to omit for the moment the proof of the strong minimum principle and to give it in a more general setting in Lemma 5.1. Note that the argument of Lemma 5.1 is purely combinatorial and does not use the geometry of nested fractals.

Before introducing the nested fractals I recall some facts on graphs. A graph is a pair (V, W) where V is a nonempty set and W is a subset of the set of the subsets of V having precisely two elements. (V, W) is said to be connected if for all $P, Q \in V$ there exist $n = 1, 2 \ldots, P_1, \ldots, P_n \in V$ such that $P_1 = P$, $P_n = Q$, and $\{P_i, P_{i+1}\} \in W$ for $i = 1, \ldots, n-1$. When V is clear from the context we can identify the graph with W, and say for example that W is connected. Also, if $\emptyset \neq V' \subseteq V$, I denote by $(V'; W)$ the graph $(V', \{S \in W: S \subseteq V'\})$.

Consider now a finite set $\Psi = \{\psi_1, \dots, \psi_k\}$ of similarities of $\mathbb{R}^\nu$ with the property that we have $\|\psi_i(x) - \psi_i(y)\| = (1/R)\|x - y\|$ for some $R > 1$ and for every $x, y \in \mathbb{R}^\nu$ and $i = 1, \dots, k$. Then, by a theorem of Hutchinson ([5]), there exists a unique nonempty compact set K in $\mathbb{R}^\nu$ such that $K = \Phi(K)$, where I set $\Phi(A) = \bigcup_{i=1}^{k} \psi_i(A)$ for every $A \subseteq \mathbb{R}^\nu$. Now, letting P_i be the fixed point of ψ_i, we see that the set $\Phi^n(\{P_i\})$ is increasing on n, and $K = \overline{\bigcup_{n=0}^{\infty} \Phi^n(\{P_i\})}$. However, in the previous formula we can replace the set of all P_i with the set of the so-called essential fixed points. Namely, P_j is said to be essential if there exist $j', i, i' = 1, \dots, k$ with $j' \neq j$, such that $\psi_i(P_j) = \psi_{i'}(P_{j'})$. Put $V^{(0)} = \{P_1, \dots, P_N\}$ to be the set of all essential fixed points (of course $N \leq k$), and $V_{i_1, \dots, i_n} = \psi_{i_1, \dots, i_n}(V^{(0)})$ where $\psi_{i_1, \dots, i_n}$ is an abbreviation for $\psi_{i_1} \circ \dots \circ \psi_{i_n}$ ($V_\emptyset = V^{(0)}$); every $V_{i_1, \dots, i_n}$ will be called an n-cell. Finally, put $V^{(n)} = \bigcup_{i_1, \dots, i_n = 1}^{k} V_{i_1, \dots, i_n}$. In this situation, according to [10], K is said to be a nested fractal if $V^{(0)}$ has at least two elements, and

i) There exists $U \subseteq \mathbb{R}^\nu$, U bounded open and nonempty such that $\bigcup_{i=1}^{k} \psi_i(U) \subseteq U$, and the sets $\psi_i(U)$ are mutually disjoint (open set condition).

ii) The graph $\left(\bigcup_{i=1}^{k} \psi_i(V^{(0)}), \{\{\psi_i(P), \psi_i(Q)\} : i = 1, \dots, k, P, Q \in V^{(0)}, P \neq Q\}\right)$ is connected.

iii) If $j_1, j_2 = 1, \dots, N$, $j_1 \neq j_2$, then the symmetry ϕ_{j_1, j_2} with respect to $W_{j_1, j_2} = \{z : \|z - P_{j_1}\| = \|z - P_{j_2}\|\}$, maps n-cells to n-cells for $n \geq 0$ and any n-cell containing elements on both sides of W_{j_1, j_2} is mapped to itself (symmetry axiom).

iv) If $(i_1, \dots, i_n) \neq (i_1', \dots, i_n')$, then $\psi_{i_1, \dots, i_n}(V^{(0)}) \neq \psi_{i_1', \dots, i_n'}(V^{(0)})$, and $\psi_{i_1, \dots, i_n}(K) \cap \psi_{i_1', \dots, i_n'}(K) = \psi_{i_1, \dots, i_n}(V^{(0)}) \cap \psi_{i_1', \dots, i_n'}(V^{(0)})$ (nesting axiom).

I define $\mathcal{D}$, $\widetilde{\mathcal{D}}$, S_n, M_n, $\overline{E}$, eigenforms, according to the introduction. I will also sometimes use the term eigenform for $E \in \mathcal{D}$, but, unless specified otherwise, an eigenform is meant to be an element of $\widetilde{\mathcal{D}}$. I define $\widetilde{\mathcal{D}}_G$ to be the set of those $E \in \widetilde{\mathcal{D}}$ satisfying $E(u \circ \sigma) = E(u)$ for all $\sigma \in G$, G denoting the group of the permutations of $V^{(0)}$ generated by all the reflections ϕ_{j_1, j_2} with respect to W_{j_1, j_2}, when P_{j_1} and P_{j_2} are different points in $V^{(0)}$.

We can easily prove that M_n maps $\mathcal{A}$ into itself when $\mathcal{A}$ is $\mathcal{D}$, $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}_G$. From the nesting axiom it easily follows that $M_{n+1} = M_n \circ M_1$ thus $M_n = (M_1)^n$ for all $n \in \mathbb{N}$.

Given $E \in \widetilde{\mathcal{D}}$, I denote by $H_{n;E}(u)$ or simply $H_n(u)$ that (unique) $v \in$

$\mathcal{L}(n, u)$ such that $M_n(E)(u) = S_n(E)(v)$. I also write $H_{n, m; E}(u)$ or $H_{n, m}(u)$ for $H_{n; M_m(E)}(u)$. Note that H_n cannot be defined in this way for $E \in \mathcal{D}$, because «v» in definition of H_n is no longer unique (cf. [14], p. 163). Note also that $H_{n; aE} = H_{n; E}$ when $a > 0$. When E is an eigenform, $H_{n; E}$ is called the harmonic continuation of u on $V^{(n)}$. The above-discussed definitions and statements will be used without mention in the following. Another useful fact is (cf. for example [6], Theorem 2.9):

PROPOSITION 2.1. – *If $E \in \widetilde{\mathcal{D}}$, $u \in \mathbb{R}^{V^{(0)}}$, then for every $P \in V^{(1)}$*

$$\min u \leqslant H_1(u)(P) \leqslant \max u$$

(*weak minimum principle*). *If $E \in \widetilde{\mathcal{D}}_G$ then the two inequalities are strict for $P \in V^{(1)} \setminus V^{(0)}$ unless u is constant* (*strong minimum principle*). ■

Let now $E \in \widetilde{\mathcal{D}}$, $j = 1, \dots, N$. Then put $T_{j; E}$ (or simply T_j) to be that map from $\mathbb{R}^{V^{(0)}}$ into itself defined by $T_j(u) = H_{1; E}(u) \circ \psi_j$. Moreover, put $T_{j, n; E}$ (or $T_{j, n}$) for $T_{j; M_n(E)}$. Obviously the same definitions could be given also when $N < j \leqslant k$, but I will use them only for $j \leqslant N$. Regarding T_j we have the following two propositions. Note that the positivity of the matrix in Prop. 2.2 is a simple consequence of the minimum principle.

PROPOSITION 2.2. – *T_j is linear and $(T_j(u))(P_j) = u(P_j)$. Also, if we consider T_j as a map from $\{u \in \mathbb{R}^{V^{(0)}} \mid u(P_j) = 0\}$ into itself, it is linear and its matrix has non-negative entries, and positive entries if $E \in \widetilde{\mathcal{D}}_G$. Finally, we have $T_j({}_j u) = {}_j T_j(u)$, where if $u\colon V^{(0)} \to \mathbb{R}$, I denote by ${}_j u$ the function from $V^{(0)}$ to $\mathbb{R}$ defined by ${}_j u(P) = u(P) - u(P_j)$.* ■

PROPOSITION 2.3. – *We have*

i) $H_{n+m}(u) \circ \psi_{i_1, \dots, i_{n+m}} = H_n(H_{m, n}(u) \circ \psi_{i_1, \dots, i_m}) \circ \psi_{i_{m+1}, \dots, i_{m+n}}$.

ii) $H_{m+n, l-n}(u) \circ \psi_{i_1, \dots, i_m, j, \dots, j} =$
$T_{j, l-n} \circ \dots \circ T_{j, l-1}(H_{m, l}(u) \circ \psi_{i_1, \dots, i_m})$, *when* $m \geqslant 0$, $l \geqslant n \geqslant 0$.

PROOF. – It follows from the nesting axiom that there exists $v \in \mathbb{R}^{V^{(m+n)}}$ such that $v \circ \psi_{i_1, \dots, i_{m+n}} = H_n(H_{m, n}(u) \circ \psi_{i_1, \dots, i_m}) \circ \psi_{i_{m+1}, \dots, i_{m+n}}$, and clearly, $v \in \mathcal{L}(m+n, u)$. We easily get

$$S_{m+n}(E)(H_{m+n}(u)) = M_{m+n}(E)(u) = M_m(M_n(E))(u) = S_{m+n}(E)(v).$$

Thus, we obtain i), and ii) follows from i). ■

As a notation for the following, given a nonempty set A (clear from the context), for every $a \in A$ I denote by e_a the function from A to $\mathbb{R}$ defined by $e_a(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$, and when $A = V^{(0)}$ I put e_j for e_{P_j}.

Now, I will investigate the eigenforms. By a theorem of Lindstrøm (see [10], Theorem V.5) we know that an eigenform in $\widetilde{\mathcal{D}}_G$ does exist. ϱ in the definition of eigenform is called the eigenvalue of the eigenform, and in the following ϱ will denote the eigenvalue of this, thus of any, eigenform, for, it is easy to see that different eigenforms have the same eigenvalue (see for example [1], this could be also deduced from Remark 3.2). Also, clearly, every positive multiple of an eigenform is an eigenform as well. I will now prove that the eigenform in $\widetilde{\mathcal{D}}_G$ is unique up to a multiplicative constant. If $E_1, E_2 \in \widetilde{\mathcal{D}}$, let $\lambda_+ (= \lambda_+(E_1, E_2)) = \max(E_2(u)/E_1(u))$, $\lambda_- (= \lambda_-(E_1, E_2)) = \min(E_2(u)/E_1(u))$, where the maximum and the minimum are taken over all nonconstant u. Note that $\lambda_\pm(E_1, E_2) = \lambda_\mp(E_2, E_1)$. Also, put

$$A^\pm (= A^\pm(E_1, E_2)) = \{u \in \mathbb{R}^{V^{(0)}} : E_2(u) = \lambda_\pm(E_1, E_2)\, E_1(u)\},$$

$$\widetilde{A}^\pm (= \widetilde{A}^\pm(E_1, E_2)) = \{u \in A^\pm : u \text{ nonconstant}\}.$$

LEMMA 2.4. – *If E_1, E_2 are eigenforms and $u \in A^\pm(E_1, E_2)$ then $T_{j;E_1}(u) = T_{j;E_2}(u) \in A^\pm(E_1, E_2)$ for all $j = 1, \dots, N$.*

PROOF. – (Cf. [17], Lemme V.8) I show only the proof for A^-, that for A^+ being analogous. We have

$$\varrho E_2(u) = M_1(E_2)(u) = \sum_{i=1}^{k} E_2(H_{1;E_2}(u) \circ \psi_i) \geqslant \tag{2.1}$$

$$\sum_{i=1}^{k} \lambda_- E_1(H_{1;E_2}(u) \circ \psi_i) \geqslant \tag{2.2}$$

$$\sum_{i=1}^{k} \lambda_- E_1(H_{1;E_1}(u) \circ \psi_i) =$$

$$\lambda_- M_1(E_1)(u) = \lambda_- \varrho E_1(u) = \varrho E_2(u).$$

Thus the inequalities in (2.1) and (2.2) are in fact equalities. By the equality in (2.1) we have $E_2(H_{1;E_2}(u) \circ \psi_i) = \lambda_- E_1(H_{1;E_2}(u) \circ \psi_i)$ for all $i = 1, \dots, k$ for, $\geqslant$ holds for all i. By the equality in (2.2), and the uniqueness of the harmonic continuation, we have $H_{1;E_2}(u) = H_{1;E_1}(u)$ and this concludes the proof. ∎

PROPOSITION 2.5. – *Let* $E_1, E_2 \in \widetilde{\mathcal{D}}$. *Then*

i) $A^{\pm}$ *is closed.*

ii) $u \in A^{\pm} \Rightarrow c_1 u + c_2 \in A^{\pm}$ $\forall c_1, c_2 \in \mathbb{R}$.

iii) E_2 *is a multiple of* $E_1 \Leftrightarrow \widetilde{A}^{+} \cap \widetilde{A}^{-} \neq \emptyset$.

iv) If $E_1, E_2 \in \widetilde{\mathcal{D}}_G$ *and* $\sigma \in G$, *then* $u \in \widetilde{A}^{\pm} \Leftrightarrow u \circ \sigma \in \widetilde{A}^{\pm}$.

PROOF. – The proof is a simple verification. ■

THEOREM 2.6. – *If* E_1 *and* E_2 *are eigenforms in* $\widetilde{\mathcal{D}}_G$, *then* $\exists a > 0$ *such that* $E_2 = aE_1$.

PROOF. – By the transitivity of G and Prop. 2.5 iv), there exist $u_{\pm} \in \widetilde{A}^{\pm}$, which take minimum at the same point, for example min $u_{\pm} = u_{\pm}(P_1)$, and we can assume, in view of Prop. 2.5 ii) that such a minimum is 0. By Prop 2.5 ii) and Lemma 2.4, by putting

$$u_{\pm, n} = \mathcal{N}(T^n_{1; E_1}(u_{\pm}))$$

(here and in the following I denote by $\mathcal{N}(v)$ the vector $v/\|v\|$ when v is a nonzero element of a normed linear space), we have $u_{\pm, n} \in \widetilde{A}^{\pm}$. On the other hand, by Prop. 2.2, using the Perron-Frobenius Theorem (see for example [18]), we see that $u_{+, n}$, $u_{-, n}$ converge to the same limit $\bar{u}$. By Prop. 2.5 i), we have $\bar{u} \in \widetilde{A}^{+} \cap \widetilde{A}^{-}$, and by Prop 2.5 iii) we conclude. ■

REMARK 2.7. – If K is the Gasket in $\mathbb{R}^{\nu}$, by symmetry we easily see that $E_1 = \overline{E}$ is an eigenform. Also, every eigenform in $\widetilde{\mathcal{D}}$ (not only in $\widetilde{\mathcal{D}}_G$) is a positive multiple of $\overline{E}$. Indeed, given E_2 eigenform, we can imitate the proof of Theorem 2.6. Now, even though here we can have $P_{j(+)} \neq P_{j(-)}$, $P_{j(\pm)}$ denoting any minimum point of $u_{\pm}$; nevertheless, by a symmetry argument, $u_{\pm, n} \xrightarrow[n \to \infty]{} u_{j(\pm)}$, where $u_k \in \mathbb{R}^{V^{(0)}}$ is defined by $u_k = \mathcal{N}(1 - e_k)$, and $u_{j(\pm)} \in \widetilde{A}^{\pm}$, but, since $u_{j(\pm)}$ attains its maximum at any $j \neq j(\pm)$, by the same argument as before, $u_j \in \widetilde{A}^{\pm}$ for all $j (\neq j(\pm))$. Therefore, $\widetilde{A}^{+} \cap \widetilde{A}^{-} \neq \emptyset$. ■

3. – Homogenization on nested fractals.

In this section I prove that, given $E \in \widetilde{\mathcal{D}}_G$, E is homogenizable, i.e., we have $\widetilde{M}_n(E) \xrightarrow[n \to \infty]{} \widetilde{E}$ for some $\widetilde{E} \in \widetilde{\mathcal{D}}_G$, where I put $\widetilde{M}_n(E) = M_n(E)/\varrho^n$ and the convergence is meant to be pointwise. However, we can make the linear space generated by $\mathcal{D}$ a Banach space, by introducing the norm $\|\ \|$ defined by $\|E\| = \sup_{\|u\| = 1,\, u(P_1) = 0} |E(u)|$. It is easy to see that in $\mathcal{D}$, the pointwise convergence amounts to the convergence in norm and to the convergence of the coefficients, for, when $E \in \mathcal{D}$ we have $c_{j_1, j_2}(E) = (1/4)(E(u_{-1, j_1, j_2}) - E(u_{1, j_1, j_2}))$, where I put

$$u_{a, j_1, j_2} = ae_{j_1} + e_{j_2}.$$

It is not difficult to verify that M_n is continuous from $\mathcal{D}$ into itself, and that the map from $\widetilde{\mathcal{D}} \times \mathbb{R}^{V^{(0)}}$ to $\mathbb{R}^{V^{(n)}}$ defined by $(E, u) \mapsto H_{n;E}(u)$, is continuous. It easily follows that if $E \in \widetilde{\mathcal{D}}$ and $\widetilde{M}_n(E) \xrightarrow[n \to \infty]{} \widetilde{E}$, then $\widetilde{E}$ is an eigenform. These facts will be used in the following without explicit mention.

In addition to the convergence of $\widetilde{M}_n(E)$ for $E \in \widetilde{\mathcal{D}}_G$, we could prove similarly an analogous result for $E \in \widetilde{\mathcal{D}}$ provided some stronger assumptions on the connectedness of the graph are satisfied. This kind of assumption permits us to conclude that for every $E \in \widetilde{\mathcal{D}}$, $M_1(E)$ satisfies the strong minimum principle, which is essential in our argument. However, the result is valid also without this assumption, but we have to modify the proof substantially. Thus, I do not discuss this assumption deeply, and in the next section I shall prove that result in a more general setting which includes every nested fractal. I will now introduce two particular cases of a certain semimetric, called Hilbert's projective metric (see for example [16] for information on this topic). The case on $\widetilde{\mathcal{D}}$, in some sense, has been tacitly used in section 2, and has been used in the uniqueness proofs of [14] and [17]. As there is no possibility of confusion I use the same letter λ in the two cases.

DEFINITION 3.1. – *Given* $E_1, E_2 \in \widetilde{\mathcal{D}}$ *let* $\lambda(E_1, E_2) = \ln(\lambda_+(E_1, E_2)) - \ln(\lambda_-(E_1, E_2))$. *If A is a finite nonempty set and* $v_1, v_2 \in \mathbb{R}^A$, *and* $(v_1)_j \geq 0$, $(v_2)_j \geq 0$ *for all* $j \in A$, *and the set of those j for which* $(v_1)_j = 0$ *is the same as the set of those j for which* $(v_2)_j = 0$, *and is nonempty, put* $\lambda_+(v_1, v_2) = \max_{(v_1)_j > 0} ((v_2)_j / (v_1)_j)$, $\lambda_-(v_1, v_2) = \min_{(v_1)_j > 0} ((v_2)_j / (v_1)_j)$, *and* $\lambda(v_1, v_2) = \ln(\lambda_+(v_1, v_2)) - \ln(\lambda_-(v_1, v_2))$.

Note that in this section A denotes a finite nonempty set, and usually $A = V^{(0)} \setminus \{P_j\}$, and $(v_1)_j > 0$, $(v_2)_j > 0$ for all j, in previous definition. As known from the general theory of Hilbert's projective metric, λ is a semimetric in the sense that it is symmetric and satisfies the triangular inequality, but $\lambda(x, y) = 0$ if and only if y is a multiple of x. Also, $\lambda(ax, by) = \lambda(x, y)$ for all $a > 0$, $b > 0$. In particular, in the argument of $\lambda_\pm$, $M_n(E)$ can be replaced by $\widetilde{M}_n(E)$. Moreover, note that λ_+ and λ_- (thus also λ) are continuous.

REMARK 3.2. – If $\widetilde{E} \in \widetilde{\mathcal{D}}$ is an eigenform with eigenvalue ϱ, then, for every $E \in \widetilde{\mathcal{D}}$ we can easily see that $\lambda_-(\widetilde{E}, E)\, \widetilde{E} \leq \widetilde{M}_n(E) \leq \lambda_+(\widetilde{E}, E)\, \widetilde{E}$. It follows that every subsequence of $\widetilde{M}_n(E)$ has a subsequence convergent to some $E' \in \widetilde{\mathcal{D}}$. In fact, it suffices to take a subsequence which is convergent, if evaluated at any $u_{\pm 1, j_1, j_2}$. ■

PROPOSITION 3.3. – *Given* $E_1, E_2 \in \widetilde{\mathcal{D}}$, *put* $\lambda_{\pm, n} = \lambda_{\pm}(M_n(E_1), M_n(E_2))$. *Then for every* $n \in \mathbb{N}$, *if* α *is one of the symbols* $+$, $-$, *we have*

$$\alpha\lambda_{\alpha, n} \leqslant \alpha\lambda_{\alpha, 0}, \tag{3.1}$$

$$\exists \lim_{n \to \infty} \lambda_{\alpha, n} \in]0, +\infty[. \tag{3.2}$$

Also, if in (3.1) *the equality holds, then for every* m *such that* $0 \leqslant m \leqslant n$ *we have* $\lambda_{\alpha, m} = \lambda_{\alpha, 0}$, *and for every* $u \in A^{\alpha}(M_n(E_1), M_n(E_2))$ *we have*

$$H_{n-m, m; E_1}(u) \circ \psi_{i_1, \dots, i_{n-m}} = H_{n-m, m; E_2}(u) \circ \psi_{i_1, \dots, i_{n-m}} \in A^{\alpha}(M_m(E_1), M_m(E_2)) .$$

PROOF. – (3.1) is easily proved (see for example [14]) and (3.2) is a simple consequence of (3.1). In order to prove the last statement, note that if $u \in A^{\alpha}(M_n(E_1), M_n(E_2))$, then

$$\lambda_{-, 0} M_n(E_1)(u) = M_n(E_2)(u) = M_{n-m}(M_m(E_2))(u) =$$

$$S_{n-m}(M_m(E_2))(H_{n-m, m; E_2}(u)) =$$

$$\sum_{i_1, \dots, i_{n-m} = 1, \dots, k} M_m(E_2)(H_{n-m, m; E_2}(u) \circ \psi_{i_1, \dots, i_{n-m}}) \geqslant$$

$$\sum_{i_1, \dots, i_{n-m} = 1, \dots, k} \lambda_{-, 0} M_m(E_1)(H_{n-m, m; E_2}(u) \circ \psi_{i_1, \dots, i_{n-m}}) \geqslant$$

$$\lambda_{-, 0} M_{n-m}(M_m(E_1))(u) = \lambda_{-, 0} M_n(E_1)(u)$$

and we can proceed like in Lemma 2.4. ■

In the proof of Theorem 2.6 I used the fact that, by the Perron-Frobenius Theorem, the iterated of a positive linear operator contracts the cone of non-negative vectors to a half-line. Here we need an improvement in this result, that is, under certain conditions, we can replace the iterated of a single positive linear operator by the composition of possibly different positive linear operators (Prop. 3.5), and in Lemma 3.6 this result will be applied to the operators $T_{j, n}$.

LEMMA 3.4. – *Let* $T \colon \mathbb{R}^A \to \mathbb{R}^A$ *be a linear operator. Suppose* T *is positive in the sense that its matrix has entries* > 0, *and* v_1, v_2 *are in* $\mathbb{R}^A$ *and have components* > 0. *Then* $\lambda(T(v_1), T(v_2)) \leqslant \lambda(v_1, v_2)$, *and the equality holds only when* $\lambda(v_1, v_2) = 0$.

PROOF. – The proof is a simple verification (see [18], Ch. 3, Lemma 3.2). ■

PROPOSITION 3.5. – *Suppose* $T_1, \dots, T_n, \dots, T_\infty$ *are positive linear operators from* $\mathbb{R}^A$ *into itself, and there exists* $\sigma\colon \mathbb{N}\to\mathbb{N}$ *strictly increasing such that* $T_{\sigma(n)} \xrightarrow[n\to\infty]{} T_\infty$. *Let* $C = \{u \in \mathbb{R}^A\colon u_a \geq 0\ \forall a \in A,\ u \neq 0\}$. *Then there exists* $\bar{v} \in \mathring{C}$ *such that* $\sup_{v\in C} \lambda(T_1 \circ \dots \circ T_n(v), \bar{v}) \xrightarrow[n\to\infty]{} 0$.

PROOF. – Clearly $T_n(C) \subseteq \mathring{C}$ for all n. Put $\operatorname{diam}(E) = \sup_{v_1, v_2 \in E} \lambda(v_1, v_2)$ for $E \subseteq \mathring{C}$, $E \neq \emptyset$, $\operatorname{diam}\emptyset = 0$. Put $\eta_n = \sup\,(T_n(e_i)_j / T_n(e_i)_{j'})$ where the sup is taken over all $i, j, j' \in A$. A simple calculation yields: $\operatorname{diam} T_n(\mathring{C}) \leq 2 \ln \eta_n$, thus there exists $\eta > 0$ such that $\operatorname{diam} T_{\sigma(n)}(\mathring{C}) \leq \eta\ \forall n$. Put now $T_{m,n} = T_m \circ \dots \circ T_n$ (for $1 \leq m \leq n$). By Lemma 3.4, we have

$$\operatorname{diam} T_{m,n}(\mathring{C}) \leq \eta \ \text{ if } \ \exists h : m \leq \sigma(h) \leq n\,,$$

$$\operatorname{diam} T_{m,n}(\mathring{C}) \leq \operatorname{diam} T_{m+1,n}(\mathring{C}) \ \text{ if } \ m+1 \leq n\,,$$

and using a standard result in Hilbert's projective metric (see [16], Theorem 2.3, and references therein) we get $\operatorname{diam} T_{\sigma(n)}(E) \leq \tanh(\eta/4)\operatorname{diam}(E)$ for every $E \subseteq \mathring{C}$, thus

$$\operatorname{diam} T_1 \circ \dots \circ T_n(C) \xrightarrow[n\to\infty]{} 0 \tag{3.3}$$

(it is not difficult, in fact, to prove (3.3) without using the results of [16]). Also, letting $F_n = \mathcal{N}(T_1 \circ \dots \circ T_n(\mathcal{N}(C)))$, we see that F_n is a decreasing sequence of nonempty compact sets. Thus, if we take $\bar{v} \in \bigcap_{n=1}^{\infty} F_n$, we easily get $\bar{v} \in \bigcap_{n=1}^{\infty} T_1 \circ \dots \circ T_n(C)$, and in view also of (3.3) we conclude the proof. ■

Note that the thesis of Prop. 3.5 easily implies that for every $v_n \in C$ we have $\mathcal{N}(T_n(v_n)) \xrightarrow[n\to\infty]{} \mathcal{N}(\bar{v})$.

LEMMA 3.6. – *Suppose* $E \in \widetilde{\mathcal{D}}_G$ *is not an eigenform. Then there exists* $n \in \mathbb{N}$ *such that* $\lambda(M_n(E), M_{n+1}(E)) < \lambda(E, M_1(E))$.

PROOF. – If on the contrary, $\lambda(M_n(E), M_{n+1}(E)) = \lambda(E, M_1(E))$ for every n, let $\sigma\colon \mathbb{N}\to\mathbb{N}$ be strictly increasing, let $j(\pm) = 1, \dots, N$, and let $u_{\pm,n} \in \widetilde{A}^{\pm}(M_{\sigma(n)}(E), M_{\sigma(n)+1}(E))$ be such that $u_{\pm,n}(P_{j(\pm)}) = \min u_{\pm,n} = 0$. Put $u_{\pm,n,l} = H_{\sigma(n)-l,\tilde{l}}(u_{\pm,n}) \circ \psi_{j(\pm),\dots,j(\pm)}$ (when $0 \leq l \leq \sigma(n)$), where $\tilde{l}$ can be l or $l+1$ (see Prop. 3.3). By Prop. 3.3 again we have

$$u_{\pm,n,l} \in A^{\pm}(M_l(E), M_{l+1}(E))\,. \tag{3.4}$$

Also, by Prop. 2.3 we have $u_{\pm,n,l} = T_{j(\pm),\hat{l}} \circ \ldots \circ T_{j(\pm),\hat{l}+\sigma(n)-l-1}(u_{\pm,n})$, and $u_{\pm,n} \in C$ where C is as in Prop. 3.5 with $A = V^{(0)} \setminus \{P_{j(\pm)}\}$. In view of Prop. 2.2 and Remark 3.2 we can apply Prop. 3.5, thus

$$\exists \lim_{n\to\infty} \mathcal{N}(u_{\pm,n,l}) = u_{\pm;l} = u_{\pm;l+1},$$

hence, there exists $u_\pm$ (obviously nonconstant) such that $\mathcal{N}(u_{\pm,n,l}) \xrightarrow[n\to\infty]{} u_\pm$ for all l, and by (3.4) $u_\pm \in \widetilde{A}^\pm(M_l(E), M_{l+1}(E))$. In conclusion, $M_{l+1}(E)(u_\pm) = \lambda_\pm(E, M_1(E))\, M_l(E)(u_\pm)$. Since E is not an eigenform, then $\lambda_+(E, M_1(E)) > \lambda_-(E, M_1(E))$, and this implies $(M_l(E)(u_+)/M_l(E)(u_-)) \xrightarrow[l\to\infty]{} +\infty$, in contrast to Remark 3.2. ■

LEMMA 3.7. – *Suppose $E \in \widetilde{\mathcal{D}}$, $\sigma\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and*

$$\widetilde{M}_{\sigma(n)}(E) \xrightarrow[n\to\infty]{} \widetilde{E}. \tag{3.5}$$

Then for every $m \in \mathbb{N}$ we have $\lambda(M_m(\widetilde{E}), M_{m+1}(\widetilde{E})) = \lim_{n\to\infty} \lambda(M_n(E), M_{n+1}(E))$. If in addition $\widetilde{E}$ is an eigenform, then $\widetilde{M}_n(E) \xrightarrow[n\to\infty]{} \widetilde{E}$.

PROOF. – Recall that by Prop. 3.3, the sequence $\lambda_{\pm,n} = \lambda_\pm(M_n(E), M_{n+1}(E))$ is convergent to some $\lambda_\pm \in]0, +\infty[$. Thus, $\lambda(M_m(\widetilde{E}), M_{m+1}(\widetilde{E})) = \lim_{n\to\infty} \lambda(M_{m+\sigma(n)}(E), M_{m+\sigma(n)+1}(E)) = \ln(\lambda_+/\lambda_-)$. If $\widetilde{E}$ is an eigenform, then, by Prop. 3.3 again, the sequence $\lambda_\pm(\widetilde{E}, M_n(E))$ is convergent and to the limit 1 by (3.5), thus, using also Remark 3.2, $\widetilde{M}_n(E) \xrightarrow[n\to\infty]{} \widetilde{E}$. ■

THEOREM 3.8. – *Given $E \in \widetilde{\mathcal{D}}_G$ there exists an eigenform $\widetilde{E} \in \widetilde{\mathcal{D}}_G$ such that $\widetilde{M}_n(E) \xrightarrow[n\to\infty]{} \widetilde{E}$.*

PROOF. – Consider $\sigma\colon \mathbb{N} \to \mathbb{N}$ strictly increasing and $\widetilde{E} \in \widetilde{\mathcal{D}}_G$ such that $\widetilde{M}_{\sigma(n)}(E) \xrightarrow[n\to\infty]{} \widetilde{E}$. By Lemmas 3.6 and 3.7 $\widetilde{E}$ is an eigenform and we conclude by Lemma 3.7 again. ■

4. – Homogenization on fractal structures.

In this section we study the homogenization in the more general setting of combinatorial fractal structures.

DEFINITION 4.1. – *A prefractal structure is a triple (V, W, Ψ) where V is a finite set having at least two elements, W is a finite set containing V, and Ψ*

is a finite set of one-to-one maps from V to W such that

$$W = \bigcup_{\psi \in \Psi} \psi(V),$$

$$\forall P \in V \exists \psi_{(P)} \in \Psi: \ \textit{for } P, Q \in V, \ \psi \in \Psi\big((\psi(P) = Q) \Leftrightarrow (P = Q, \ \psi = \psi_{(P)})\big),$$

$$P \neq Q \Rightarrow \psi_{(P)} \neq \psi_{(Q)}.$$

REMARK 4.2. – Let (V, W, Ψ) be a prefractal structure, $V = \{P_1, \dots, P_N\}$, $\Psi = \{\psi_1, \dots, \psi_k\}$, and $\psi_i = \psi_{(P_i)}$ for $i = 1, \dots, N$. Let $\mathcal{R} = \big\{((i_1, j_1), (i_2, j_2)) : \psi_{i_1}(P_{j_1}) = \psi_{i_2}(P_{j_2})\big\}$. We have

i) $k \geqslant N \geqslant 2$,

ii) $\mathcal{R}$ is an equivalence relation on $\{1, \dots, k\} \times \{1, \dots, N\}$,

iii) $(i, j_1)\, \mathcal{R}(i, j_2) \Leftrightarrow j_1 = j_2$,

iv) $(i, i)\, \mathcal{R}(i', j) \Leftrightarrow i = i' = j$.

If conversely we have an equivalence relation $\mathcal{R}$ on $\{1, \dots, k\} \times \{1, \dots, N\}$ with $k \geqslant N \geqslant 2$ such that i), ii), iii), iv) hold, we get a prefractal structure (V, W, Ψ) with $V = \{P_1, \dots, P_N\}$, $\Psi = \{\psi_1, \dots, \psi_k\}$, $\psi_i = \psi_{(P_i)}$, and $\psi_{i_1}(P_{j_1}) = \psi_{i_2}(P_{j_2}) \Leftrightarrow (i_1, j_1)\,\mathcal{R}(i_2, j_2)$; also, this prefractal structure is unique up to an isomorphism. For example we can take $V = \{P_j = (j, j) : j = 1, \dots, N\}$, $W = (\{1, \dots, k\} \times \{1, \dots, N\})/\mathcal{R}$, $\psi_i(j, j) = \overline{(i, j)}$. In the following I will often identify a prefractal structure with the triple $(N, k, \mathcal{R})$, and I will define $\mathcal{R}$ by exhibiting a set of relations generating it. Here, an isomorphism between two prefractal structures $F_1 = (V_1, W_1, \Psi_1)$ and $F_2 = (V_2, W_2, \Psi_2)$ is meant to be a pair (T, τ) where T is a bijection from W_1 to W_2 such that $T_{|V_1}$ is a bijection from V_1 to V_2 and τ is a bijection from Ψ_1 to Ψ_2 such that $(\tau(\psi)) \circ T_{|V_1} = T \circ \psi \ \ \forall \psi \in \Psi_1$. ■

DEFINITION 4.3. – *A fractal structure is a triple (V, X, Ψ) where V is a finite set, X is a set containing V, and Ψ is a set of one-to-one maps from X to X such that $(V, \bigcup_{\psi \in \Psi} \psi(V), \{\psi_{|V} : \psi \in \Psi\})$ is a prefractal structure, $X = \bigcup \psi_{i_1} \circ \dots \circ \psi_{i_n}(V)$ where the union is taken over all $n \in \mathbb{N}$, $\psi_{i_1}, \dots, \psi_{i_n} \in \Psi$, and, when $\psi, \psi' \in \Psi$ and $\psi \neq \psi'$, then $\psi_{|V} \neq \psi'_{|V}$.*

DEFINITION 4.4. – *Given two fractal structures $\mathcal{F} = (V, X, \Psi)$ and $\mathcal{F}' = (V', X', \Psi')$, an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$ is a pair (T, τ) where T is a bijection from X to X' such that $T_{|V}$ is a bijection from V to V', and τ is a bijection from Ψ to Ψ' such that $\tau(\psi) \circ T = T \circ \psi$ for all $\psi \in \Psi$. I say that two fractal structures $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic if there exists an isomorphism from $\mathcal{F}$ to $\mathcal{F}'$.*

In the following, given a fractal structure (V, X, Ψ) I enumerate V as $V = V^{(0)} = \{P_j : j = 1, \dots, N\}$ and $\Psi = \{\psi_i : i = 1, \dots, k\}$, in such a way that $\psi_{(P_j)} = \psi_j$. I define $V_{i_1, \dots, i_n}$ and $V^{(n)}$ as in case of nested fractals. It is immediate to verify that $V^{(n)} \subseteq V^{(n+1)}$ for all n.

DEFINITION 4.5. – *I say that the fractal structure* (V, X, Ψ) *satisfies the (combinatorial) nesting axiom if* $(i_1, \dots, i_n) \neq (i'_1, \dots, i'_n) \Rightarrow V_{i_1, \dots, i_n, i} \cap V_{i'_1, \dots, i'_n, i'} \subseteq V_{i_1, \dots, i_n}$.

It is not difficult to see that, if F is a prefractal structure, then there exists a (unique up to an isomorphism) fractal structure $\mathcal{F} = (V, X, \Psi)$ satisfying the nesting axiom such that $(V^{(0)}, V^{(1)}, \{\psi_{|V^{(0)}} : \psi \in \Psi\}) \simeq F$. Thus, in the following, to define a fractal structure satisfying the nesting axiom, I will describe only the corresponding prefractal structure, by simply exhibiting $N, k, \mathcal{R}$, using notation of Remark 4.2.

DEFINITION 4.6. – *I say that a prefractal structure* (V, W, Ψ) *is connected if the graph* $(W, \{\{\psi(P), \psi(Q)\} : \psi \in \Psi, P, Q \in V, P \neq Q\})$ *is connected. I say that a fractal structure is connected if the corresponding prefractal structure is connected.*

Given a nested fractal, but this construction is valid for a more general class of fractals, we can define a fractal structure in the following way. Let $V^{(0)}$ be the set of all essential fixed points and let $X = \cup\, \psi_{i_1, \dots, i_n}(V^{(0)})$ where the union is taken over all $n \in \mathbb{N}$, $i_1, \dots, i_n = 1, \dots, k$ and let $\Psi = \{\psi_i|_X : i = 1, \dots, k\}$. Then, $(V^{(0)}, X, \Psi)$ is a fractal structure and I say that it is the fractal structure of K and similarly for the prefractal structure (here ψ_i are the similitudes defining the fractal; note that it is possible in fact that a fractal is defined by different sets of similitudes, so to be precise I should specify that the fractal structure is related not only to the fractal, but also to the set of similitudes). Recall (see [10], Prop. VI.13 and Corollary 4.14) that if K is a nested fractal, then with the above notation we have $V^{(0)} \cap V_i = \{P_i\}$, so that the properties in Def. 4.1 are in fact satisfied.

From now on, I fix a fractal structure $\mathcal{F} = (V, X, \Psi)$ and assume it is connected and satisfies the nesting axiom. In this situation we can repeat the same definitions as in the case of nested fractals. A remarkable difference is that, as known using slight modifications of nested fractals (see [4] for example), in fractal structures the existence of an eigenform is in general not guaranteed. However, in case there exists an eigenform, the results of sections 2 and 3 are still valid, apart from those related to the group G, which here is, a priori, not defined. Also, many results do not depend on the existence of an eigenform, and are thus valid for every fractal structure. Thus I will refer when convenient to results of sections 2 and 3 (obviously, in this case P_j, $V^{(0)}$, and so on, are those defined in this section).

In the rest of this section I will prove that, if there exists an eigenform, then every $E \in \widetilde{\mathcal{D}}$ is homogenizable. Note that, clearly, every eigenform is homogenizable, so, in particular, if $N = 2$, every $E \in \widetilde{\mathcal{D}}$, being an eigenform, is homogenizable. Note also that it is not difficult to verify that the eigenvalue ϱ is necessarily < 1 (see [17]). The difficulty in imitating the proof of homogenizability of section 3 is that in this case E in general does not satisfy the strong minimum principle [1] thus the operators $T_{j,n}$ are not necessarily positive. Thus, I first prove a variant of Prop. 3.5 (Prop. 4.7), then, but this requires some preliminary results, I prove that such a variant is suitable for operators $T_{j,n}$ (Lemma 4.21), and finally, in Theorem 4.22, I imitate with slight modifications the proof of section 3.

PROPOSITION 4.7. – *Suppose* $T_1, \dots, T_n, \dots, T_\infty$ *are linear maps from* $\mathbb{R}^A$ *to* $\mathbb{R}^A$ *where* A *is a finite nonempty set, and there exists* $B \subseteq A$ *such that*

i) $(T_n(e_j))_{j'} = 0$ *if* $j \in B, j' \notin B$, $n \in \mathbb{N} \cup \{\infty\}$.

ii) $B = B_{1,n} \cup B_{2,n}$ *where* $B_{1,n} = \{j \in B : (T_n(e_j))_{j'} > 0 \forall j' \in B\}$, $B_{2,n} = \{j \in B : (T_n(e_j))_{j'} = 0 \forall j' \in B\}$, *and* $B_{1,n} \neq \emptyset$, *for every* $n \in \mathbb{N} \cup \{\infty\}$.

iii) There exists $\sigma : \mathbb{N} \to \mathbb{N}$ *strictly increasing with* $T_{\sigma(n)} \xrightarrow[n \to \infty]{} T_\infty$, *and* $B_{1,\sigma(n)} = B_{1,\infty}$ *for every* $n \in \mathbb{N}$.

Then, putting $C = \{v \in \mathbb{R}^A : v_j > 0 \forall j \in B, v_j = 0 \forall j \notin B\}$, *there exists* $\bar{v} \in C$ *such that* $\mathcal{N}(T_1 \circ \dots \circ T_n(v_n)) \xrightarrow[n \to \infty]{} \bar{v}$, *if* $v_n \in C$ *for every* n.

PROOF. – We can imitate the proof of Prop. 3.5. The proof of formula (3.3) in this case is very similar. Here in the definition of $\mathring{\eta}_n$ the sup is taken over $i \in B_{1,n}, j, j' \in B$, and C plays the role of $\mathring{C}$. It remains to prove that $\bigcap_{n=1}^{\infty} T_1 \circ \dots \circ T_n(C) \neq \emptyset$. Here we have to modify the proof of Prop. 3.5 because T_n maps $\overline{C} \setminus \{0\}$ (which corresponds to C of proof of Prop. 3.5), into a set containing (possibly) 0, that therefore we cannot normalize. Thus, we consider $C' = \{v \in \mathbb{R}^A \setminus \{0\} : v_j \geq 0 \forall j, v_j = 0 \ \forall j \notin B_{1,\infty}\}$, and by the hypothesis we easily get

$$\mathcal{N}(T_1 \circ \dots \circ T_{\sigma(n+1)}(\mathcal{N}(C'))) \subseteq \mathcal{N}(T_1 \circ \dots \circ T_{\sigma(n)}(\mathcal{N}(C))) \subseteq \mathcal{N}(T_1 \circ \dots \circ T_{\sigma(n)}(\mathcal{N}(C')))$$

for all n. Then, put $F_n = \mathcal{N}(T_1 \circ \dots \circ T_{\sigma(n)}(\mathcal{N}(C')))$, and take $\bar{v} \in \bigcap_{n=1}^{\infty} F_n$.

[1] For example, the strong minimum principle is not satisfied in the fractal structure described in Remark 5.12, cf. Prop. 4.10.

It follows that $\bar{v} \in \bigcap_{n=1}^{\infty} T_1 \circ \ldots \circ T_n(C)$, and we conclude as in proof of Prop. 3.5. ■

In case of nested fractals the coefficients of $M_1(E)$ are strictly positive when $E \in \widetilde{\mathcal{D}}_G$, but for fractal structures and $E \in \widetilde{\mathcal{D}}$, this may fail. Thus, we are led to investigate deeplier some graphs related to the fractal structures. For $E \in \mathcal{D}$, I put

$$gr_n = \{\{\psi_{i_1, \ldots, i_n}(P_{j_1}), \psi_{i_1, \ldots, i_n}(P_{j_2})\} : j_1 \neq j_2\},$$

$$gr_n(E) = \{\{\psi_{i_1, \ldots, i_n}(P_{j_1}), \psi_{i_1, \ldots, i_n}(P_{j_2})\} : j_1 \neq j_2,\ c_{j_1, j_2}(E) > 0\},$$

$$\widetilde{gr}_0 = \{\{P_{j_1}, P_{j_2}\} : j_1 \neq j_2, \text{ and } \exists n \geq 2,\ Q_1, \ldots, Q_n \in V^{(1)} : Q_1 = P_{j_1},\ Q_n = P_{j_2},$$
$$Q_i, Q_{i+1} \in V_{\gamma(i)} \text{ for some } \gamma(i) \text{ if } 1 \leq i \leq n-1,\ \gamma(i) > N \text{ if } 1 < i < n-1\}.$$

From our hypotheses it easily follows that the graphs gr_n, $\widetilde{gr}_0$ are connected, and, in case $E \in \widetilde{\mathcal{D}}$, $gr_n(E)$ is also connected. At first glance, the graph $\widetilde{gr}_0$ does not seem to be very natural. However, it is useful because it is a connected graph contained in all $gr_0(M_1(E))$ (Corollary 4.12). This graph has also been studied in [17] where it is called L_0, and its connectedness is proved, as well as the statement of Corollary 4.12. Now, if (V, W) is a graph and $V' \subseteq V$, we say that $P, Q \in V$ are connected in $(V'; W)$ or simply in V', if there exist $P_1, \ldots, P_m \in V$ such that, $P_1 = P$, $P_m = Q$, $P_j \in V'$ if $1 < j < m$, and $\{P_j, P_{j+1}\} \in W$ for $1 \leq j < m$. We say that a subset of V' is a component of $(V'; W)$ if it is the set of all $P \in V'$ that are connected in $(V'; W)$ to a given $\bar{P} \in V'$ (thus V is connected if and only if it has precisely one component). We have the following lemma.

LEMMA 4.8. – *Suppose that (V, W) is a connected graph. Let $Q_1, \ldots, Q_n \in V$ with $Q_n = Q_1$, and $Q_i \neq Q_{i+1}$ for $i = 1, \ldots, n-1$, $n \geq 3$. Then for at least one $i = 1, \ldots, n-2$, Q_i and Q_{i+2} are connected in $(V \setminus \{Q_{i+1}\}; W)$.*

PROOF. – If the lemma is false we can take the smallest $n \geq 3$ for which there exist $Q_1, \ldots, Q_n \in V$ satisfying the hypotheses of the Lemma, but such that Q_i and Q_{i+2} are not connected in $V \setminus \{Q_{i+1}\}$, for every $i = 1, \ldots, n-2$. Clearly, $n \geq 4$, and also we have $Q_i \neq Q_j$ for $i \neq j$, unless $\{i, j\} = \{1, n\}$. I will now prove that Q_i and Q_{i+1} are connected in $V \setminus \{Q_2\}$ for $i = 3, \ldots, n-1$. If Q_i and Q_{i+1} are not connected in $V \setminus \{Q_2\}$ for some $i = 3, \ldots, n-1$, then Q_{i+1} and Q_2 are connected in $V \setminus \{Q_i\}$ (because V is connected), thus Q_{i-1} and Q_2 are not connected in $V \setminus \{Q_i\}$. Therefore, by considering $Q_2, \ldots, Q_i, Q_2$ we have contradicted the definition of n. Since Q_i and Q_{i+1} are connected in $V \setminus \{Q_2\}$, for $i = 3, \ldots, n-1$, it is easy to conclude that Q_3 and $Q_n = Q_1$ are connected in $V \setminus \{Q_2\}$, contrary to our assumption. ■

In order to prove Lemma 4.21, we need a more precise version of the minimum principle, which will lead us to determine the zeroes of the matrix of T_j. To obtain this I will introduce the set of those points that are in some sense reachable from $P \in V^{(1)}$ in E.

DEFINITION 4.9. – *Given* $\overline{P} \in V^{(1)}$ *and* $E \in \widetilde{\mathcal{D}}$, *let* $R_{;E}(\overline{P})(=R(\overline{P})) = \{P \in V^{(0)}: P$ *and* $\overline{P}$ *are connected in* $(V^{(1)} \setminus V^{(0)}; gr_1(E))\}$. *Put also* $R_{;M_n(E)}(\overline{P}) = R_{n;E}(\overline{P}) = R_n(\overline{P})$.

PROPOSITION 4.10. – *Given* $\overline{P} \in V^{(1)} \setminus V^{(0)}$ *and* $E \in \widetilde{\mathcal{D}}$, *we have*

$$\min_{P \in R(\overline{P})} u(P) \leqslant H_1(u)(\overline{P}) \leqslant \max_{P \in R(\overline{P})} u(P),$$

and the inequalities are strict unless u *is constant on* $R(\overline{P})$ (*hence, if for some* $\overline{P} \in V^{(1)} \setminus V^{(0)}$ *we have* $R(\overline{P}) \neq V^{(0)}$, *then the strong minimum principle does not hold). If* $(V^{(0)} \setminus \{P\}; gr_0(E))$ *is connected for all* $P \in V^{(0)}$, *then*

$$\min_{P \in V^{(0)}} u(P) < H_1(u)(\overline{P}) < \max_{P \in V^{(0)}} u(P)$$

unless u *is constant on* $V^{(0)}$.

PROOF. – The proof is a light variant of that of the minimum principle. ■

PROPOSITION 4.11. – *Let* $E \in \widetilde{\mathcal{D}}$. *Then, if* $j_1 \neq j_2$,

$$\{P_{j_1}, P_{j_2}\} \in gr_0(M_1(E)) \Leftrightarrow P_{j_1} \in R_{;E}(P_{j_2}).$$

PROOF. – As noted in [17], this property can be seen as a simple consequence of the probabilistic interpretation of $M_1(E)$. It would also be possible to give a non-probabilistic proof. ■

COROLLARY 4.12. – $\widetilde{gr}_0 \subseteq gr_0(M_1(E))\ \forall E \in \widetilde{\mathcal{D}}$. ■

COROLLARY 4.13. – *There exist* $n_1, n_2 \geqslant 1$ *such that for every* $E \in \widetilde{\mathcal{D}}$ *and for every* $n \geqslant n_1$ *we have* $gr_0(M_n(E)) = gr_0(M_{n+n_2}(E))$. *Moreover, if* $E = \lim_{n \to \infty} \widetilde{M}_{\sigma(n)}(E')$ *with* $E' \in \widetilde{\mathcal{D}}$, $\sigma: \mathbb{N} \to \mathbb{N}$ *strictly increasing, and there exists an eigenform, then the same holds for every* n, *i.e.*, $gr_0(M_n(E)) = gr_0(M_{n+n_2}(E))$ *for all* $n \in \mathbb{N}$.

PROOF. – The first statement follows from the fact that, by Prop. 4.11, $gr_0(M_{n+1}(E))$ depends only on $gr_0(M_n(E))$. For the second statement, it suffices to note that, by Remark 3.2 there exists $E_1 \in \widetilde{\mathcal{D}}$ such that $E = \widetilde{M}_{n_1}(E_1)$, with E_1 limit point of $\widetilde{M}_{\sigma(n)-n_1}(E')$. ■

In order to study the positivity of the coefficients of the matrix of T_j, I now define some subsets of $V^{(0)}$ related to $gr_0(M_n(E))$.

DEFINITION 4.14. – *Given $E \in \widetilde{\mathcal{D}}$, $j_1 \neq j_2$, let $C_{j_1, j_2, n; E}$ (or simply $C_{j_1, j_2, n}$) be the component of $(V^{(0)} \setminus \{P_{j_1}\}, gr_0(M_n(E)))$ containing P_{j_2}. Let $C'_{j_1, j_2, n; E} (= C'_{j_1, j_2, n}) = \{P_j \in V^{(0)} \setminus \{P_{j_1}\}: P_{j_2} \in R_n(\psi_{j_1}(P_j))\}$. Let $C'_{j_1, A, n; E} (= C'_{j_1, A, n}) = \bigcup_{P_{j_2} \in A} C'_{j_1, j_2, n; E}$ for $A \subseteq V^{(0)} \setminus \{P_{j_1}\}$. When $n = 0$ I omit n and write simply C_{j_1, j_2} and similar.*

REMARK 4.15. – We can easily verify that C'_{j_1, j_2} is a union of components of $(V^{(0)} \setminus \{P_{j_1}\}, gr_0(E))$. ■

REMARK 4.16. – We easily see that $C'_{j_1, j_2} \neq \emptyset$ if and only if $P_{j_2} \in R(P_{j_1})$. ■

PROPOSITION 4.17. – *For all $j, j_1, j_2 = 1, \dots, N$, $j_1 \neq j_2$, $E \in \widetilde{\mathcal{D}}$ we have $(T_{j_1}(e_{j_2}))(P_j) \geq 0$ and the inequality is strict if and only if $P_j \in C'_{j_1, j_2}$. Moreover, $T_{j_1; E}(e_{j_2})$ and $T_{j_1; E'}(e_{j_2})$ have the same zeroes if $gr_0(E) = gr_0(E')$.*

PROOF. – This easily follows from Prop. 4.10. ■

A simple consequence of Prop. 4.17 is that if u takes its minimum at P_j, then $T_j(u)$ also takes its minimum at P_j. However, it is important to note that it is sufficient that P_j is a minimum point for u restricted to $R(P_j)$, to have that $T_j(u)$ takes its minimum at P_j, and similarly for the maximum. This motivates the following definition.

DEFINITION 4.18. – *Given $E \in \widetilde{\mathcal{D}}$ and $u \in \mathbb{R}^{V^{(0)}}$, for $a = 1, -1$ I define $D^a_{;E}(u) (= D^a(u))$ to be the set of those $P_j \in V^{(0)}$ such that $au(P_j) \leq au(P)$ for all $P \in R(P_j)$ and the inequality is strict for at least one $P \in R(P_j)$. Put also $D_{;E}(u) = D^1_{;E}(u) \cup D^{-1}_{;E}(u)$, $D^a_{n;E} = D^a_n = D^a_{;M_n(E)}$ and $D_{n;E} = D_n = D_{;M_n(E)}$.*

REMARK 4.19. – Note that if $E \in \widetilde{\mathcal{D}}$, and A is a component of $(V^{(0)} \setminus \{P_j\}; gr_0(M_1(E)))$ and $u \in \mathbb{R}^{V^{(0)}}$ is not constant on $A \cup \{P_j\}$, then $D(u) \cap A \neq \emptyset$. Indeed we can choose $P \in A$ such that for some $a = 1, -1$, $au(P) \geq au(Q)$ for every $Q: \{P, Q\} \in gr_0(M_1(E))$, and the inequality is strict for some $Q: \{P, Q\} \in gr_0(M_1(E))$. By Prop. 4.11, $P \in D^a(u)$. ■

LEMMA 4.20. – *Let $E \in \widetilde{\mathcal{D}}$, $u \in \mathbb{R}^{V^{(0)}}$, $P_j \in D^a_1(u)$. Then*

$$a({}_jT_{j,1}(u)) \geq 0 . \tag{4.1}$$

Moreover, if we put $A = \{P \in V^{(0)}: a_j u(P) > 0\}$, $A' = \{P \in V^{(0)}: a_j T_{j,1}(u)(P) > 0\}$, we have $A' = C'_{j, A, 1}$, and $P_j \in D^a_{;E'}(T_{j,1}(u))$ for every $E' \in \widetilde{\mathcal{D}}$.

PROOF. – Since ${}_jT_{j,1}(u) = T_{j,1}({}_ju)$, (4.1) and the formula $A' = C'_{j,A,1}$ follow from Prop. 4.17. Also, given $P_h \in R_1(P_j) \cap A$, by Remark 4.16 $C'_{j,h,1} \neq \emptyset$. By Remark 4.15 $C'_{j,h,1}$ contains a component of $\big(V^{(0)} \setminus \{P_j\};\, gr_0(M_1(E))\big)$. Hence, in view of Corollary 4.12, $C'_{j,h,1}$ contains some $P \in V^{(0)}$ such that $\{P, P_j\} \in \widetilde{gr}_0 \subseteq gr_0(M_1(E'))$. In conclusion, since $P_h \in A$, then $P \in C'_{j,A,1} = A'$, and by Prop. 4.11, $P \in R_{;E'}(P_j)$. Therefore, by using also (4.1), $P_j \in D^a_{;E'}(T_{j,1}(u))$. ■

LEMMA 4.21. – *There exists $n_3 \geq 1$ such that, if $h \geq n_1 + n_2 + n_3$, $E \in \widetilde{\mathcal{D}}$, u nonconstant, and $M_{h+1}(E)(u) = \lambda_\pm(E, M_1(E))M_h(E)(u)$, then there exist m with $0 \leq m \leq n_3$, $i_1, \dots, i_m = 1, \dots, k$, $j = 1, \dots, N$, $B \subseteq V^{(0)} \setminus \{P_j\}$, $a = 1, -1$, such that*

i) $a\,{}_jH_{m,h-m}(u) \circ \psi_{i_1,\dots,i_m}(P) \geq 0$ and the inequality is strict if only if $P \in B$.

ii) Putting $T_n = T_{j,n+n_1}$, then for suitable $B_{1,n}$, $B_{2,n}$, i) and ii) of Prop. 4.7 are satisfied.

PROOF. – Suppose

$$h \geq (n_2 + 2)(N-1) + n_1. \tag{4.2}$$

Then there exist μ with $1 \leq \mu \leq (n_2+2)(N-2)+1$, j, $j' \neq j$, $i_1, \dots, i_{\mu-1} = 1, \dots, k$ such that

$$\begin{cases} P_j \in \widetilde{D}_{h-\mu}(H_{\mu-1,h-(\mu-1)}(u) \circ \psi_{i_1,\dots,i_{\mu-1}}) \\ {}_jH_{l,h-l}(u) \circ \psi_{i_1,\dots,i_{\mu-1},j,\dots,j}(P) = 0 \;\; \forall P \notin \widetilde{C}_{j,j',h-l};\; l = \mu, \dots, \mu+n_2+1 \end{cases} \tag{4.3}$$

where I write $\widetilde{D}_i(v)$ for $D_i(v) \cup D_{i+1}(v)$, and $\widetilde{C}_{j,j',i}$ for $C_{j,j',i} \cap C_{j,j',i+1}$. To see this, let $P_{j_0} \in V^{(0)}$; by Remark 4.19 there exists $j_1 \neq j_0$ such that $P_{j_1} \in D_{h-1}(u)$. If (4.3) is false, using Remark 4.19, we inductively find $P_{j_2}, \dots, P_{j_N}$, $\beta(0), \dots, \beta(N-1)$, and $i_1, \dots, i_{\beta(N-1)}$ such that $\beta(0) = 0$, $\beta(s+1) - \beta(s) \in \{1, \dots, n_2+2\}$ for $s = 0, \dots, N-2$, and

$$P_{j_{s+1}} \in \widetilde{D}_{h-\beta(s)-1}(H_{\beta(s),h-\beta(s)}(u) \circ \psi_{i_1,\dots,i_{\beta(s)}}) \setminus (\widetilde{C}_{j_s,j_{s-1},h-\beta(s)} \cup \{P_{j_s}\})$$

for $s = 1, \dots, N-1$. Thus $P_{j_{s+1}}$ and $P_{j_{s-1}}$ are not connected in $(V^{(0)} \setminus \{P_{j_s}\}; \widetilde{gr}_0)$ (see Corollary 4.12), and since, clearly two of the indices $j_0, \dots, j_N$ are equal, this contradicts Lemma 4.8, thus (4.3) holds. Put

$$\widetilde{H}_{l,s} = H_{l,s}(u) \circ \psi_{i_1,\dots,i_{\mu-1},j,\dots,j}$$

when $l \geq \mu - 1$, $s \geq 0$ and where the index j appears $l - (\mu - 1)$ times. We have

$$\widetilde{H}_{l,\, h-l} = \widetilde{H}_{l,\, h+1-l} \tag{4.4}$$

if $\mu - 1 \leq l \leq h$ by Prop. 3.3. Also, if $s \geq 1$, $l \geq \mu - 1$, by Prop. 2.3

$$\widetilde{H}_{l+1,\, s-1} = T_{j,\, s-1}(\widetilde{H}_{l,\, s}). \tag{4.5}$$

By (4.3) there exists $a = 1, -1$ such that

$$P_j \in D^a_{h-\mu}(\widetilde{H}_{\mu-1,\, h-(\mu-1)}) \cup D^a_{h+1-\mu}(\widetilde{H}_{\mu-1,\, h-(\mu-1)}).$$

By Lemma 4.20, (4.4) and (4.5), using a recursive argument, when $\mu \leq l \leq h-1$ we have

$$a_j \widetilde{H}_{l,\, h-l} \geq 0 \tag{4.6}$$

and, putting $A_l = \{P \in V^{(0)}\colon a_j \widetilde{H}_{l,\, h-l}(P) > 0\}$, for $\tilde{h} = h,\, h+1$ we get

$$A_{l+1} = C'_{j,\, A_l,\, \tilde{h}-(l+1)} \neq \emptyset \qquad \text{if } \mu \leq l \leq h-2\,, \tag{4.7}$$

and by (4.3) we have $A_l \subseteq C_{j,\, j',\, \tilde{h}-l}$ if $\mu \leq l \leq \mu + n_2 + 1$, thus by Remark 4.15 we have

$$A_l = C_{j,\, j',\, \tilde{h}-l} \qquad \text{if } \mu \leq l \leq \mu + n_2 + 1\,. \tag{4.8}$$

Since (4.8) holds both for $\tilde{h} = h$ and for $\tilde{h} = h+1$, putting $m = \mu + 1$ we have $A_m = A_{m+1} = \ldots = A_{m+n_2}$. Now, putting $n_3 = (n_2 + 2)(N-1) - n_2$, we have $1 \leq m \leq n_3$, and (4.2) amounts to $h \geq n_1 + n_2 + n_3$. Also, by Corollary 4.13 and (4.7) there exists B such that $A_l = B$ if $m \leq l \leq h - n_1$, and, thanks to (4.6), we get i). Also, in view of Corollary 4.13 and (4.8) we see that $B = C_{j,\, j',\, h-l}$ if $l \leq h - n_1$. Thus, by (4.7) and Remark 4.15 we see that for every $P_{j_1} \in B$, $C'_{j,\, j_1,\, h-l}$ is either $\emptyset$ or B, and is B for at least one $P_{j_1} \in B$. By Prop. 4.17 we easily get ii). ■

THEOREM 4.22. – *Suppose there exists an eigenform in $\widetilde{\mathcal{D}}$. Then every $E \in \widetilde{\mathcal{D}}$ is homogenizable.*

PROOF. – By Lemma 3.7 it is sufficient to find E_1 limit point of $\widetilde{M}_n(E)$ for which the formula

$$\lambda(M_n(E_1),\, M_{n+1}(E_1)) = \lambda(E_1,\, M_1(E_1)) > 0 \ \ \forall n \in \mathbb{N} \tag{4.9}$$

does not hold. I will prove that (4.9) does not hold if we take E_1 to be a limit point of $\widetilde{M}_n(E)$ for which $gr_0(E_1)$ has the minimal number of elements. In view of Remark 3.2, there exists $\sigma\colon \mathbb{N}\to\mathbb{N}$ strictly increasing such that

$$\widetilde{M}_{\sigma(n)}(E_1) \underset{n\to\infty}{\longrightarrow} E_2 \in \widetilde{\mathcal{D}}$$

and we can assume that $\sigma(n) \equiv \sigma(m) \pmod{n_2}$ for all n, m, and by replacing $\sigma(n)$ by $\sigma(n)+h$ for some $h\in\mathbb{N}$, we can assume that for all n, $\sigma(n)$ is an integer multiple of n_2. Note that, if $\sigma'\colon \mathbb{N}\to\mathbb{N}$ strictly increasing is such that $\widetilde{M}_{\sigma'(n)}(E) \underset{n\to\infty}{\longrightarrow} E_1$, we have

$$\lambda(\widetilde{M}_{\sigma'(n)+\sigma(n)}(E), E_2) \leqslant$$

$$\lambda\big(\widetilde{M}_{\sigma(n)}(\widetilde{M}_{\sigma'(n)}(E)), \widetilde{M}_{\sigma(n)}(E_1)\big) + \lambda(\widetilde{M}_{\sigma(n)}(E_1), E_2) \leqslant$$

$$\lambda(\widetilde{M}_{\sigma'(n)}(E), E_1) + \lambda(\widetilde{M}_{\sigma(n)}(E_1), E_2) \underset{n\to\infty}{\longrightarrow} 0$$

and, using Remark 3.2, there exists a limit point of $\widetilde{M}_n(E)$ which is a multiple of E_2. Thus, by our assumption on E_1, using also Corollary 4.13, we easily get

$$gr_0\widetilde{M}_{\sigma(n)}(E_1) = gr_0(E_1) = gr_0(E_2)\,. \tag{4.10}$$

Now, if (4.9) holds, there exist $u_{\pm,n} \in \widetilde{A}^{\pm}(M_{\sigma(n)}(E_1), M_{\sigma(n)+1}(E_1))$ and, using Lemma 4.21, by passing to a subsequence we can assume that $\sigma(n) \geqslant n_1 + n_2 + n_3$, and there exist $m(\pm) \leqslant n_3$, $i_{1,\pm}, \ldots, i_{m(\pm),\pm} = 1, \ldots, k$, $j_\pm = 1, \ldots, N$, $B_\pm \subseteq V^{(0)}\setminus\{P_{j_\pm}\}$, $a_\pm = 1, -1$ such that

$$_{j(\pm)}H_{m(\pm),\tilde{n}-m(\pm);E_1}(a_\pm u_{\pm,n}) \circ \psi_{i_{1,\pm},\ldots,i_{m(\pm),\pm}}(P) \geqslant 0$$

and the inequality is strict if and only if $P\in B_\pm$ for $\tilde{n} = \sigma(n), \sigma(n)+1$ (see Prop. 3.3). Putting $T_{n,\pm} = T_{j(\pm),n+n_1;E_1}$, then $T_{n,\pm}$, $n\geqslant 1$, satisfy the hypotheses of Prop. 4.7 with $T_\infty = T_{j(\pm);E_2}$ by Lemma 4.21 and (4.10), in view also of Prop. 4.17. Put now

$$u_{\pm,n,l} = {}_{j(\pm)}H_{\sigma(n)-l,\tilde{l};E_1}(a_\pm u_{\pm,n}) \circ \psi_{i_{1,\pm},\ldots,i_{m(\pm),\pm},j_\pm,\ldots,j_\pm}$$

where $\tilde{l}$ can be l or $l+1$, if $1\leqslant l\leqslant \sigma(n)-m(\pm)$, in view of Prop. 3.3, and we derive a contradiction as in proof of Lemma 3.6. ∎

REMARK 4.23. – So far, we have considered only forms with nonnegative coefficients, but we could also consider forms E with possibly negative coefficients (cf. for example [14]), but such that $E(u)\geqslant 0$ for every u and the equality holds only when u is constant. This set $\mathcal{P}$ of forms fits Hilbert's projective metric λ better than $\widetilde{\mathcal{D}}$, and in view of a result of [16] (Theorem 4.2), if there

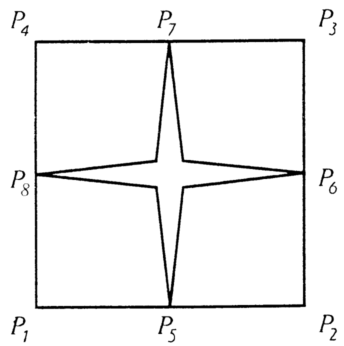

Fig. 1. – See Remark 4.24.

are no eigenforms in $\mathcal{P}$ then every limit point of $\mathcal{N}(M_n(E))$ is reducible when $E \in \widetilde{\mathcal{D}}$. On the other hand, if we know that there exists an eigenform in $\mathcal{P}$, then the argument of Theorem 4.22 works, as easily verified, for $E \in \widetilde{\mathcal{D}}$ (it appears to be nontrivial to extend it for $E \in \mathcal{P}$), thus obviously there exists an eigenform in $\widetilde{\mathcal{D}}$, namely the limit of $\widetilde{M}_n(E)$. In conclusion, there are two possibilities: either there exists an eigenform and every $E \in \widetilde{\mathcal{D}}$ is homogenizable, or there are no eigenforms and every limit point of $\mathcal{N}(M_n(E))$ is reducible for every $E \in \widetilde{\mathcal{D}}$. ∎

REMARK 4.24. – I do not know whether, when there are no eigenforms, every $E \in \widetilde{\mathcal{D}}$ is homogenizable even in the weaker sense that $\mathcal{N}(M_n(E))$ is convergent (the limit would be reducible by Remark 4.23). However, if $E \in \mathcal{D} \setminus \widetilde{\mathcal{D}}$, E may fail to be homogenizable, even in this weaker sense. Consider the prefractal structure given by, $N = k = 4$, $\mathcal{R}$ defined by the relations $(i_1, j_1)\,\mathcal{R}(i_2, j_2)$ for $j_1 \equiv i_1 + \theta \pmod 4$, $i_2 \equiv i_1 - \theta \pmod 4$, $j_2 \equiv i_1 - 2\theta \pmod 4$, $\theta = \pm 1$ (see Figure 1: $P_5 = \psi_1(P_4) = \psi_2(P_3)$, $P_6 = \psi_2(P_1) = \psi_3(P_4)$, $P_7 = \psi_3(P_2) = \psi_4(P_1)$, $P_8 = \psi_4(P_3) = \psi_1(P_2)$) (2). For $A \geqslant 0$, $B \geqslant 0$, let $E_{A,B}$ be defined by $E_{A,B}(u) = A(u(P_1) - u(P_2))^2 + A(u(P_3) - u(P_4))^2 + B(u(P_1) - u(P_4))^2 + B(u(P_2) - u(P_3))^2$. Then it is easy to see that, when $A > 0$

$$M_n(E_{A,0}) = \begin{cases} \dfrac{1}{2^n} E_{A,0} & \text{if } n \text{ even} \\[2ex] \dfrac{1}{2^n} E_{0,A} & \text{if } n \text{ odd}\,. \end{cases} \qquad \blacksquare$$

(2) In this figure, as well as in Figures 2 and 3, I describe the fractal structure by picturing $V^{(1)}$.

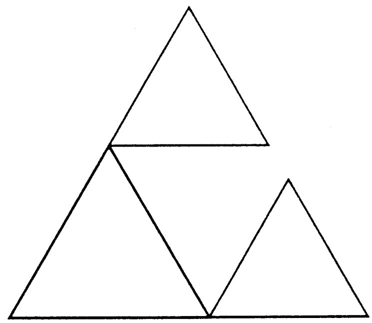

Fig. 2. – See Remark 5.12.

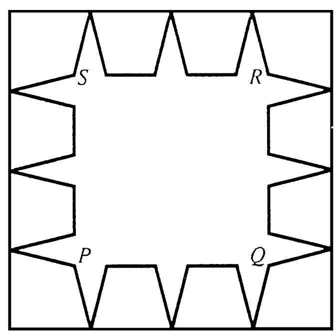

Fig. 3. – See Remark 5.13. Here, $P = R$, $Q = S$.

5. – Uniqueness of eigenforms in fractal structures.

In this section I generalize the uniqueness results of section 2 to the case of fractal structures. In the following G denotes a transitive subgroup of the group $Per(V^{(0)})$ of all permutations of $V^{(0)}$ (more generally, by $Per(A)$ I mean the group of all permutations of a nonempty set A), and of course $\widetilde{\mathcal{D}}_G$ (resp. $\mathcal{D}_G$) denotes the set of those $E \in \widetilde{\mathcal{D}}$ (resp. $E \in \mathcal{D}$) such that $E(u \circ \sigma) = E(u)$ for every $\sigma \in G$. In order to extend Theorem 2.6 to fractal structures, the only nontrivial fact to prove is the strong minimum principle (s.m.p.). Note, in fact, that the uniqueness argument of Theorem 2.6, unlike those of [14] and [17], does not use any specific property of nested fractals. We see, in view of Prop. 4.10, that $E \in \widetilde{\mathcal{D}}$ satisfies s.m.p. provided $(V^{(0)} \setminus \{P\}; gr_0(E))$ is connected for all $P \in V^{(0)}$. Note that this is trivially true if $c_{j_1, j_2} > 0$. We will now see that every $E \in \widetilde{\mathcal{D}}_G$ satisfies s.m.p., in particular we will finally prove completely Prop. 2.1.

LEMMA 5.1. – *Let* $E \in \widetilde{\mathcal{D}}_G$. *Then* E *satisfies s.m.p..*

PROOF. – Suppose $P \in V^{(0)}$ is such that $(V^{(0)} \setminus \{P\}; gr_0(E))$ is disconnected. Since $E \in \widetilde{\mathcal{D}}_G$ and G is transitive, it is easy to see that $(V^{(0)} \setminus \{Q\}; gr_0(E))$ is disconnected for every $Q \in V^{(0)}$. Now, we can easily find $Q_0, \dots, Q_N \in V^{(0)}$ such that $Q_i \neq Q_{i+1}$, and Q_i and Q_{i+2} are not connected in $(V^{(0)} \setminus \{Q_{i+1}\}; gr_0(E))$, and this contradicts Lemma 4.8. ■

THEOREM 5.2. – *Any two eigenforms in* $\widetilde{\mathcal{D}}_G$ *are multiples of each other.* ■

PROOF. – We can proceed in the exactly same way as in Theorem 2.6. ■

I state explicitly the following result which is implicit in the proofs of Theorem 2.6 and Remark 2.7.

LEMMA 5.3. – *If* $E \in \widetilde{\mathcal{D}}$ *satisfies s.m.p., then there exists a unique eigenvector* $u_j (= u_{j;E})$ *of* T_j *such that* $\|u_j\| = 1$, $u_j(P_j) = 0$, $u_j(P) \geq 0$ *for all P. If in addition,* $E' \in \widetilde{\mathcal{D}}$ *and* E, E' *are eigenforms, and* $u \in \widetilde{A}^{\pm}(E, E')$ *and u attains its minimum or its maximum at* P_j, *then* $u_j \in \widetilde{A}^{\pm}(E, E')$; *if in addition,* u_j *takes its maximum at* $P_{j'}$ *then* $u_{j'} \in \widetilde{A}^{\pm}(E, E')$. ■

In order to understand better the questions concerning the existence and uniqueness of eigenforms, I will introduce the notion of symmetry with respect to a transitive group of permutations of $V^{(0)}$. The following definitions, for the case $G = Per(V^{(0)})$, have been introduced in [15]. In fact, the definitions of [15] slightly differ (a priori) from those in Def. 5.4, but they are equivalent in the practical cases. Notions like that of strongly G-symmetric structure have been introduced in other articles (using a different terminology), (cf. for example [9], [17]). Instead, the notion of weakly G-symmetric structure, to my knowledge, has not appeared in other papers (apart from [15] for the case $G = Per(V^{(0)})$). One of the reasons of introducing it is that the structure of every nested fractal is weakly G-symmetric (Remark 5.5). Prop. 5.6 is, in some sense, the natural generalization of the analogous and well known result for nested fractals, but I prefer to prove it in detail in order to clarify the role of G-symmetry. Corollary 5.7 has been stated in [15]. The uniqueness results in Corollary 5.7 and Theorem 5.10 are obtained by using the method of section 2, essentially Lemma 2.4, and, at first glance, they do not appear to be simply obtainable by the methods of [14] and [17]; instead the existence result in Theorem 5.10 and the non-existence example of Remark 5.13 can also be obtained using the method of [17].

DEFINITION 5.4. – *a)* $\mathcal{F}$ *is weakly G-symmetric if for every* $\sigma \in G$, *there exist* $\tilde{\sigma} \in Per(V^{(1)})$ *with* $\tilde{\sigma}_{|V^{(0)}} = \sigma$, $\beta \in Per(\{1, \dots, k\})$, $\tau_{i,\sigma} \in G$ *for* $i = 1, \dots, k$, *such that* $\tilde{\sigma} \circ \psi_i = \psi_{\beta(i)} \circ \tau_{i,\sigma}$ *for all* $i = 1, \dots, k$.

b) $\mathcal{F}$ *is strongly G-symmetric if in the preceding definition we can take* $\tau_{i,\sigma} = \sigma$.

I omit «G» when $G = Per(V^{(0)})$ *and write simply weakly symmetric, strongly symmetric.*

Note that, a priori, $\mathcal{F}$ can be weakly symmetric but not weakly G-symmetric.

REMARK 5.5. – The Gasket in $\mathbb{R}^\nu$ is strongly symmetric. The Vicsek set in $\mathbb{R}^\nu$ is weakly symmetric, and strongly G_1-symmetric for suitable G_1. These facts can be easily verified and are suggested by geometrical considerations. Moreover, it is not difficult to see that the Vicsek set in $\mathbb{R}^\nu$ is not strongly symmetric and the snowflake is not weakly symmetric. If $\mathcal{F}$ is the fractal structure of a nested fractal, then $\mathcal{F}$ is weakly G-symmetric where G is the group described in section 2. Indeed, from the properties of the nested fractals it easily follows that, for every $i = 1, \dots, k$, $\phi_{j_1, j_2} \circ \psi_i = \psi_{\beta(i)} \circ \tau_{i, j_1, j_2}$ on $V^{(0)}$ for suitable $\beta(i)$, where τ_{i, j_1, j_2} is an isometric transformation of $V^{(0)}$, and it is easy to prove that every isometric transformation of $V^{(0)}$ is in G. If, in addition, every ψ_i has the form $\psi_i(x) = Q_i + (x/R)$ for some $Q_i \in \mathbb{R}^\nu$, $R > 1$ (this occurs for the most usual nested fractals, in particular for those considered in this paper), then it is easy to verify that $\tau_{i, j_1, j_2} = \phi_{j_1, j_2}$, so $\mathcal{F}$ is strongly G-symmetric. Here, I have used the simple fact that it is sufficient that a) (resp. b)) of Def. 5.4 holds when σ is in a set of generators of G, to have that $\mathcal{F}$ is weakly (resp. strongly) G-symmetric. ■

PROPOSITION 5.6. – *If* $\mathcal{F}$ *is weakly G-symmetric, then* $M_1(\tilde{\mathcal{D}}_G) \subseteq \tilde{\mathcal{D}}_G$.

PROOF. – Given $E \in \tilde{\mathcal{D}}_G$ and $\sigma \in G$, we have

$$M_1(E)(u \circ \sigma) = \min_{v \in \mathcal{L}(1, u \circ \sigma)} \sum_{i=1}^k E(v \circ \psi_i) =$$

$$\min_{v \in \mathcal{L}(1, u)} \sum_{i=1}^k E(v \circ \tilde{\sigma} \circ \psi_i) = \min_{v \in \mathcal{L}(1, u)} \sum_{i=1}^k E(v \circ \psi_{\beta(i)} \circ \tau_{i,\sigma}) =$$

$$\min_{v \in \mathcal{L}(1, u)} \sum_{i=1}^k E(v \circ \psi_{\beta(i)}) = \min_{v \in \mathcal{L}(1, u)} \sum_{i=1}^k E(v \circ \psi_i) = M_1(E)(u). \qquad ■$$

COROLLARY 5.7. – *If $\mathcal{F}$ is weakly symmetric, then $\overline{E}$ is an eigenform. If in addition $\mathcal{F}$ is strongly symmetric, then every eigenform is a multiple of $\overline{E}$.*

PROOF. – For the first statement it is sufficient to note that $\widetilde{\mathcal{D}}_{Per(V^{(0)})} = \{t\overline{E}: t>0\}$, for the second it is easy to see that strong symmetry permits us to imitate the argument of Remark 2.7. ■

REMARK 5.8. – If there exists an eigenform on $\mathcal{F}$ and $\mathcal{F}$ is weakly G-symmetric, then there exists an eigenform in $\widetilde{\mathcal{D}}_G$. Indeed, by Theorem 4.22, given $E \in \widetilde{\mathcal{D}}_G$, then E is homogenizable, and by Prop. 5.6 and a trivial argument, $\lim_{n\to\infty} \widetilde{M}_n(E)$ is in $\widetilde{\mathcal{D}}_G$ and, also is an eigenform. ■

REMARK 5.9. – I have not investigated deeply the problem of the existence of an eigenform. Apart from the Theorem of Lindstrøm and Corollary 5.7, rather general results are given by C. Sabot in [17], and it is well known that not all fractal structures have eigenforms (in $\widetilde{\mathcal{D}}$). However, we now see that there is always a non-zero eigenform in $\mathcal{D}$. A similar result is given in [13], but there it is not proved that the eigenform is not 0. Clearly, there exists $c>0$ such that the set $P = \left\{E \in \mathcal{D}: M_1(E) \geqslant cE,\ \sum_{i=1}^{N} E(e_i) = 1\right\}$ contains some $E \in \widetilde{\mathcal{D}}$, thus is nonempty. It is not difficult, using the convexity of the elements of $\mathcal{D}$ and the fact that P is equibounded on the compact subsets of $\mathbb{R}^{V^{(0)}}$, to prove that P is a compact and convex subset of $\mathcal{D}$. Let $\widehat{M}_1: P \to P$ be defined by

$$\widehat{M}_1(E) = \frac{M_1(E)}{\sum_{i=1}^{N} M_1(E)(e_i)}.$$

We can easily see that in fact $\widehat{M}_1(E) \in P\ \forall E \in P$, and that $\widehat{M}_1$ is continuous. By the Schauder Theorem there exists $E \in P$ such that $\widehat{M}_1(E) = E$. ■

We will now see that when N is prime we have existence and uniqueness of eigenform in $\widetilde{\mathcal{D}}$, provided it is sufficiently symmetric. As seen in the introduction, it is known that the Vicsek set shows that for $N=4$ this could not occur. Some particular cases of this uniqueness result (either $N=3$ or $N=5$ with an additional condition, for nested fractals) have been proved in [9].

THEOREM 5.10. – *If $\mathcal{F}$ is weakly G-symmetric, and N is prime, then there exists an eigenform. If in addition $\mathcal{F}$ is strongly G-symmetric, then the eigenform is unique up to a multiplicative constant.*

PROOF. – For the existence we can imitate the proof in Remark 5.9. Let $P = \{E \in \widetilde{\mathcal{D}}_G: E(e_1) = 1\}$. Let $\widehat{M}_1: P \to P$ be defined by $\widehat{M}_1(E) = (M_1(E)/M_1(E)(e_1))$. Clearly, P is convex. Also, if $E \in P$, then $E(e_1) = 1$, thus

$E(e_i) = 1$ for every i for, if $\sigma \in G$ satisfies $\sigma(P_1) = P_i$, then $E(e_i) = E(e_i \circ \sigma) = E(e_1) = 1$. By convexity, P is equibounded on every bounded set, thus is relatively compact. It remains to prove that P is closed. For this it suffices to prove that every $E \in \mathcal{D}_G$ for which $E(e_1) = 1$ is in $\widetilde{\mathcal{D}}$. Now, there exist $j_1, j_2 : j_1 \neq j_2$, and $c_{j_1, j_2}(E) > 0$. Note that G being transitive, it is easy to see that ord$G$ is an integer multiple of N, thus since N is prime, G contains a cyclic permutation σ, and we can choose σ in G, such that $\sigma(P_{j_1}) = P_{j_2}$. Let $P_{j_n} = \sigma^{n-1}(P_{j_1})$ for $n = 1, \dots, N$. Since $E \in \mathcal{D}_G$, we easily deduce $c_{j_n, j_{n+1}}(E) = \underset{\sim}{c}_{j_1, j_2}(E) > 0$ for $n = 1, \dots, N-1$. Thus $(V^{(0)}, gr_0(E))$ is connected and $E \in \widetilde{\mathcal{D}}$. Suppose now $\mathcal{F}$ is strongly G-symmetric and prove the uniqueness of the eigenform up to a multiplicative constant. Let $E_1 \in \widetilde{\mathcal{D}}_G$, $E_2 \in \widetilde{\mathcal{D}}$ be eigenforms. Put $u_j = u_{j, E_1}$ and

$$S(P_j) = \{P_{j'} \in V^{(0)} : u_j(P_{j'}) = \max u_j\}$$

for $j = 1, \dots, N$. Since $\mathcal{F}$ is strongly G-symmetric it is easy to see that

$$S(\sigma(P_j)) = \sigma(S(P_j)) \ \forall \sigma \in G . \tag{5.1}$$

Also, by Lemma 5.3, there exists $u_{j_1} \in \widetilde{A}^{\pm}(E_1, E_2)(= \widetilde{A}^{\pm})$, and, if $u_j \in \widetilde{A}^{\pm}$ and $P_{j'} \in S(P_j)$, then $u_{j'} \in \widetilde{A}^{\pm}$. Let $P_{j_2} \in S(P_{j_1})$, and let $\sigma \in G$ be a cyclic permutation such that $\sigma(P_{j_1}) = P_{j_2}$. By (5.1) we see that $u_j \in \widetilde{A}^{\pm}$ for all $j = 1, \dots, N$, thus $\widetilde{A}^{+} \cap \widetilde{A}^{-} \neq \emptyset$. ■

REMARK 5.11. – The case $N = 3$ can be discussed rather in detail. The first fact to note is that if $E \in \widetilde{\mathcal{D}}$ is an eigenform satisfying s.m.p., then every eigenform is a multiple of E. Indeed, if $E' \in \widetilde{\mathcal{D}}$ is an eigenform, by Lemma 5.3 we find $u_{j_1}, u_{j_2} \in \widetilde{A}^{+}(E, E')$ with $j_1 \neq j_2$, $u_{j_1'}, u_{j_2'} \in \widetilde{A}^{-}(E, E')$ with $j_1' \neq j_2'$. Since $N = 3$, $\widetilde{A}^{+}(E, E') \cap \widetilde{A}^{-}(E, E') \neq \emptyset$.

It follows that if there exist two eigenforms which are not multiples of each other, then there exist $j_1, j_2 = 1, 2, 3$ such that $\underset{\sim}{c}_{j_1, j_2}(E) = 0$ for every E eigenform. Indeed, if there exists an eigenform $E \in \widetilde{\mathcal{D}}$ such that $c_{j_1, j_2}(E) > 0$ for all j_1, j_2, then E satisfies s.m.p. and thus, as seen above, every eigenform is a multiple of E. So, for every eigenform $E \in \widetilde{\mathcal{D}}$, we have $c_{j_1, j_2}(E) = 0$ for some j_1, j_2 (depending on E). On the other hand the set Ei of all eigenforms in $\widetilde{\mathcal{D}}$ is connected, for it is easy to deduce from Theorem 4.22 that the map $E \mapsto \|E\| \lim\limits_{n \to \infty} \mathcal{N}(M_n(E))$ is a continuous surjection from $\widetilde{\mathcal{D}}$ to Ei. Now, it is easy to conclude that we can find j_1, j_2 such that $c_{j_1, j_2}(E) = 0$ for every $E \in Ei$.

In view of Prop. 4.11, a case in which every $E \in \widetilde{\mathcal{D}}$ satisfies $c_{j, j'}(M_1(E)) > 0$ for $j \neq j'$, $E \in \widetilde{\mathcal{D}}$, is when any two points of $V^{(0)}$ can be connected in $V^{(1)} \setminus V^{(0)}$, using only any two of the three coefficients, more precisely: Given j_1, j_2, j_3, j_4, j_5; P_{j_1} and P_{j_2} are connected in $(V^{(1)} \setminus V^{(0)}; gr_1)$ by a path whose edges have the form $\{\psi_i(P_{j_3}), \psi_i(P_{j_4})\}$ or $\{\psi_i(P_{j_3}), \psi_i(P_{j_5})\}$. Thus, under this condition, any two eigenforms are multiples of each other. Note that this condition is satisfied by many fractal structures, for example by many kinds of «asymmetric gasket» (some kinds of asymmetric gasket are considered in [4] and [17]). ■

REMARK 5.12. – The following example shows that under the sole condition $N = 3$, we can have infinitely many eigenforms which are, mutually, not multiples of each other. Suppose $N = k = 3$, and $\mathcal{R}$ is the equivalence relation defined by $(1, 2)\mathcal{R}(2, 1)$, $(1, 3)\mathcal{R}(3, 1)$ (see Figure 2). Then, it is easy to see that every E_a of the form $E_a(u) = a(u(P_1) - u(P_2))^2 + (u(P_1) - u(P_3))^2$, $a > 0$, is an eigenform in $\widetilde{\mathcal{D}}$. ■

REMARK 5.13. – By Corollary 5.7 there exists an eigenform if the structure is weakly symmetric. Here I give an example of a strongly G-symmetric fractal structure without eigenforms (cf. also Theorem 5.10). Consider the prefractal structure with $N = 4$, $k = 4m, (m = 2, 3, 4 \ldots)$, and $\mathcal{R}$ defined by the following relations:

$$(1, 3)\ \mathcal{R}(3, 1), (2, 4)\ \mathcal{R}(4, 2), \big(\Delta(\overline{am + k}), \Delta(\overline{(a+1)m + 1})\big)\ \mathcal{R}$$

$$\big(\Delta(\overline{am + k + 1}), \Delta(\overline{am + 1})\big)$$

for $a = 0, 1, 2, 3$, $k = 1, \ldots, m$, where $\Delta : \mathbb{Z}_{4m} \to \{1, \ldots, 4m\}$ is defined by $\Delta(\overline{am + 1}) = a + 1$, $\Delta(\overline{am + k}) = a(m - 1) + k + 3$, for $a = 0, 1, 2, 3$, $k = 2, \ldots, m$ (see Figure 3). Then it is easy to see that $\mathcal{F}$ is strongly G-symmetric, where G is the group generated by the «rotation of $\pi/2$». To see that $\mathcal{F}$ has no eigenforms (for sufficiently large m), consider $u_1, u_2 : V^{(0)} \to \mathbb{R}$ defined by $u_1 = e_1 + e_3$, $u_2 = e_1 + e_2$. If $E \in \mathcal{D}$, then $E \in \widetilde{\mathcal{D}}_G$ if and only if we have $c_{1,2}(E) = c_{2,3}(E) = c_{3,4}(E) = c_{1,4}(E)$, $c_{1,3}(E) = c_{2,4}(E)$. Moreover, if $E \in \widetilde{\mathcal{D}}_G$, then $c_{1,2}(E) = (1/4)\, E(u_1)$, $c_{1,3}(E) = (1/2)\, E(u_2) - (1/4)\, E(u_1)$, thus by simple calculations it is possible to see that for sufficiently large m, if $E \in \widetilde{\mathcal{D}}_G$, and $(c_{1,3}(E)/c_{1,2}(E)) \geq A$ where A is a suitable constant > 0, we have $\beta_n \geq 2^n \beta_0$, where $\beta_n = c_{1,3}(M_n(E))/c_{1,2}(M_n(E))$. Therefore there are no eigenforms (for example by Remark 3.2). ■

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Roberto Peirone: Università di Roma - Tor Vergata
Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail: peirone@mat.uniroma2.it

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