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On Blow-Up and Asymptotic Behavior of Solutions for some Semilinear Parabolic Systems of Second Order.

THÉODORE K. BONI

Sunto. – *In questo lavoro sotto queste ipotesi si ottiengono alcune condizioni di non esistenza e di esistenza delle soluzioni per alcuni sistemi parabolici semilineari del secondo ordine. Inoltre si studia il comportamento asintotico di alcune soluzioni.*

1. – Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following boundary value problems:

(I)

(1.1) $$\frac{\partial u_i}{\partial t} = L_i u_i + f_i(u_{i+1})\, f_{*i}(u_i) \quad \text{in } \Omega \times (0, T),$$

(1.2) $$\mu_i \frac{\partial u_i}{\partial N_i} + (1-\mu_i)\, u_i = 0 \quad \text{on } \partial\Omega \times (0, T),$$

(1.3) $$u_i(x, 0) = u_0^{(i)}(x) \quad \text{in } \Omega,$$

(II)

(1.4) $$\frac{\partial u_i}{\partial t} = L_0 u_i - a(x)\, u_i \quad \text{in } \Omega \times (0, T),$$

(1.5) $$\frac{\partial u_i}{\partial N_0} + b(x)\, u_i = g_i(u_{i+1}) \quad \text{on } \partial\Omega \times (0, T),$$

(1.6) $$u_i(x, 0) = u_0^{(i)}(x) \quad \text{in } \Omega,$$

where $i = 1, \ldots, m$, $u_{m+1} = u_1$, μ_i and $b(x)$ are nonnegative functions on $\partial\Omega$

with $\mu_i \leqslant 1$, $a(x)$ is a nonnegative function in Ω. For $l \in \{0, 1, \dots, m\}$,

$$L_l u_i = \sum_{k,j=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kj}^{(l)}(x)\frac{\partial u_i}{\partial x_j}\right), \qquad \frac{\partial u_i}{\partial N_l} = \sum_{k,j=1}^{n} \cos(\nu, x_k)\, a_{kj}^{(l)}(x)\frac{\partial u_i}{\partial x_j}.$$

Here, the coefficients $a_{kj}^{(l)}(x) \in C^1(\Omega)$ satisfy the following inequalities

$$\lambda_1^{(l)}|\xi|^2 \geqslant \sum_{k,j=1}^{n} a_{kj}^{(l)}(x)\,\xi_k\xi_j \geqslant \lambda_2^{(l)}|\xi|^2$$

for any $\xi \in \mathbb{R}^n$ and $x \in \Omega$ with positive constants $\lambda_1^{(l)}$, $\lambda_2^{(l)}$. ν is the exterior normal unit vector on $\partial\Omega$, $f_{*i}(s)$, $f_i(s)$, $g_i(s)$ are nonnegative and increasing functions for positive values of s with $f_i(0) = g_i(0) = 0$. $u_0^{(i)}(x)$ are positive and continuous functions in Ω.

In this note, if $h_1(s)$ and $h_2(s)$ are two positive functions defined in $(0, \infty)$, we put $h_1 \circ h_2(s) = h_1[h_2(s)]$.

We want to determine when the nonnegative solutions are global, i.e defined for every $t \in (0, \infty)$.

DEFINITION 1.1. – *We say that a solution* $(u_1, \dots, u_m)$ *of the problem* (1.1)-(1.3) *or* (1.4)-(1.6) *blows up in a finite time if there exists a finite time* T_0 *such that*

$$\lim_{t\to T_0}\left\{\sum_{i=1}^{m}\|u_i(x,t)\|_{L^\infty(\Omega)}\right\} = \infty .$$

T_0 *is the blow up time of the solution* $(u_1, \dots, u_m)$. *A point* $x \in \overline{\Omega}$ *is a blow up point of the solution* $(u_1, \dots, u_m)$ *if there exists a sequence* (x_n, t_n) *such that* $x_n \to x$, $t_n \to T_0$ *and* $\lim_{n\to\infty}\left\{\sum_{i=1}^{m}|u_i(x_n,t_n)|\right\} = \infty$. *The set*

$$E_B = \{x \in \overline{\Omega} \text{ such that } x \text{ is a blow up point of the solution } (u_1, \dots, u_m)\}$$

is the blow up set of the solution $(u_1, \dots, u_m)$.

The global existence and blow-up of solutions for parabolic systems of second order have been the subject of investigation of many authors (see, for instance [1], [3], [4], [5], [6], [7], [10], [12]). In [4], Escobedo and Herrero have considered the following system:

$$\frac{\partial u}{\partial t} = \Delta u + v^p \quad \text{in} \quad \Omega \times (0, T),$$

$$\frac{\partial v}{\partial t} = \Delta v + u^q \quad \text{in} \quad \Omega \times (0, T),$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \qquad v = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$u(x,0)=u_0(x) \quad \text{in} \quad \Omega, \qquad v(x,0)=v_0(x) \quad \text{in} \quad \Omega.$$

They have shown that if $pq>1$, there are global and blow up nonnegative solutions. In [12], Rossi and Wolanski have studied the following system:

$$\frac{\partial u}{\partial t}=\Delta u+v^p e^u \quad \text{in} \quad \Omega\times(0,T),$$

$$\frac{\partial v}{\partial t}=\Delta v+u^q e^v \quad \text{in} \quad \Omega\times(0,T),$$

$$u=0 \quad \text{on} \quad \partial\Omega\times(0,T), \qquad v=0 \quad \text{on} \quad \partial\Omega\times(0,T),$$

$$u(x,0)=u_0(x) \quad \text{in} \quad \Omega, \qquad v(x,0)=v_0(x) \quad \text{in} \quad \Omega.$$

They have also shown that if $pq>1$, there are global and blow up nonnegative solutions. In their analysis, they remark that the phenomenon of global existence and blow up depends on the nature of the domain. In this paper, we generalize these results considering the problem of the form (1.1)-(1.3). We also give some conditions under which the solutions of the problem (1.1)-(1.3) tend to zero and describe their asymptotic behavior. Finally, we study the asymptotic behavior of some global solutions. For the problem (1.4)-(1.6), some authors have studied the blow up of the solutions under some conditions (see, for instance [6], [10]). An interesting question of the problem (1.4)-(1.6) is the localization of the blow up set. This problem has been studied by some authors in the case where $m=2$, $L_0=\Delta$, $a(x)=0$, $b(x)=0$, $g_1(u_2)=u_2^p$, $g_2(u_1)=u_1^q$ with $p>1$, $q>1$ (see, for instance [3]). In this paper, we give another characterization of the blow up of solutions for the problem (1.4)-(1.6) and describe their blow up set. The paper is written in the following manner. In Section 2, we give some conditions of global existence of solutions for the problem (1.1)-(1.3). In Section 3, we obtain some conditions under which the solutions of (1.1)-(1.3) tend to zero as $t\to\infty$ and describe their asymptotic behavior. In Sections 4 and 5, we obtain some blow up conditions of solutions for the problem (1.1)-(1.3). In Section 6, we give the asymptotic behavior of some global solutions for the problem (1.1)-(1.3) and finally, in Section 7, we study the blow up set of some blow up solutions for the problem (1.4)-(1.6).

We recall that in this work, we consider the nonnegative solutions.

2. – Global existence.

In this section, we give some conditions under which the solutions of the problem (1.1)-(1.3) exist globally.

If $f_i(s)$ are locally Lipschitz continuous, local existence and uniqueness of nonnegative solution are well known (see, for instance [9]). Now consider the general case. Let $(u_{1n}, \dots, u_{mn})$ satisfying $u_{in} \geqslant 1/n$ be the maximum solution of the following system

$$\frac{\partial u_i}{\partial t} = L_i u_i + f_{in}(u_{i+1})\, f_{*i}(u_i) \quad \text{in} \quad \Omega \times (0, T),$$

$$\mu_i \frac{\partial u_i}{\partial N_i} + (1-\mu_i)\, u_i = \frac{1}{n} \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$u_i(x, 0) = u_0^{(i)}(x) + \frac{1}{n} \quad \text{in} \quad \Omega,$$

where $f_{in}(s) = f_i(s)$ for $s \geqslant 1/n$. f_{in} are locally Lipschitz in $\mathbb{R}$. Using the maximum principle, we see that u_{in} $(i = 1, \dots, m)$ are nonincreasing sequences such that $u_{in} \geqslant 0$. Therefore $u_i = \lim_{n\to\infty} u_{in}$ $(i = 1, \dots, m)$ exist. Using the «variation of constant formula», we obtain the result.

The following lemma which will be useful later.

COMPARISON LEMMA 2.1. – *Let* $(\overline{u_1}, \dots, \overline{u_m})$ *satisfying the following inequalities*:

$$\frac{\partial \overline{u_i}}{\partial t} \geqslant L_i \overline{u_i} + f_i(\overline{u_{i+1}}) f_{*i}(\overline{u_i}) \quad \text{in} \quad \Omega \times (0, T),$$

$$\mu_i \frac{\partial \overline{u_i}}{\partial N_i} + (1-\mu_i)\overline{u_i} > 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$\overline{u_i}(x, 0) > u_0^{(i)}(x) \quad \text{in} \quad \Omega, \quad i = 1, \dots, m,$$

where $\overline{u_{m+1}} = \overline{u_1}$ *and* $\overline{u_i}(x, 0)$ *are continuous up to* $t = 0$. *If* $(u_1, \dots, u_m)$ *is a solution of the problem* (1.1)-(1.3) *with initial data* $(u_0^{(1)}, \dots, u_0^{(m)})$, *then we have*

$$u_i(x, t) < \overline{u_i}(x, t) \quad \text{in} \quad \Omega \times (0, T), \quad i = 1, \dots, m.$$

We call $(\overline{u_1}, \dots, \overline{u_m})$ *supersolution of the problem* (1.1)-(1.3).

PROOF. – We have $\overline{u_i}(x, 0) - u_0^{(i)}(x) > \delta$ in Ω and $\mu_i(\partial\overline{u_i}/\partial N_i) + (1-\mu_i)\,\overline{u_i} > \delta$ on $\partial\Omega \times (0, T)$ for some $\delta > 0$. Let

$$T_0 = \sup\left\{ t \text{ such that } \overline{u_i}(x, t) - u_i(x, t) > \frac{\delta}{2} \text{ for all } i \right\}.$$

$T_0 > 0$ because the function $\overline{u_i}(x, 0) - u_i(x, 0)$ is continuous up to $t = 0$ and $\overline{u_i}(x, 0) - u_i(x, 0) > \delta$. We also have $\overline{u_j}(x_0, T_0) = u_j(x_0, T_0) + \delta/2$ for some

$j \in \{1, \ldots, m\}$ and some $x_0 \in \Omega$. Therefore we get

$$\frac{\partial(\overline{u_j} - u_j)}{\partial t} - L_j(\overline{u_j} - u_j) \geq f_j(\overline{u_{j+1}}) f_{*j}(\overline{u_j}) - f_j(u_{j+1})\, f_{*j}(u_j) \geq 0 \text{ in } \Omega \times (0, T_0),$$

because the functions $f_j(s)$ and $f_{*j}(s)$ are nonnegative, increasing for positive values of s. We also have

$$\mu_j \frac{\partial(\overline{u_j} - u_j)}{\partial N_j} + (1 - \mu_j)(\overline{u_j} - u_j) > \delta \quad \text{on} \quad \partial\Omega \times (0, T_0),$$

$$\overline{u_j}(x, 0) - u_j(x, 0) > \delta \quad \text{in} \quad \Omega .$$

From the maximum principle, we deduce that $\overline{u_j}(x, t) - u_j(x, t) \geq \delta$ in $\Omega \times (0, T_0)$. This implies that $\overline{u_j}(x_0, T_0) - u_j(x_0, T_0) > \delta/2$, which is a contradiction. Then we have the result. ■

THEOREM 2.2. – *Suppose that*

$$\lim_{s \to 0} \frac{f_1 \circ (c_2 f_2) \circ \ldots \circ (c_m f_m)(s)}{s} = 0 ,$$

where c_j $(j = 2, \ldots, m)$ are positive constants. Then there exists a positive constant a_0 such that any solution $(u_1, \ldots, u_m)$ of the problem (1.1)-(1.3) *with initial data $(u_0^{(1)}, \ldots, u_0^{(m)})$ exists globally for $u_0^{(i)}(x) < a_0$.*

REMARK 2.3. – Suppose that the functions $f_m(s)$, $f_{m-1} \circ f_m(s)$, $\ldots$, $f_2 \circ \ldots \circ f_m(s)$ are convex for small positive values of s and $f_m(0) = 0$, $f_{m-1} \circ f_m(0) = 0, \ldots, f_2 \circ \ldots \circ f_m(0) = 0$. If

$$\lim_{s \to 0} \frac{f_1 \circ \ldots \circ f_m(s)}{s} = 0 ,$$

then we have

$$\lim_{s \to 0} \frac{f_1 \circ (c_2 f_2) \circ \ldots \circ (c_m f_m)(s)}{s} = 0 ,$$

where c_j $(j = 2, \ldots, m)$ are positive constants.

In fact, since $f_i(s)$ are increasing functions, we obtain

$$\lim_{s \to 0} \frac{f_1 \circ (c_2 f_2) \circ \ldots \circ (c_m f_m)(s)}{s} \leq \lim_{s \to 0} \frac{f_1 \circ (c_{*2} f_2) \circ \ldots \circ (c_{*m} f_m)(s)}{s}$$

where $c_{*j} \geqslant \sup\{1, c_j\}$. It follows that

$$\lim_{s\to 0} \frac{f_1 \circ (c_2 f_2) \circ \ldots \circ (c_m f_m)(s)}{s} \leqslant \lim_{s\to 0} \frac{f_1 \circ f_2 \circ \ldots \circ f_m(c_{*2} \ldots c_{*m} s)}{s} = 0\,,$$

because the functions $f_m(s)$, $f_{m-1} \circ f_m(s)$, $\ldots$, $f_2 \circ \ldots \circ f_m(s)$ are convex for small positive values of s with $f_m(0) = 0$, $f_{m-1} \circ f_m(0) = 0$, $\ldots$, $f_2 \circ \ldots \circ f_m(0) = 0$.

PROOF OF THEOREM 2.2. – For $k \in \{1, \ldots, m\}$, let $\Phi_k(x)$ be a solution of the following problem:

$$L_k \Phi_k(x) = -1 \quad \text{in} \quad \Omega\,, \tag{2.1}$$

$$\mu_k \frac{\partial \Phi_k(x)}{\partial N_k} + (1-\mu_k)\, \Phi_k(x) = 0 \quad \text{on} \quad \partial\Omega\,, \tag{2.2}$$

$$\Phi_k(x) > 0 \quad \text{in} \quad \Omega\,. \tag{2.3}$$

Let

$$\overline{u_i} = a_i(\Phi_i(x) + \delta)\,, \tag{2.4}$$

where δ is a positive constant, and a_i $(i = 1, \ldots, m)$ are positive constants which will be indicated later. Put $K_i = \sup_{x\in\Omega}\{\Phi_i(x) + \delta\}$. We have

$$\frac{\partial \overline{u_i}}{\partial t} - L_i \overline{u_i} - f_i(\overline{u_{i+1}})\, f_{*i}(\overline{u_i}) \geqslant a_i - f_i(a_{i+1} K_{i+1})\, f_{*i}(a_i K_i)\,, \tag{2.5}$$

$$\mu_i \frac{\partial \overline{u_i}}{\partial N_i} + (1-\mu_i)\, \overline{u_i} = \tag{2.6}$$

$$a_i\left(\mu_i \frac{\partial \Phi_i(x)}{\partial N_i} + (1-\mu_i)\, \Phi_i(x)\right) + a_i \delta(1-\mu_i) = a_i \delta (1-\mu_i), \qquad i = 1, \ldots, m\,,$$

where $\overline{u_{m+1}} = \overline{u_1}$, $a_{m+1} = a_1$, $K_{m+1} = K_1$. Show that there exist a_i $(i = 1, \ldots, m)$ such that

$$a_i \geqslant f_i(a_{i+1} K_{i+1})\, f_{*i}(a_i K_i), \qquad i = 1, \ldots, m-1\,, \tag{2.7}$$

$$a_m \geqslant f_m(a_1 K_1)\, f_{*m}(a_m K_m)\,. \tag{2.8}$$

Let a_l $(l = 2, \ldots, m)$ satisfy the following relations

$$f_i(a_{i+1} K_{i+1}) = \frac{a_i}{f_{*i}(a_i K_i)}\,, \qquad i = 1, \ldots, m-1\,. \tag{2.9}$$

(2.9) may be written in the following form

$$a_i K_i = c_{a_i} f_i(a_{i+1} K_{i+1}), \qquad i = 1, \dots, m-1, \tag{2.10}$$

where for $k \in \{1, \dots, m-1\}$, c_{a_k} is a positive constant which depends on a_k. Therefore, we have

$$a_2 K_2 = (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})(a_m K_m). \tag{2.11}$$

Now, show that we can determine a_1 such that the inequality (2. 8) be satisfied. Since $f_1 \circ (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})$ is an increasing function, multiplying inequality (2.8) by K_m, we obtain

$$\begin{aligned} &f_1 \circ (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})(a_m K_m) \\ &\qquad \geqslant f_1 \circ (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})[K_m f_m(a_1 K_1)\, f_{*m}(a_m K_m)]. \end{aligned} \tag{2.12}$$

From (2.9), (2.11) and (2.12), it follows that

$$\begin{aligned} \frac{1}{f_{*1}(K_1 a_1)} = \frac{f_1(a_2 K_2)}{a_1} = \frac{f_1 \circ (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})(a_m K_m)}{a_1} \geqslant \\ \frac{f_1 \circ (c_{a_2} f_2) \circ \dots \circ (c_{a_{m-1}} f_{m-1})[f_m(a_1 K_1)\, K_m f_{*m}(a_m K_m)]}{a_1}. \end{aligned} \tag{2.13}$$

By hypothesis, the last term of (2.13) tends to zero as a_1 tends to zero. Then take a_1 so small that (2.13) holds. This implies that (2.8) is satisfied. Put $K_i' = \inf_{x \in \Omega} \{\Phi_i(x) + \delta\}$. Since (2.7) and (2.8) are valid, taking $a_0 = \inf_{l \in \{1, \dots, m\}} a_l K_l'$ from (2.4)-(2.6), we see that $(\overline{u_1}, \dots, \overline{u_m})$ is a supersolution of the problem (1.1)-(1.3). Therefore $(u_1, \dots, u_m)$ exists globally, which gives the result. ■

COROLLARY 2.4. – *Let* $f_i(u_{i+1}) = u_{i+1}^{p_i}$, $f_{*i}(u_i) = e^{u_i}$ *or* $f_{*i}(u_i) = 1$ *where* p_i *are positive numbers. If* $\prod_{i=1}^m p_i > 1$, *then there exists a positive constant* a_0 *such that any solution* $(u_1, \dots, u_m)$ *of the problem* (1.1)-(1.3) *with initial data* $(u_0^{(1)}, \dots, u_0^{(m)})$ *exists globally for* $\sum_{i=1}^m \|u_0^{(i)}(x)\|_{L^\infty(\Omega)} \leqslant a_0$.

THEOREM 2.5. – *Suppose that* $\mu_i = 0$ $(i = 1, \dots, m)$ *and there exists* $j \in \{1, \dots, n\}$ *such that* $\Omega \subset\subset \Omega_1 \times (0, l) \times \Omega_2$ *where* $\Omega_1 \subset \mathbb{R}^{j-1}$ *and* $\Omega_2 \subset \mathbb{R}^{n-j}$. *Then if* l *is small enough, there exists a positive constant* a_0 *such that any solution* $(u_1, \dots, u_m)$ *of the problem* (1.1)-(1.3) *with initial data* $(u_0^{(1)}, \dots, u_0^{(m)})$ *exists globally for* $u_0^{(i)}(x) < a_0$.

PROOF. – Put $\overline{u_i} = a_i(\Phi_i(x) + \delta)$ where a_i are positive numbers. As in the proof of Theorem 2.2, it is sufficient to show that

$$a_i \geqslant f_i(a_{i+1}K_{i+1})\, f_{*i}(a_i K_i), \qquad i = 1, \ldots, m, \tag{2.14}$$

where $a_{m+1} = a_1$, $K_{m+1} = K_1$ with $K_i = \sup_{x \in \Omega}\{\Phi_i(x) + \delta\}$. Since Ω_1 and Ω_2 are two bounded domains, there exist numbers l_k $(k = 1, \ldots, j-1, j+1, \ldots, n)$ such that $\Omega \subset\subset \prod_{k=1}^{j-1}[0, l_k] \times (0, l) \times \prod_{k=j+1}^{n}[0, l_k] = I$. Let $\psi_i(x_1, x_j, x_2)$ functions defined in I by

$$\psi_i(x_1, x_j, x_2) = \frac{1}{2a_0^{(i)}} x_j(l - x_j), \qquad i = 1, \ldots, m, \tag{2.15}$$

where $a_0^{(i)} = \inf_{x \in \Omega} a_{jj}^{(i)}(x) > 0$, with $x_j \in (0, l)$, $x_1 \in \prod_{k=1}^{j-1}[0, l_k]$ and $x_2 \in \prod_{k=j+1}^{n}[0, l_k]$. We have

$$L_i \psi_i(x_1, x_j, x_2) + 1 \leqslant 0 \quad \text{in} \quad I, \qquad \psi_i(x_1, x_j, x_2) \geqslant 0 \quad \text{on} \quad \partial I. \tag{2.16}$$

Since $\psi_i(x_1, x_j, x_2) > 0$ in $\overline{\Omega}$, from the maximum principle, $\psi_i \geqslant \Phi_i$ in Ω, where for $i \in \{1, \ldots, m\}$, $\Phi_i(x)$ is the solution of the following problem

$$L_i \Phi_i + 1 = 0 \quad \text{in} \quad \Omega, \qquad \Phi_i = 0 \quad \text{on} \quad \partial\Omega. \tag{2.17}$$

Since $\|\psi_i\|_{L^\infty(I)} \leqslant l^2/8a_0^{(i)}$, we also have $w_{i0} = \|\Phi_i\|_{L^\infty(\Omega)} \leqslant l^2/8a_0^{(i)}$. It follows that K_i tends to zero as δ and l tend to zero. Since $f_i(0) = 0$, choose δ and l so small that the inequalities (2.14) hold. Hence the result. ■

3. – Asymptotic behavior of solutions which tend to zero.

In this section, we suppose that $L_i = L_0$, $\mu_i = \mu_0$. We give some conditions under which the solutions of the problem (1.1)-(1.3) tend to zero as $t \to \infty$. We also describe the asymptotic behavior of these solutions. We suppose that for positive values of s, $f_i(s)$ $(i = 1, \ldots, m)$ are functions of class C^1 such that $f_i(0) = f_i'(0) = 0$. Suppose that for any interval $[0, A]$ with $A > 0$, there exist a constant C_* depending on A and $p > 1$ such that

$$f_i(s) \leqslant C_* s^p \qquad \text{for} \qquad s \in [0, A]. \tag{3.1}$$

Let $\varphi(x)$ and λ, be respectively, the first eigenfunction and the first eigenvalue of the following boundary value problem:

$$\text{(P1)} \qquad -L_0\varphi(x) = \lambda\varphi \quad \text{in} \quad \Omega\,,$$

$$\text{(P2)} \qquad \mu_0 \frac{\partial\varphi(x)}{\partial N_0} + (1-\mu_0)\,\varphi(x) = 0 \quad \text{on} \quad \partial\Omega\,,$$

$$\text{(P3)} \qquad \varphi(x) > 0 \quad \text{in} \quad \Omega, \qquad \int_\Omega \varphi(x)\,dx = 1\,.$$

Define for $r_i > 0$ $(i = 1, \ldots, m)$,

$$U^*(r_i) = \inf\{s > 0 \quad \text{such that} \quad f_i(s) = r_i s\},$$

and put

$$\alpha = \alpha(r_1, \ldots, r_m) = \sup_{l \in \{1, \ldots, m\}} r_l f_{*l}(U^*(r_{l-1}))\,,$$

where $r_0 = r_m$.

REMARK 3.1. – We have $U^*(r_i) > 0$ for $r_i > 0$.

THEOREM 3.2. – *Suppose that there are constants C, r_i such that*:

$$0 < \alpha(r_1, \ldots, r_m) < \lambda, \qquad u_i(x, 0) < C\varphi(x) < U^*(r_{i-1}) \quad \text{in} \quad \Omega\,.$$

Then any solution $(u_1, \ldots, u_m)$ *of the problem* (1.1)-(1.3) *exists globally and*

$$\lim_{t\to\infty} e^{\lambda t} u_i(x, t) = C_i\varphi(x)\,,$$

uniformly in Ω, where C_i $(i = 1, \ldots, m)$ are positive constants.

The proof of Theorem 3.2 is based on the following lemmas

LEMMA 3.3. – *Under the hypotheses of Theorem* 3.2, *any solution* $(u_1, \ldots, u_m)$ *of the problem* (1.1)-(1.3) *exists globally and*

$$0 \leqslant u_i(x, t) < C\varphi(x)\, e^{(-\lambda + \alpha(r_1, \ldots, r_m))t} \quad \textit{in} \quad \Omega \times (0, \infty), \qquad i = 1, \ldots, m$$

where C is a positive constant.

PROOF. – Put

$$\text{(3.2)} \qquad v_i(x, t) = C\varphi(x)\, e^{(-\lambda + \alpha)t}.$$

We obtain

$$\frac{\partial v_i}{\partial t} - L_0 v_i = \alpha v_{i+1} \geqslant r_i f_{*i}(U^*(r_{i-1}))\, v_{i+1} \ \text{ in } \ \Omega \times (0, T), \quad i = 1, \dots, m, \tag{3.3}$$

where $v_{m+1} = v_1$. Since $0 \leqslant u_i(x, 0) < U^*(r_{i-1})$, let t_1 be the first $t > 0$ such that

$$0 \leqslant u_i(x, t) < U^*(r_{i-1}) \quad \text{in} \quad \Omega \times (0, t_1), \tag{3.4}$$

but $u_j(x_1, t_1) = U^*(r_{j-1})$ for some $j \in \{1, \dots, m\}$ and x_1 in Ω. Therefore by the definition of $U^*(r_i)$, we have

$$f_i(u_{i+1}) < r_i u_{i+1} \quad \text{in} \quad \Omega \times (0, t_1). \tag{3.5}$$

We deduce that

$$\frac{\partial u_i}{\partial t} - L_0 u_i < r_i f_{*i}(U^*(r_{i-1}))\, u_{i+1} \quad \text{in} \quad \Omega \times (0, t_1). \tag{3.6}$$

We also have

$$u_i(x, 0) < C\varphi(x) = v_i(x, 0) \quad \text{in} \quad \Omega. \tag{3.7}$$

From the maximum principle for parabolic systems(see for instance [11]), it follows that

$$u_i(x, t) < v_i(x, t) \quad \text{in} \quad \Omega \times (0, t_1),$$

that is

$$0 \leqslant u_i(x, t) < C\varphi(x)\, e^{(-\lambda + \alpha)t} \quad \text{in} \quad \Omega \times (0, t_1). \tag{3.8}$$

We conclude that $t_1 = \infty$. In fact suppose that $t_1 < \infty$. Then we have

$$u_j(x_1, t_1) \leqslant C\varphi(x_1)\, e^{(-\lambda + \alpha)t_1}.$$

Therefore, we deduce that

$$U^*(r_{j-1}) = u_j(x_1, t_1) < C\varphi(x_1).$$

This is a contradiction because by hypothesis $C\varphi(x_1) < U^*(r_{j-1})$. Then we conclude that $t_1 = \infty$ and

$$0 \leqslant u_i(x, t) \leqslant C\varphi(x)\, e^{(-\lambda + \alpha)t} \quad \text{in} \quad \Omega \times (0, \infty),$$

which gives the result. ■

LEMMA 3.4. – *Under the hypotheses of Theorem* 3.2, *there exists a positive constant* $M(r)$ *depending on* r *such that for any solution* $(u_1, \dots, u_m)$ *of the* *problem* (1.1)-(1.3), *the following estimates hold*

$$|u_i(x, t)| \leqslant M(r)e^{-\lambda t} \quad in \quad \Omega \times (0, \infty), \quad i = 1, \dots, m.$$

PROOF. – Assume at first that $\lambda \neq p^n(\lambda - r)$ for any $n \geqslant 1$. Let $(S_*(t))_{t \geqslant 0}$ the semigroup of contractions of $L^2(\Omega)$ generated by $-L_0$ with (1.2) as boundary data. Let $(S(t))_{t \geqslant 0}$ the restriction of $(S_*(t))_{t \geqslant 0}$ to $L^\infty(\Omega)$. It is well known that there is a positive constant M such that

$$|S(t)| \leqslant Me^{-\lambda t} \tag{3.9}$$

for any $t \geqslant 0$. Moreover, u_i may be written in the following form

$$u_i(., t) = S(t)\, u_i(., 0) - \int_0^t S(t-s)\, f_i(u_{i+1}(., s))\, f_{*i}(u_i(., s))\, ds. \tag{3.10}$$

Since $|f_i(s)| \leqslant C_* |s|^p$ for $s \in [0, C]$, by Lemma 3.3, there is a positive constant C_1 such that

$$\|u_i(., t)\|_{L^\infty(\Omega)} \leqslant Me^{-\lambda t} \|u_i(., 0)\|_{L^\infty(\Omega)} + MC_1 \int_0^t e^{-\lambda(t-s) - p(\lambda - r)s} ds. \tag{3.11}$$

Since $\lambda \neq p(\lambda - r)$, there are two positive constants A and B such that

$$\|u_i(., t)\|_{L^\infty(\Omega)} \leqslant Ae^{-\lambda t} + Be^{-p(\lambda - r)t}.$$

Iterating this process we have the result. If there is $n \geqslant 1$ such that $\lambda = p^n(\lambda - r)$, there exists $p_1 \in]1, p[$ such that

$$p_1^n(\lambda - r) < \lambda < p_1^{n+1}(\lambda - r),$$

that is to say

$$p_1^m(\lambda - r) \neq \lambda,$$

for any $m \geqslant 1$. Moreover there exists a positive constant K such that $|f_i(s)| \leqslant K|s|^{p_1}$ for $|s| \leqslant C$. Applying the above method, we obtain the result.

PROOF OF THEOREM 3.2. – Let $w_i(x, t) = e^{\lambda t} u_i(x, t)$. We have

$$\frac{\partial w_i}{\partial t} - L_0 w_i = \lambda w_i + e^{\lambda t} f_i(e^{-\lambda t} w_{i+1})\, f_{*i}(e^{-\lambda t} w_i).$$

Put $w_i(x, t) = C_i^*(t)\,\varphi(x) + w_{1i}(x, t)$, where for $j \in \{1, \dots, m\}$, w_{1j} is the projection of w_j on $[\mathrm{Ker}\,(L_0 + \lambda I)]^{\perp}$. Then there exists a positive constant C_2 such that

$$\left| \frac{dC_i^*(t)}{dt} \right| \leq C_2 e^{-(p-1)\lambda t}$$

for any $t > 0$. Therefore $(dC_i^*(t)/dt) \in L^1(0, \infty)$ and $\lim_{t \to \infty} C_i^*(t)$ $(i = 1, \dots, m)$ exist. Let $S_2(t)$ the restriction of $S(t)$ to $[Ker(L_0 + \lambda I)]^{\perp}$. It is well known that there is a positive constant M_2 such that

$$\|S_2(t)\| \leq M_2 e^{-\lambda_2 t},$$

where $\lambda_2 > \lambda$ is the second eigenvalue of the problem (P1)-(P3). Put $u_{1i} = e^{-\lambda t} w_{1i}$. It follows that

$$u_{1i}(.\,, t) = S_2(t)\, u_{1i}(.\,, 0) - \int_0^t S_2(t-s)\, g_{2i}(u_{i+1}(.\,, s))\, g_{2*i}(u_i(.\,, s))\, ds\,,$$

where for $j \in \{1, \dots, m\}$, $g_{2j}(u_{j+1})$ is the projection of $f_j(u_{j+1})$ on $[\mathrm{Ker}\,(L_0 + \lambda I)]^{\perp}$. Since $|g_{2i}(u_{i+1}(.\,, s))| \leq Ce^{-p\lambda s}$, we obtain

$$\|u_{1i}(.\,, t)\|_{L^\infty(\Omega)} \leq Me^{-\lambda_2 t} + \int_0^t e^{-\lambda_2(t-s)} e^{-p\lambda s}\, ds.$$

Therefore

$$\|w_{1i}(.\,, t)\|_{L^\infty(\Omega)} \leq Me^{(\lambda - \lambda_2)t} + M_2 e^{-(p-1)\lambda t}.$$

Then we have

$$\lim_{t \to \infty} e^{\lambda t} u_i(x, t) = C_i\, \varphi(x), \qquad i = 1, \dots, m$$

uniformly in Ω, where C_i $(i = 1, \dots, m)$ are positive constants, which yields the result. ■

COROLLARY 3.4. – *Suppose that* $f_i(u_{i+1}) = u_{i+1}^{p_i}$, $f_{*i}(u_i) = e^{u_i}$, *with* $p_i > 1$. *Then there exists a positive constant* b *such that any solution* $(u_1, \dots, u_m)$ *of the problem* (1.1)-(1.3) *exists globally and*

$$\lim_{t \to \infty} e^{\lambda t} u_i(x, t) = C_i\, \varphi(x)$$

uniformly in Ω *for* $u_0^{(i)}(x) \leq b$ *where* C_i $(i = 1, \dots, m)$ *are positive constants.*

4. – Blow up solutions.

In this section, we give some conditions under which the solutions of the problem (1.1)-(1.3) blow up in a finite time.

Let z be the solution of the following problem:

$$\text{(Q1)} \qquad \frac{\partial z}{\partial t} = L_k z + \lambda f_{*k}(z) \quad \text{in} \quad \Omega \times (0, T),$$

$$\text{(Q2)} \qquad \mu_k \frac{\partial z}{\partial N_k} + (1-\mu_k) z = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$\text{(Q3)} \qquad z(x, 0) = u_0^{(k)}(x) \geqslant 0 \quad \text{in} \quad \Omega,$$

where $k \in \{1, \ldots, m\}$.

LEMMA 4.1. – *Let w_0 be the maximum of the solution for the following boundary value problem*

$$L_k w + 1 = 0 \quad in \quad \Omega, \qquad \mu_k \frac{\partial w}{\partial N_k} + (1-\mu_k) w = 0 \quad on \quad \partial\Omega,$$

where $\mu_k < 1$. Suppose that $f_{*k}(s)$ is positive and increasing for positive values of s with $f_{*k}(0) > 0$. If

$$\lambda > \frac{1}{w_0} \int_0^\infty \frac{ds}{f_{*k}(s)},$$

then the solution z of the problem (Q1)-(Q3) *blows up in a finite time.*

PROOF. – Assume at first that $u_0^{(k)}(x) = 0$. Let $(0, T_{\max})$ be the maximum time interval in which the classical solution z of the problem (Q1)-(Q3) exists. From the maximum principle, $z(x, t) \geqslant 0$ in $\Omega \times (0, T_{\max})$. Put

$$\text{(4.1)} \qquad v(x, t) = F(z(x, t)) = \int_0^z \frac{ds}{\lambda f_{*k}(s)}.$$

We have

$$\text{(4.2)} \qquad \frac{\partial v}{\partial t} - L_k v = \frac{1}{\lambda f_{*k}(z)} \left(z_t - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}^{(k)}(x) \frac{\partial z}{\partial x_j} \right) \right) + \left[\sum_{i,j=1}^n a_{ij}^{(k)}(x)\, z_{x_i} z_{x_j} \right] \frac{f'_{*k}(z)}{\lambda f^2_{*k}(z)}.$$

Since $f_{*k}(z)$ is an increasing function, we also have

$$v(x, t) = \int_0^z \frac{ds}{\lambda f_{*k}(s)} \geqslant \frac{z}{\lambda f_{*k}(z)}. \tag{4.3}$$

From (Q1) and (4.2) we deduce that

$$\frac{\partial v}{\partial t} - L_k v - 1 \geqslant 0 \quad \text{in} \quad \Omega \times (0, T_{\max}). \tag{4.4}$$

From (4.3), we also have

$$\mu_k \frac{\partial v}{\partial N_k} = \frac{1}{\lambda f_{*k}(z)} \mu_k \frac{\partial z}{\partial N_k} = \frac{-(1-\mu_k) z}{\lambda f_{*k}(z)} \geqslant -(1-\mu_k) v, \tag{4.5}$$

that is to say

$$\mu_k \frac{\partial v}{\partial N_k} + (1-\mu_k) v \geqslant 0 \quad \text{on} \quad \partial\Omega \times (0, T_{\max}). \tag{4.6}$$

Since $w_0 > \int_0^\infty \frac{ds}{\lambda f_{*k}(s)}$ and $z < \infty$ in $\Omega \times (0, T_{\max})$, we have

$$\sup_{(x,t) \in \Omega \times (0, T_{\max})} v(x, t) < w_0. \tag{4.7}$$

Let z be the solution of the following problem:

$$\frac{\partial z}{\partial t} = L_k z + 1 \quad \text{in} \quad \Omega \times (0, \infty), \tag{4.8}$$

$$\mu_k \frac{\partial z}{\partial N_k} + (1-\mu_k) z = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{4.9}$$

$$z(x, 0) = 0 \quad \text{in} \quad \Omega. \tag{4.10}$$

From the maximum principle, we obtain

$$v(x, t) \geqslant z(x, t) \quad \text{in} \quad \Omega \times (0, T_{\max}). \tag{4.11}$$

We also have

$$\lim_{t \to \infty} z(x, t) = w(x). \tag{4.12}$$

Therefore from (4.7) and (4.12), there exist $x_0 \in \Omega$ and a finite t_0 such that

$$z(x_0, t_0) > \sup_{(x,t) \in \Omega \times (0, T_{\max})} v(x, t), \tag{4.13}$$

which implies that $t_0 \geqslant T_{\max}$. In fact, suppose that $t_0 < T_{\max}$. From (4.11), we have $v(x_0, t_0) \geqslant z(x_0, t_0)$ which contradicts (4.13). Consequently, $T_{\max}$ is finite and z blows up in a finite time.

Now, suppose that $u_0^{(k)}(x) \geqslant 0$. From the maximum principle

$$z(x, t) \geqslant u_1(x, t) \quad \text{in} \quad \Omega \times (0, T_1) \tag{4.14}$$

where u_1 is a solution of the problem (Q1)-(Q2) with $u_1(x, 0) = 0$ and $(0, T_1)$ is the maximum time interval in which the solutions z and u_1 exist. From the above result, we know that u_1 blows up in a finite time because

$$w_0 > \int_0^b \frac{ds}{\lambda f_{*k}(s)}\,. \tag{4.15}$$

Therefore, from (4. 14), z blows up in a finite time, which yields the result. ■

THEOREM 4.2. – *Suppose that there exists $k \in \{1, \ldots, m\}$ such that $f_{*k}(0) > 0$, $\int_0^\infty (ds/f_{*k}(s)) < \infty$ and $\lim_{s \to \infty} f_k(s) = \infty$. Fix $(u_0^{(1)}, \ldots, u_0^{(m)})$. There exists $\gamma_0 > 0$ such that, if $\gamma > \gamma_0$ then the solution $(u_{1\gamma}, \ldots, u_{m\gamma})$ of the problem* (1.1)-(1.3) *with initial data $(u_0^{(1)}, \ldots, \gamma u_0^{(k+1)}, \ldots, u_0^{(m)})$ blows up in a finite time.*

PROOF. – Since $u_0^{(k+1)}(x) > 0$ in Ω, there exists a ball B such that $\overline{B} \subset\subset \Omega$ and $u_0^{(k+1)}(x) \geqslant \varepsilon > 0$ in B (this is possible because $u_0^{(k+1)}(x)$ is continuous in Ω). Let z be the solution of the following problem

$$\frac{\partial z}{\partial t} = L_k z + \lambda_0 f_{*k}(z) \quad \text{in} \quad B \times (0, T), \tag{4.16}$$

$$z = 0 \quad \text{on} \quad \partial B \times (0, T), \tag{4.17}$$

$$z(x, 0) = u_0^{(k)}(x) \geqslant 0 \quad \text{in} \quad B, \tag{4.18}$$

where λ_0 is such that z blows up in a finite time T_0 (this is possible because of Lemma 4.1). Let $w(x, t)$ be the solution of the following problem:

$$\frac{\partial w}{\partial t} - L_{k+1} w = 0 \quad \text{in} \quad \Omega \times (0, T_0), \tag{4.19}$$

$$w = 0 \quad \text{on} \quad \partial\Omega \times (0, T_0), \tag{4.20}$$

$$w(x, 0) = u_0^{(k+1)}(x) \quad \text{in} \quad \Omega. \tag{4.21}$$

Then we have

$$\alpha = \inf_{x \in \overline{B} \times (0,\, T_0)} w(x, t) > 0 \tag{4.22}$$

because $u_0^{(k+1)}(x) > 0$ in $\overline{B}$. From the maximum principle,

$$u_{(k+1)\gamma}(x, t) \geqslant \gamma w(x, t) \quad \text{in} \quad \Omega \times (0, T_0). \tag{4.23}$$

Therefore we obtain

$$\inf_{(x,\, t) \in B \times (0,\, T_0)} u_{(k+1)\gamma}(x, t) \geqslant \gamma \inf_{(x,\, t) \in B \times (0,\, T_0)} w(x, t) = \gamma\alpha. \tag{4.24}$$

Since f_k is increasing and $\lim_{t \to \infty} f_k(t) = \infty$, from (4.24), take $\gamma_0 > 0$ such that $f_k(u_{(k+1)\gamma}) > \lambda_0$ for $\gamma > \gamma_0$. Therefore if $\gamma > \gamma_0$, $u_{k\gamma}$ satisfies the following problem

$$\frac{\partial u_{k\gamma}}{\partial t} > L_k u_{k\gamma} + \lambda_0 f_{*k}(u_{k\gamma}) \quad \text{in} \quad B \times (0, T_0), \tag{4.25}$$

$$u_{k\gamma} > 0 \quad \text{on} \quad \partial B \times (0, T_0), \tag{4.26}$$

$$u_{k\gamma}(x, 0) = u_0^{(k)}(x) \geqslant 0 \quad \text{in} \quad B. \tag{4.27}$$

From the maximum principle

$$u_{k\gamma}(x, t) \geqslant z(x, t) \quad \text{in} \quad \Omega \times (0, T_0) \quad \text{for} \quad \gamma > \gamma_0.$$

Therefore if $\gamma > \gamma_0$, the solution $(u_{1\gamma}, \ldots, u_{m\gamma})$ blows up in a finite time $T' \leqslant T_0$. ■

COROLLARY 4.3. – *Suppose that there exists* $k \in \{1, \ldots, m\}$ *such that* $f_k(u_{k+1}) = u_{k+1}^{p_k}$, $f_{*k}(u_k) = e^{u_k}$ *or* $f_{*k}(u_k) = u_k^{p_{*k}} + \varepsilon$, *with* $\varepsilon > 0$, $p_k > 0$ *and* $p_{*k} > 1$. *Fix* $(u_0^{(1)}, \ldots, u_0^{(m)})$. *There exists* $\gamma_0 > 0$ *such that, if* $\gamma > \gamma_0$ *then the solution* $(u_{1\gamma}, \ldots, u_{m\gamma})$ *of the problem* (1.1)-(1.3) *with initial data* $(u_0^{(1)}, \ldots, \gamma u_0^{(k+1)}, \ldots, u_0^{(m)})$ *blows up in a finite time.*

THEOREM 4.4. – *Let* $L_i = d_i \Delta$ *where* d_i $(i = 1, \ldots, m)$ *are positive constants and suppose that there exists* $k_1 \in \{1, \ldots, m\}$ *such that* $f_{*k_1}(0) > 0$, $\int_0^\infty \frac{ds}{f_{*k_1}(s)} < \infty$ *and* $\lim_{s \to \infty} f_{k_1}(s) = \infty$. *Suppose also that* $\liminf_{t \to \infty} (\mathrm{f_i(s)/f_i'(s)}) > 0$ $(i = 1, \ldots, m)$,

$$\liminf_{s \to 0} \frac{f_{*m}(s)\, f_m \circ f_1 \circ \ldots \circ f_{m-1}(s)}{s} > 0$$

and

$$\liminf_{s\to\infty} \frac{f_{*m}(s)\, f_m \circ f_1 \circ \ldots \circ f_{m-1}(s)}{s} > 0\,.$$

Then if Ω contains a large ball, any positive solution of the problem (1.1)-(1.3) *blows up in a finite time.*

PROOF. – Let $\phi_1 > 0$ be a solution of the following problem

$$\Delta\phi_1(x) \geq \alpha > 0 \quad \text{if} \quad \phi_1(x) \leq c_1, \qquad \phi_1 = 0 \quad \text{on} \quad \partial B_1, \tag{4.28}$$

where B_1 is a ball of radius 1. Put $\phi_k(x) = \phi(x/k)$. Then ϕ_k satisfies the following relations

$$\Delta\phi_k(x) \geq \frac{\alpha}{k^2} > 0 \quad \text{if} \quad \phi_k \leq c_1, \qquad \phi_k = 0 \quad \text{on} \quad \partial B_k, \tag{4.29}$$

$$\Delta\phi_k(x) \geq \frac{-L}{k^2} > 0 \qquad (-L = \inf_{x\in B_1} \Delta\phi_1(x))\,, \tag{4.30}$$

where B_k is a ball of radius k. Let $\overline{u_i} = a_i(t)\,\phi_k(x)$, where $a_i(t)$ $(i = 1, \ldots, m)$ are increasing functions which will be determined later. Our aim is to show that $(\overline{u_1}, \ldots, \overline{u_m})$ is a subsolution of the problem (1.1)-(1.3). Then, it is sufficient to show that the following inequalities hold

$$a_i'(t)\,\phi_k(x) \leq a_i(t)\, d_i \Delta\phi_k(x) + f_i(a_{i+1}(t)\,\phi_k(x))\, f_{*i}(a_i(t)\,\phi_k(x))\,, \tag{4.31}$$

$$a_m'(t)\,\phi_k(x) \leq a_m(t)\, d_m \Delta\phi_k(x) + f_m(a_1(t)\,\phi_k(x))\, f_{*m}(a_m(t)\,\phi_k(x)), \tag{4.32}$$

where $(i = 1, \ldots, m-1)$. If $\phi_k \leq c_1$, the inequalities (4.31) and (4.32) are valid if

$$a_i'(t)\, c_1 \leq \frac{\alpha}{k^2} d_i a_i(t), \qquad i = 1, \ldots, m-1, \tag{4.33}$$

$$a_m'(t)\, c_1 \leq \frac{\alpha}{k^2} d_m a_m(0). \tag{4.34}$$

For $\phi_k \geq c_1$, let $c_2 = \sup \phi_k$. Then the inequalities (4.31) and (4.32) are true if

$$a_i'(t) c_2 \leq -a_i(t)\, d_i \frac{L}{k^2} + f_i(a_{i+1}(t)\, c_1)\, f_{*i}(a_i(t)\, c_1), \quad i = 1, \ldots, m-1, \tag{4.35}$$

$$a_m'(t)\, c_2 \leqslant -a_m(t)\, d_m \frac{L}{k^2} + f_m(a_1(t)\, c_1)\, f_{*m}(a_m(t)\, c_1)\,. \tag{4.36}$$

Thus our new aim is to show that we may determine the functions $a_i(t)$ $(i = 1, \ldots, m)$ for that the inequalities (4.33), (4.34), (4.35) and (4.36) be true. Take $a_m(t) = \varepsilon t + a_m(0)$, $a_i(t)\, c_1 = f_i(c_1 a_{i+1}(t))$ $(i = 1, \ldots, m-1)$ and put $\delta_i = \inf_{s \geqslant c_1 a_{i+1}(0)} (f_i(s)/f_i'(s))$. Then the inequalities (4.33) and (4.34) hold if

$$c_1 \varepsilon \leqslant \frac{\alpha}{c_1 k^2} d_i \delta_i, \qquad i = 1, \ldots, m-1, \tag{4.37}$$

$$\varepsilon c_1 \leqslant \frac{\alpha}{k^2} d_m a_m(0) \tag{4.38}$$

and the inequalities (4.35) and (4.36) are true if

$$\varepsilon f_i'(c_1 a_{i+1}(t)) \leqslant -\frac{1}{c_1} f_i(c_1 a_{i+1}(t))\, d_i \frac{L}{k^2} + f_i(a_{i+1}(t)\, c_1) f_{*i}(a_i(t)\, c_1), \qquad i = 1, \ldots, m-1, \tag{4.39}$$

$$\varepsilon c_2 \leqslant -a_m(t)\, d_m \frac{L}{k^2} + f_m \circ f_1 \circ f_2 \circ \ldots \circ f_{m-1}(c_1 a_m(t))\, f_{*m}(a_m(t)\, c_1). \tag{4.40}$$

Let k be so large that $(L d_i / c_1 k^2) < (1/2)\, f_{*i}(c_1 a_i(0))$ $(i = 1, \ldots, m-1)$. The inequalities (4.39) hold if

$$\varepsilon \leqslant \delta_i \left[-\frac{L d_i}{c_1 k^2} + f_{*i}(c_1 a_i(0)) \right], \qquad i = 1, \ldots, m-1. \tag{4.41}$$

Let k_* be such that $f_{*m}(s) \geqslant k_* \dfrac{s}{f_m \circ f_1 \circ \ldots \circ f_{m-1}(s)}$ for $s > a_m(0)\, c_1$. Then the inequality (4.40) is true if

$$\varepsilon c_2 \leqslant a_m(0) \left[-d_m \frac{L}{k^2} + k_* \right]. \tag{4.42}$$

Let k again be such that $d_m (L/k^2) < k_*/2$. Thus we may choose ε small enough that the inequalities (4.41) and (4.42) be valid. Take $a_i(0)$ be sufficiently small that $\overline{u_i}(x, 0) < u_0^{(i)}(x)$ in B_k. Therefore, there exists a ball B_k such that

$$\frac{\partial \overline{u_i}}{\partial t} - d_i \Delta \overline{u_i} \leqslant f_i(\overline{u_{i+1}}) f_{*i}(\overline{u_i}) \quad \text{in} \quad B_k \times (0, T),$$

$$\overline{u_i} = 0 \quad \text{on} \quad \partial B_k \times (0, T),$$

$$\overline{u_i}(x, 0) < u_0^{(i)}(x) \quad \text{in} \quad B_k, \qquad i = 1, \ldots, m,$$

where $\overline{u_{m+1}} = \overline{u_1}$. Since $(u_1, \ldots, u_m)$ is a positive solution of the problem (1.1)-(1.3), by Comparison lemma 2.1, we deduce that $u_i(x, t) \geqslant \overline{u_i}(x, t)$. Therefore we have

$$\lim_{t \to \infty} u_i(x, t) = \infty .$$

By Theorem 4.2, we obtain the result. ■

COROLLARY 4.5. – *Let $L_i = d_i \Delta$ where d_i $(i = 1, \ldots, m)$ are positive constants and suppose that there exists $k \in \{1, \ldots, m\}$ such that $f_{*k}(u_k) = e^{u_k}$ or $f_{*k}(u_k) = u_k^{p_{*k}} + \varepsilon$. Suppose also that $f_i(u_{i+1}) = u_{i+1}^{p_i}$ $(i = 1, \ldots, m)$, $f_{*m}(u_m) = e^{u_m}$ or $f_{*m}(u_m) = u_m^{p_{*m}} + \varepsilon$ with $\varepsilon > 0$, $p_i > 0$ and $p_{*m} > 1 - \prod_{i=1}^{m} p_i \geqslant 0$. Then if Ω contains a large ball, any positive solution $(u_1, \ldots, u_m)$ of the problem* (1.1)-(1.3) *blows up in a finite time.*

5. – Other blow up solutions.

In this section, we give other conditions under which the solutions of the problem (1.1)-(1.3) blow up in a finite time in the case where $m = 2$, $\mu_i = 1$, $L_i = L_0$, $f_1 = f$, $f_{1*} = f_*$, $f_2 = g$ and $f_{2*} = g_*$. If $\int^{\infty} (ds/f_*(s)) < \infty$ or $\int^{\infty} (ds/g_*(s)) < \infty$, we easily show that any solution (u, v) of the problem (1.1)-(1.3) with initial data (u_0, v_0) blows up in a finite time. In fact, suppose that $\int^{\infty} (ds/f_*(s)) < \infty$. From the maximum principle, we have $v(x, t) \geqslant c > 0$. Then u is a solution of the following problem

$$\frac{\partial u}{\partial t} \geqslant L_0 u + f(c)\, f_*(u) \quad \text{in} \quad \Omega \times (0, T),$$

$$\frac{\partial u}{\partial N_0} = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$u(x, 0) = u_0(x) > 0 \quad \text{in} \quad \Omega .$$

It is well known that any solution of the above problem blows up in a finite time (see, for instance [9]). Hence the result. Thus, in this section, we assume that $\int^{\infty} (ds/f_*(s)) = \infty$ and $\int^{\infty} (ds/g_*(s)) = \infty$. Consider the following system:

(R1) $$\alpha_1'(t) = f_*(\alpha_1(t))\, f(\alpha_2(t)),$$

(R2) $$\alpha_2'(t) = g(\alpha_1(t))\, g_*(\alpha_2(t))\,.$$

We have

$$\frac{d\alpha_1}{d\alpha_2} = \frac{f_*(\alpha_1(t))\, f(\alpha_2(t))}{g(\alpha_1(t))\, g_*(\alpha_2(t))}\,,$$

that is to say

$$\frac{g(\alpha_1)\, d\alpha_1}{f_*(\alpha_1)} = \frac{f(\alpha_2)\, d\alpha_2}{g_*(\alpha_2)}\,.$$

Let $G(s)$ be a primitive of $g(s)/f_*(s)$ and $F(s)$ that of $f(s)/g_*(s)$ with $F(0) = G(0) = 0$. Then we have $G(\alpha_1) = F(\alpha_2)$, that is to say $\alpha_2 = F^{-1}[G(\alpha_1)] = k(\alpha_1)$, where F^{-1} is the inverse function of F. We suppose that $k(z) = F^{-1} \circ G(z)$ is an increasing function for positive values of z.

THEOREM 5.1. – *Suppose that* $k(0) = f(0) = g(0) = 0$ *and*

$$\int^{+\infty} \frac{dz}{f_*(z)\, f(k(z))} < \infty \quad \text{or} \quad \int^{+\infty} \frac{dz}{g(k^{-1}(z))\, g_*(z)} < \infty\,.$$

Then any solution (u, v) *of the problem* (1.1)-(1.3) *initial data* (u_0, v_0) *blows up in a finite time.*

PROOF. – Put $c_0 = \inf_{x \in \Omega} u_0(x) > 0$, $d_0 = \inf_{x \in \Omega} v_0(x) > 0$. Let

$$\overline{u} = \alpha_1(\tau), \qquad \overline{v} = \alpha_2(\tau)$$

with $\tau = \varepsilon t - \varepsilon w(x) + \varepsilon c$, $\alpha_1(0) = c_0/2^*$, where 2^* is big enough that $k(2(c_0/2^*)) < d_0/2$ and $w(x)$ satisfies the following problem:

(5.1) $$L_0 w(x) = d \quad \text{in} \quad \Omega, \qquad \frac{\partial w}{\partial N_0} = 1 \quad \text{on} \quad \partial\Omega,$$

with $d = |\partial\Omega|/|\Omega|$, c is such that $c - w(x) > 0$. Since $\alpha_1'(t) \geqslant 0$ and $\alpha_1(0) > 0$, there is t_1 such that $\alpha_1(t_1) = 2(c_0/2^*)$. Take $\varepsilon > 0$ so small that

$$-\varepsilon w(x) + \varepsilon c \leqslant t_1, \qquad \varepsilon + \varepsilon d < 1\,.$$

Therefore, we obtain

(5.2) $$\overline{u}(x, 0) \leqslant \alpha_1(t_1) < u(x, 0) \quad \text{in} \quad \Omega\,.$$

Similarly since $k(z)$ is an increasing function, we get

$$\overline{v}(x,0) \leqslant \alpha_2(t_1) = k\left(2\frac{c_0}{2^*}\right) < v(x,0) \quad \text{in} \quad \Omega. \tag{5.3}$$

We also have

$$\frac{\partial \overline{u}}{\partial t} - L_0 \overline{u} = \alpha_1'(\tau)(\varepsilon + \varepsilon L_0 w) - \varepsilon^2 \alpha_1''(\tau) \sum_{i,j=1}^{n} a_{ij}^{(0)}(x) \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}, \tag{5.4}$$

$$\frac{\partial \overline{v}}{\partial t} - L_0 \overline{v} = \alpha_2'(\tau)(\varepsilon + \varepsilon L_0 w) - \varepsilon^2 \alpha_2''(\tau) \sum_{i,j=1}^{n} a_{ij}^{(0)}(x) \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}. \tag{5.5}$$

Since $f(s), f_*(s), g(s), g_*(s)$ are nonnegative and increasing for positive values of s, we have $\alpha_1''(\tau) \geqslant 0$, $\alpha_2''(\tau) \geqslant 0$. From (5.4) and (5.5) it follows that

$$\frac{\partial \overline{u}}{\partial t} - L_0 \overline{u} < f(\overline{v})\, f_*(\overline{u}) \quad \text{in} \quad \Omega \times (0,T), \tag{5.6}$$

$$\frac{\partial \overline{v}}{\partial t} - L_0 \overline{v} < g(\overline{u})\, g_*(\overline{v}) \quad \text{in} \quad \Omega \times (0,T). \tag{5.7}$$

We also have

$$\frac{\partial \overline{u}}{\partial N_0} = -\varepsilon \frac{\partial w}{\partial N_0} \alpha_1'(\tau) < 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{5.8}$$

$$\frac{\partial \overline{v}}{\partial N_0} = -\varepsilon \frac{\partial w}{\partial N_0} \alpha_2'(\tau) < 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{5.9}$$

Applying Comparison lemma 2.1, we deduce that

$$u(x,t) \geqslant \overline{u}(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{5.10}$$

$$v(x,t) \geqslant \overline{v}(x,t) \quad \text{in} \quad \Omega \times (0,T). \tag{5.11}$$

On the other hand, $\alpha_1(t)$ and $\alpha_2(t)$ satisfy the following relations:

$$\int_{c_0/2^*}^{\alpha_1(t)} \frac{dz}{f_*(z)\, f(k(z))} = t \quad \text{and} \quad \int_{k(c_0/2^*)}^{\alpha_2(t)} \frac{dz}{g(k^{-1}(z))\, g_*(z)} = t.$$

This implies that $(\overline{u}, \overline{v})$ blows up in a finite time because

$$\int^{+\infty} \frac{dz}{f_*(z)\, f(k(z))} < \infty \quad \text{or} \quad \int^{+\infty} \frac{dz}{g(k^{-1}(z))\, g_*(z)} < \infty\,,$$

which leads to the result. ■

REMARK 5.2. – Let $f(s) = s^{p_1}$, $f_*(s) = s^{p_2}$, $g(s) = s^{q_1}$, $g_*(s) = s^{q_2}$. We have

$$k(s) = \left\{ \frac{p_1 - q_2 + 1}{q_1 - p_2 + 1} \right\}^{1/(p_1 - q_2 + 1)} s^{(q_1 - p_2 + 1)/(p_1 - q_2 + 1)},$$

$$f_*(s)\, f(k(s)) = \left\{ \frac{p_1 - q_2 + 1}{q_1 - p_2 + 1} \right\}^{p_1/(p_1 - q_2 + 1)} s^{(p_1 q_1 - p_2 q_2 + p_1 + p_2)/(p_1 - q_2 + 1)},$$

$$g_*(s)\, g(k^{-1}(s)) = \left\{ \frac{q_1 - p_2 + 1}{p_1 - q_2 + 1} \right\}^{q_1/(p_1 - q_2 + 1)} s^{(p_1 q_1 - p_2 q_2 + q_1 + q_2)/(q_1 - p_2 + 1)}.$$

If $p_2 > 1$ or $q_2 > 1$, then any solution of the problem (1.1)-(1.3) with initial data (u_0, v_0) blows up in a finite time.

If $p_2 \leqslant 1$, $q_2 \leqslant 1$ and $p_1 q_1 - p_2 q_2 + p_2 + q_2 > 1$, then any solution (u, v) of the problem (1.1)-(1.3) with initial data (u_0, v_0) blows up in a finite time.

6. – Asymptotic behavior of global solutions.

In this section, we suppose that the functions $\alpha_1(t)$ and $\alpha_2(t)$ of the system (R1)-(R2) are replaced by $\alpha(t)$ and $\beta(t)$ respectively. We also suppose that $f_*(s) = g_*(s) = 1$. Under the conditions in below, we obtain the asymptotic behavior of any solution for the problem (1.1)-(1.3). Thus we have the following theorem:

THEOREM 6.1. – *Suppose that for positive values of s, the functions $f(s)$ and $g(s)$ are concave with $f(0) = g(0) = 0$,*

$$\int^{\infty} \frac{ds}{f[k(s)]} = \int^{\infty} \frac{ds}{g[k^{-1}(s)]} = \infty\,,$$

$$\lim_{t \to \infty} \frac{f'[k(t)]\, g(t)}{f(k(t))} = \lim_{t \to \infty} \frac{g'(t)\, f[k(t)]}{g(t)} = 0\,.$$

Then if (u, v) is a solution of the problem (1.1)-(1.3) *with initial data (u_0, v_0) we have*:

(i) (u, v) *exists globally and*

$$\lim_{t\to\infty} u(x, t) = \lim_{t\to\infty} v(x, t) = \infty$$

uniformly in $x \in \Omega$.

(ii) *Moreover if*

$$\lim_{s\to\infty} \frac{sf[k(H(s))]}{H(s)} \leqslant c_2 \qquad \text{or} \qquad \lim_{s\to\infty} \frac{sg[k^{-1}(K(s))]}{K(s)} \leqslant c_3,$$

where c_2 *and* c_3 *are two positive constants, we also have*

$$u(x, t) = \alpha(t)(1 + o(1)) \quad \text{as} \quad t \to \infty,$$

or

$$v(x, t) = \beta(t)(1 + o(1)) \quad \text{as} \quad t \to \infty,$$

where $H(s)$ *and* $K(s)$ *are the inverse functions of*

$$G(s) = \int_1^s \frac{d\sigma}{f[k(\sigma)]} \quad \textit{and} \quad M(s) = \int_1^s \frac{d\sigma}{g[k^{-1}(\sigma)]}$$

respectively, $\alpha'(t) = f(\beta(t))$, $\beta'(t) = g(\alpha(t))$ *with* $\alpha(0) = 1$, $\beta(0) = k(1)$.

PROOF. – (i) Put

$$w(x, t) = \alpha(t) + \psi(x)\, f(\beta(t)), \qquad z(x, t) = \beta(t) + \psi(x)\, g(\alpha(t)),$$

with

$$\alpha'(t) = \lambda f(\beta(t)), \qquad \alpha(0) = 1,$$

$$\beta'(t) = \lambda g(\alpha(t)), \qquad \beta(0) = k(1),$$

where ψ and λ will be determined later. Since

$$\int^{\infty} \frac{ds}{f[k(s)]} = \int^{\infty} \frac{ds}{g[k^{-1}(s)]} = \infty,$$

we have

(6.1) $$\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \infty.$$

We also have

$$\frac{\partial w}{\partial t} - L_0 w - f(z) =$$

$$f(\beta(t))(\lambda - L_0\psi) + \beta'(t)\, f'(\beta(t))\, \psi(x) - f(\beta(t)) - \psi(x)\, g(\alpha(t))\, f'(y),$$

$$\frac{\partial z}{\partial t} - L_0 z - g(w) =$$

$$g(\alpha(t))(\lambda - L_0 \psi) + \alpha'(t)\, g'(\alpha(t))\, \psi(x) - g(\alpha(t)) - \psi(x)\, f(\beta(t))\, g'(z),$$

with $y \in [\beta(t) + \psi(x)\, g(\alpha(t))]$ and $z \in [\alpha(t), \alpha(t) + \psi(x)\, f(\beta(t))]$. Let ψ be a positive solution of the following problem

$$\lambda - L_0 \psi = 1 - \delta, \qquad \frac{\partial \psi}{\partial N_0} = -\delta.$$

Take $\lambda \leqslant 1/2$ and $\delta = |\Omega|/(|\Omega| + |\partial\Omega|) - |\Omega|/(|\Omega| + |\partial\Omega|)\,\lambda$. Therefore the function ψ exists. Then, we obtain

$$\frac{\partial w}{\partial t} - L_0 w - f(z) = -\delta f(\beta(t)) + \beta'(t)\, f'(\beta(t))\, \psi(x) - \psi(x)\, g(\alpha(t))\, f'(y),$$

$$\frac{\partial z}{\partial t} - L_0 z - g(w) = -\delta g(\alpha(t)) + \alpha'(t)\, g'(\alpha(t))\, \psi(x) - \psi(x)\, f(\beta(t))\, g'(z),$$

$$\frac{\partial w}{\partial N_0} = -\delta f(\beta(t)), \qquad \frac{\partial z}{\partial N_0} = -\delta g(\alpha(t)).$$

Since $\lim\limits_{t \to \infty} \dfrac{f'[k(t)]\, g(t)}{f(k(t))} = \lim\limits_{t \to \infty} \dfrac{g'(t)\, f[k^{-1}(t)]}{g(t)} = 0$, there exists $t_1 \geqslant 0$ such that

$$\frac{\partial w}{\partial t} - L_0 w - f(z) < 0 \quad \text{in} \quad \Omega \times (t_1, \infty),$$

$$\frac{\partial z}{\partial t} - L_0 z - g(w) < 0 \quad \text{in} \quad \Omega \times (t_1, \infty).$$

Since f and g are concave, there exists l so small that

$$\frac{\partial lw}{\partial t} - L_0 lw - f(lz) < 0 \quad \text{in} \quad \Omega \times (t_1, \infty),$$

$$\frac{\partial lz}{\partial t} - L_0 lz - g(lw) < 0 \quad \text{in} \quad \Omega \times (t_1, \infty),$$

$$u(x, 0) > lw(x, t_1), \qquad v(x, 0) > lz(x, t_1).$$

From the maximum principle we deduce that

$$\lim_{t \to \infty} u(x, t) = \lim_{t \to \infty} v(x, t) = \infty$$

uniformly in $x \in \Omega$.

(ii) Put

$$w_1(x,t)=\alpha_1(t)+\psi_1(x)\,f(\beta_1(t)),\qquad z_1(x,t)=\beta_1(t)+\psi_1(x)\,g(\alpha_1(t))$$

with

$$\alpha_1'(t)=\left(1-\frac{\varepsilon}{2}\right)f(\beta_1(t)),\quad \alpha_1(0)=1$$

and

$$\beta_1'(t)=\left(1-\frac{\varepsilon}{2}\right)g(\alpha_1(t)),\quad \beta_1(0)=k(1).$$

We have

$$\frac{\partial w_1}{\partial t}-L_0w_1-f(z_1)=f(\beta_1(t))\left(1-\frac{\varepsilon}{2}-L_0\psi_1\right)+$$

$$\left(1-\frac{\varepsilon}{2}\right)\psi_1(x)\,f'(\beta_1(t))\,g(\alpha_1(t))-f(\beta_1(t))-\psi_1(x)\,f'(y_1)\,g(\alpha_1(t)),$$

$$\frac{\partial z_1}{\partial t}-L_0z_1-g(w_1)=g(\alpha_1(t))\left(1-\frac{\varepsilon}{2}-L_0\psi_1\right)+$$

$$\left(1-\frac{\varepsilon}{2}\right)\psi_1(x)\,g'(\alpha_1(t))\,f(\beta_1(t))-g(\alpha_1(t))-\psi_1(x)\,g'(z_1)\,f(\beta_1(t)),$$

$$\frac{\partial w_1}{\partial N_0}=f(\beta_1(t))\frac{\partial\psi_1}{\partial N_0},\qquad \frac{\partial z_1}{\partial N_0}=g(\alpha_1(t))\frac{\partial\psi_1}{\partial N_0},$$

with

$y_1\in[\beta_1(t),\,\beta_1(t)+\psi_1(x)\,g(\alpha_1(t))]$

and

$$z_1\in[\alpha_1(t),\,\alpha_1(t)+\psi_1(x)\,f(\beta_1(t))].$$

Let ψ_1 be a positive solution of the following problem:

$$-\frac{\varepsilon}{2}-L_0\psi_1=-\delta,\qquad \frac{\partial\psi_1}{\partial N_0}=-\delta\,.$$

ψ_1 exists if and only if $\delta=|\Omega|/(|\Omega|+|\partial\Omega|)(\varepsilon/2)$. If $\varepsilon=0$ then $\delta=0$. Put $\delta(r)=|\Omega|/(|\Omega|+|\partial\Omega|)\,r$. We have $\delta'(0)>0$. Then for any $\varepsilon>0$ small enough, it follows that $\delta(\varepsilon/2) > 0$. Therefore, we obtain

$$\frac{\partial w_1}{\partial t} - L_0 w_1 - f(z_1) = -\delta f(\beta_1(t)) + \left(1-\frac{\varepsilon}{2}\right)\psi_1(x)\, f'(\beta_1(t))\, g(\alpha_1(t)) - \psi_1(x)\, f'(y_1)\, g(\alpha_1(t)),$$

$$\frac{\partial z_1}{\partial t} - L_0 z_1 - g(w_1) = -\delta g(\alpha_1(t)) + \left(1-\frac{\varepsilon}{2}\right)\psi_1(x)\, g'(\alpha_1(t))\, f(\beta_1(t)) - \psi_1(x)\, g'(z_1)\, f(\beta_1(t)),$$

$$\frac{\partial w_1}{\partial N_0} = -\delta f(\beta_1(t)), \qquad \frac{\partial z_1}{\partial N_0} = -\delta g(\alpha_1(t)).$$

Then there exists $T_1 > 0$ such that

$$\frac{\partial w_1}{\partial t} - L_0 w_1 - f(z_1) < 0 \quad \text{in} \quad \Omega \times (T_1, \infty),$$

$$\frac{\partial z_1}{\partial t} - L_0 z_1 - g(w_1) < 0 \quad \text{in} \quad \Omega \times (T_1, \infty),$$

$$\frac{\partial w_1}{\partial N_0} < 0 \quad \text{on} \quad \partial\Omega \times (T_1, \infty),$$

$$\frac{\partial z_1}{\partial N_0} < 0 \quad \text{on} \quad \partial\Omega \times (T_1, \infty).$$

Since $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x,t) = \infty$ uniformly in $x \in \Omega$, there exists $\tau > 0$ such that

$$u(x,\tau) > w_1(x, T_1), \qquad v(x,\tau) > z_1(x, T_1).$$

From the maximum principle, we get

$$u(x, t+\tau) \geqslant w_1(x, t+T_1) = \alpha_1(t+T_1) + \psi_1(x)\, f(\beta_1(t+T_1)), \tag{6.2}$$

$$v(x, t+\tau) \geqslant z_1(x, t+T_1) = \beta_1(t+T_1) + \psi_1(x)\, g(\alpha_1(t+T_1)). \tag{6.3}$$

Put $w_2(x,t) = \alpha_2(t) + \psi_2(x)\, f(\beta_2(t))$, $z_2(x,t) = \beta_2(t) + \psi_2(x)\, g(\alpha_2(t))$ with $\alpha_2'(t) = (1+\varepsilon/2)\, f(\beta_1(t))$, $\alpha_2(0) = 1$ and $\beta_2'(t) = (1+\varepsilon/2)\, g(\alpha_1(t))$, $\beta_2(0) =$

$k(1)$. We have

$$\frac{\partial w_2}{\partial t} - L_0 w_2 - f(z_2) = f(\beta_1(t))\left(1+\frac{\varepsilon}{2} - L_0\psi_2\right) + \left(1+\frac{\varepsilon}{2}\right)\psi_2(x)\, f'(\beta_2(t))g(\alpha_2(t)) - f(\beta_2(t)) - \psi_2(x)\, f'(y_2)\, g(\alpha_2(t)),$$

$$\frac{\partial z_2}{\partial t} - L_0 z_2 - g(w_2) = g(\alpha_2(t))\left(1+\frac{\varepsilon}{2} - L_0\psi_2\right) + \left(1+\frac{\varepsilon}{2}\right)\psi_2(x) g'(\alpha_1(t))\, f(\beta_2(t)) - g(\alpha_2(t)) - \psi(x)\, g'(z_2)\, f(\beta_2(t)),$$

$$\frac{\partial w_2}{\partial N_0} = f(\alpha_1(t))\frac{\partial \psi_2}{\partial N_0}, \qquad \frac{\partial z_2}{\partial N_0} = g(\alpha_1(t))\frac{\partial \psi_2}{\partial N_0},$$

with

$$y_2 \in [\beta_2(t),\ \beta_2(t) + \psi_2(x)\, g(\alpha_2(t))]$$

and

$$z_2 \in [\alpha_2(t),\ \alpha_2(t) + \psi_2(x)\, f(\beta_2(t))].$$

Let ψ_2 be a positive solution of the following problem:

$$\frac{\varepsilon}{2} - L_0\psi_2 = -\mu, \qquad \frac{\partial \psi_2}{\partial N_0} = -\mu.$$

ψ_2 exists if and only if $\mu = -(\varepsilon/2)|\Omega|/(|\Omega| + |\partial\Omega|)$. If $\varepsilon = 0$ then $\delta = 0$. Put $\mu(r) = -r(|\Omega|/(|\Omega| + |\partial\Omega|))$. Since $\mu(\varepsilon/2) = \delta(-\varepsilon/2)$ and $\delta'(0) > 0$, it follows that $\mu(\varepsilon/2) < 0$. Therefore, we obtain

$$\frac{\partial w_2}{\partial t} - L_0 w_2 - f(z_2) = -\mu f(\beta_2(t)) + \left(1+\frac{\varepsilon}{2}\right)\psi_2(x)\, f'(\beta_2(t))g(\alpha_2(t)) - \psi_2(x)\, f'(y_2)\, g(\alpha_2(t)),$$

$$\frac{\partial z_2}{\partial t} - L_0 z_2 - g(w_2) = -\mu g(\alpha_2(t)) + \left(1+\frac{\varepsilon}{2}\right)\psi(x)\, g'(\alpha_2(t))\, f(\beta_2(t)) - \psi(x)\, g'(z_2)\, f(\beta_2(t)),$$

$$\frac{\partial w_2}{\partial N_0} = -\mu f(\beta_2(t)), \qquad \frac{\partial z_2}{\partial N_0} = -\mu g(\alpha_2(t)).$$

Since $\lim_{t\to\infty} w_2(x,t) = \lim_{t\to\infty} z_2(x,t) = \infty$ uniformly in $x \in \Omega$, there exists $T_2 > 0$

such that

$$u(x,\tau) < w_2(x,T_2), \qquad v(x,\tau) < z_2(x,T_2).$$

From the maximum principle, we get

$$u(x,t+\tau) \leqslant w_2(x,t+T_2) = \alpha_2(t+T_2) + \psi_2(x)\, f(\beta_2(t+T_2)), \tag{6.4}$$

$$v(x,t+\tau) \leqslant z_2(x,t+T_2) = \beta_2(t+T_2) + \psi_2(x)\, g(\alpha_2(t+T_2)). \tag{6.5}$$

Therefore (u,v) exists globally. For any $\gamma > 0$, we have

$$\lim_{t\to\infty} \frac{\alpha(t-\gamma)}{\alpha(t)} = 1. \tag{6.6}$$

In fact, since $\alpha(t)$ is increasing and convex, we obtain

$$\alpha(t) - \gamma f(k(\alpha(t))) \leqslant \alpha(t-\gamma) \leqslant \alpha(t).$$

Moreover since by hypothesis we have $0 \leqslant \lim_{t\to\infty} f[k(\alpha(t))]/\alpha(t) \leqslant c_2 \lim_{t\to\infty} 1/t = 0$, we deduce that $\lim_{t\to\infty} \alpha(\gamma - t)/\alpha(t) = 1$. On the other hand, show that for all $\varepsilon > 0$ small enough, we have

$$1 - \frac{c_2\varepsilon}{2} \leqslant \liminf_{t\to\infty} \frac{\alpha_1(t)}{\alpha(t)} \leqslant \limsup_{t\to\infty} \frac{\alpha_1(t)}{\alpha(t)} \leqslant 1. \tag{6.7}$$

In fact

$$1 \geqslant \frac{\alpha_1(t)}{\alpha(t)} = \frac{H(t-(\varepsilon/2)\,t)}{H(t)} \geqslant \frac{H(t) - (\varepsilon/2)\, tf[k(H(t))]}{H(t)}.$$

Since $\lim_{s\to\infty} sf(k(H(s)))/H(s) \leqslant c_2$, we have the result. We also have

$$1 \leqslant \liminf_{t\to\infty} \frac{\alpha_2(t)}{\alpha(t)} \leqslant \limsup_{t\to\infty} \frac{\alpha_2(t)}{\alpha(t)} \leqslant 1 + \frac{3c_2\varepsilon}{2}. \tag{6.8}$$

In fact

$$1 \leqslant \liminf_{t\to\infty} \frac{\alpha_2(t)}{\alpha(t)} \leqslant \limsup_{t\to\infty} \frac{\alpha_2(t)}{\alpha(t)} \leqslant \frac{1}{1-(c_2\varepsilon/2(1-\varepsilon/2))} \leqslant 1 + \frac{3c_2\varepsilon}{2}.$$

From (6.2)-(6.8), we deduce that for any $\varepsilon > 0$ small enough, we get

$$1 - k_1\varepsilon \leqslant \liminf_{t\to\infty} \frac{u(x,t)}{\alpha(t)} \leqslant \limsup_{t\to\infty} \frac{u(x,t)}{\alpha(t)} \leqslant 1 + k_2\varepsilon, \tag{6.9}$$

where k_1 and k_2 are two positive constants. Then we deduce that

$$u(x,t) = \alpha(t)(1+o(1)) \quad \text{as} \quad t\to\infty.$$

Making the same reasoning for v, we obtain

$$v(x,t)=\beta(t)(1+o(1)) \quad \text{as} \quad t\to\infty,$$

which yields the result. ■

REMARK 6.2. – Let $f(s)=s^{p_1}$, $g(s)=s^{q_1}$ with $p_1\leq 1$, $q_1\leq 1$. $p_1 q_1<1$. We have

$$k(s)=\left\{\frac{p_1+1}{q_1+1}\right\}^{1/(p_1+1)} s^{(q_1+1)/(p_1+1)},$$

$$f(k(s))=\left\{\frac{p_1+1}{q_1+1}\right\}^{p_1/(p_1+1)} s^{(p_1 q_1+p_1)/(p_1+1)},$$

$$g(k^{-1}(s))=\left\{\frac{q_1+1}{p_1+1}\right\}^{q_1/(p_1+1)} s^{(p_1 q_1+q_1)/(q_1+1)}.$$

Moreover any solution (u,v) of the problem (1.1)-(1.3) initial data (u_0,v_0) exists globally and

$$\lim_{t\to\infty}\frac{u(x,t)}{t^{(p_1+1)/(1-p_1q_1)}}=\left[\left(\frac{p_1+1}{q_1+1}\right)^{p_1/(p_1+1)}\left(\frac{1-p_1q_1}{1+p_1}\right)\right]^{(1+p_1)/(1-p_1q_1)},$$

$$\lim_{t\to\infty}\frac{v(x,t)}{t^{(q_1+1)/(1-p_1q_1)}}=\left[\left(\frac{q_1+1}{p_1+1}\right)^{q_1/(p_1+1)}\left(\frac{1-p_1q_1}{1+q_1}\right)\right]^{(1+q_1)/(1-p_1q_1)}.$$

7. – Blow up set.

In this section, we suppose that for positive values of s, the functions $g_i(s)$ are positive, increasing and convex with $g_i(0)=0$. Under our hypotheses, local existence and uniqueness of a classical solution for the problem (1.4)-(1.6) up to some time T_0 can be found in [1]. Here, we are interested in the blow up and blow up set of the solutions for the problem (1.4)-(1.6). We give some conditions under which the solutions of the problem (1.4)-(1.6) blow up in a finite time and describe their blow up set.

DEFINITION 7.1. – *A function $g(s)$ is called the convex minimal function of the functions $g_i(s)$ if $g(s)$ is positive, continuous, and piecewise convex with $g_i(s)\geq g(s)$ in $(0,\infty)$ and $g'(s)$ is positive and continuous in $(0,\infty)$. We write $g(s)=cm(g_1(s),\dots,g_m(s))$.*

In an interval (α, β) with $\alpha < \beta$, $\alpha \in [0, \infty[$ and $\beta \in]0, \infty]$, $g(s)$ may be constructed in the following manner:

If $g_i(s) \geqslant g_{i_0}(s)$ in (α, β), $i = 1, \ldots, m$ for a certain $i_0 \in \{1, \ldots, m\}$ then $g(s) = g_{i_0}(s)$.

If $m = 2$ and $g_1(s) < g_2(s)$ in $]\alpha, s_0[$, $g_1(s_0) = g_2(s_0)$, $g_1(s) > g_2(s)$ in $]s_0, \beta[$, then a line $z = as - b$ with positive a, b may be taken to be tangent to $g_1(s)$ at $s_1 \in (\alpha, s_0)$ and to $g_2(s)$ at $s_2 \in (s_0, \beta)$ for some s_1, s_2. Then $g(s)$ is given by:

$$g(s) = g_1(s) \quad \text{in} \quad (\alpha, s_1),$$

$$g(s) = as - b \quad \text{in} \quad (s_1, s_2),$$

$$g(s) = g_2(s) \quad \text{in} \quad (s_2, \beta).$$

If $g_1(s) = g_2(s)$ has more than one solution in (α, β), then $cm(g_1, g_2)$ may be constructed by repeated use of the above construction. If $m \geqslant 2$, we construct at first $g_{12} = cm(g_1, g_2)$. After, we construct $g_{123} = cm(g_{12}, g_3)$ by the method described above and by iteration, we obtain $g_{12\ldots m} = cm(g_{12\ldots m-1}, g_m)$. Therefore we take $g = g_{12\ldots m}$.

Let $m = 2$, $g_1(s) = s^p$, $g_2(s) = s^q$. If $p > q$, then

$$cm(s^p, s^q) = s^p \quad \text{for} \quad 0 \leqslant s < s_1,$$

$$cm(s^p, s^q) = bs - c \quad \text{for} \quad s_1 \leqslant s < s_2,$$

$$cm(s^p, s^q) = s^q \quad \text{for} \quad s_2 \leqslant s$$

where

$$s_1 = \left(\frac{q}{p}\right)^{q/(p-q)} \left(\frac{p-1}{q-1}\right)^{(q-1)/(p-q)},$$

$$s_2 = \left(\frac{q}{p}\right)^{q/(p-q)} \left(\frac{p-1}{q-1}\right)^{(p-1)/(p-q)},$$

$$s_1 < 1 < s_2,$$

$$b = \frac{q^{(q(p-1)/(p-q))}}{p^{(p(q-1)/(p-q))}} \left(\frac{p-1}{q-1}\right)^{((q-1)(p-1)/(p-q))},$$

$$c = \left(\frac{q}{p}\right)^{qp/(p-q)} \frac{(p-1)^{(q(p-1)/(p-q))}}{(q-1)^{(p(q-1)/(p-q))}},$$

$$c < b < c + 1.$$

If $p = q$, then $cm(s^p, s^q) = s^p$.

THEOREM 7.2. – *Suppose that* $L_0 u_0^{(i)}(x) - a(x) u_0^{(i)}(x) > 0$ *and*

$$\int^{\infty} \frac{ds}{cm(g_1(s), \dots, g_m(s))} < \infty .$$

Then, any solution $(u_1, \dots, u_m)$ *of the problem* (1.4)-(1.6) *blows up in a finite time* T *and there exists a positive constant* δ *such that*

$$\sum_{i=1}^{m} \frac{1}{m} \sup_{x \in \Omega} u_i(x, t) \leq G_p(\delta(T-t))$$

where G_p *is the inverse function of* $G_*(s) = \int_s^{\infty} \frac{d\sigma}{cm(g_1(\sigma), \dots, g_m(\sigma))}$.

PROOF. – Let $w_i = u_{it}$. Since $L_0 u_0^{(i)}(x) - a(x) u_0^{(i)}(x) > 0$, we have $w_i(x, 0) > 0$. Therefore w_i $(i = 1, \dots, m)$ satisfy the following relations

$$w_{it} - L_0 w_i = -a(x) w_i \quad \text{in} \quad \Omega \times (0, T), \tag{7.1}$$

$$\frac{\partial w_i}{\partial N_0} + b(x) w_i = g_i'(u_{i+1}) w_{i+1} \quad \text{on} \quad \partial\Omega \times (0, T), \tag{7.2}$$

$$w_i(x, 0) > 0 \quad \text{in} \quad \Omega. \tag{7.3}$$

From the maximum principle, there exists a number c such that

$$u_{it}(x, t) \geq c > 0 \quad \text{in} \quad \Omega \times (\varepsilon_0, T) \tag{7.4}$$

for $\varepsilon_0 > 0$. Put

$$J_i(x, t) = u_{it} - \delta g_i(u_{i+1}). \tag{7.5}$$

We have

$$\begin{aligned} J_{it} - L_0 J_i &= (u_{it} - L_0 u_i)_t - \delta g_i'(u_{i+1})(u_{(i+1)t} - L_0 u_{i+1}) + \\ &\delta g_i''(u_{i+1}) \sum_{k,j=1}^{n} a_{kj}^{(0)}(x) \frac{\partial u_{i+1}}{\partial x_k} \frac{\partial u_{i+1}}{\partial x_j} = -a(x) J_i + \\ &a(x) \delta[g_i'(u_{i+1}) u_{i+1} - g_i(u_{i+1})] + \delta g_i''(u_{i+1}) \sum_{k,j=1}^{n} a_{kj}^{(0)}(x) \frac{\partial u_{i+1}}{\partial x_k} \frac{\partial u_{i+1}}{\partial x_j}, \end{aligned} \tag{7.6}$$

$$\begin{aligned} &\frac{\partial J_i}{\partial N_0} + b(x) J_i = \\ &g_i'(u_{i+1}) J_{i+1} + \delta b(x)[g_i'(u_{i+1}) u_{i+1} - g_i(u_{i+1})] \quad \text{on} \quad \partial\Omega \times (0, T). \end{aligned} \tag{7.7}$$

Since for positive values of s, the functions $g_i(s)$ are convex with $g_i(0) = 0$, from (7.6) and (7.7), we obtain

$$J_{it} - L_0 J_i + a(x) J_i \geqslant 0 \quad \text{in} \quad \Omega \times (0, T), \tag{7.8}$$

$$\frac{\partial J_i}{\partial N_0} + b(x) J_i \geqslant g_i'(u_{i+1}) J_{i+1} \quad \text{on} \quad \partial\Omega \times (0, T). \tag{7.9}$$

From (7.4) and (7.5), take δ small enough that

$$J_i(x, \varepsilon_0) > 0 \quad \text{in} \quad \Omega. \tag{7.10}$$

From the maximum principle, we have

$$u_{it} \geqslant \delta g_i(u_{i+1}) \quad \text{in} \quad \Omega \times (\varepsilon_0, T). \tag{7.11}$$

Put $w(x, t) = \dfrac{1}{m} \sum_{i=1}^{m} u_i$ and $g(s) = cm(g_1(s), \dots, g_m(s))$. From (7.11) and by the

definition of $g(s)$, we get

$$w_t \geqslant \delta \sum_{i=1}^{m} \frac{1}{m} g(u_{i+1}) \geqslant \delta g(w). \tag{7.12}$$

The inequality (7.12) implies that

$$-(G_*(w))_t = \frac{w_t}{g(w)} \geqslant \delta. \tag{7.13}$$

Integrating (7.13) over (ε_0, T), it follows that

$$\infty > G_*(w(x, \varepsilon_0)) \geqslant G_*(w(x, \varepsilon_0)) - G_*(w(x, T)) \geqslant \delta(T - \varepsilon_0). \tag{7.14}$$

This implies that T is finite and w blows up in a finite time T. On the other hand, integrating (7.13) over (t, T), we see that

$$G_*(w(x, t)) \geqslant G_*(w(x, t)) - G_*(w(x, T)) \geqslant \delta(T - t). \tag{7.15}$$

Since G_* is decreasing, then its inverse function G_p is also decreasing and from (7.15), we obtain

$$w(x, t) \leqslant G_p[\delta(T - t)],$$

which gives the result. ■

THEOREM 7.3. – *Under the hypotheses of Theorem* 7.2, *suppose that there exists a positive constant* C_0 *such that*

$$s g'(G_p(s)) \leqslant C_0 \quad \textit{for} \quad s > 0$$

where $g(s) = cm(g_1(s), \dots, g_m(s))$. *Then any solution* $(u_1, \dots, u_m)$ *of the* *problem* (1.4)-(1.6) *blows up in a finite time* T *and* $E_B \subset \partial\Omega$, *where* E_B *is the blow up set of the solution* $(u_1, \dots, u_m)$.

PROOF. – By Theorem 7.2, we know that $(u_1, \dots, u_m)$ blows up in a finite time T. Thus our aim in this proof is to show that $E_B \subset \partial\Omega$. Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $v(x) = d^2(x)$ for $x \in N_\varepsilon(\partial\Omega)$ where

$$N_\varepsilon(\partial\Omega) = \{x \in \Omega \text{ such that } d(x) < \varepsilon\}.$$

Since $\partial\Omega$ is of class C^2, then the function $v(x) \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ if ε is sufficiently small. On $\partial\Omega$, we have

$$L_0 v - \frac{C_0}{v}\sum_{i,j=1}^n a_{ij}^{(0)}(x)\, v_{x_i} v_{x_j} =$$

$$\sum_{i=1}^n a_{ii}^{(0)}(x)\, v_{x_i x_i} + \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial a_{ij}^{(0)}(x)}{\partial x_j}\right) v_{x_i} - \frac{C_0}{v}\sum_{i,j=1}^n a_{ij}^{(0)} v_{x_i} v_{x_j} =$$

$$2\sum_{i=1}^n a_{ii}^{(0)}(x) + 2d\sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial a_{ij}^{(0)}(x)}{\partial x_j}\right) d_{x_i} - 4C_0 \sum_{i,j=1}^n a_{ij}^{(0)}(x)\, d_{x_i} d_{x_j} \geqslant$$

$$-2\sum_{i=1}^n |a_{ii}^{(0)}(x)| - 2d'\sum_{i=1}^n \left|\sum_{j=1}^n \frac{\partial a_{ij}^{(0)}(x)}{\partial x_j}\right| |\nabla d| - 4C_0 \lambda_2^{(0)} |\nabla d|^2$$

where $d' = \sup_{x \in \overline{\Omega},\, y \in \overline{\Omega}} \|x - y\|$. Therefore, there exists a positive constant C_1 such that

$$L_0 v - \frac{C_0}{v}\sum_{i,j=1}^n a_{ij}^{(0)} v_{x_i} v_{x_j} \geqslant -C_1 \quad \text{on} \quad \partial\Omega. \tag{7.16}$$

Since $v \in C^2(\overline{N_\varepsilon(\partial\Omega)})$ for ε sufficiently small, let ε_0 be so small that

$$L_0 v - \frac{C_0}{v}\sum_{i,j=1}^n a_{ij}^{(0)} v_{x_i} v_{x_j} \geqslant -2C_1 \quad \text{in} \quad \overline{N_{\varepsilon_0}(\partial\Omega)}. \tag{7.17}$$

We extend v to a function of class $C^2(\overline{\Omega})$ such that $v \geqslant C_0^* > 0$ in $\overline{\Omega - N_{\varepsilon_0}(\partial\Omega)}$. Therefore, we deduce that

$$L_0 v - \frac{C_0}{v}\sum_{i,j=1}^n a_{ij}^{(0)}(x)\, v_{x_i} v_{x_j} \geqslant -C^* \quad \text{in} \quad \overline{\Omega} \tag{7.18}$$

for some $C^* > 0$. Multiplying (7.18) by ε small enough, we may assume without loss of generality that $C^* < 1$. Put $w_*(x, t) = C_1 G_p(\tau)$ where $\tau = \delta(v(x) +$

$C^*(T-t))$ and $C_1 > 1$ is a constant which will be indicated later. We get

(7.19) $$w_{*t} - L_0 w_* = -\delta C_1 G_p'(\tau)\left[C^* + L_0 v + \delta \frac{G_p''(\tau)}{G_p'(\tau)} \sum_{i,j=1}^{n} a_{ij}^{(0)}(x)\, v_{x_i} v_{x_j}\right].$$

Since $G_p(s)$ is the inverse function of $G(s)$, we have $G_p'(s) = -g(G_p(s))$ and $G_p''(s) = -G_p'(s)\, g'(G_p(s))$. Consequently

(7.20) $$w_{*t} - L_0 w_* = \delta C_1 g(G_p(s))\left[C^* + L_0 v - \delta g'(G_p(\tau)) \sum_{i,j=1}^{n} a_{ij}^{(0)}(x)\, v_{x_i} v_{x_j}\right].$$

Since $sg'(G_p(s)) \leqslant C_0$ for $s > 0$, using the fact that $g'(G_p(s))$ is a decreasing function (g' is increasing and G_p is decreasing), we have

(7.21) $$w_{*t} - L_0 w_* \geqslant \delta C_1 g(G_p(\tau))\left[C^* + L_0 v - \frac{C_0}{v} \sum_{i,j=1}^{n} a_{ij}^{(0)}(x)\, v_{x_i} v_{x_j}\right].$$

Therefore from (7.18), we deduce that

(7.22) $$w_{*t} - L_0 w_* + a(x)\, w_* \geqslant 0 \quad \text{in} \quad \Omega \times (\varepsilon_0, T).$$

On $\partial\Omega$, we have $w_*(x, t) = C_1 G_p(\delta C^*(T-t)) > G_p(\delta(T-t))$ because $C_1 > 1$ and $C^* < 1$. Then by Theorem 7.2, we obtain

(7.23) $$w_*(x, t) > \sum_{i=1}^{m} \frac{1}{m} u_i(x, t) \quad \text{on} \quad \partial\Omega \times (\varepsilon_0, T).$$

Choose C_1 large enough that

(7.24) $$w_*(x, \varepsilon_0) = C_1 G_p(\delta(v(x) + C^*(T-\varepsilon_0))) > \sum_{i=1}^{m} \frac{1}{m} u_i(x, \varepsilon_0).$$

Consequently, from the maximum principle we deduce that

$$\sum_{i=1}^{m} \frac{1}{m} u_i(x, t) < w_*(x, t) \quad \text{in} \quad \Omega \times (\varepsilon_0, T).$$

Then if $\Omega' \subset\subset \Omega$ we have

$$\sum_{i=1}^{m} \frac{1}{m} u_i(x, t) \leqslant C_1 G_p(\delta(v(x) + C^*(T-t))) \leqslant C_1 G_p(\delta v(x)).$$

It follows that

$$\sum_{i=1}^{m} \frac{1}{m} \sup_{x \in \Omega',\, t \in [\varepsilon_0, T)} u_i(x, t) \leqslant \sup_{x \in \Omega'} C_1 G_p(\delta v(x)) < \infty,$$

which yields the result. ■

COROLLARY 7.4. – *Let* $g_i(s) = s^{p_i}$, *where* $p_i > 1$. *Suppose that* $L_0 u_0^{(i)}(x) - a(x) u_0^{(i)}(x) > 0$. *Then any solution* $(u_1, \ldots, u_m)$ *of the problem* (1.4)-(1.6) *blows up in a finite time and we have* $E_B \subset \partial\Omega$ *where* E_B *is the blow up set of the solution* $(u_1, \ldots, u_m)$.

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