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Total Negation under Constraint: Pre-anti Properties.

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Sunto. – *L'operazione «anti( )» di Paul Bankston fu introdotta in contesto della famiglia di tutti gli spazii topologici. Però, per molte ricerche ci conviene lavorare esclusivamente in una classe costretta di spazii di cui la struttura e ricca abbastanza di facilitare il ragionamento. In quest'articolo descriviamo come trasferire anti( ), e concetti allacciati, dentro una tale classe costretta; con riferimento speciale all'esistenza di «pre-antis».*

1. – Introduction.

The total negation operator anti( ) was formulated by Paul Bankston in [1] and is a process for generating, from each given topological invariant $\mathcal{P}$, another written as anti($\mathcal{P}$) and described as follows. First, one identifies the class spec($\mathcal{P}$) of non-zero cardinal numbers λ which are such that *every* topology on a set of cardinality λ makes it into a $\mathcal{P}$ space. Then a space X is declared to be anti($\mathcal{P}$) provided that a subspace Y of X can be a $\mathcal{P}$ space *only* in the unavoidable case where its cardinality $|Y|$ belongs to spec($\mathcal{P}$). For example, an anti(separable) space is one which has no separable subspaces except, inevitably, the finite or denumerable ones. For an account of this topic see, for example, [6], [11] and the articles referenced in their bibliographies.

In one respect this approach to total negation is not perfectly compatible with the working practices of topologists or, particularly, of other mathematicians who make use of topology. There are many areas in which the full generality of the definition of «topological space» is not appropriate and in which researchers find it convenient to work exclusively with spaces that are — for example — completely regular, or compact Hausdorff, or separable metrizable. The present article arises from an investigation of what effects, upon the total negation process, there can be of such a decision to work solely within a constrained class of topological spaces.

The particular problem which we address here is that of the existence of what are called pre-anti properties. Returning for a moment to the «unconstrained» classical case of total negation, given an invariant $\mathcal{Q}$ it may or may not be possible to find another invariant $\mathcal{P}$ such that anti($\mathcal{P}$) = $\mathcal{Q}$. If it is possible, any such $\mathcal{P}$ is referred to as a *pre-anti* for $\mathcal{Q}$. Bankston's original paper [1] includes a simple description of which invariants possess pre-antis, and there has been subsequent work on which of these have pre-antis of a particular type (hereditary, for example) and on how many pre-antis a given property may possess [4], [8]. The imposition of a constraint considerably complicates the discussion of the existence of pre-anti properties, and it is this which we primarily investigate in the present article. Section 2 sets out the definitions and notations for constrained total negation and the relevant elementary results, while section 3 presents criteria for the existence of pre-anti properties under constraint. Throughout we shall identify a space with all spaces that are homeomorphic to it, and we shall identify an invariant with the class of all spaces possessing it; thus, the terms «property», «topological invariant», «class of spaces» will be used interchangeably. Other aspects of this study may be found in [9], [10], [12].

2. – Constrained total negation.

Let $\mathcal{C}$, $\mathcal{P}$ denote arbitrary classes of topological spaces. We make the following definitions:

1) $\mathcal{C}$-spec$(\mathcal{C} \cap \mathcal{P}) = \{\lambda : \lambda$ is a non-zero cardinal, there are spaces in $\mathcal{C}$ having cardinality λ and *all* of them are $\mathcal{P}$ spaces$\}$.

2) $\mathcal{C}$-proh$(\mathcal{C} \cap \mathcal{P}) = \{\lambda : \lambda$ is a non-zero cardinal, there are spaces in $\mathcal{C}$ having cardinality λ and *none* of them is a $\mathcal{P}$ space$\}$.

3) $\mathcal{C}$-ind$(\mathcal{C} \cap \mathcal{P}) = \{\lambda : \lambda$ is a non-zero cardinal, there are spaces in $\mathcal{C}$ having cardinality λ and *some but not all* of them are $\mathcal{P}$ spaces$\}$.

$\mathcal{C}$-anti$(\mathcal{C} \cap \mathcal{P}) = \{X \in \mathcal{C}$: whenever Y is a subspace of X and $Y \in \mathcal{C} \cap \mathcal{P}$ then $|Y| \in \mathcal{C}$-spec$(\mathcal{C} \cap \mathcal{P})\}$.

We call $\mathcal{C}$ the *constraint*. When $\mathcal{C}$ is the universal class of all topological spaces, these four definitions specialize to the terms spec$(\mathcal{P})$, proh$(\mathcal{P})$, ind$(\mathcal{P})$ and anti$(\mathcal{P})$ of classical total negation.

Whenever $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{Q}$ are classes of spaces for which $\mathcal{C}$-anti$(\mathcal{C} \cap \mathcal{P}) = \mathcal{Q}$, we shall call $\mathcal{P}$ a *pre-anti for* $\mathcal{Q}$ *within* $\mathcal{C}$ or a $\mathcal{C}$*-pre-anti for* $\mathcal{Q}$. Clearly then $\mathcal{Q} \subseteq \mathcal{C}$, and $\mathcal{C} \cap \mathcal{P}$ is also a $\mathcal{C}$-pre-anti for $\mathcal{Q}$. For this reason, it will generally be the case in this article that each class $\mathcal{R}$ of spaces examined is a subclass of the current constraint $\mathcal{C}$; naturally we shall then abbreviate $\mathcal{C} \cap \mathcal{R}$ to $\mathcal{R}$ unless the emphasis seems useful.

The following basic facts concerning unconstrained total negation can be found in [1] and [3].

LEMMA 1. (i) For any invariant $\mathcal{P}$, anti($\mathcal{P}$) is a hereditary class.

(ii) All spaces are anti($\mathcal{P}$) if and only if ind($\mathcal{P}$) is empty.

(iii) If a space X is both $\mathcal{P}$ and anti($\mathcal{P}$) then its cardinality $|X|$ belongs to spec($\mathcal{P}$).

LEMMA 2. – Let $\mathcal{Q}$ be a non-empty hereditary class such that there is no positive integer n for which

$$\operatorname{spec}(\mathcal{Q}) = [1, n] \text{ and } \operatorname{proh}(\mathcal{Q}) = [n+1, \infty).$$

Then $\mathcal{Q}$ has a pre-anti. The converse is also true.

The first of these lemmas transfers easily to the constrained scenario. If $\mathcal{C}$ is a class of spaces and $\mathcal{Q}$ is a subclass of $\mathcal{C}$, let us say that $\mathcal{Q}$ is *$\mathcal{C}$-hereditary* if, whenever Y is a subspace of X and $X \in \mathcal{Q}$ and $Y \in \mathcal{C}$, we find $Y \in \mathcal{Q}$ also. Then:

LEMMA 3. – (i) For any $\mathcal{P} \subseteq \mathcal{C}$, $\mathcal{C}$-anti($\mathcal{P}$) is $\mathcal{C}$-hereditary. Consequently, if $\mathcal{C}$ is hereditary then so is $\mathcal{C}$-anti($\mathcal{P}$).

(ii) $\mathcal{C}$-anti($\mathcal{P}$) = $\mathcal{C}$ if and only if $\mathcal{C}$-ind($\mathcal{P}$) is empty.

(iii) If $X \in \mathcal{C}$ is both $\mathcal{P}$ and $\mathcal{C}$-anti($\mathcal{P}$) then $|X| \in \mathcal{C}$-spec($\mathcal{P}$).

By way of illustration, we shall next show that the questions of existence of pre-anti properties with and without constraint are independent.

EXAMPLES. – (i) Take $\mathcal{C}$ to encompass all finite spaces and all discrete spaces, and $\mathcal{Q}$ to be the class of finite spaces. Note that both $\mathcal{C}$ and $\mathcal{Q}$ are hereditary. It is known [3] that $\mathcal{Q}$ has pre-anti properties, for instance anti(first-countable) = anti(completely separable) = $\mathcal{Q}$. If, however, there were to exist $\mathcal{P} \subseteq \mathcal{C}$ such that $\mathcal{C}$-anti($\mathcal{P}$) = $\mathcal{Q}$, then the countably infinite discrete space $\boldsymbol{N}$ is not $\mathcal{C}$-anti(P) so there is $M \subseteq \boldsymbol{N}$ such that $M \in \mathcal{P}$ and $|M| \in \mathcal{C}$-ind($\mathcal{P}$). Since there is only one $\mathcal{C}$ space of cardinality $\aleph_0$, this forces $|M|$ to be finite; but then M is both $\mathcal{Q} = \mathcal{C}$-anti($\mathcal{P}$) and $\mathcal{P}$, yielding the contradiction $|M| \in \mathcal{C}$-spec($\mathcal{P}$). Thus $\mathcal{Q}$ has no $\mathcal{C}$-pre-anti.

(ii) Let $\mathcal{Q}$ be the class of compact T_2 spaces, $\mathcal{P}$ be the class of all spaces that are not completely regular ($= T_{3(1/2)}$) and $\mathcal{C}$ be the union $\mathcal{P} \cup \mathcal{Q}$. Each member of $\mathcal{P}$ shares its cardinality with a $\mathcal{Q}$ space, so $\mathcal{C}$-spec($\mathcal{P}$) is empty; therefore no member of $\mathcal{P}$ is $\mathcal{C}$-anti($\mathcal{P}$). No member of $\mathcal{Q}$ can contain a member of $\mathcal{P}$ irrespective of cardinality considerations. Therefore $\mathcal{C}$-anti($\mathcal{P}$) coincides precisely with $\mathcal{Q}$. On the other hand, $\mathcal{Q}$ has no pre-anti in the unconstrained sense, since it is not hereditary.

Notice that in (ii) above, unlike in (i), the constraint was not hereditary. This was unavoidable because, as we shall now show, the existence of a pre-anti *within a hereditary constraint* guarantees existence in the unconstrained sense also (with one obvious exception).

PROPOSITION 4. – Let $\mathcal{Q}$ be a *proper* subclass of a hereditary class $\mathcal{C}$. If $\mathcal{Q}$ has a $\mathcal{C}$-pre-anti then it also has a pre-anti.

PROOF. – If not, then by Lemma 2 either $\mathcal{Q}$ is not hereditary or there is a positive integer n such that the $\mathcal{Q}$ spaces are precisely those on n or fewer points. The first possibility is ruled out by Lemma 3(i). In the second eventuality, since $\mathcal{Q}$ is *strictly* contained in $\mathcal{C}$ there is a $\mathcal{C}$ space on more than n elements and therefore, as $\mathcal{C}$ is hereditary, there is a $\mathcal{C}$ space Y on precisely $n+1$ elements. Writing $\mathcal{Q} = \mathcal{C}$-anti$(\mathcal{P})$, we see that Y is not $\mathcal{C}$-anti$(\mathcal{P})$ so there is a subspace Z of Y with $Z \in \mathcal{C} \cap \mathcal{P}$ but $|Z| \in \mathcal{C}$-ind$(\mathcal{P})$. If $|Z| \leqslant n$ then Z is both $\mathcal{P}$ and $\mathcal{C}$-anti$(\mathcal{P})$ so the contradiction $|Z| \in \mathcal{C}$-spec$(\mathcal{P})$ is obtained. If not, Z is the entirety of Y and so Y is a $\mathcal{P}$ space and $n+1 \in \mathcal{C}$-ind$(\mathcal{P})$; choose therefore a non-$\mathcal{P}$ space Y' in $\mathcal{C}$ on $n+1$ points. Then Y' is $\mathcal{C}$-anti$(\mathcal{P})$ since its only possible $\mathcal{P}$ subspaces have n or fewer points and are therefore $\mathcal{C}$-anti$(\mathcal{P})$ also, whence each has cardinality in $\mathcal{C}$-spec$(\mathcal{P})$. This contradicts the choice of n and completes the demonstration.

We have often found it necessary, as in the above proposition, to focus attention on *hereditary* constraints in order to get satisfactory results. Whereas this certainly represents some loss of generality, it is broadly true that most of the topological environments within which research takes place are hereditary; besides, it reflects the fundamental role played by hereditary invariants throughout the discussion of total negation. Somewhat more surprisingly, a condition much weaker than hereditariness, and its dual, are important in the exploration of $\mathcal{C}$-pre-antis. These are the UP and DOWN properties introduced by Matthews [7], [8] for quasiordered classes which, for convenience, we shall redescribe here for topological spaces. A class $\mathcal{C}$ of topological spaces is said to satisfy DOWN (respectively, UP) if, whenever λ and μ are cardinal numbers such that $\lambda \leqslant \mu$, every $\mathcal{C}$ space on μ elements has a $\mathcal{C}$ subspace on λ elements (respectively, every $\mathcal{C}$ space on λ elements is embeddable into some $\mathcal{C}$ space on μ elements). Clearly, every hereditary class satisfies DOWN.

The last preliminary idea we need concerns minimality of topological spaces. The turning point of the previous proof was the fact that the spaces Y and Y' were minimal *in the sense that* a subspace of one of these was either the entire space, or had strictly smaller cardinality and different topological characteristics. This is an utterly trivial idea for finite spaces but is somewhat more subtle for infinite ones. Following Matthews [7], [8] we say that a topological space X belonging to a class $\mathcal{F}$ of spaces is *strictly quasiminimal in* $\mathcal{F}$ (sqm in $\mathcal{F}$) if, whenever Y is a subspace of X and $Y \in \mathcal{F}$, then X and Y are homeomorphic. The simplest non-trivial example concerns the five «Ginsburg and

Sands» spaces [2] which are sqm in the class of infinite spaces and which occur as subspaces of every infinite space.

3. – Criteria for existence of $\mathcal{C}$-pre-antis.

The existence question for pre-antis of a class $\mathcal{P}$ in a constraint $\mathcal{C}$ is involved with two simple cardinality conditions — $\mathcal{C}$-proh$(\mathcal{P}) = \phi$ and $\mathcal{C}$-ind$(\mathcal{P}) \neq \phi$ — and a third involving cardinality and topology which, to prevent unnecessary repetition, we shall label as condition $(*)$:

$$\text{there is a } \mathcal{C} \text{ space } Z \text{ of cardinality } \lambda = \min(\mathcal{C}\text{-proh}(\mathcal{P}))$$

$$\text{that is not sqm in the class not } \mathcal{P} \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

where min$(\mathcal{D})$ represents the smallest member of a (non-empty) class $\mathcal{D}$ of cardinals. The following proposition collects together the connections:

PROPOSITION 5. – Let $\mathcal{P}$ be a subclass of a constraint $\mathcal{C}$.

I (*a*) Provided that $\mathcal{P}$ is $\mathcal{C}$-hereditary, if $\mathcal{C}$-proh$(\mathcal{P}) = \phi$ then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti.

(*b*) Provided that $\mathcal{P}$ is $\mathcal{C}$-hereditary and $\mathcal{C}$ satisfies UP and DOWN, if $\mathcal{C}$-ind$(\mathcal{P}) \neq \phi$ then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti.

(*c*) Provided that $\mathcal{P}$ is $\mathcal{C}$-hereditary and $\mathcal{C}$ satisfies DOWN, if $(*)$ holds then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti.

II (*a*) If $\mathcal{P}$ has a $\mathcal{C}$-pre-anti then $\mathcal{C}$-proh$(\mathcal{P}) = \phi$ or $(*)$ holds.

(*b*) Provided that $\mathcal{P}$ is a proper subclass of $\mathcal{C}$, if $\mathcal{P}$ has a $\mathcal{C}$-pre-anti then $\mathcal{C}$-ind$(\mathcal{P}) \neq \phi$ or $(*)$ holds.

PROOF. – [I (*a*)] If $\mathcal{P}$ is $\mathcal{C}$-hereditary and $\mathcal{C}$-proh$(\mathcal{P}) = \phi$, it is routine to confirm that $\mathcal{C}$-anti$(\mathcal{C} \setminus \mathcal{P}) = \mathcal{P}$.

[I (*b*)] Suppose that $\mathcal{P}$ is $\mathcal{C}$-hereditary, $\mathcal{C}$ satisfies UP and DOWN, $\mathcal{C}$-ind$(\mathcal{P}) \neq \phi$ and, without loss of generality, $\mathcal{C}$-proh$(\mathcal{P}) \neq \phi$ also. Put $\lambda = \min(\mathcal{C}\text{-proh}(\mathcal{P}))$ and define

$$\mathcal{W} = \{X \in \mathcal{C} : (X \text{ is not } \mathcal{P} \text{ and } |X| \in \mathcal{C}\text{-ind}(\mathcal{P})) \text{ or}$$

$$(|X| = \lambda \text{ and } (Y \subseteq X,\ |Y| < \lambda) \Rightarrow Y \text{ is } \mathcal{P})\}.$$

It is apparent that no cardinal $< \lambda$ can belong to $\mathcal{C}$-spec$(\mathcal{W})$. Now since DOWN holds, every $\mathcal{C}$ space on more than λ points contains (necessarily non-$\mathcal{P}$) subspaces on λ points and cannot therefore be $\mathcal{P}$; thus, the cardinals above λ either belong to $\mathcal{C}$-proh$(\mathcal{P})$ or else appertain to no $\mathcal{C}$ spaces at all. Choose μ in $\mathcal{C}$-ind$(\mathcal{P})$ and a space Y in $\mathcal{C} \setminus \mathcal{P}$ of that cardinality. Since $\mu < \lambda$, UP assures us that Y is embeddable into a $\mathcal{C}$ space X on λ points. Then $X \notin \mathcal{W}$, so $\lambda \notin \mathcal{C}$-spec($\mathcal{W}$) and we deduce that $\mathcal{C}$-spec($\mathcal{W}$) is empty.

A $\mathcal{P}$ space cannot contain a $\mathcal{W}$ space and thus $\mathcal{P}$ implies $\mathcal{C}$-anti($\mathcal{W}$). Conversely, let A belong to $\mathcal{C} \setminus \mathcal{P}$. If $|A| < \lambda$ then $A \in \mathcal{W}$ and $|A| \in \mathcal{C}$-ind($\mathcal{W}$) so A cannot be $\mathcal{C}$-anti($\mathcal{W}$). If $|A| \geq \lambda$, use DOWN to find a subspace B of A with $|B| = \lambda$; then either $B \in \mathcal{W}$ (and $|B| \in \mathcal{C}$-ind($\mathcal{W}$)) or B contains a $\mathcal{W}$ subspace on fewer than λ-many points. In both cases, then, $A \notin \mathcal{C}$-anti($\mathcal{W}$). We now have that $\mathcal{W}$ is a $\mathcal{C}$-pre-anti for $\mathcal{P}$.

[I (c)] Next, let $\mathcal{P}$ be $\mathcal{C}$-hereditary, $\mathcal{C}$ satisfy DOWN, ($*$) hold, and Z and λ be as described in ($*$). Put

$$\mathcal{W} = \{X \in \mathcal{C} : (X \text{ is not } \mathcal{P} \text{ and } |X| \in \mathcal{C}\text{-ind}(\mathcal{P})) \text{ or } (|X| = \lambda \text{ and } X \neq Z)\} .$$

Now if there is a space X that is $\mathcal{P}$ but not $\mathcal{C}$-anti($\mathcal{W}$), then X has a subspace $Y \in \mathcal{W}$ with $|Y| \in \mathcal{C}$-ind($\mathcal{W}$); yet $\mathcal{P}$ is $\mathcal{C}$-hereditary, so Y is both $\mathcal{P}$ and $\mathcal{W}$ which is impossible. On the other hand, if V is $\mathcal{C}$-anti($\mathcal{W}$) but not $\mathcal{P}$, then V has no $\mathcal{W}$ subspaces since $\mathcal{C}$-spec($\mathcal{W}$) is empty. We must therefore have $|V| < \lambda$ since, in view of DOWN, every $\mathcal{C}$ space on λ or more points contains one on λ points and therefore — whether this one be Z or not — contains also a member of $\mathcal{W}$. Thus we have $|V| \in \mathcal{C}$-ind($\mathcal{P}$); but then V is a $\mathcal{W}$ subspace of itself, yielding another contradiction. We conclude that $\mathcal{W}$ is a $\mathcal{C}$-pre-anti for $\mathcal{P}$ in this case.

[II (a)] Suppose if possible that $\mathcal{P}$ takes the form $\mathcal{C}$-anti($\mathcal{W}$) but that $\mathcal{C}$-proh($\mathcal{P}$) $\neq \phi$ and ($*$) fails. So λ is defined, and the $\mathcal{C}$ spaces on λ-many elements are all sqm in not $\mathcal{P}$. Let F denote a typical such space. Since F is not $\mathcal{C}$-anti($\mathcal{W}$), there is a subspace G of F with $G \in \mathcal{W}$ and $|G| \in \mathcal{C}$-ind($\mathcal{W}$). If G were $\mathcal{P}$ we would, as usual, get the contradiction $|G| \in \mathcal{C}$-spec($\mathcal{W}$). Therefore G also is not $\mathcal{P}$, and the sqm property of F shows that F and G are homeomorphic. We conclude that every $\mathcal{C}$ space of cardinality λ is $\mathcal{W}$, that is, $\lambda \in \mathcal{C}$-spec($\mathcal{W}$) contradicting its definition.

[II (b)] Suppose if possible that $\mathcal{P} \neq \mathcal{C}$, $\mathcal{P}$ is of the form $\mathcal{C}$-anti($\mathcal{W}$), $\mathcal{C}$-ind($\mathcal{P}$) $= \phi$ and ($*$) fails. Then $\mathcal{C}$-proh($\mathcal{P}$) cannot be empty as well, or else every $\mathcal{C}$ space would after all be $\mathcal{P}$, so II (a) generates the desired contradiction. (Of course, if $\mathcal{P} = \mathcal{C}$ then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti, $\mathcal{C}$ itself for example, irrespective of whether or not $\mathcal{C}$-ind($\mathcal{P}$) is empty or ($*$) holds.)

Of the varying ways to assemble «necessary and sufficient» criteria from the above components, we mention the following:

THEOREM 6. – Suppose that $\mathcal{C}$ satisfies DOWN and that $\mathcal{P} \subseteq \mathcal{C}$ is $\mathcal{C}$-hereditary. Then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti if and only if $\mathcal{C}$-proh($\mathcal{P}$) $= \phi$ or ($*$) holds.

THEOREM 7. – Suppose that $\mathcal{C}$ satisfies DOWN and UP and that $\mathcal{P}$ is a $\mathcal{C}$-hereditary proper subclass of $\mathcal{C}$. Then $\mathcal{P}$ has a $\mathcal{C}$-pre-anti if and only if $\mathcal{C}$-ind$(\mathcal{P}) \neq \phi$ or $(*)$ holds.

EXAMPLES. – (i) In the light of these results, the observations on constrained pre-antis in the Example of section 2 are now trivial. To proceed a little further, let us briefly explore which constraints give the class $\mathcal{S}$ of separable metrizable spaces a pre-anti property.

There are separable metrizable spaces of every cardinality from 1 to $c = 2^{\aleph_0}$ inclusive but none larger than that. Also $\mathcal{S}$ is hereditary, and therefore $\mathcal{C}$-hereditary for any $\mathcal{C} \supseteq \mathcal{S}$. If $\mathcal{C}$ were to comprise, say, only the members of $\mathcal{S}$ and the discrete, the cofinite and the trivial spaces on c^+ elements, then Proposition 5(II)(a) or (b) shows that there can be no $\mathcal{C}$-pre-anti for $\mathcal{S}$. By Proposition 5(I)(a), if $\mathcal{C}$ contains no space of cardinality in excess of c then pre-antis for $\mathcal{S}$ in $\mathcal{C}$ will exist; but otherwise, and assuming now that DOWN holds for $\mathcal{C}$, c^+ will be the least member of $\mathcal{C}$-proh$(\mathcal{S})$ and Theorem 6 asserts that the necessary and sufficient condition is that some member of $\mathcal{C}$ on c^+ points contains either another distinct such space or a $\mathcal{C}$ space on c or fewer points that is not separable metrizable. For instance, if the discrete space on c^+ points belongs to $\mathcal{C}$ then (using DOWN) so must the discrete space on c points, so $\mathcal{C}$-pre-antis of $\mathcal{S}$ are guaranteed. The same conclusion will follow if $\mathcal{C}$ contains, for example, both the Tychonoff cube $[0, 1]^{c^+}$ and this cube with an isolated point adjoined.

(ii) Since the condition UP is used only once in Proposition 5 and, unlike DOWN, is not naturally related to hereditary properties, it may be worthwhile to present a small counterexample to illustrate its needfulness. Suppose we use D_n and T_n to denote the discrete and trivial spaces on n elements, and put $\mathcal{C} = \{D_1, D_2, D_3, D_4, T_2, T_3\}$, $\mathcal{P} = \{D_1, D_2, D_3, T_2\}$. All conditions of Proposition 5(I)(b) except UP are satisfied, and $\mathcal{C}$-ind$(\mathcal{P}) = \{3\}$, and yet it is readily confirmed that $\mathcal{P}$ has no $\mathcal{C}$-pre-anti.

Lastly we turn to the question of whether an invariant which possesses a hereditary pre-anti (in the universal context) must also possess such a one under constraint. It appears to be necessary to stipulate that the constraint $\mathcal{C}$ be hereditary also, to make progress; but then, under a slight additional assumption, it transpires that the sought $\mathcal{C}$-pre-anti may be taken as simply the relativization of the known pre-anti onto $\mathcal{C}$.

PROPOSITION 8. – Let $\mathcal{C}$ and $\mathcal{Q}$ be topological invariants such that

(i) $\mathcal{C}$ is hereditary,

(ii) $\mathcal{C}$-proh$(\mathcal{C} \cap \mathcal{Q}) = \phi$,

(iii) there exists hereditary $\mathcal{P}$ such that anti$(\mathcal{P}) = \mathcal{Q}$.

Then $\mathcal{C} \cap \mathcal{P}$ is a hereditary $\mathcal{C}$-pre-anti for $\mathcal{C} \cap \mathcal{Q}$,

PROOF. – Let Y be a subspace of X where $X \in \mathcal{C} \cap Q$ and $Y \in \mathcal{C} \cap \mathcal{P}$. Since Y is a $\mathcal{P}$ subspace of an anti($\mathcal{P}$) space, we know $|Y| \in$ spec($\mathcal{P}$); therefore $|Y| \in \mathcal{C}$-spec($\mathcal{C} \cap \mathcal{P}$) and X is $\mathcal{C}$-anti($\mathcal{C} \cap \mathcal{P}$).

Conversely, suppose that X is $\mathcal{C}$-anti($\mathcal{C} \cap \mathcal{P}$) but not $\mathcal{Q}$. Then we can select a $\mathcal{P}$ subspace Y of X whose cardinality λ belongs to ind($\mathcal{P}$). From (i), Y is $\mathcal{C}$ as well as $\mathcal{P}$ so, since X is $\mathcal{C}$-anti($\mathcal{C} \cap \mathcal{P}$), $\lambda \in \mathcal{C}$-spec($\mathcal{C} \cap \mathcal{P}$). From (ii), there is a $\mathcal{C} \cap \mathcal{Q}$ space Z of cardinality λ; but then Z is both $\mathcal{P}$ and $\mathcal{Q}$, so the contradiction $|Z| \in$ spec($\mathcal{P}$) arises. The proof is therefore complete.

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