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Relationship of Certain Rings of Infinite Matrices over Integers.

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Sunto. – *Sia N l'insieme degli interi non negativi e Z l'anello degli interi. Sia $\mathcal{A}$ l'anello delle matrici $N \times N$ su Z che hanno solo un numero finito di cifre non nulle in ogni linea ed in ogni colonna. Sia $\mathcal{B}$ il sottoanello generato da X e Y, dove X (rispettivamente Y) è ottenuto dalla matrice identità muovendo gli 1 una posizione a destra (rispettivamente in giù). Sia pure $\mathcal{C}$ il sottoanello di $\mathcal{A}$ generato da $1-X$ e $1-Y$. Infine sia $\mathcal{F}$ il sottoanello delle matrici di $\mathcal{A}$ che hanno solo un numero finito di cifre non nulle. Consideriamo estensioni essenziali di anelli e l'immersione usuale di un anello in un anello unitario (chiamata qui l'immersione di Dorroh). Usando questi concetti, mostriamo che $\mathcal{A}$ è una estensione essenziale massimale di tutti i suoi ideali che contengono $\mathcal{F}$. Mostriamo inoltre che $\mathcal{B}$ è una estensione di Dorroh di C, che è l'idealizer di $\mathcal{C}$ in $\mathcal{A}$ e anche che è una estensione essenziale massimale di tutti i suoi ideali che contengono $\mathcal{C}$.*

1. – Introduction and summary.

A ring R with identity 1 is said to be directly finite if for any $a, b \in R$, $ab = 1$ implies $ba = 1$; otherwise R is directly infinite. Let N be the set of nonnegative integers. We are primarily interested in the ring $\mathcal{B}$ of $N \times N$ matrices over the ring Z of integers generated by two elements: one obtained by shifting the ones of the identity matrix one position to the right, call it X, and the other one one position down, call it Y. This ring appears in ([3], p. 263). If in a unitary ring R we have a pair of elements a, b such that $ab = 1 \neq ba$, then as we have seen in [8] there exists a homomorphism of $\mathcal{B}$ onto the ring $R(a, b)$ generated by a and b whose kernel does not contain the matrix with 1 in the (0,0)-position and 0 elsewhere. Conversely, every homomorphism of this kind produces a pair of elements $a, b \in R$ such that $ab = 1 \neq ba$. Therefore such homomorphisms characterize directly infinite rings.

Another interesting ring is obtained by taking the subring $\mathcal{C}$ of $\mathcal{B}$ generated by $P = 1 - X$ and $Q = 1 - Y$, with X and Y defined above. This ring plays the same role as $\mathcal{B}$ for directly infinite rings but for the following class of

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rings. A ring R is said to be quasi directly finite if for any $a, b \in R$, $a + b = ab$ implies that $ab = ba$; otherwise R is quasi directly infinite. In the case of a unitary ring, the concepts of directly finite and quasi directly finite coincide. This notion and result are due to Munn [4]. For further characterizations and the result for quasi directly infinite rings corresponding to the one quoted above for directly infinite rings, consult [9].

There are two further rings that arise naturally in this study, see [8] and [9]. The first one is the ring $\mathcal{F}$ of all $N \times N$ matrices over Z with only a finite number of nonzero entries. The second one is the ring $\mathcal{A}$ of all $N \times N$ matrices over Z with only a finite number of nonzero entries in each row and each column. In comparison with the rings discussed above, we have the hierarchy

$$\mathcal{F} \subset \mathcal{C} \subset \mathcal{B} \subset \mathcal{A},$$

and, obviously, $\mathcal{F}$ is an ideal of $\mathcal{A}$ (and thus of both $\mathcal{B}$ and $\mathcal{C}$). Note that addition of members of $\mathcal{A}$ is by components while the multiplication is that of usual (finite) matrices where the sum of infinitely many zeros is set equal to zero. We have devoted [10] to the properties of the ring $\mathcal{B}$.

Essential extensions of rings with trivial annihilator are treated in Section 2 in some detail. The usual embedding of a ring into a unitary one, here called the Dorroh extension, is discussed in Section 3. Section 4 contains a minimum of notation and terminology needed in the remainder of the paper. These sections form preparation for the main body of the paper dealing with the rings mentioned above. The mutual relationship of the rings $\mathcal{A}$ and $\mathcal{F}$ is investigated in Section 5 with some statements concerning all nonzero ideals of $\mathcal{A}$. We conclude in Section 6 with exploration of the mutual relationship between rings $\mathcal{B}$ and $\mathcal{C}$. In both of the last two sections, the translational hull of a ring plays a central role.

2. – Essential extensions.

We prove here a general result concerning these extensions. In Sections 5 and 6, we shall deduce the relevant conclusions concerning the rings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{F}$. Partly analogous to the semigroup case (see [5], Chapter III), we state the following items.

Let R be a ring. A transformation λ of R is a *left translation* of R if it is an additive homomorphism and $\lambda(xy) = (\lambda x)\, y$ for all $x, y \in R$; a *right translation* of R is an additive homomorphism of R with the property $(xy)\,\varrho = x(y\varrho)$ for all $x, y \in R$. The set $\Lambda(R)$ (respectively, $P(R)$) of all left (respectively, right) translations of R under composition of operations on the left (respectively, right) is a unitary ring. For $\lambda \in \Lambda(R)$ and $\varrho \in P(R)$, the pair (λ, ϱ) is a *bitranslation* of R if $x(\lambda y) = (x\varrho)y$ for all $x, y \in R$. The set $\Omega(R)$ of all bitranslations is a unitary subring of $\Lambda(R) \times P(R)$, the *translational hull* of R.

For every $r \in R$, define the mappings λ_r and ϱ_r by $\lambda_r x = rx$ and $x\varrho_r = xr$ for all $x \in R$, respectively. Then $\pi_r = (\lambda_r, \varrho_r)$ is the *inner bitranslation* of R induced by r. The set $\Pi(R)$ of all inner bitranslations of R is an ideal of $\Omega(R)$.

Let R be an ideal of a ring E. Regarding E as an extension of R, we shall need the following. For every $a \in E$, let

$$\lambda^a = \lambda_a|_R, \qquad \varrho^a = \varrho_a|_R, \qquad \tau^a = (\lambda^a, \varrho^a).$$

Then the mapping

$$\tau(E : R) : E \to \Omega(R), \qquad a \mapsto \tau^a(E : R) = \tau^a$$

is the *canonical homomorphism* of E into $\Omega(R)$. Its image is denoted by $T(E : R)$ and called the *type* of the extension E of R. The kernel of $\tau(R : R)$, namely the set

$$\mathcal{Q}(R) = \{r \in R \,|\, rx = xr = 0 \text{ for all } x \in R\}$$

is the *annihilator* of R. If $\mathcal{Q}(R) = (0)$, then R has *trivial annihilator.* The ideal R of E is *large* in E if R has nonzero intersection with every nonzero ideal of E. In such a case, E is an *essential extension* of R. If in addition R has no essential extension strictly containing E, then E is a *maximal essential extension* of R. If E' is another extension of R, then the extensions E and E' are *equivalent* if there exists an isomorphism $\varphi : E \to E'$ which leaves all elements of R fixed.

The semigroup version of the following theorem is known (see, e.g., [5], Chapter III) but a ring theoretical version does not seem to have a convenient reference, see ([7], Proposition 5) and the relevant discussion.

THEOREM 2.1. – Let R be an ideal of a ring E and assume that $\mathcal{Q}(R) = (0)$.

(i) The ring E is an essential extension of R if and only if $\tau(E : R)$ is injective.

(ii) Let E' be another extension of R. If E and E' are equivalent extensions of R, then $T(E : R) = T(E' : R)$. The converse holds if both E and E' are essential extensions of R.

(iii) The ring E is a maximal essential extension of R if and only if $\tau(E : R)$ is an isomorphism of E onto $\Omega(R)$.

(iv) The ring R has a maximal essential extension. Any two maximal essential extensions are equivalent.

PROOF. – (i) *Necessity.* Let $\tau = \tau(E : R)$ and $I = \ker \tau$. If $a \in I \cap R$, then for any $r \in R$, we have $ar = \lambda^a r = \lambda_a|_R r = 0$ and similarly $ra = 0$ so that $a \in \mathcal{Q}(R) = (0)$. Hence $I \cap R = (0)$ and the hypothesis implies that $I = (0)$. Therefore τ is injective.

Sufficiency. Let J be an ideal of E for which $R \cap J = (0)$. Then $RJ = JR = (0)$ and thus, for any $a \in J$, $ar = ra = 0$ for all $r \in R$ which yields $\tau^a(E : R) = 0$ and the hypothesis gives that $a = 0$. Therefore $J = (0)$, that is R is large in E, and E is an essential extension of R.

(ii) Let $\varphi : E \to E'$ be an isomorphism which fixes all elements of R and let $a \in E$. Then for $(\lambda_a, \varrho_a) \in \Pi(E)$ and $(\lambda'_{a\varphi}, \varrho'_{a\varphi}) \in \Pi(E')$, we have

$$\tau^a(E : R) = (\lambda_a|_R, \varrho_a|_R), \qquad \tau^{a\varphi}(E' : R) = (\lambda'_{a\varphi}|_R, \varrho'_{a\varphi}|_R).$$

It follows that for any $r \in R$, we have

$$ar = (ar)\,\varphi = (a\varphi)(r\varphi) = (a\varphi)\,r$$

which proves that $\lambda_a|_R = \lambda'_{a\varphi}|_R$; similarly we get $\varrho_a|_R = \varrho'_{a\varphi}|_R$. Hence $\tau^a(E : R) = \tau^{a\varphi}(E' : R)$ whence follows the assertion.

Conversely, assume that both E and E' are essential extensions of R and that $T(E : R) = T(E' : R)$. Then by part (i),

$$\tau = \tau(E : R)(\tau(E' : R))^{-1}$$

is an isomorphism of E onto E' such that for any $r \in R$, we have

$$r\tau = \tau^r(E : R)(\tau^{r\varphi}(E' : R))^{-1} = r\,.$$

Therefore E and E' are equivalent extensions of R.

(iii) *Necessity.* By part (i), the mapping $\tau(E : R)$ is an isomorphism of R onto a subring Y of $\Omega(R)$ containing $\Pi(R)$. Assume that $T \neq \Omega(R)$. We can then let $E' = E \cup (\Omega(R) \backslash T)$ and define the ring structure on E' in a standard way. It follows by direct verification that E' is an essential extension of R with the property that $E \subset E'$. This contradicts the maximality of E. Therefore $T = \Omega(R)$.

Sufficiency. By hypothesis $T(E : R) = \Omega(R)$ and by part (i), E is an essential extension of R. Let E' be an essential extension of R such that $E \subseteq E'$. Then clearly $T(E : R) \subseteq T(E' : R)$ whence $T(E' : R) = \Omega(R)$. Also by part (i), $\tau(E' : R)$ is injective which then yields that $E = E'$. Consequently E is a maximal essential extension of R.

(iv) This follows directly from parts (ii) and (iii). ■

There are further useful statements we can make about rings with trivial annihilator.

PROPOSITION 2.2. – Let R be a ring with $\mathcal{Q}(R) = (0)$.

(i) The types of extensions of R coincide with subrings of $\Omega(R)$ containing $\Pi(R)$.

(ii) Classes of equivalent essential extensions of R are in one-to-one correspondence with subrings of $\Omega(R)$ containing $\Pi(R)$ and thus also with subrings of $\Omega(R)/\Pi(R)$.

(iii) If R is unitary ring, then $\Omega(R) = \Pi(R)$ and R has no proper essential extensions.

PROOF. – (i) First note that a type of extension is a subring of $\Omega(R)$ containing $\Pi(R)$. Conversely, let K be a subring of $\Omega(R)$ containing $\Pi(R)$. We can define the structure of a ring on the set $E = R \cup (K \backslash \Pi(R))$ in a standard way. Direct checking will show that $T(E:R) = K$ so that K is a type of extension of R.

(ii) This follows directly from part (i) and Theorem 2.1(ii).

(iii) Let $(\lambda, \varrho) \in \Omega(R)$ and $r \in R$. Then

$$\lambda r = \lambda(1r) = (\lambda 1)r = \lambda_{\lambda 1} r$$

so that $\lambda = \lambda_{\lambda 1}$; analogously $\varrho = \varrho_{1\varrho}$. Further,

$$\lambda 1 = 1(\lambda 1) = (1\varrho)1 = 1\varrho$$

and thus $(\lambda, \varrho) = \pi_{\lambda 1} \in \Pi(R)$. The assertion now follows from part (ii). ■

LEMMA 2.3. – Let R be a ring in which every element has a left and a right identity. Let E be a maximal essential extension of R and I be an ideal of E containing R. Then E is a maximal essential extension of I.

PROOF We know by Theorem 2.1(iv) that $\mathcal{Q}(R) = (0)$ implies that E exists. Since R is large in E so is I. Hence E is an essential extension of I. Also note that $\mathcal{Q}(R) = (0)$ implies that $\mathcal{Q}(I) = (0)$. We must show that $\tau(E : I)$ is surjective. Let $\omega = (\lambda, \varrho) \in \Omega(I)$. For any $a \in R$, let l be a left identity and r be a right identity of a. Then

$$\lambda a = \lambda(ar) = (\lambda a)\, r \in R\,, \qquad a\varrho = (la)\,\varrho = l(a\varrho) \in R$$

since R is an ideal of I. Hence λ and ϱ map R onto R so that $(\lambda|_R, \varrho|_R) \in \Omega(R)$. Let $\lambda' = \lambda|_R$, $\varrho' = \varrho|_R$ and $\omega' = (\lambda', \varrho')$.

Since E is a maximal essential extension of R, there exists $e \in E$ such that $\tau^e(E:R) = \omega'$. This means that

$$\lambda' x = ex\,, \qquad x\varrho' = xe \qquad (x \in R)\,,$$

so by the definition of λ' and ϱ', we have

$$\lambda x = ex\,, \qquad x\varrho = xe \;\; (x \in R)\,.$$

Now let $a \in I$. For any $x \in R$, we obtain

$$(\lambda a)\, x = \lambda(ax) = e(ax) = (ea)\, x$$

$$x(\lambda a) = (x\varrho)\, a = (xe)\, a = x(ea)$$

and thus $\lambda a - ea \in \mathcal{Q}(R) = (0)$ whence $\lambda a = ea$. Analogously $a\varrho = ae$ which shows that $\omega = \tau^e(E : I)$. Therefore $\tau(E : I)$ is surjective and E is a maximal essential extension of I. ∎

Lemma 2.3 directly implies ([6], Theorems I.7.10 and I.7.13). The first of these references is valid for dense extensions of semigroups but Lemma 2.3 has an obvious analogue in that case.

The following converse of Theorem 2.1(iv) was proved by Shevrin [12].

LEMMA 2.4. – If a ring R has nontrivial annihilator, then R has no maximal essential extension.

Combining this with Theorem 2.1(iv), we obtain that a ring R has a maximal essential extension if and only if $\mathcal{Q}(R) = (0)$.

3. – The Dorroh extension.

That every ring can be embedded into a unitary ring was first proved by Dorroh [1]. This embedding is now standard which we formalize as follows. Given a ring R, let $\boldsymbol{D}(R) = R \times Z$ with coordinatewise addition and multiplication defined by

$$(a, m)(b, n) = (ab + mb + na, mn)\,.$$

We call $\boldsymbol{D}(R)$ the *Dorroh extension* of R and refer to the mapping

$$\delta(R) \colon R \to \boldsymbol{D}(R)\,, \qquad r \mapsto (r, 0)$$

as the *canonical embedding* of R into $\boldsymbol{D}(R)$.

Note that $(0, 1)$ is the identity of $\boldsymbol{D}(R)$. We may identify R with its image under $\delta(R)$. We do this in the following modification of the definition of equivalence of extensions. Let R be an ideal of a (unitary) ring U. Then U is equivalent to $\boldsymbol{D}(R)$ if there exists an isomorphism $\varphi : U \to \boldsymbol{D}(R)$ such that $\varphi|_R = \delta(R)$.

With these concepts, we have the following characterization of the Dorroh extension.

PROPOSITION 3.1. – Let E be a ring with identity 1, R an ideal of E, U the subring of E generated by the set $R \cup \{1\}$ and G the additive group generated by 1. Then

$$U = \{r + n1 \mid r \in R,\ n \in Z\}.$$

Moreover, U is equivalent to $\boldsymbol{D}(R)$ if and only if G is torsion free and $G \cap R = (0)$.

PROOF. – The first assertion as well as the necessity part of the second require straightforward verification. Assume that G is torsion free and $G \cap R = (0)$. First let $a, b \in R$ and $m, n \in Z$ be such that $a + m1 = b + n1$. It follows from the hypothesis that

$$(m - n)1 = b - a \in G \cap R = (0)$$

so that $(m - n)1 = 0$ and $a = b$. The hypothesis further implies that $m - n = 0$ and thus $m = n$.

Hence we may define a function

$$\varphi : U \to \boldsymbol{D}(R), \qquad r + m1 \mapsto (r, m).$$

We get

$$\begin{aligned}(a + m1)\varphi(b + n1)\varphi &= (a, m)(b, n) = (ab + mb + na, mn)\\ &= (ab + mb + na + (mn)1)\varphi = ((a + m1)(b + n1))\varphi\end{aligned}$$

and similarly for addition. Hence φ is a homomorphism. Clearly, φ is both injective and surjective, and $\varphi|_R = \delta(R)$. Therefore U and $\boldsymbol{D}(R)$ are equivalent. ■

We now prove the main result of this section.

THEOREM 3.2. – Let $D = \boldsymbol{D}(R)$ be the Dorroh extension of a ring R. For $n \in N$, let

$$I_n = \{(r, nk) \mid r \in R,\ k \in Z\}.$$

(i) $\{I_n \mid n \in N\}$ is the set of all proper ideals of D containing I_0.

(ii) If D is a maximal essential extension of I_0, then it is also a maximal essential extension of I_n for all $n > 0$.

PROOF. – (i) The function

$$\varphi : D \to Z, \qquad (r, n) \mapsto n$$

is essentially the natural homomorphism of D onto D/I_0. It is easy to verify that for any $n \in N$ we have $(nZ)\varphi^{-1} = I_n$. By the correspondence of ideals of

D containing I_0 and ideals of D/I_0 and the knowledge of ideals of Z, we conclude the correctness of the assertion of that part of the theorem.

(ii) Suppose that D is a maximal essential extension of I_0. By Lemma 2.4, we have $\mathcal{Q}(I_0) = (0)$ and by the isomorphism $\delta(R)$, also $\mathcal{Q}(R) = (0)$.

Let $n \in N$. By hypothesis I_0 is large in D and hence so is I_n and thus D is an essential extension of I_n. Also $\mathcal{Q}(I_0) = (0)$ implies that $\mathcal{Q}(I_n) = (0)$. Let $\tau = \tau(D : I_n)$. By Theorem 2.1(i), τ is injective. To prove the claim it remains to show that τ is also surjective.

Let $(\lambda, \varrho) \in \Omega(I_n)$. For any $(a, 0), (b, m) \in I_n$ we obtain

$$(a, 0)(\lambda(b, m)) = (c, 0), \quad ((a, 0)\varrho)(b, m) = (d, n)(b, m)$$

for some $c, d \in R$ and $n \in Z$. The equality of these two expressions implies that $nm = 0$. Since we may take $m \neq 0$, it follows that $n = 0$. Thus $(a, 0)\varrho = (d, 0)$ for some $d \in R$. One similarly obtains that $\lambda(a, 0) = (e, 0)$ for some $e \in R$. It follows that λ and ϱ map I_0 into itself and thus $(\lambda|_{I_0}, \varrho|_{I_0}) \in \Omega(I_0)$.

By hypothesis, D is a maximal essential extension of I_0 which by Theorem 2.1(iii) yields that there exists an element $(a, k) \in D$ such that

$$\tau^{(a, k)}(D : I_0) = (\lambda|_{I_0}, \varrho|_{I_0}).$$

This means that for any $r \in R$, we have

$$\lambda(r, 0) = (a, k)(r, 0), \quad (r, 0)\varrho = (r, 0)(a, k).$$

Now let $(b, m) \in I_n$. Then for any $r \in R$, we obtain

$$(\lambda(b, m))(r, 0)(r, 0)(\lambda(b, m)) =$$

$$= \lambda((b, m)(r, 0)) = (a, k)((b, m)(r, 0)) = ((a, k)(b, m))(r, 0),$$

$$((r, 0)\varrho)(b, m) = ((r, 0)(a, k))(b, m) = (r, 0)((a, k)(b, m))$$

and thus $\lambda(b, m) - (a, k)(b, m) \in \mathcal{Q}(I_0)$. We have remarked above that $\mathcal{Q}(I_0) = (0)$ and hence $\lambda(b, m) = (a, k)(b, m)$. One shows similarly that $(b, m)\varrho = (b, m)(a, k)$. Therefore $\tau^{(a, k)} = (\lambda, \varrho)$ and τ is surjective.

We have proved that τ is an isomorphism of D onto $\Omega(I_n)$ and have seen also that $\mathcal{Q}(I_n) = (0)$. Theorem 2.1(iii) now yields that D is a maximal essential extension of I_n. ∎

We conclude this section with the following simple statement.

LEMMA 3.3. – Let R be a ring and I a proper ideal of $\boldsymbol{D}(R)$ containing I_0.

(i) $E(I) = E(I_0)$.

(ii) Every ideal of I_0 is also an ideal of I.

PROOF. – (i) Let $(r, n) \in E(\boldsymbol{D}(R))$. Then

$$(r, n)^2 = (r^2 + 2nr, n^2) = (r, n)$$

if and only if $n = 0$ and $r^2 = r$ or $n = 1$ and $r^2 + r = 0$. In the first case $(r, 0) \in I_0$ and in the second case $(r, 1) \notin I$ since I is proper. The assertion follows.

(ii) Let J be an ideal of I_0, $(a, 0) \in J$ and $(x, n) \in I$. Then

$$(a, 0)(x, n) = (ax + na, 0) = (a, 0)(x, 0) + n(a, 0) \in J$$

and so $JI \subseteq J$. Similarly, $IJ \subseteq J$ and J is an ideal of I. ■

4. – Terminology and notation.

For symbolism and concepts in rings, we follow [2]. In addition, we shall need the following.

Given a ring R, we denote by $E(R)$ the set of its idempotents. Let $N = \{0, 1, 2, \ldots\}$. For $n \in N$, we write $\overline{n} = \{0, \ldots, n\}$. The letter Z stands for the ring of integers; $M_n(Z)$ for the ring of $\overline{n} \times \overline{n}$ matrices over Z; I_n for the identity of $M_n(Z)$; $\mathcal{A}$ for the ring of all $N \times N$ matrices over Z with only a finite number of nonzero entries in each row and each column; $\langle m, n \rangle$ for the matrix in $\mathcal{A}$ with 1 in the $(m+t, n+t)$-position for $t = 0, 1, 2, \ldots$ and 0 elsewhere. In particular, $\langle 0, 0 \rangle = 1$. If $A = (a_{ij})$ is any matrix over Z and $k \in Z$, we define $kA = (ka_{ij})$ as usual. A *ray matrix* is a matrix of the form $k\langle m, n \rangle$ with $k \in Z \backslash \{0\}$. We also write $[m, n]$ for the matrix in $\mathcal{A}$ with 1 in the (m, n)-position and 0 elsewhere. Note that

$$[m, n] = \langle m, n \rangle - \langle m+1, n+1 \rangle .$$

Two ray matrices without a nonzero entry in the same position are said to be *disjoint*. A usual $\overline{m} \times \overline{n}$ matrix A over Z is a *finite matrix*; we denote by A^0 the matrix in $\mathcal{A}$ in which A takes up the upper left corner and the rest is filled with zeros. Let

$$\mathcal{F} = \{A^0 \mid A \text{ is a finite matrix over } Z\}$$

and $\mathcal{B}$ be the subring of $\mathcal{A}$ generated by the matrices $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$.

We start with two lemmas from ([8], Section 3).

LEMMA 4.1. – For any $m, n, p, q \in N$, we have

$$\langle m, n \rangle \langle p, q \rangle = \langle m + p - r, n + q - r \rangle ,$$

where $r = \min\{n, p\}$. In particular,

$$B = \{\langle m, n\rangle \mid m, n \in N\}$$

is a bicyclic semigroup.

LEMMA 4.2. – The ring $\mathcal{B}$ consists precisely of the elements of the form

$$A^0 + \sum_{i=1}^{p} k_i \langle m_i, n_i\rangle$$

where A is a finite matrix over Z, $k_i \in Z$, $m_i, n_i \in N$, $i = 1, \ldots, p$ and $p \geq 0$. Moreover, the rays $\langle m_i, n_i\rangle$ may be assumed pairwise disjoint.

Let $\mathcal{C}$ be the subring of $\mathcal{B}$ generated by the elements

$$P = 1 - \langle 0, 1\rangle, \qquad Q = 1 - \langle 1, 0\rangle.$$

In ([9], Proposition 3.1(ii)) we have determined the form of elements of $\mathcal{C}$.

LEMMA 4.3. – The ring $\mathcal{C}$ consists precisely of the elements

$$A^0 + \sum_{i=1}^{p} k_i \langle m_i, n_i\rangle$$

where A is a finite matrix over Z, $k_i \in Z$, $m_i, n_i \in N$, $i = 1, \ldots, p$, $p \geq 0$ and $\sum_{i=1}^{p} k_i = 0$. Moreover, the rays $\langle m_i, n_i\rangle$ may be assumed pairwise disjoint.

We shall use these lemmas without explicit reference.

5. – The rings $\mathcal{A}$ and $\mathcal{F}$.

After some general statements about all (nonzero) ideals of $\mathcal{A}$, we prove here that $\mathcal{A}$ is a maximal essential extension of $\mathcal{F}$.

PROPOSITION 5.1. – Let I be an ideal of $\mathcal{A}$ and U the subring of $\mathcal{A}$ generated by the set $I \cup \{1\}$. As an extension of I, U is equivalent to $\boldsymbol{D}(I)$ if and only if $n1 \in I$ implies that $n = 0$ for any $n \in Z$.

PROOF. – Note that the additive group generated by 1 is torsion free in this case. Now apply Proposition 3.1. ∎

LEMMA 5.2. – Let $0 \neq X \in \mathcal{A}$ and $k \geq 1$. Then there exists $Y \in \mathcal{F}_k$ such that $XY \neq 0$.

PROOF. – Let $x \neq 0$ be the (m, n)-entry of X. Then for $Y = k[n, m]$, the (m, m)-entry of XY is $kx \neq 0$ and $Y \in \mathcal{F}_k$. ∎

COROLLARY 5.3. – Every subring of $\mathcal{A}$ which contains $\mathcal{F}_k$ for some $k \geqslant 1$ has trivial annihilator.

We are now able to establish some properties of all nonzero ideals of $\mathcal{A}$.

PROPOSITION 5.4. – Let I be a nonzero ideal of $\mathcal{A}$. Then $\mathcal{F}_k \subseteq I$ for some $k \geqslant 1$, $\mathcal{Q}(I) = (0)$ and I is large in $\mathcal{A}$.

PROOF. – By Lemma 5.2, $\mathcal{F}_k$ is large in $\mathcal{A}$ for every $k \geqslant 1$. Since $\mathcal{F} = \mathcal{F}_1$, we obtain that also $\mathcal{F}$ is large in $\mathcal{A}$. Hence $I \cap \mathcal{F} \neq (0)$ which in view of ([8], Lemma 4.2) implies that $I \cap \mathcal{F} = \mathcal{F}_k$ for some $k \geqslant 1$. Now Corollary 5.3 yields that $\mathcal{Q}(I) = (0)$. Since $\mathcal{F}_k \subseteq I$ and $\mathcal{F}_k$ is large in $\mathcal{A}$, so is I. ∎

We now prove the principal result of this section.

THEOREM 5.5. – The canonical homomorphism $\tau(\mathcal{A} : \mathcal{F})$ is an isomorphism of $\mathcal{A}$ onto $\Omega(\mathcal{F})$.

PROOF. – Let $A \in \mathcal{A}$, say $A = (a_{ij})$, and assume that $a_{mn} \neq 0$. Then the product $A[n, m]$ has a_{mn} in the (m, m)-position so that $A[n, m] \neq 0$. It follows that

$$\ker \tau(\mathcal{A} : \mathcal{F}) = (0)$$

and thus $\tau(\mathcal{A} : \mathcal{F})$ is injective.

Next let $(\lambda, \varrho) \in \Omega(\mathcal{F})$. For $n \in N$, we have

$$\lambda[n, n] = (a_{ij}), \quad [n, n]\,\varrho = (b_{ij})$$

for some $a_{ij}, b_{ij} \in Z$; denote by A_i the i-th column of (a_{ij}) and by B_i the i-th row of (b_{ij}). Then

$$\lambda[n, n] = \lambda[n, n]^2 = (\lambda[n, n])[n, n] = [0 \dots 0 A_n 0 \dots]$$

that is $a_{ij} = 0$ for all $j \neq n$, and

$$[n, n]\,\varrho = [n, n]^2\varrho = [n, n]([n, 1, n]\,\varrho) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_n \\ 0 \\ \vdots \end{bmatrix}$$

that is $b_{ij} = 0$ for all $i \neq n$. Let

$$A = [A_0 A_1 A_2 \ldots],\ B = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \end{bmatrix}$$

so that A is a column finite and B is a row finite matrix. For any $m, n \geqslant 0$, we obtain

$$[m, m](\lambda[n, n]) = a_{mn}[m, n], \quad ([m, m]\,\varrho)[n, n]) = b_{mn}[m, n]$$

and thus $a_{mn} = b_{mn}$ whence $A = B$. But then A is both row and column finite, that is $A \in \mathcal{F}$. On the one hand,

$$\lambda[m, n] = \lambda([m, m][m, n]) = (\lambda[m, m])[m, n] = [0 \ldots 0 A_m 0 \ldots]$$

where the A_m column vector is now in the n-th column, and on the other hand,

$$A[m, n] = [A_0 A_1 A_2 \ldots][m, n] = [0 \ldots 0 A_m 0 \ldots]$$

where the A_m column vector is again in the n-th column. Therefore $\lambda[m, n] = A[m, n]$. Since λ is an additive homomorphism, we conclude that $\lambda X = AX$ for all $X \in \mathcal{F}$. Analogously we obtain that $X\varrho = XA$ for all $X \in \mathcal{F}$. Therefore $\tau^A(\mathcal{A} : \mathcal{F}) = (\lambda, \varrho)$ and $\tau(\mathcal{A} : \mathcal{F})$ is also surjective. ■

COROLLARY 5.6. – The ring $\mathcal{A}$ is a maximal essential extension of every ideal of $\mathcal{A}$ which contains $\mathcal{F}$.

PROOF. – This follows directly from Theorem 2.1, Lemma 2.3 and Theorem 5.5. ■

6. – The rings $\mathcal{B}$ and $\mathcal{C}$.

We give here a threefold description of the relationship between the rings $\mathcal{B}$ and $\mathcal{C}$: by means of the Dorroh extension, in terms of idealizers and in the language of maximal essential extensions. We start with the first of these which is the simplest.

PROPOSITION 6.1. – As an extension of $\mathcal{C}$, the ring $\mathcal{B}$ is equivalent to $\boldsymbol{D}(\mathcal{C})$.

PROOF. – This follows directly from ([9], Corollary 3.5). ■

COROLLARY 6.2. – Let I be a proper ideal of $\mathcal{B}$ which contains $\mathcal{C}$. Then $E(I)=E(\mathcal{F})$.

PROOF. – By ([9], Proposition 3.6), $E(\mathcal{C})=E(\mathcal{F})$, and the result follows from Lemma 3.3(i) and Proposition 6.1. ■

Let S be a subring of a ring R. We denote by $i_R(S)$ the *idealizer* of S in R, that is the greatest subring of R having S as an ideal. It is easily verified that

$$i_R(S)=\{r\in R\,|\,rs,\ sr\in S \text{ for all } s\in S\}.$$

Note that $i_{\mathcal{A}}(\mathcal{B})=\mathcal{B}$ since $\mathcal{B}$ contains the identity of $\mathcal{A}$, and $i_{\mathcal{A}}(\mathcal{F})=\mathcal{A}$ since $\mathcal{F}$ is an ideal of $\mathcal{A}$. We consider next $i_{\mathcal{B}}(\mathcal{C})$. As a preparation for it, we make explicit in the next lemma the effect of pre- and postmultiplying by $P=1-\langle 0,1\rangle$ and $Q=1-\langle 1,0\rangle$, namely

LEMMA 6.3. – Let $X\in\mathcal{B}$. Then

(i) PX is the matrix obtained from X by subtracting from each row the next row.

(ii) QX is the matrix obtained from X by keeping the first row and subtracting from each other row the preceding one.

(iii) XP is the matrix obtained from X by keeping the first column and subtracting from each other column the preceding one.

(iv) XQ is the matrix obtained from X by subtracting from each column the next column.

PROOF. – (i) Let $PX=(y_{ij})$. For all $i,j\geq 0$, we have

$$\begin{aligned} y_{ij}[i,j]&=[i,i]\,PX[j,j]=[i,i]\,X[j,j]-[i,i]\langle 0,1\rangle X[j,j]\\ &=x_{ij}[i,j]-[i,i+1]\,X[j,j]=x_{ij}[i,j]-x_{i+1,j}[i,j]=(x_{ij}-x_{i+1,j})[i,j]\end{aligned}$$

and so $y_{ij}=x_{ij}-x_{i+1,j}$.

The remaining assertions are proved similarly. ■

The next lemma characterizes elements of $\mathcal{B}$ within $\mathcal{A}$. Given $X=(x_{ij})\in\mathcal{A}$ and $n\in Z$, the n-th *diagonal* of X is the sequence

$$\begin{cases}(x_{0n},\,x_{1,\,n+1},\,x_{2,\,n+2}\ldots) & \text{if } n\geq 0\\ (x_{-n,\,0},\,x_{-n+1,\,1},\,x_{-n+2,\,2}\ldots) & \text{if } n<0\,.\end{cases}$$

A diagonal of X is *quasi constant* if all but a finite number of its terms are equal.

LEMMA 6.4. – Let $X \in \mathcal{A}$. Then $X \in \mathcal{B}$ if and only if the following conditions hold:

(i) All nonzero entries of X lie in finitely many diagonals of X.
(ii) All diagonals of X are quasi constant.

PROOF. – Straightforward. ■

We are now ready for the second characterization.

THEOREM 6.5. – The ring $\mathcal{B}$ is the idealizer in $\mathcal{A}$ of every ideal of $\mathcal{B}$ containing $\mathcal{C}$.

PROOF. – We show first that $\mathcal{B} = i_{\mathcal{A}}(\mathcal{C})$.

Since $\mathcal{C}$ is an ideal of $\mathcal{B}$, we have $\mathcal{B} \subseteq i_{\mathcal{A}}(\mathcal{C})$. Therefore we need to show that the implication

$$X\mathcal{C},\ \mathcal{C}X \subseteq \mathcal{C} \Rightarrow X \in \mathcal{B}$$

holds for every $X \in \mathcal{A}$. Since $\mathcal{C}$ is generated by $P = 1 - \langle 0, 1\rangle$ and $Q = 1 - \langle 1, 0\rangle$, the condition $X\mathcal{C}, \mathcal{C}X \subseteq \mathcal{C}$ is equivalent to

$$PX,\ QX,\ XP,\ XQ \in \mathcal{C}. \tag{1}$$

Let $X = (x_{ij}) \in \mathcal{A} \setminus \mathcal{B}$. We must show that (1) is not satisfied. By Lemma 6.4, either there exist infinitely many diagonals of X with nonzero entries or some diagonal of X is not quasi constant.

Consider first the case where X possesses infinitely many diagonals with nonzero entries. Then there exist infinitely many such diagonals labelled by positive integers or there exist infinitely many such diagonals labelled by negative integers. We shall assume the first possibility, the other case being similar.

Since $X \in \mathcal{A}$, only finitely many nonzero entries can arise in each row (or column). Therefore we can define a sequence (r_k) by:

$$r_k = \max(\{n > k \mid x_{kn} \neq 0\} \cup \{0\}).$$

Since X possesses infinitely many positive diagonals with nonzero entries, there exists a subsequence $(r_{i_1}, r_{i_2}, \ldots,)$ of (r_k) such that $r_{i_k} - i_k < r_{i_{k+1}} - i_{k+1}$ for every $k \geq 1$. Let $XQ = (y_{ij})$. By definition of (r_k), we have $x_{i_k, r_{i_k+1}} = 0$ and so $y_{i_k, r_{i_k+1}} = x_{i_k, r_{i_k+1}} \neq 0$ for every $k \geq 1$. Since $\{r_{i_k} - i_k \mid k \geq 1\}$ is infinite, it follows that XQ possesses infinitely many diagonals with nonzero entries and so $XQ \notin \mathcal{B}$ by Lemma 6.4.

We consider now the case where X possesses only finitely many diagonals with nonzero entries, but some of the diagonals are not quasi constant. Let m be the minimum $n \in Z$ such that the n-th diagonal of X is not quasi constant.

By minimality of m, the $(m-1)$-th diagonal of X is quasi constant, hence there exist $k, u \in Z$ such that

$$x_{k,\, k+m-1} = x_{k+1,\, k+m} = x_{k+2,\, k+m+1} = \ldots = u\ ,$$

with $k, k+m-1 \geqslant 0$. Let $PX = (z_{ij})$. For every $i \geqslant 0$, Lemma 6.3 yields that

$$z_{k+i,\, k+m+i} = x_{k+i,\, k+m+i} - x_{k+i+1,\, k+m+i} = x_{k+i,\, k+m+i} - u$$

and so the m-th diagonal of PX is not quasi constant either. Thus by Lemma 6.4, $PX \notin \mathcal{B}$ and $i_{\mathcal{C}}(\mathcal{C}) \subseteq \mathcal{B}$. Therefore $i_{\mathcal{C}}(\mathcal{C}) = \mathcal{B}$.

Now let I be an ideal of $\mathcal{B}$ containing $\mathcal{C}$. The claim for $I = \mathcal{C}$ has just been established and for $I = \mathcal{B}$ the assertion is trivial since $\mathcal{B}$ has an identity element. Hence it remains to consider the case $\mathcal{C} \neq I \neq \mathcal{B}$. Since I is an ideal of $\mathcal{B}$, we have $\mathcal{B} \subseteq i_{\mathcal{C}}(I)$.

In view of Proposition 6.1, we may replace the ring $\mathcal{B}$ by the Dorroh extension of $\mathcal{C}$. Now Theorem 3.2(i) implies that we may set $I = I_n$ for some positive integer n. Let $X \in i_{\mathcal{C}}(I)$ and $Y \in \mathcal{C}$. With this identification, we may write $Y = (a, 0)$ for some $a \in \mathcal{C}$. Since $Y \in I$, we have XY, $YX \in I$ and in the other notation $X(a, 0), (a, 0)X \in I_n$. Then $(a, 0)X = (b, kn)$ for some $b \in \mathcal{C}$ and $k \in Z$. In particular,

$$((a, 0)X)(0, n) = (b, kn)(0, n) = (nb, kn^2)\,,$$

with $X(0, n) \in I_n$ so that $((a, 0)X)(0, n)$ is of the form $(c, 0)$ for some $c \in \mathcal{C}$. But then $kn^2 = 0$ whence $k = 0$. It follows that $(a, 0)X = (b, 0)$ and similarly $X(a, 0) = (d, 0)$ for some $d \in \mathcal{C}$. This means that $X \in i_{\mathcal{C}}(\mathcal{C})$ and hence $X \in \mathcal{B}$ as proved above. Therefore $i_{\mathcal{C}}(I) \subseteq \mathcal{B}$ and equality prevails. ■

We now establish the third characterization.

THEOREM 6.6. – The canonical homomorphism $\tau(\mathcal{B} : \mathcal{C})$ is an isomorphism of $\mathcal{B}$ onto $\Omega(\mathcal{C})$.

PROOF. – Let $\tau = \tau(\mathcal{B} : \mathcal{C})$ and $X = (x_{ij}) \in \ker \tau$. Suppose that $X \neq 0$. Then there exist $i, j \in N$ such that $x_{ij} \neq 0$ and $x_{i,\, j+1} = 0$. By Lemma 6.3, the (i, j)-th entry of XQ is $x_{ij} - x_{i,\, j+1} \neq 0$, and so $\lambda^X Q = XQ \neq 0$, contradicting $\tau X = 0$. Hence $X = 0$ and τ is injective.

It remains to prove surjectivity. Let $(\lambda, \varrho) \in \Omega(\mathcal{C})$. Since $\mathcal{C}$ is generated by P and Q, the translations λ and ϱ are determined by the images of P and Q. Moreover, $P + Q = PQ$ yields

$$\lambda P + \lambda Q = (\lambda P)\, Q\ , \qquad P\varrho + Q\varrho = P(Q\varrho)$$

and so

$$\lambda Q = (\lambda P)(Q-1), \qquad P\varrho = (P-1)(Q\varrho).$$

Hence λ is determined by $\lambda P = (a_{ij})$ and ϱ is determined by $Q\varrho = (b_{ij})$. Our aim is then to show that there exists $X = (x_{ij}) \in \mathcal{B}$ such that

$$\lambda P = XP, \qquad Q\varrho = QX.$$

Indeed, this implies that $(\lambda, \varrho) = (\lambda^X, \varrho^X) = X\tau$, proving the theorem.

Since $(\lambda, \varrho) \in \Omega(\mathcal{C})$, we have $Q(\lambda P) = (Q\varrho)P = Y$ for some $Y = (y_{ij}) \in \mathcal{C}$. Fix $j \in N$. By Lemma 6.3, $a_{0j} = y_{0j}$ and $a_{ij} - a_{i-1,j} = y_{ij}$ for every $i \geqslant 1$. It follows that

$$a_{ij} = \sum_{k=0}^{i} y_{kj} \tag{2}$$

for all $i, j \in N$. Since $Y \in \mathcal{C}$, Y has only finitely many nonzero entries in each column. If the sum of all nonzero entries of Y in a given column is $m \neq 0$, (2) implies that λP has infinitely many entries m in the same column, a contradiction. Thus

$$\sum_{k \geqslant 0} y_{kj} = 0 \tag{3}$$

for every $j \in N$. Similarly, we show that

$$b_{ij} = \sum_{t=0}^{j} y_{it} \tag{4}$$

and

$$\sum_{t \geqslant 0} y_{it} = 0 \tag{5}$$

for all $i, j \in N$. Let $X = (x_{ij})$ be the $N \times N$ matrix over Z defined by

$$x_{ij} = \sum_{t=0}^{j} \sum_{k=0}^{i} y_{kt}.$$

Since $Y \in \mathcal{C}$, there exist $m, n \in Z$ with $m \leqslant n$ such that all nonzero entries of Y lie between its m-th and n-th diagonals. If the entry (i, j) lies below the diagonal m, then x_{ij} is the sum of all nonzero entries of Y in columns $0, 1, \ldots, j$. By (3), it follows that $x_{ij} = 0$. If (i, j) lies above the diagonal n, then x_{ij} is the sum of all nonzero entries of Y in rows $0, 1, \ldots, i$. By (5), we also obtain $x_{ij} = 0$. Hence all nonzero entries of X lie between its m-th and n-th diagonals. In particular, $X \in \mathcal{A}$ and in order to show that $X \in \mathcal{B}$, by Lemma 6.4 we only have to prove that all diagonals of X are quasi constant. Since $Y \in \mathcal{C}$, there exists a nonnegative integer $M > -m$ such that all diagonals of Y are constant below row M.

Let $i > M$ and $j > M + n$ be such that $m \leqslant j - i \leqslant n$. We claim that $x_{i+1, j+1} = x_{ij}$. Let

$$\alpha = \sum_{t=0}^{j} \sum_{k=0}^{j-n-1} y_{kt}, \quad \beta = \sum_{t=0}^{j} \sum_{k=j-n}^{i} y_{kt}, \quad \gamma = \sum_{t=0}^{j+1} \sum_{k=0}^{j-n} y_{kt}, \quad \delta = \sum_{t=0}^{j+1} \sum_{k=j-n+1}^{i} y_{kt}.$$

It follows from the definition of X that $x_{ij} = \alpha + \beta$ and $x_{i+1, j+1} = \gamma + \delta$. First, we claim that α is the sum of all nonzero entries of Y in rows $0, 1, \ldots, j - n - 1$. Indeed, if $0 \leqslant k \leqslant j - n - 1$ and $t > j$, then

$$t - k > j - (j - n - 1) = n + 1,$$

hence the entry y_{kt} lies above the n-th diagonal and so $y_{kt} = 0$. Thus $\alpha = 0$ by (5). Replacing j by $j + 1$, the same argument yields that $\gamma = 0$. It remains to prove that $\beta = \delta$. We have $j - n > M$. Thus the diagonals of Y are constant in the submatrices Y_1 and Y_2 of Y considered in β and δ, respectively. Since $M > -m$, there are no nonzero entries in the first column of both Y_1 and Y_2 and so the nonzero positions of Y_1 can be mapped onto the nonzero positions of Y_2 by shifting one column to the right and one row downwards. Thus $\beta = \delta$ and $x_{i+1, j+1} = x_{ij}$. Therefore the diagonals of X are quasi constant and $X \in \mathcal{B}$.

It remains to check that $\lambda P = XP$ and $Q\varrho = QX$. By Lemma 6, the (i, j)-th entry of XP is x_{ij} if $j = 0$ and $x_{ij} - x_{i, j-1}$ otherwise. In either case, we obtain

$$\sum_{t=0}^{j} \sum_{k=0}^{i} y_{kt} - \sum_{t=0}^{j-1} \sum_{k=0}^{i} y_{kt} = \sum_{k=0}^{i} y_{kj}$$

which is the (i, j)-th entry of λP by (2). Hence $\lambda P = XP$. Similarly, the (i, j)-th entry of QX is x_{ij} if $i = 0$ and $x_{ij} - x_{i-1, j}$ otherwise. In either case, we obtain

$$\sum_{t=0}^{j} \sum_{k=0}^{i} y_{kt} - \sum_{t=0}^{j} \sum_{k=0}^{i-1} y_{kt} = \sum_{t=0}^{j} y_{it}$$

which is the (i, j)-th entry of $Q\varrho$ by (4). Hence $Q\varrho = QX$. Thus $(\lambda, \varrho) = X\tau$ and so τ is also surjective. ■

We can not apply Lemma 2.3 to $\mathcal{C}$ since $\mathcal{C}$ does not satisfy the hypotheses of that lemma. Indeed, let $X \in \mathcal{C} \setminus \mathcal{F}$, and suppose that $YX = X$ for some $Y \in \mathcal{C}$. By ([8], Lemma 6.1), $\mathcal{B} \setminus \mathcal{F}$ is a multiplicative semigroup, therefore $(1 - Y)X = 0$ implies that $1 - Y \in \mathcal{F}$. Hence $1 = A + Y$ for some $A \in \mathcal{F}$ and $1 \in \mathcal{C}$, a contradiction. Thus X has no left identity in $\mathcal{C}$. However, we have Theorem 3.2 at our disposal.

COROLLARY 6.7. – The ring $\mathcal{B}$ is a maximal essential extension of every ideal of $\mathcal{B}$ which contains $\mathcal{C}$.

PROOF. – By Theorems 2.1(iii) and 6.6, $\mathcal{B}$ is a maximal essential extension of $\mathcal{C}$. The assertion now follows from Proposition 6.1 and Theorem 3.2. ∎

We may summarize the results of this section as follows. The ring $\mathcal{B}$ is:

(i) equivalent to the Dorroh extension $\boldsymbol{D}(\mathcal{C})$,

(ii) the idealizer of $\mathcal{C}$ in $\mathcal{A}$,

(iii) a maximal essential extension of $\mathcal{C}$.

In particular, from Corollaries 5.6 and 6.7, we deduce that th rings $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{F}$ are essentially embedded in $\mathcal{A}$ in the sense that their idealizers in $\mathcal{A}$ are their maximal essential extensions.

We conclude with a simple statement that involves $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{F}$. Recall that an ideal I of a ring R is *completely prime* if for any $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$.

PROPOSITION 6.8. – The ring $\mathcal{F}$ is the least completely prime ideal of $\mathcal{B}$ and $\mathcal{C}$ but is not a completely prime ideal of $\mathcal{A}$.

PROOF. – By ([8], Lemma 6.1), $\mathcal{B} \setminus \mathcal{F}$ is a multiplicative subsemigroup of $\mathcal{B}$, hence $\mathcal{F}$ is a completely prime ideal of $\mathcal{B}$ and $\mathcal{C}$.

Let I be a completely prime ideal of $\mathcal{B}$. By ([8], Lemma 4.2), $I \cap \mathcal{F} = \mathcal{F}_k$ for some $k \geqslant 1$. If $k > 1$, then $[0, 0], [1, 1] \notin \mathcal{F}_k$ and so $[0, 0], [1, 1] \notin I$; however, $[0, 0][1, 1] = 0 \in I$, contradicting I being completely prime. Thus $k = 1$ and $\mathcal{F} \subseteq I$.

Now let I be a completely prime ideal of $\mathcal{C}$. By Lemma 3.3(ii) and Proposition 6.1, we have that I is an ideal of $\mathcal{B}$ and so $I \cap \mathcal{F} = \mathcal{F}_k$ for some $k \geqslant 1$. We can now apply the above argument to deduce that $\mathcal{F} \subseteq I$.

Finally, we show that $\mathcal{F}$ is not a completely prime ideal of $\mathcal{A}$. Define $X = (x_{ij})$, $Y = (y_{ij}) \in \mathcal{A}$ by

$$x_{ij} = \begin{cases} 1 & \text{if } i = j \in 2Z \\ 0 & \text{otherwise}, \end{cases} \qquad y_{ij} = \begin{cases} 1 & \text{if } i = j \notin 2Z \\ 0 & \text{otherwise}, \end{cases}$$

respectively. Clearly, $X, Y \notin \mathcal{F}$ and $XY = 0 \in \mathcal{F}$. Thus $\mathcal{F}$ is not a completely prime ideal of $\mathcal{A}$. ∎

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