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Modular invariant theory and the iterated total power operation


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1. – Introduction.

Fix an odd prime \( p \) and let \( H^* \) be the reduced ordinary cohomology theory over \( F_p \) the Galois field of order \( p \). The Steenrod algebra \( \mathcal{A}_p \) is the algebra of all stable operations in \( H^* \). Its generators \( \beta, P^i, i \geq 0 \), can be defined through the ring homomorphism

\[
T : H^*(X) \to H^*(\mathbb{Z}/p) \otimes H^*(X),
\]

where \( X \) is a CW complex. As it is well known, the cohomology ring of an elementary abelian \( p \)-group of rank \( m \) is

\[
H^*((\mathbb{Z}/p)^m) = E[x_1, \ldots, x_m] \otimes F_p[y_1, \ldots, y_m],
\]

where \( E[x_1, \ldots, x_m] \) is the exterior algebra on \( m \) generators \( x_1, \ldots, x_m \), each having degree 1, and \( F_p[y_1, \ldots, y_m] \) is a polynomial ring with generators \( y_1, \ldots, y_m \) in grading 2.

\( T \) is known as the total power operation and it has been extensively studied by Steenrod in [10]:

\[
T(z) = \mu(q) \sum_{\epsilon = 0, 1} (-1)^{\epsilon + i} x_1^\epsilon y_1^{(q - 2i)h - \epsilon} \otimes \beta^\epsilon P^i(z),
\]

where \( z \in H^q(X), \ h = (p - 1)/2, \ \mu(q) = (h!)^q(-1)^{hq(q - 1)/2} \). Other operations

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are obtained by iterating $T$. For each $m \geq 1$, we have
\[ T_m : H^*(X) \to H^* ((\mathbb{Z}/p)^m) \otimes H^*(X), \]
which multiplies the degrees by $p^m$. There is a natural action of $GL_m = GL(m, \mathbb{F}_p)$ upon $H^* ((\mathbb{Z}/p)^m)$ and the invariant elements rings are closely related to $\mathcal{C}_p$; in fact, from the geometric construction of $T_m$, it follows that
\[ \text{Im}(T_m) \subset (H^* ((\mathbb{Z}/p)^m))^{\bar{SL}_m}, \]
$\bar{SL}_m$ being the subgroup consisting of those matrices $\omega \in GL_m$ such that $(\det \omega)^h = 1$. Fixed any linear basis $\mathcal{B}$ in $\mathcal{C}_p$, we get an expression of the form
\[ T_m(z) = \sum_{b \in \mathcal{B}} f(b) \otimes b(z), \]
with $f(b) \in H^* ((\mathbb{Z}/p)^m)^{\bar{SL}_m}$. The coefficients $f(b)$ have been computed when $\mathcal{B} = \mathcal{B}_{\text{Mil}}$, the Milnor basis of $\mathcal{C}_p$ (see [8]). After recalling some basic facts about the geometric setting of $\mathcal{C}_p$ and the modular invariant theory in Section 1, in Section 2 we consider the basis $\mathcal{B}_{\text{Adm}}$ of admissible monomials and show how the coefficients $f(b)$ appear when $m = 2$. (The case $p = 2$ has been treated in [4]). The last Section is devoted to providing another proof of the normalized version of Mùi’s Theorem [3, Th. 2.9]. We proceed in a way analogous to [6], where the case $p = 2$ has been dealt with. In our case, the corresponding sequence of maps is $\delta_m : \mathcal{C}_p^* \to \Delta_m$, where $\Delta_m = \Phi_m^{H_m}$. Here $\Phi_m$ is the localization of $H^* ((\mathbb{Z}/p)^m)$ out of its Euler class $e_m$ and $B_m$ is the Borel subgroup of $GL_m$.

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2. – Preliminaries.

Let $A_p$ be the alternating group on $\mathbb{F}_p^m$, $G$ an even permutation group containing an elementary abelian $p$ – group of rank $m$, and $X$ a based $CW$ complex. So we have the Steenrod power map
\[ P_G : H^q(X) \to H^{p^m q} (EG^+ \wedge_G X^{(p^m)}), \]
which sends \( z \) to \( 1 \otimes z^m \) at the cochain level, the diagonal homomorphism:

\[
d^G_\delta : H^*(EG^+ \wedge_G X^{(p^m)}) \to H^*(BG) \otimes H^*(X),
\]

induced by the \( G \)-homomorphism

\[
EG^+ \wedge_G X \to EG^+ \wedge_G X^{(p^m)}
\]

via the diagonal \( X \to X^{(p^m)} \) \((H^*(EG^+ \wedge_G X) = H^*(BG) \otimes H^*(X) \) by the Künneth formula,) and the restriction homomorphism

\[
\text{Res}((\mathbb{Z}/p)^m, G) : H^*(G) \to H^*(\mathbb{Z}/p)^m)
\]

induced by the inclusion \((\mathbb{Z}/p)^m \subset G\). The resulting composition of these three homomorphisms does not depend on the group \( G \) containing \((\mathbb{Z}/p)^m\) and contained in \( A_{p^m} \); it gives rise to the iterated total power operation \( T_{p^m} \). The fact that \( \text{Im}(T_{p^m}) \subset H^*((\mathbb{Z}/p)^m)^{SL_{p^m}} \otimes H^*(X) \) comes from the construction above.

We need to recall some facts about modular invariant theory. Let

\[
V_k = \prod_{i=1}^k (\lambda_1 y_1 + \cdots + \lambda_{k-1} y_{k-1} + y_k),
\]

\[
L_m = V_1 \cdots V_m, \quad \tilde{L}_m = L_m^k, \quad Q_{m,s} = Q_{m-1,s} V_m^{p^{-k}} + Q_{m-1,s-1};
\]

conventionally, \( Q_{s,s} = 1 \) for each \( s \geq 0 \) and \( Q_{m,s} = 0 \) if either \( s < 0 \) or \( s > m \). The \( Q_{m,s} \), called Dickson’s invariants, arise when we consider the polynomial part of \( H^*((\mathbb{Z}/p)^m) \). Concerning with the exterior part, we set

\[
[k; e_{k+1}, \ldots, e_m] = \frac{1}{k!} \det \begin{pmatrix}
x_1 & \cdots & x_m \\
\vdots & \ddots & \vdots \\
x_1 & \cdots & x_m \\
y_1^{p_{e_k+1}} & \cdots & y_m^{p_{e_k+1}} \\
\vdots & \ddots & \vdots \\
y_1^{p_{e_m}} & \cdots & y_m^{p_{e_m}}
\end{pmatrix},
\]

where \( e_{k+1}, \ldots, e_m \) are non negative integers, \( 0 \leq k \leq m \), and \( M_m; s_1, \ldots, s_k = [k; 0, 1, \ldots, \bar{s}_1, \ldots, \bar{s}_k, \ldots, m-1]. \) As usual, \( \bar{s}_j \) means that \( s_j \) is omitted. We have

\[
M_m^2; s_1 = 0; \quad M_m; s_1 \cdots M_m; s_k = (-1)^{k(k-1)/2} M_m; s_1, \ldots, s_k L_m^{k-1},
\]
where $0 \leq s_1 < \ldots < s_k \leq m - 1$. We set

$$M_m; s_1, \ldots, s_k = M_m; s_1, \ldots, s_k L_m^{k-1}, \quad R_m; s_1, \ldots, s_k = M_m; s_1, \ldots, s_k L_m^{p-2}$$

and

$$e_m = \prod (\lambda_1 y_1 + \ldots + \lambda_m y_m) \quad \text{(the Euler class)},$$

where the product runs over all nontrivial $m$-tuples of elements of $\mathbb{F}_p$. We observe that

$$Q_m, 0 = \tilde{L}_m^{p-1} = \tilde{L}_m^2 = (-1)^m e_m.$$ 

We invert the Euler class in $H^*((\mathbb{Z}/p)^m)$ and get the ring

$$\Phi_m = H^*((\mathbb{Z}/p)^m)[e_m^{-1}]$$

upon which the action of $GL_m$ on $H^*((\mathbb{Z}/p)^m)$ extends. As it is well known:

$$\Gamma_m = \Phi_m^{GL_m} = E[R_m; 0, \ldots, R_m; m-1] \otimes F_p[Q_m, 0, Q_m, 1, \ldots, Q_m, m-1],$$

$$\tilde{\Gamma}_m = \Phi_m^{SL_m} = E[\tilde{M}_m; 0, \ldots, \tilde{M}_m; m-1] \otimes F_p[\tilde{L}_m^2, Q_m, 1, \ldots, Q_m, m-1].$$

In $\Phi_m$, we have defined particular elements which can be assumed as generators of $\Phi_m^{B_m}$. We set:

$$v_1 = V_1, \quad v_{k+1} = V_{k+1}/Q_k, 0, \quad k \geq 0$$

$$u_k = M_{k, k-1}/(v_1^{p-1} v_2^{p-2} \ldots v_k v_k), \quad k \geq 1;$$

the gradings of $v_k$ and $u_k$ are 2 and $-1$ respectively.

The following relations hold:

$$V_k = v_1^{(p-1)p^{k-2}} v_2^{(p-1)p^{k-3}} \ldots v_k^{(p-1)} v_k$$

$$L_k = v_1^{p-1} v_2^{p-2} \ldots v_k^{p-1} v_k.$$ 

Further, let $w_k$ be $v_k^{p-1}$.

**Proposition 1.** — $\Phi_m^{B_m} \cong E[u_1, \ldots, u_m] \otimes F_p[w_1^{\pm 1}, \ldots, w_m^{\pm 1}].$ 

**Proof.** — From [5, Prop. 7.5], we know that $\Phi_m^{B_m} \cong E[N_1, \ldots, N_m] \otimes F_p[W_1^{\pm 1}, \ldots, W_m^{\pm 1}],$
where $N_k = L_k^{p-1} M_{k; k-1}$ and $W_k = V_k^{p-1}$. Easy calculations lead to
\[ W_1 = w_1 \]
\[ W_k = (W_1 \ldots W_{k-1})^{p-1} w_k \]
\[ N_k = u_k W_k. \]

From [9, Lemma 5.4], we know that
\[ M_{m; s} = \sum_{r = s+1}^{m} M_{r; r-1} V_{r+1} \ldots V_m Q_{r-1}, s. \]
Combining this relation with the second of (1) and the (2), we get:
\[ R_{m; s} = M_{m; s} L_m^{p-1} = Q_{m, 0} \sum_{r = s+1}^{m} \frac{M_{r; r-1}}{v_1^{p-1} v_2^{p-2} \ldots v_{r-1}^{p-1} v_r} Q_{r-1}, s \]
\[ = Q_{m, 0} \sum_{r = s+1}^{m} u_r \frac{V_r}{v_r} Q_{r-1}, s = Q_{m, 0} \sum_{r = s+1}^{m} u_r Q_{r-1}^{-1} Q_{r-1}, s. \]

3. – On the double power operation.

From [8], we know the coefficients $f(b)$ when $B = B_{Mil}$. Mûi’s Theorem reads as follows:

**Theorem 2. – ([8, 1.3])** Let $z \in H^q(X), s = (s_0, \ldots, s_k), 1 \leq s_0 < \ldots < s_k \leq m, R = (r_1, \ldots, r_m)$. Then
\[ T_m(z) = \mu(q)^m \sum_{S, R} (-1)^{r(S, R)} R_{m, s_0} \ldots R_{m, s_k} Q_{m, 0}^{r_0} \ldots Q_{m, m-1}^{r_m} \otimes St^S, R(z), \]
where $r_0 = -k - (r_1 + \ldots + r_m)$, $r(S, R) = k + s_1 + \ldots + s_k + r_1 + 2r_2 + \ldots + mr_m$ and $St^S, R \in B_{Mil}$ (see below).

In [2] the coefficients $f(b)$ in the double iterated total power operation are computed when we choose in $\mathfrak{cl}_p$ the classical basis $B_{Adm}$. We adopt the abbreviated notation $P^I = \beta_{\varepsilon_1} P^I_{t_1} \beta_{\varepsilon_2} P^I_{t_2}$ for a typical monomial in $\mathfrak{cl}_p$, where $I = (\varepsilon_1, t_1, \ldots, \varepsilon_k, t_k)$ is a multi-index whose entries $\varepsilon_i$ are 0 or 1 and $t_i$ are positive integers (possibly $t_k = 0$ if $\varepsilon_k = 1$). The length of $P^I$ is $k$ if $t_k \neq 0$; it is $k - 1$ if $t_k = 0$ and $\varepsilon_k = 1$. A monomial $P^I$ belongs to $B_{Adm}$ if $t_j \geq pt_{j+1} + \varepsilon_{j+1}$ for each $1 \leq j \leq k - 1$. Then an admissible monomial of length 2 is of the form $\beta_{\varepsilon_1} P^{pt+\varepsilon_2+\alpha} \beta_{\varepsilon_2} P^I$, where $\alpha, t \geq 0$ and $\varepsilon_1, \varepsilon_2 = 0, 1$. Leading to the admissible basis, the Adem relations play an important role in determining the $f(b)$, together with comparisons of coefficients in suitable power series.
THEOREM 3. – ([2]) For each \( z \in H^q(X) \), \( X \) a CW complex, \( q \geq 0 \), we have:

\[
T_2(z) = \mu(q)^2 T_2^q \sum_{i, a, i} (-1)^{a+i} Q_{2,0}^{\alpha-p_i-t-a-1} Q_{2,1}^{a-p_i-t-i-1}.
\]

As we can see, the combinatorics involved is complicated since the double iteration. Consider \( \mathbf{c}_p \) as graded by the length of monomials. In grading 2, it suffices to apply once the Adem relations in order to get the admissible expression of any monomial. A similar procedure does not apply to upper length monomials, since there are not explicit non-recursive formulas, neither to obtain an admissible expression of any monomial of length \( k > 2 \), nor to convert a Milnor basis element to the basis \( B_{\text{Adm}} \) (see [7]).

4. – An alternative proof of the normalized total power operation.

We start from the ring homomorphism

\[
S_m : H^*(X) \to \Phi_m^B \otimes H^*(X).
\]

For each \( z \in H^*(X) \), \( S_m(z) \) is:

\[
S_m(z) = \sum_{i, J} u^{(i, J)} \Theta^{\varepsilon, J}(z),
\]

where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \), \( \varepsilon_i = 0, 1 \), \( u^\varepsilon = u_1^{\varepsilon_1} \ldots u_m^{\varepsilon_m} \), \( J = (j_1, \ldots, j_m) \), \( i = 1, \ldots, m \), \( w^{-j} = w_1^{-j_1} \ldots w_m^{-j_m} \), \( \Theta^{(i, J)} = \beta^{\varepsilon_1} P_j^{j_1} \ldots \beta^{\varepsilon_m} P_j^{j_m} \). Up to a sign, \( S_m(z) \) has the same dimension as \( z \). Following the idea in [6] for \( p = 2 \), we construct a sequence of maps:

\[
\delta_m : \mathbf{c}_p^* \to \Lambda_m = \Phi_m^B,
\]

where \( \mathbf{c}_p^* \) denotes the \( F_p \)-dual of \( \mathbf{c}_p \), and we will use them to give an alternative proof of a normalized version of a result of Mùi (it is quoted here for the \( \mathbf{c}_p \)-module \( H^*(X) \)).
THEOREM 4. – ([3, Th. 2.9])

$$S_m(z) = \sum_{S,R} (-1)^{r(S,R)} R_m; s_1 \cdots R_m; s_k Q_{m,0}^{r_0} \cdots Q_{m,m-1}^{r_m} \otimes St^{S,R}(z),$$

where $r_0 = -k - r_1 - \cdots - r_m$, $r(S,R) = k + s_1 + \cdots + s_k + r_1 + 2r_2 + \cdots + mr_m$.

We recall that $\mathfrak{c}_p^*$ is isomorphic to:

$$E[\tau_0, \tau_1, \ldots, \tau_k, \ldots] \otimes F_p[\xi_1, \ldots, \xi_k, \ldots].$$

Here $\xi_k$ and $\tau_k$ are dual to $P^{p^k-1} P^{p^{k-2}} \cdots P^1$ and $P^{p^k-1} P^{p^{k-2}} \cdots P^1 \beta$ respectively, with respect to the basis of admissible monomials. For sequences $S = (s_1, \ldots, s_k)$, $0 \leq s_1 < s_2 < \cdots < s_k$, $k \geq 0$ and $R = (r_1, \ldots, r_l)$, $r_i \geq 0$, let

$$St^{S,R} = (\tau_S \xi^R)^* = (\tau_{s_1} \cdots \tau_{s_k} \xi_{s_1}^r \cdots \xi_{s_k}^r)^*$$

with respect to the basis $\{\tau_S \xi^R\}_{S,R}$ of $\mathfrak{c}_p^*$. These elements form the so called Milnor basis of $\mathfrak{c}_p^*$. We are going to show that

$$S_m(z) = \sum_{R,S} \delta_m(\tau_S \xi^R) \otimes St^{S,R}(z).$$

Then we prove that $\delta_m(\tau_S \xi^R)$ is just equal to

$$(-1)^{r(S,R)} R_m; s_1 \cdots R_m; s_k Q_m^{r_0 - k - (r_1 + \cdots + r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_m - 1},$$

hence $S_m$ is the normalized iterated total power operation. We first introduce a map which is formally identical to $S_m$:

$$S_m : \mathfrak{c}_p \to \Delta_m \otimes \mathfrak{c}_p$$

$$\Theta \mapsto \sum_{\xi, J} u^\delta w^{-J} \otimes \Theta^{(\xi, J)} \circ \Theta.$$

DEFINITION 5. – $\delta_m : \mathfrak{c}_p^* \to \Delta_m$ has the following definition: for $\tau_S \xi^R \in \mathfrak{c}_p^*$, we set

$$\delta_m(\tau_S \xi^R) := (-1)^{r(S,R)} ((id \otimes \tau_S \xi^R) \circ S_m)(1),$$

that is $\delta_m(\tau_S \xi^R)$ is the image of $1 \in \mathfrak{c}_p$ under the following composition:

$$\mathfrak{c}_p \xrightarrow{id} \Delta_m \otimes \mathfrak{c}_p \xrightarrow{id \otimes \tau_S \xi^R} \Delta_m \otimes F_p \equiv \Delta_m.$$
As \( S_m(1) = \sum u^j w^{-j} \otimes \Theta^{(i, J)} \) (an infinite sum!), we have that:

\[
\delta_m(\tau_S \xi^R) = (-1)^{r(S, R)} (\text{id} \otimes \tau_S \xi^R) \left( \sum u^j w^{-j} \otimes \Theta^{(i, J)} \right)
\]

\[
= \sum (-1)^{r(S, R)} u^j w^{-j} \langle \tau_S \xi^R, \Theta^{(i, J)} \rangle,
\]

where \( \langle \tau_S \xi^R, \Theta^{(i, J)} \rangle \) is the value of \( \tau_S \xi^R \) on \( \Theta^{(i, J)} \). It is easy to check that \( \delta_m \) is a ring homomorphism.

**Lemma 6.** – Let \( a < pb \) and \( a + b = p^n + p^{n-1} \). Then

(i) the coefficient of \( P^a P^{p^n-1} \) in

\[
P^a P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \left( \frac{(p-1)(b-t) - 1}{a-pt} \right) P^{a+b-t} P^t
\]

is zero;

(ii) the coefficient of \( P^{p^n} = P^p P^0 \) in (5) is zero.

**Corollary 7.** – Let \( a_1 + \ldots + a_m = p^n-1 + p^{n-2} + \ldots + 1 = p^n - 1 \) (\( m \geq n \)). Then the coefficient of \( P^{p^{n-1}} P^{p^{n-2}} \ldots P^1 \) in the admissible expression of \( P^{a_1} P^{a_2} \ldots P^{a_m} \) is zero.

The same argument works to show that \( P^{p^{k-1}} P^{p^{k-2}} \ldots P^1 \beta \) does not appear in the admissible expression of any nonadmissible monomial \( P^{a_1} P^{a_2} \ldots P^{a_m} \beta \).

**Corollary 8.**

(i) \( \langle \xi_k, P^{i_1} \ldots P^{i_n} \rangle = 1 \) if and only if \( n = k \) and \( (i_1, \ldots, i_n) = (p^{k-1}, p^{k-2}, \ldots, 1) \);

(ii) \( \langle \tau_k, P^{i_1} \ldots P^{i_n} \beta \rangle = 1 \) if and only if \( n = k \) and \( (i_1, \ldots, i_n) = (p^{k-1}, p^{k-2}, \ldots, 1) \).

**Proposition 9.** – \( \delta_n(\xi_k) = (-1)^k \sum w^{-J} \), where \( J \) is a multi-index of the form \( (0, \ldots, 0, p^{k-1}, \ldots, 0, \ldots, p, 0, \ldots, 1, 0, \ldots) \) with \( n - k \) zeros inserted.

**Proof.** – \( \xi_k = (P^{p^{k-1}} \ldots P^p P^1)^\delta \). From Cor. 3(i), \( \langle \xi_k, \Theta^{(i, J)} \rangle = 1 \) if and only if \( \Theta^{(i, J)} = P^{p^{k-1}} \ldots P^p P^1 \). The corresponding coefficient is \( \sum w^{-J} \), where \( J \) is as above. ■
PROPOSITION 10. \(- \delta_n(\tau_k) = (-1)^{k+1} \sum_{t=k+1}^n u_tw^{-j_t}\) for all \(k = 0, \ldots, n-1\), where the sequence \(J_t\) is of the following type:

\[ J_t = (j_1, j_2, \ldots, j_{t-1}) = (0, \ldots, P^{p_{k-1}}, 0, \ldots, P^p, 0, \ldots P^1, 0, \ldots, 0) \]

with \(t-1-k\) zeros inserted.

PROOF. \(- \tau_k = (P^{p_{k-1}}P^{p_{k-2}} \ldots P^p P^1)^{\beta}_{\ast}. \) Applying Cor. 3 (ii), we have \(\langle \tau_k, \Theta^{(\xi, J)}\rangle = 1\) if and only if \(\Theta^{(\xi, J)} = P^{p_{k-1}}P^{p_{k-2}} \ldots P^p P^1\) and the corresponding summands are those indicated in the statement. \(\Box\)

PROPOSITION 11. \(- \delta_n(\xi_k) = (-1)^k Q_{n,0}^{-1}Q_{n,k} \in \Gamma_n \subset \Delta_n\) for each \(k \geq 1\).

PROOF. \(- The relation above holds for \(n = k\) since \(Q_n, n = 1\) and \(Q_{n,0}^{-1}Q_{n,n} = Q_{n,0}^{-1}Q_{n,k} = w_1^{-p_{n-1}}w_2^{-p_{n-2}} \ldots w_n^{-1}.\) If \(k > n\), then \(\delta_n(\xi_k) = 0 = Q_{n,0}^{-1}Q_{n,k}\) as, by convention, \(Q_{n,k} = 0\) in this case. So let \(n > k\) and suppose that \(\delta_n(\xi_k) = Q_{n-1,0}^{-1}Q_{n-1,k}.\) The following relations hold:

\[
Q_{n,s} = Q_{n-1,0}Q_{n-1,s}w_n + Q_{n-1,s-1}w_n^0 \\
Q_{n,0} = Q_{n-1,0}w_n = w_1^{-p_{n-1}}w_2^{-p_{n-2}} \ldots w_n^{-1} \\
V_n^{p-1} = Q_{n-1,0}w_n 
\]

Hence,

\[
Q_{n-1,0}Q_{n,k} = (Q_{n-1,0}w_n^{-1})(Q_{n-1,k}w_n + Q_{n-1,k-1}w_0^0) \\
= Q_{n-1,0}Q_{n-1,k} + (Q_{n-1,0}Q_{n-1,k-1})^p w_n^{-1}.
\]

By the induction hypothesis, we know that

\[
Q_{n-1,0}Q_{n-1,k} = \sum w_1^{-p_{k-1}}w_2^{-p_{k-2}} \ldots w_{j_k}^{-1},
\]

where the sum runs over all integers \(j_i\) such that \(1 \leq j_1 < \ldots < j_k \leq n-1.\)

Thus:

\[
Q_{n-1,0}Q_{n-1,k} + (Q_{n-1,0}Q_{n-1,k-1})^p w_n^{-1} = \\
(\sum w_1^{-p_{k-1}}w_2^{-p_{k-2}} \ldots w_{j_k}^{-1}) + (\sum w_1^{-p_{k-1}}w_2^{-p_{k-2}} \ldots w_{j_k}^{-1}) w_n^{-1} = \\
\sum_J w^{-J} + \sum w_1^{-p_{k-1}}w_2^{-p_{k-2}} \ldots w_{j_k}^{-1} = \sum_J w^{-J} + \sum_J w^{-J'},
\]

where the symbol \(J\) denote sequences of length \(n\) with the last element zero and others \(n-1-k\) zeros are inserted among places from 1 to \(n-1\), and the symbols \(J'\) denote sequences of length \(n\) with the last element equal to 1 and
others $n-k$ zeros are inserted among places from 1 to $n-1$. Then we get

$$(-1)^k Q_{n,0} - Q_{n,k} = (-1)^k \sum_{j_1} w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \cdots w_{j_k}^{-1} = \delta_n(\xi_k),$$

the sum being over $(j_1, \ldots, j_k)$, where $1 \leq j_1 < \ldots < j_k \leq n$.

**Proposition 12.** \(- \delta_n(\tau_k) = (-1)^{k+1} R_{n,k}^{-1} Q_{n,0}^{-1} \) for each $0 \leq k \leq n-1$.

**Proof.** From (3), $R_{n,k}^{-1} Q_{n,0}^{-1} = \sum_{r=k+1}^{n} u_r Q_{r-1,0}^{-1}, Q_{r-1,k}$.

We want to prove that:

$$R_{n,k}^{-1} Q_{n,0}^{-1} = \sum_{r=k+1}^{n} u_r w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \cdots w_{j_k}^{-1},$$

with $1 \leq j_1 < \ldots < j_k \leq r-1$. But this directly follows from the previous Proposition, since we have shown that

$$Q_{r-1,0} Q_{r-1,k} = \sum_{1 \leq j_1 < \ldots < j_k \leq r-1} w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \cdots w_{j_k}^{-1}.$$ 

**Corollary 13.** \(- For S = (s_1, \ldots, s_k), 1 \leq s_1 < \ldots < s_k and R = (r_1, \ldots, r_l), r_i \geq 0, l \geq 1,$

$$\delta_n(\tau_{S,R}^{\xi}) = (-1)^{r(S,R)} R_{n,s_1} \cdots R_{n,s_k} Q_{n,0}^{r_1} Q_{n,1}^{r_1} \cdots Q_{n,l}^{r_l},$$

where $r_0 = -k - (r_1 + \ldots + r_l)$.

We have proved the following

**Theorem 14.** \(- S_n(z) = \sum_{S,R} (-1)^{r(S,R)} R_{n;s_1} \cdots R_{n;s_k} Q_{n,0}^{r_1} \cdots Q_{n,l}^{r_l} \otimes St^{S,R}(z).$$

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