Massimo Gobbino

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Non-local Approximation of Functionals: Variational and Evolution Problems.

MASSIMO GOBBINO (*)

Sunto. – *Questa nota tratta dell'approssimazione di funzionali, usati in problemi con discontinuità libere, mediante famiglie di funzionali non locali in cui il gradiente è sostituito dal rapporto incrementale. Vengono inoltre presentate alcune applicazioni di questa teoria a problemi variazionali e di evoluzione.*

1. – Introduction.

In last years many non-local functionals have been introduced in order to approximate local ones. The simplest example is the approximation of a classical integral functional

$$I(u) = \int_{\mathbb{R}} \varphi(\nabla u(x))\, dx$$

by non-local functionals where the gradient is replaced by a finite difference, like

$$I_\varepsilon(u) = \int_{\mathbb{R}} \varphi\left(\frac{u(x+\varepsilon) - u(x)}{\varepsilon}\right) dx\,.$$

A less trivial case is the approximation of free discontinuity problems in any dimension, which was indeed the main motivation of this theory.

1.1. – *Free discontinuity problems.*

The canonical examples of free discontinuity problems are the minimum problems related to the so called Mumford-Shah functional, defined by

$$MS(u) = \int_{\Omega} |\nabla u(x)|^2\, dx + \mathcal{H}^{n-1}(S_u)\,, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, u belongs to the space $GSBV(\Omega)$ of generalized special functions with bounded variation, ∇u is the approximate

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gradient of u, S_u is the set of essential discontinuity points of u, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

This functional is the weak formulation of the functional introduced by D. MUMFORD and J. SHAH in [8] to approach image segmentation problems.

A natural generalization of (1.1) are the functionals

$$\mathcal{F}_{\varphi,\psi}(u) = \int_{\Omega} \varphi(|\nabla u(x)|)\,dx + \int_{S_u} \psi(|u^+(x) - u^-(x)|)\,d\mathcal{H}^{n-1}(x), \tag{1.2}$$

where $\varphi, \psi : [0, +\infty[\to [0, +\infty]$ are given functions, and $u^+(x)$ and $u^-(x)$ are the approximate (in the measure theoretic sense) lim sup and lim inf of u at the point x.

1.2. – *Existence and regularity.*

Variational problems involving $\mathcal{F}_{\varphi,\psi}$ can be solved using the direct methods of the calculus of variations. The fundamental tool is the following lower semicontinuity and compactness theorem, proved by L. AMBROSIO in [1].

THEOREM 1.1. – *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\varphi : [0, +\infty[\to [0, +\infty]$ be a non-decreasing convex function such that*

$$\lim_{r \to +\infty} \frac{\varphi(r)}{r} = +\infty, \tag{1.3}$$

and let $\psi :]0, +\infty] \to [0, +\infty]$ be a non-decreasing concave function such that

$$\lim_{r \to 0^+} \frac{\psi(r)}{r} = +\infty. \tag{1.4}$$

Then the functional $\mathcal{F}_{\varphi,\psi}(u)$ defined in (1.2) *is lower semicontinuous in $L^1_{\mathrm{loc}}(\Omega)$.*

If moreover $\{u_n\} \subseteq L^\infty(\Omega)$ is a sequence such that

$$\sup_{n \in \mathbb{N}} \{\mathcal{F}_{\varphi,\psi}(u_n) + \|u_n\|_\infty\} < +\infty,$$

then $\{u_n\}$ is relatively compact in $L^1_{\mathrm{loc}}(\Omega)$.

The regularity of minimizers has been deeply studied in the case of the Mumford-Shah functional: the interested reader can find appropriated references in the survey [2].

1.3. – *Approximation.*

The research on free discontinuity problems has developed along several directions, among them

— giving non-trivial examples of minimizers;

— providing numerical algorithms to approximate such minimizers;

— finding a reasonable definition of gradient flow associated with $\mathcal{F}_{\varphi,\psi}$.

A natural approach to these problems is to approximate $\mathcal{F}_{\varphi,\psi}$ by functionals $\mathcal{F}_\varepsilon$ defined in better spaces, *e.g.* Sobolev spaces or finite dimensional vector spaces. These functionals $\{\mathcal{F}_\varepsilon\}$ should converge to $\mathcal{F}_{\varphi,\psi}$ in the sense of Γ-convergence, since this notion is stable under continuous perturbations, and guarantees that any limit point of minimizers of $\mathcal{F}_\varepsilon$ is a minimizer for $\mathcal{F}_{\varphi,\psi}$. Moreover, one can hope to define the gradient flow associated to $\mathcal{F}_{\varphi,\psi}$ as the limit of the gradient flows associated to $\mathcal{F}_\varepsilon$ (if this limit exists, of course!).

It is easy to see (cf. [3]) that $\mathcal{F}_{\varphi,\psi}$ can *not* be approximated in the sense of Γ-convergence by local integral functionals like

$$\int_\Omega f_\varepsilon(\nabla u(x))\,dx\,, \tag{1.5}$$

defined in the Sobolev space $W^{1,2}(\Omega)$.

This difficulty has been overcome in different ways (see [3] for a complete list of statements, proofs and references):

— by introducing an auxiliary variable;

— by considering non-local functionals depending on the average of the gradient in small balls;

— by adding to (1.5) a singular perturbation depending on higher order derivatives of u;

— by using finite elements approximations, *i.e.* local functionals like (1.5) defined in suitable spaces of piecewise affine functions;

— by considering non-local functionals where the gradient is replaced by finite differences.

The last approach was suggested in 1996 by E. De Giorgi, who conjectured the convergence of the family

$$\mathcal{D}\mathcal{G}_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n\times\mathbb{R}^n} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) e^{-|\xi|^2}\,d\xi\,dx\,, \tag{1.6}$$

to the Mumford-Shah functional in $\mathbb{R}^n$ (up to some constants), both in the sense of pointwise convergence, and in the sense of Γ-convergence. This conjecture has been proved in [5] by reducing, via an integral-geometric approach, to the simpler family of one-dimensional functionals

$$DG_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right) dx\,. \tag{1.7}$$

This result has been extended in [7], where (1.7) and (1.6) are generalized, respectively, by

$$F_\varepsilon(u) = \int_{\mathbb{R}} \varphi_\varepsilon\left(\frac{|u(x+\varepsilon)-u(x)|}{\varepsilon}\right) dx\,. \tag{1.8}$$

and

$$\mathcal{F}_\varepsilon(u) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi_{\varepsilon|\xi|}\left(\frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon|\xi|}\right)\eta(\xi)\,d\xi\,dx\,, \tag{1.9}$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is a family of Borel functions, and $\eta\in L^1(\mathbb{R}^n)$ is a non-negative function.

This note is organized as follows: in § 2 we summarize the results obtained in [5, 7] concerning the finite difference approximation of (1.1) and (1.2), with applications to variational problems; in § 3 we describe the strategy followed in [6] in order to introduce a notion of gradient flow for the Mumford-Shah functional, and we discuss which parts of this machinery can be easily extended in higher dimension or to the general functional $\mathcal{F}_{\varphi,\psi}$; in § 4 we present a possible future development of this approach to evolution problems, with applications to the Perona-Malik equation.

We refer to the quoted literature for a review of the standard theory of bounded variation functions and Γ-convergence.

2. – Variational approximation.

We state here two results concerning the approximation of (1.2). We recall that, in the context of Γ-convergence, it is convenient to have all the functionals defined in the same space: this is usually obtained by extending them to $+\infty$ outside their «natural domain». All the functional introduced in this note are thought as defined in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The following result deals with the one dimensional case.

THEOREM 2.1. – *Let φ and ψ be as in the lower semicontinuity Theorem* 1.1.

Then there exists a family $\{\varphi_\varepsilon\}$ such that, defining $\{F_\varepsilon\}$ as in (1.8), *we have that*:

(c1) *Pointwise estimate:* $F_\varepsilon(u) \leq \mathcal{F}_{\varphi,\psi}(u)$ *for every* $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, *and every* $\varepsilon > 0$;

(c2) *Pointwise convergence:* $\lim_{\varepsilon \to 0^+} F_\varepsilon(u) = \mathcal{F}_{\varphi,\psi}(u)$ *for every* $u \in L^1_{\mathrm{loc}}(\mathbb{R})$;

(c3) *Γ-convergence:* $\mathcal{F}_{\varphi,\psi}(u)$ *is the* Γ^-*-limit of* $\{F_\varepsilon(u)\}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R})$.

The proof of Theorem 2.1 easily follows from the theory developed in [7], and a possible choice of $\{\varphi_\varepsilon\}$ turns out to be

$$\varphi_\varepsilon(r) := \min\left\{\varphi(l) + \frac{1}{\varepsilon}\psi(\varepsilon(r-l)) : l \in [0, r]\right\}, \qquad \forall r \geq 0 .$$

In dimension $n > 1$ the situation is more delicate. In [7] it is proved that $\mathcal{F}_{\varphi,\psi}$ can be approximated by functionals of the form (1.9), provided that φ satisfies a technical condition, called «sectionability», which depends on the dimension. We don't recall here the precise definition (the interested reader is referred to [7, Definition 6.1]). We just recall that this condition is satisfied in dimension one by any convex function $\varphi : [0, +\infty[\to [0, +\infty]$, and in any dimension *e.g.* by the function $\varphi(r) = |r|^p$ (with $p \geq 1$).

THEOREM 2.2 (cf. [7, Theorem 6.3]). – *Let* φ *and* ψ *be as in the lower semicontinuity Theorem* 1.1, *and let* $\eta \in L^1(\mathbb{R}^n)$ *be a non-negative radial function such that* $\{\xi \in \mathbb{R}^n : \eta(\xi) > c\}$ *has non-empty interior for some* $c > 0$. *Let us assume that* φ *is sectionable in dimension n.*

Then there exists a family $\{\varphi_\varepsilon\}$ *such that, defining* $\{\mathcal{F}_\varepsilon\}$ *as in* (1.9), *we have that*

(C1) *Pointwise estimate:* $\mathcal{F}_\varepsilon(u) \leq \mathcal{F}_{\varphi,\psi}(u)$ *for every* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *and every* $\varepsilon > 0$;

(C2) *Pointwise convergence:* $\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u) = \mathcal{F}_{\varphi,\psi}(u)$ *for every* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$;

(C3) *Γ-convergence:* $\mathcal{F}_{\varphi,\psi}(u)$ *is the* Γ^-*-limit of* $\{\mathcal{F}_\varepsilon(u)\}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$;

(C4) *Compactness: if* $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R}^n)$ *and*

$$\sup_{\varepsilon > 0}\{\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < +\infty ,$$

then $\{u_\varepsilon\}$ *is relatively compact in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

REMARK 2.3. – The advantage of $\{\mathcal{F}_\varepsilon\}$ with respect to the family $\{F_\varepsilon\}$ is twofold:

— it can be defined in every space dimension;

— it fulfills the compactness property (C4) (the family $\{F_\varepsilon\}$, on the contrary, satisfies no compactness properties: just remark that $F_\varepsilon(u) = 0$ for every ε-periodic function u!). For this reason it may be useful to use $\{\mathcal{F}_\varepsilon\}$ to approximate free discontinuity problems also in dimension one (see Theorem 2.5 below).

REMARK 2.4. – Pointwise estimates like (c1) and (C1) are one of the main advantages of this approach, for at least two reasons.

— Thanks to such estimates, the passage from the one-dimensional to the n-dimensional case is a simple application of Fatou's lemma and standard integral geometric equalities.

For this reason the finite difference approach is, at the present, the only approach which has been proved to work also in the case where $\varphi(r) = r^2$ and $\psi(r) = \sqrt{r}$ (note that in this case $\mathcal{F}_{\varphi,\psi}(u)$ can be finite even if $\mathcal{H}^{n-1}(S_u) = +\infty$).

— Pointwise convergence and Γ-convergence together imply stability under lower semicontinuous perturbations, in the following sense: if $\{\mathcal{F}_\varepsilon\}$ converges to $\mathcal{F}$ (in both senses) and $\mathcal{G}$ is lower semicontinuous, then $\{\mathcal{F}_\varepsilon + \mathcal{G}\}$ Γ-converges to $\mathcal{F} + \mathcal{G}$ (Γ-convergence alone is stable only under continuous perturbations).

This is particularly useful when considering minimum problems with lower order terms and/or convex constraints.

Thanks to the general properties of Γ-convergence, Theorem 2.2 leads to the following approximation result.

THEOREM 2.5 (cf. [5, Theorem 6.1]). – *Let φ and ψ be as in the lower semicontinuity Theorem* 1.1, *and let $\{\mathcal{F}_\varepsilon\}$ be the family given by Theorem* 2.2.

Let $1 \leq p < +\infty$, and let $g \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for every $\varepsilon > 0$ there exists a solution u_ε to the minimum problem

$$m_\varepsilon = \min\left\{\mathcal{F}_\varepsilon(u) + \int_{\mathbb{R}^n} |u-g|^p\,dx : u \in BV(\mathbb{R}^n),\ |Du|(\mathbb{R}^n) \leq 1/\varepsilon\right\}. \tag{2.1}$$

Moreover every sequence $\{u_{\varepsilon_j}\}$ with $\{\varepsilon_j\} \to 0^+$ has a subsequence converging in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a solution of the minimum problem

$$m_0 = \min\left\{\mathcal{F}_{\varphi,\psi}(u) + \int_{\mathbb{R}^n} |u-g|^p\,dx : u \in GSBV(\mathbb{R}^n)\right\}.$$

Furthermore

$$\lim_{\varepsilon \to 0} m_\varepsilon = m_0.$$

The *a-priori* restrictions $u \in BV(\mathbb{R}^n)$, $|Du|(\mathbb{R}^n) \leq 1/\varepsilon$ guarantee the existence of the minimum (2.1).

Without such restrictions, the existence of minimizers for (2.1) with a non-convex φ_ε is still an open problem, also in the case of the Mumford-Shah functional.

OPEN PROBLEM. – Does the problem

$$\min\left\{\mathcal{D}\mathcal{G}_\varepsilon(u)+\int_{\mathbb{R}^n}|u-g|^p\,dx : u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\right\} \tag{2.2}$$

have a solution for every $\varepsilon>0$, and every $g\in L^p(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$?

Looking at the Euler equation for (2.2), one can prove that if (2.2) has a minimizer which is continuous (resp. differentiable, BV), then g is necessarily continuous (resp. differentiable, BV). This shows that if $g\notin BV(\mathbb{R}^n)$, then the minimizers of (2.1) are not minimizers of (2.2).

REMARK 2.6. – For the sake of simplicity we stated here all the results for functionals defined in $\mathbb{R}^n$, but with some minor changes all the theory works in any open set $\Omega\subseteq\mathbb{R}^n$: the interested reader is referred to [7].

2.1. – *Examples.*

Let us show some examples of families of functionals defined as in (1.9), corresponding to different choices of $\{\varphi_\varepsilon\}$ ($\eta\in L^1(\mathbb{R}^n)$ is a non-negative *radial* function).

In the examples below we don't give the explicit expressions for the constants λ and μ which may appear in the limit functionals: for a precise computation see section 7 of [7].

EXAMPLE 1. – Let $\varphi_\varepsilon(r)=|r|^p$ with $p>1$. Then the limit functional is

$$\mathcal{E}_1(u)=\lambda\int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx\,.$$

This corresponds to $\mathcal{F}_{\varphi,\psi}$ with $\varphi(r)=\lambda|r|^p$ and $\psi(r)=+\infty$.

EXAMPLE 2. – Let $\varphi_\varepsilon(r)=\varepsilon^{-1}|\varepsilon r|^{1/p}$ with $p>1$. Then the limit functional is

$$\mathcal{E}_2(u)=\lambda\int_{S_u}|u^+-u^-|^{1/p}\,d\mathcal{H}^{n-1}$$

if $u\in GSBV(\mathbb{R}^n)\cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla u(x)=0$ for a.e. $x\in\mathbb{R}^n$; and $+\infty$ otherwise.

This corresponds to $\mathcal{F}_{\varphi,\psi}$ with $\varphi(r)=+\infty$ and $\psi(r)=\lambda|r|^{1/p}$.

EXAMPLE 3. – Let $\varphi_\varepsilon(r)=|r|$. In this case the limit functional is a multiple of the total variation of u. This is the limit case $p=1$ both of Example 1 and of Example 2.

EXAMPLE 4. – Let $\varphi_\varepsilon(r) = \varepsilon^{-1}\arctan(\varepsilon r^2)$. In this case the family $\{\mathcal{F}_\varepsilon\}$ is similar to the family $\{\mathcal{D}\mathcal{G}_\varepsilon\}$ of (1.6). The limit functional is $\mathcal{F}_{\varphi,\psi}$ with $\varphi(r) = \lambda|r|^2$ and $\psi(r) = \mu$, *i.e.* the Mumford-Shah functional, up to some constants.

EXAMPLE 5. – Let $\varphi_\varepsilon(r) = r^2(\sqrt{\varepsilon}|r|^{3/2} + 1)^{-1}$. Then the limit functional is $\mathcal{F}_{\varphi,\psi}$ with $\varphi(r) = \lambda|r|^2$ and $\psi(r) = \mu\sqrt{r}$ (cf. Remark 2.4).

3. – Evolution problems.

Evolution problems with free discontinuities seem to be a still unexplored research field, despite of the possible applications to fracture dynamic. The prototype of these evolution problems is the gradient flow for the Mumford-Shah functional, or more generally for the functional $\mathcal{F}_{\varphi,\psi}$.

A first difficulty is to establish what «gradient flow» means in this case, since $\mathcal{F}_{\varphi,\psi}$ is neither regular nor convex, and therefore it is not possible to apply standard theories, such as maximal monotone operators.

A possible approach to this problem has been proposed in [6], according to the following strategy.

(S1) We approximate $\mathcal{F}_{\varphi,\psi}$ with families of non-local functionals such as (1.9) (or (1.8)), with φ_ε regular.

(S2) Given an initial datum $u_0 \in L^\infty(\mathbb{R}^n)$, with $\mathcal{F}_{\varphi,\psi}(u_0) < +\infty$, we consider a family $\{u_{0\varepsilon}\} \subseteq L^2(\mathbb{R}^n)$ such that $\{u_{0\varepsilon}\} \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and

$$\{\mathcal{F}_\varepsilon(u_{0\varepsilon}) + \|u_{0\varepsilon}\|_\infty\} \to \mathcal{F}_{\varphi,\psi}(u) + \|u_0\|_\infty .$$

(S3) We solve for every $\varepsilon > 0$ the evolution problem in $L^2(\mathbb{R}^n)$

$$u_\varepsilon'(t) = -[\nabla \mathcal{F}_\varepsilon](u_\varepsilon(t)) , \qquad u_\varepsilon(0) = u_{0\varepsilon} . \tag{3.1}$$

Since $\mathcal{F}_\varepsilon$ is regular, the standard theory of ODEs in Hilbert spaces provides a unique solution $u_\varepsilon(t)$ of (3.1), defined for all $t \geq 0$.

(S4) We show that, up to subsequences, $\{u_\varepsilon(t)\}$ converges to a continuous function $u(t)$ such that $u(0) = u_0$. The possible limits of $\{u_\varepsilon(t)\}$ are our candidates to be the gradient flow for $\mathcal{F}_{\varphi,\psi}$.

This strategy has been applied in [6] in the case of the one-dimensional Mumford-Shah functional, using non-local approximations based on the family DG_ε defined in (1.7). In this particular case the possible limits in (S4) have been characterized as follows (see section 5 of [6]).

— For large classes of initial data, the whole family $\{u_\varepsilon(t)\}$ converges to a limit $u(t)$, which does not depend on $\{u_{0\varepsilon}\}$.

Roughly speaking, $u(t)$ can be obtained by evolving u_0, outside its singular set, according to the (rescaled) heat equation with homogeneous Neumann boundary conditions, and restarting the evolution (with the new initial datum) whenever a singularity «disappears».

Finally, the Mumford-Shah functional is decreasing along the trajectory.

— For some «pathological» choice of u_0 there is a continuum of possible limit points in (S4), depending on the sequence $\{\varepsilon_n\}$ and on the family $\{u_{0\varepsilon}\}$. However, *only one* of these limit points has the property that the Mumford-Shah functional is decreasing along the trajectory, and this limit can be characterized as above.

The compactness of the family $\{u_\varepsilon(t)\}$ relies on the standard Ascoli theorem, due to the compactness property (C4) and the following estimates for the solutions of (3.1) (for a proof see section 4 of [6]).

(E1) *Energy estimate:* $\mathcal{F}_\varepsilon(u_\varepsilon(t)) \leqslant \mathcal{F}_\varepsilon(u_{0\varepsilon})$.

(E2) L^∞-estimate: $\|u_\varepsilon(t)\|_\infty \leqslant \|u_{0\varepsilon}\|_\infty$.

(E3) *Hölder estimate*: $\|u_\varepsilon(t) - u_\varepsilon(s)\|_{L^2(\mathbb{R}^n)} \leqslant |t-s|^{1/2}\{\mathcal{F}_\varepsilon(u_{0\varepsilon})\}^{1/2}$.

For these reasons, the construction described in (S1)-(S4) works also in the general case (functional $\mathcal{F}_{\varphi,\psi}$ in any dimension), and with equations perturbed by lower order terms; however a precise characterization of the possible limits seems to be a challenging problem.

We hope to approach in a similar way evolution problems (with free discontinuities) involving second order time derivatives.

4. – The Perona-Malik equation.

The strategy described in section 3 can provide non trivial results also if the Γ-limit of the family considered in (S1) is trivial.

The example suggested by E. De Giorgi in [4] is the family $\{PM_\varepsilon\}$ defined as in (1.8) with

$$\varphi_\varepsilon(r) = \frac{1}{2}\log(1+r^2) =: \varphi(r).$$

The *formal* limit of $\{PM_\varepsilon\}$ is the functional

$$PM(u) = \frac{1}{2}\int_{\mathbb{R}} \log(1+|\nabla u(x)|^2)\,dx\,,$$

which is not lower semicontinuous, since $\varphi(r)$ is not convex. Moreover, the convex hull of $\varphi(r)$ is the constant zero, and therefore the Γ-limit of $\{PM_\varepsilon\}$ is trivially zero.

The gradient flow for PM is the so called Perona-Malik equation

$$u_t = \frac{1 - u_x^2}{(1 + u_x^2)^2} u_{xx}. \tag{4.1}$$

The Cauchy problem for this partial differential equation is a standard parabolic problem in the region where $|u_x| < 1$, and a backward parabolic problem where $|u_x| > 1$.

Only few rigorous results are known for equation (4.1); on the other hand, numerical experiments suggest the existence of solutions, at least in some weak sense.

A standard way to approximate (4.1) is to discretize in the space variable: this corresponds to apply the strategy of section 3 to the family $\{PM_\varepsilon\}$. As in the case of free discontinuity problems, it is possible to prove the compactness of the approximated gradient flows $\{u_\varepsilon(t)\}$, and the possible limits in (S4) are the candidates to be the weak solutions of (4.1).

Once again, the main problem is to characterize such limits, especially when the Cauchy datum has an interval in the «forward region», and an interval in the «backward region».

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Università degli Studi di Pisa, Dipartimento di Matematica Applicata «Ulisse Dini»
Via Bonanno 25B, 56126 PISA (Italy); E-mail: m.gobbino@dma.unipi.it

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