Bollettino Unione Matematica Italiana

Nicola Guglielmi

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 95–116.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_95_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 95-116

An Analytic Proof of Numerical Stability of Gaussian Collocation for Delay Differential (*).

NICOLA GUGLIELMI (**)

Sunto. – *In questo articolo si investigano le proprietà di stabilità asintotica dei metodi numerici per equazioni differenziali con ritardo, prendendo in esame l'equazione test:*

$$U'(t) = aU(t) + bU(t-\tau), \tag{0.1}$$

dove $a,\ b \in \mathbb{R}$, $\tau > 0$, *e* $g(t)$ *è una funzione a valori reali e continua. In particolare, viene analizzata la dipendenza dal ritardo della stabilità numerica dei metodi di collocazione Gaussiana. Nel recente lavoro* [GH99], *la stabilità di questi metodi è stata dimostrata facendo uso di un approccio geometrico, basato sul legame tra la proprietà di stabilità in esame e la geometria della* order star *della funzione razionale di A-stabilità dei metodi considerati (si veda* [HW96] *per una trattazione generale della teoria delle* order stars). *In questo lavoro, invece, viene fornita una dimostrazione puramente analitica, che poggia le proprie basi su alcuni risultati che legano le approssimanti di Padè della funzione esponenziale con certe serie ipergeometriche.*

1. – Introduction.

In this paper we investigate the asymptotic stability properties of numerical methods for delay differential equations of the form (0.1), where $a,\ b \in \mathbb{R}$, $\tau > 0$, and $g(t)$ is a continuous real-valued function. We direct attention to delay-dependent stability of one-step Gaussian collocation for the considered equation. In a recent paper Guglielmi and Hairer [GH99] proved stability of such methods by means of a geometric approach, based on the link between stability and *order stars* theory [HW96]. Here, instead, we revisit the subject and give a purely analytic proof, which is based on some connections between Padè approximants to the exponential and hypergeometric series.

There is a number of applications (see, for example Kuang [Kua93]) where it is proper to consider differential equations with a dependence on the solution in the past. In particular the use of such models is essential when ODE-based models are not efficient. An actual considerable subject, which receives

(*) This work was supported by the Italian M.U.R.S.T. and C.N.R.
(**) Comunicazione presentata a Napoli in occasione del XVI Congresso U.M.I.

the attention of researchers, is the qualitative behaviour of DDEs. In this ambit has relevance the analysis of the capabilities of difference methods to provide good numerical approximations over long integration intervals.

The last decade has seen a relatively large number of papers concerned with the study of the stability of numerical methods, using several test model problems. A common feature of those papers is that they typically make assumptions such that the solutions are asymptotically stable for every delay $\tau > 0$. Very seldom, instead, has been taken into consideration the dependence of the asymptotic stability on the delay. Although seeming to be quite difficult, even on apparently simple model problems like (0.1), such analysis would allow for a better understanding of the requirements for DDE methods to be stable. This paper aims to do this and is organized as follows. In Section 2, after a short introduction to the subject, we set the stability framework and provide it of some preliminary results. In Section 3 we analyze the root locus and give some results to characterize it, which are essential to prove the stability result, stated in Section 4. Finally, in Section 5 we draw some conclusions.

2. – Background and stability framework.

Following [Gug97], in order to simplify the notation, but without losing generality, we assume that t is scaled in the model problem (0.1) so that the delay $\tau = 1$. Hence, in the sequel, we shall consider the test equation

$$\begin{cases} U'(t) = aU(t) + bU(t-1), & t > 0, \\ U(t) = g(t), & -1 \leq t \leq 0. \end{cases} \tag{2.1}$$

By classical arguments [BC63], based on Laplace transform theory, the stability analysis for the true solution of (2.1) leads to the exam of the quasipolynomial characteristic equation

$$\mathcal{G}_\star(a, b; \lambda) = \lambda - a - b\exp(-\lambda) = 0. \tag{2.2}$$

In particular, the asymptotic stability region turns out to be given by

$$\Sigma_\star = \{(a, b) \mid \mathcal{G}_\star(a, b; \lambda) = 0 \Rightarrow \Re(\lambda) < 0\}.$$

Whenever $(a, b) \in \Sigma_\star$, the asymptotic stability of $U(t)$ is assured independently of the initial function $g(t)$.

In the considered case, that is $a, b \in \mathbb{R}$, the stability region $\Sigma_\star$ turns out to be given by the open connected domain included inside the half-plane $a < 1$

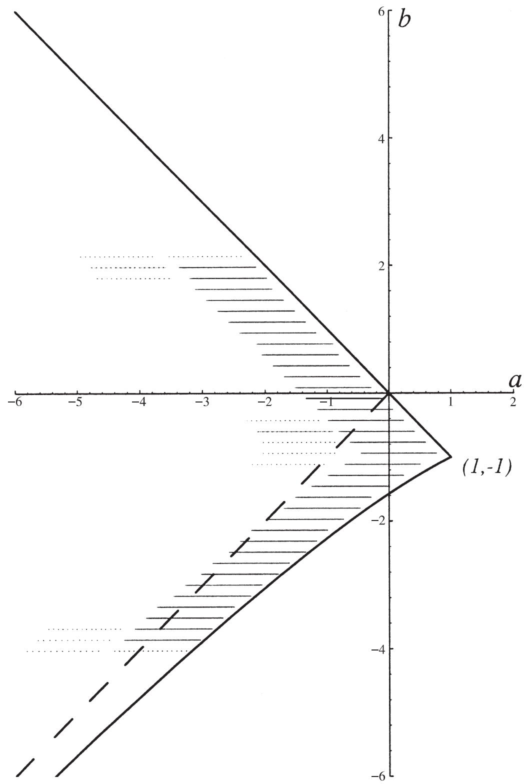

Figure 1. – Asymptotic stability region for the true solutions of equation (2.1).

and bounded to the right by the line

$$l^0 = \{(a, b) \in \mathbb{R}^2 \,|\, b = -a\}, \tag{2.3}$$

and by the transcendental curve

$$\gamma_\star = \{(a, b) \in \mathbb{R}^2 \,|\, a = a_\star(\phi),\ b = b_\star(\phi);\ \phi \in (0, \pi)\}, \tag{2.4}$$

where $a_\star(\phi) = \phi \cot(\phi)$ and $b_\star(\phi) = -\phi/\sin(\phi)$.

2.1. *The stability function for the Gaussian collocation.*

We consider here the Gaussian collocation method at s points, which can be represented by an equivalent s-level implicit Runge-Kutta process (see e.g. [HW96]), characterized by the abscissæ c_i, the weights w_i and the parameters

a_{ij}, and represented by the Butcher tableau:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & w_1 & w_2 & \cdots & w_s \end{array} \qquad \equiv \qquad \begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{w}^T \end{array}.$$

By classical results the c_i are the zeros of the shifted Legendre polynomial of degree s; hence we have $0 < c_i < 1$ and $c_i \neq c_j$, $i, j = 1, \ldots, s$. Moreover, $a_{ij} = \int_0^{c_i} \mathcal{L}_j(\xi)\, d\xi$ and $w_i = \int_0^1 \mathcal{L}_i(\xi)\, d\xi$ (where $\mathcal{L}_i$ denote Lagrange polynomials corresponding to the abscissæ $\{c_k\}$, $k = 1, \ldots, s$, $k \neq i$).

As is well-known, collocation at s Gaussian points is characterized by the nice feature that allows the highest order of non-stiff accuracy inside the class of s-stage Runge-Kutta methods. In particular, this property still holds when applying this method to equation (0.1) on particularly constrained meshes (see [Bel84]), like the one we are going to consider. Actually, in the present work we shall assume that the stepsize is constant and equal to an integer submultiple of the delay, that is

$$h = \frac{1}{m}, \qquad m \text{ positive integer}. \tag{2.5}$$

According to this hypothesis (see [Zen85]), we can use the stage values computed in the past to compute suitable approximations for the terms $U(t_n + c_i h - 1)$. Doing this yields the numerical scheme (for $n \geqslant m$)

$$\begin{cases} K_{n+1}^{(i)} = a\left(U_n + h\sum_{j=1}^{s} a_{ij} K_{n+1}^{(j)}\right) + b\left(U_{n-m} + h\sum_{j=1}^{s} a_{ij} K_{n-m+1}^{(j)}\right), \qquad i = 1, \ldots, s, \\ U_{n+1} = U_n + h\sum_{i=1}^{s} w_j K_{n+1}^{(i)}, \end{cases}$$

which can be written, with $\boldsymbol{K}_{n+1} = (K_{n+1}^{(1)}, \ldots, K_{n+1}^{(s)})^T$, in the more compact form:

$$\begin{cases} \boldsymbol{K}_{n+1} = a\left(U_n \boldsymbol{e} + \frac{1}{m}\boldsymbol{A}\boldsymbol{K}_{n+1}\right) + b\left(U_{n-m}\boldsymbol{e} + \frac{1}{m}\boldsymbol{A}\boldsymbol{K}_{n-m+1}\right), \\ U_{n+1} = U_n + \frac{1}{m}\boldsymbol{w}^T\boldsymbol{K}_{n+1}, \end{cases} \tag{2.6}$$

where $\boldsymbol{e} = [1, \ldots, 1]^T$. In the first interval $I_1 = [0, \tau]$, that is for $l \leq m$, the vectors $\boldsymbol{K}_l$ and the scalars U_l are determined by the initial function g, by applying the numerical method to the standard ODE

$$U'(t) = aU(t) + bg(t-1), \qquad U(0) = g(0).$$

Following Zennaro [Zen85], to the constant coefficient system of difference equations (2.6) is associated the characteristic equation

$$\mathcal{G}_m(a, b; \xi) = \mathrm{R}\left(\frac{1}{m}(a + b\xi^{-m})\right) - \xi = 0, \tag{2.7}$$

where

$$\mathrm{R}(z) = 1 + z\boldsymbol{w}^T(I - z\boldsymbol{A})^{-1}\boldsymbol{e}$$

is the rational A-stability function of the method.

With reference to equation (2.1), we first recall the definition of $\tau(0)$-stability of a numerical method, recently given by the author [Gug97] (we denote by $\mathbb{N}$ the set of the positive integers).

DEFINITION 2.1. – *A numerical step-by-step method for DDEs is* $\tau(0)$-*stable if*

$$\Sigma_m \supseteq \Sigma_\star \qquad \forall m \in \mathbb{N}.$$

The introduced property of $\tau(0)$-stability is stronger than the so-called P(0)-stability property, which is a property holding «for all delays» (see [Bar75].

3. – The use of the root locus for stability analysis.

As is well-known, checking the asymptotic stability of the numerical solution $\{U_n\}_{n \geq 0}$ of (2.1), for a given pair (a, b) and a given stepsize fulfilling assumption (2.5), means determining whether all the zeros $\{\xi_k\}$ of $\mathcal{G}_m(a, b; \xi)$ lie inside the open unit disk [Hen74].

First recall that the A-stability function $R(z)$ of the s-stage Gaussian collocation method is the (s, s)-Padè approximant to the exponential (see, for example, [HW96]), which we denote by $\mathcal{P}_{s,s}(z)$. We can write such Padè approximant by making use of the well-known representation

$$\mathcal{P}_{s,s}(z) = \frac{M_s(z)}{M_s(-z)}, \qquad z \in \mathbb{C}, \tag{3.1}$$

where

$$M_s(z) = \sum_{k=0}^{s} \mu_k z^k, \qquad \mu_k = \frac{(2s-k)!}{(2s)!}\binom{s}{k}. \tag{3.2}$$

We remark that all zeros of $M_s(z)$ lie inside the complex left half-plane, $\Re(z) < 0$ (see [HW96]). Next Lemma, which is concerned with the continuity of the zeros of $\mathcal{G}_m(a, b; \xi)$, establishes an important result to the following discussion.

Whenever $b = 0$, that is in the ODE case, all is known; hence, in the lemma, we focus our attention to the case $b \neq 0$.

LEMMA 3.1. – *Let m be a fixed positive integer and $b \neq 0$. The zeros of $\mathcal{G}_m(a, b; \xi)$ continuously depend on the parameters a and b.*

PROOF. – First of all we shall prove that $\mathcal{G}_m(a, b; \xi)$ can be written in the following way:

$$\mathcal{G}_m(a, b; \xi) \equiv \frac{P_m(a, b; \xi)}{Q_m(a, b; \xi)},$$

where, for every (a, b)-pair (with $b \neq 0$), the polynomials $P_m(a, b; \xi)$ and $Q_m(a, b; \xi)$ have no common zeros. To see this, set

$$z = \frac{1}{m}(a + b\xi^{-m})$$

and rewrite $\mathcal{G}_m(a, b; \xi)$ as

$$\frac{(M_s(z) - \xi M_s(-z))\,\xi^{ms}}{M_s(-z)\,\xi^{ms}},$$

$\xi = 0$ being a removable singularity of $\mathcal{G}_m$. Then we are able to define the polynomials

$$\begin{cases} P_m(a, b; \xi) = (M_s(z) - \xi M_s(-z))\,\xi^{ms}, \\ Q_m(a, b; \xi) = M_s(-z)\,\xi^{ms}. \end{cases} \tag{3.3}$$

Being $b \neq 0$, we easily get that $\xi = 0$ is not a zero of $\mathcal{G}_m$ and neither of the polynomials P_m and Q_m. As a consequence we can assume $\xi \neq 0$. Then, let us suppose $P_m(a, b; \xi) = 0$; we want to show that $Q_m(a, b; \xi) \neq 0$.

According to the previous assumption, we get, by (3.3),

$$M_s(z) = \xi M_s(-z).$$

If ξ were also a zero of $Q_m(a, b; \xi)$, $M_s(-z)$ should be zero, which would imply, by (3.4), $M_s(z) = 0$. But since the zeros $\{z_k\}$ of $M_s(z)$ lie in the complex left half-plane, this leads to a contradiction. Thus, for all $b \neq 0$, the zeros of $\mathcal{G}_m(a, b; \xi)$ coincide with those of the polynomial $P_m(a, b; \xi)$.

Further, with some algebraic manipulations, we get that the coefficient of the principal term of $P_m(a, b; \xi)$ is given by $\kappa M_s(-a/m)$, with $\kappa \neq 0$ suitable constant. Since $M_s(-a/m)$ does not vanish for any $a \in \mathbb{R}$, $m \in \mathbb{N}$, it holds that P_m fulfils the assumptions of the well-known theorem on the continuity of the zeros of a polynomial with respect to its coefficients. As a consequence, we are in a position to state the continuity of the zeros of $\mathcal{G}_m(a, b; \cdot)$ as functions of the parameters a and b, in the half-planes $b < 0$ and $b > 0$. ■

By virtue of previous lemma, an important tool for finding the asymptotic stability regions of the method, consists in the determination of the root locus, which is the set of (a, b)-pairs such that (2.7) has at least one zero of unit modulus. Setting $\mathrm{i} = \sqrt{-1}$, for an arbitrary but fixed $m \in \mathbb{N}$ we denote the root locus curve as:

$$\overline{U}_m := \{(a, b) \in \mathbb{R}^2 \mid \mathcal{G}_m(a, b; \exp(\mathrm{i}\varphi)) = 0;\ \varphi \in (-\pi, \pi]\}\ . \tag{3.5}$$

The determination of $\overline{U}_m$ allows for the partition of the (a, b)-plane into a number of different regions, each of them characterized by the property that the number of roots of $\mathcal{G}_m$ lying inside/outside the unit disk is constant. This is because any variation of this number would involve a transition of the parameters (a, b) across the root locus. Hence we can state the following result, which is a direct consequence of Lemma 3.1 and of A-stability of the method (see e.g. [HW96]).

LEMMA 3.2. – *If the condition $\overline{U}_m \cap \Sigma_\star = \emptyset$ holds for all $m \geq 1$, then the method is $\tau(0)$-stable.*

3.1. *Investigating the root locus.*

A convenient implicit representation of the root locus curve $\overline{U}_m$ is given as follows:

$$\mathcal{P}_{s,s}(z) = \exp(\mathrm{i}\varphi)\,, \tag{3.6-a}$$

$$z = \frac{1}{m}(a + b\exp(-\mathrm{i}m\varphi))\,, \tag{3.6-b}$$

with $\varphi \in (-\pi, \pi]$.

Concerning the analysis of (3.6-a), we state the following result, which yields a slight generalization of a result given by Dekker *et al.* [DKS86, Lemma 4.2]. The proof, which is based on the principle of the argument, is quite similar to that given there and is omitted for sake of conciseness.

LEMMA 3.3. – *Let* $\varphi \in (-\pi, \pi]$ *be given. Then there are precisely s different numbers* $z_1, \ldots, z_s$, *with* $z_k \in \mathbb{C} \cup \{\infty\}$ *such that*

$$\mathcal{P}_{s,s}(z_k) = \exp(i\varphi).$$

Moreover $\Re(z_k) = 0$ *for* $k = 1, \ldots, s$.

When we look at the roots of equation (3.6-a), where the number s of stages is given (but suppressed from the notation), we get, by Lemma 3.3

$$z_k(\varphi) = i y_k(\varphi), \qquad k = 1, \ldots, s,$$

where $y_k(\varphi) \in \mathbb{R} \cup \{\infty\}$. Thus, for any given $\varphi \in (-\pi, \pi]$, the set

$$\{i y_1(\varphi), i y_2(\varphi), \ldots, i y_s(\varphi)\} \tag{3.7}$$

provides the s simple imaginary roots of equation (3.6-a) (in the extended complex field).

REMARK 3.1. – By simply rewriting equation (3.6-a) with $\zeta = 1/z$, i.e.

$$\mathcal{P}_{s,s}\left(\frac{1}{\zeta}\right) = \exp(i\varphi), \tag{3.8}$$

on account of formulæ (3.1) and (3.2), it is easy to prove that, when s is even, a root ζ of (3.8) becomes 0 at $\varphi = 0$ (and hence the corresponding root z of (3.6-a) becomes ∞). Similarly, when s is odd, a root ζ of (3.8) becomes 0 at $\varphi = \pi$ (and hence a root z of (3.6-a) becomes ∞).

In the next lemma and theorem we provide a *partial* parametrization of the considered roots y_k as functions of φ.

LEMMA 3.4. – *Let* $\varphi \in (0, \pi)$. *Then the s roots of Equation* (3.6-a) *can be parametrized through s distinct functions which are continuous with respect to* φ.

PROOF. – By virtue of Lemma 3.3 and because of the well-known analiticity of the function $\mathcal{P}_{s,s}(z)$ in a neighbourhood of the imaginary axis (see, for example, [HW96]), we can rewrite (3.6-a), with $z = iy$ ($y \in \mathbb{R}$), in the following equivalent polynomial form (w.r.t. y):

$$p_s(y; \varphi) = M_s(iy) - \exp(i\varphi)\, M_s(-iy) = 0\,. \tag{3.9}$$

With this position, the effect of a root of (3.6-a) going to infinity for certain values of the parameter φ, would reveal itself through a reduction of the degree of the polynomial p_s for those values of φ.

Now take into consideration equation (3.9). First observe that, on account of (3.2), the coefficient of the principal term of the polynomial $p_s(y; \varphi)$ does not vanish for any φ considered. By virtue of this, we fulfil the assumptions for the continuous dependence of the zeros of a polynomial as functions of its coefficients (see, for example, [Hen74, Theorem 4.10.c]). Hence, we are able to parametrize every root y of (3.6-a) by a continuous function of φ, in $(0, \pi)$. On account of Lemma 3.3, such functions are distinct for every φ. ■

As a consequence of this we set

$$\mathcal{K} = \{1, \dots, s\}$$

and denote the considered set of roots as $\{\mathrm{i}\widehat{y}_1(\varphi), \mathrm{i}\widehat{y}_2(\varphi), \dots, \mathrm{i}\widehat{y}_s(\varphi)\}$ where

$$\widehat{y}_k(\varphi) : (0, \pi) \to \mathbb{R}\,, \tag{3.10}$$

is a continuous function $\forall k \in \mathcal{K}$.

Now we are in a position to state the following theorem, which provides an important property to the functions $\{\widehat{y}_k\}$.

THEOREM 3.1. – *For every $k \in \mathcal{K}$, the function $\widehat{y}_k(\varphi)$ is smooth and monotonically increasing (in $(0, \pi)$). Furthermore, it fulfils the inequality*

$$\widehat{y}_k(\varphi)\,\frac{d^2\widehat{y}_k}{d\varphi^2}(\varphi) > 0 \qquad \forall \varphi \in (0, \pi)\,. \tag{3.11}$$

In the proof we need the following lemma.

LEMMA 3.5. – *The following relations hold*:

(i) $M_{s-1}(-\mathrm{i}y)\,M_s(\mathrm{i}y) - M_{s-1}(\mathrm{i}y)\,M_s(-\mathrm{i}y) = \eta_{2s-1}\,y^{2s-1}$,

(ii) $$M_s(-\mathrm{i}y)\,M_s(\mathrm{i}y) = \sum_{j=0}^{s} \delta_{2j}\,y^{2j},$$

where

$$\eta_{2s-1} = 2\,\mathrm{i}s!(2s)!\,\frac{(s-1)!}{(2s-2)!}\,, \tag{3.12}$$

$$\delta_t = \left[\frac{s!}{(2s)!}\right]^2 \binom{2s - t/2}{t/2} \left[\frac{(2s-t)!}{(s-t/2)!}\right]^2. \tag{3.13}$$

PROOF. – For what concerns the first relation, formula (i) is a direct consequence of a result obtained by Hairer [Hai82].

Now direct attention at the second relation. By exploiting formulæ (3.2) we get

$$M_s(-\mathrm{i}y)\,M_s(\mathrm{i}y) = \sum_{r=0}^{2s} \delta_r y^r,$$

where

$$\delta_r = (-\mathrm{i})^r \left(\frac{s!}{(2s)!}\right)^2 \sum_{k=0}^{r} \frac{(2s-k)!}{k!(s-k)!} \frac{(2s-r+k)!}{(r-k)!\,(s-r+k)!} (-1)^k. \tag{3.14}$$

By making use of the Pochammer symbol,

$$(c)_n = \frac{(c+n-1)!}{(c-1)!},$$

(3.14) can be rewritten, after some manipulations, as

$$\delta_r = (-\mathrm{i})^r \frac{s!\,(2s-r)!}{(2s)!\,(s-r)!\,r!} \beta_r,$$

with

$$\beta_r = \sum_{k=0}^{r} \frac{1}{k!} \frac{(-r)_k (2s-r+1)_k (-s)_k}{(-2s)_k (s-r+1)_k}. \tag{3.15}$$

Now introduce the following generalized hypergeometric series (see, for example, [Sla66]):

$${}_3F_2\left[\begin{matrix} r,\ 2s-r+1,\ -s \\ -2s,\ s-r+1 \end{matrix} ; z\right] := \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-r)_k (2s-r+1)_k (-s)_k}{(-2s)_k (s-r+1)_k z^k}.$$

Since $(-r)_k = 0$ for $k > r$, the summation (3.15) can be written in terms of the previous generalized hypergeometric series as

$$\beta_r = {}_3F_2\left[\begin{matrix} -r,\ 2s-r+1,\ -s \\ -2s,\ s-r+1 \end{matrix} ; 1\right].$$

Then, applying Dixon's theorem (see, for example, [Hen74], p. 43) yields:

$$\beta_r = \begin{cases} \dfrac{r!}{(r/2)!} \dfrac{(s-r/2+1)_{r/2}}{(2s-r/2+1)_{r/2}(-s+r/2)_{r/2}} & \text{if } r \text{ is even}, \\ 0 & \text{if } r \textit{ is odd}. \end{cases} \tag{3.16}$$

By setting $r = 2j$ $(1 \leqslant j \leqslant s)$, we get

$$\beta_{2j} = (-1)^j \frac{(2s-j)!\,(s-2j)!}{[(s-j)!]^2} \frac{(2j)!}{j!} \frac{s!}{(2s)!}. \tag{3.17}$$

Finally, by (3.16) and (3.17), we are in a position to evaluate (3.15) and to state (ii). ■

PROOF OF THEOREM 3.1. – We proceed now to prove the theorem. Consider (3.9); since the zeros of $M_s(z)$ are located inside the complex left half-plane, we get

$$\frac{\partial p_s(y;\varphi)}{\partial \varphi} = -\,\mathrm{i}\exp(\mathrm{i}\varphi)\, M_s(-\mathrm{i}y) \neq 0 \qquad \forall(\varphi, y) \in \mathbb{R}^2. \tag{3.18}$$

Thus, for every $k \in \mathcal{K}$ the smoothness of $\widehat{y}_k(\varphi)$ in $(0, \pi)$ is a consequence of the implicit function theorem.

Now fix $k \in \mathcal{K}$ arbitrarily and direct attention at the function $\widehat{y}_k(\varphi)$. After setting $\widehat{\Phi} = (0, \pi)$, let us denote by $\widehat{y}_k(\widehat{\Phi})$ the image of $\widehat{\Phi}$ under $\widehat{y}_k(\cdot)$.

Then, for proving the second part of the theorem, it is convenient to take into consideration the inverse mapping $y \to \varphi(y)$. To this aim we write (3.6-a) as

$$\varphi = -\,\mathrm{i}\log\left(\mathcal{P}_{s,s}(\mathrm{i}y)\right), \qquad y \in \widehat{y}_k(\widehat{\Phi}), \tag{3.19}$$

where $\log(\cdot)$ is a certain determination of the natural logarithm (whose choice depends on the considered root $\widehat{y}_k$ and is such that the condition $\varphi \in (0, \pi)$ is fulfilled). By differentiating (3.19) with respect to y, we get

$$\frac{d\varphi}{dy} = \frac{\mathcal{P}'_{s,s}(\mathrm{i}y)}{\mathcal{P}_{s,s}(\mathrm{i}y)}.$$

Moreover, by (3.1) we get

$$\frac{\mathcal{P}'_{s,s}(\mathrm{i}y)}{\mathcal{P}_{s,s}(\mathrm{i}y)} = \frac{M_s'(-\mathrm{i}y)}{M_s(-\mathrm{i}y)} + \frac{M_s'(\mathrm{i}y)}{M_s(\mathrm{i}y)}.$$

Now, make use of the classical recurrence relation (see, for example, [IN91])

$$M_s'(z) = \frac{1}{2}M_s(z) - \frac{z}{4(2s-1)}M_{s-1}(z),$$

being $M_0(z) \equiv 1$. Then the following relation holds:

$$\frac{d\varphi}{dy} = 1 + \frac{\mathrm{i}y}{4(2s-1)}\,\Upsilon_s(y), \tag{3.20}$$

with

$$\Upsilon_s(y) = \frac{M_{s-1}(-\mathrm{i}y)\, M_s(\mathrm{i}y) - M_{s-1}(\mathrm{i}y)\, M_s(-\mathrm{i}y)}{M_s(-\mathrm{i}y)\, M_s(\mathrm{i}y)}$$

rational function. Then, applying Lemma 3.5 yields

$$\Upsilon_s(y) = \frac{\eta_{2s-1} y^{2s-1}}{\sum_{j=0}^{s} \delta_{2j} y^{2j}},$$

where η_{2s-1} and δ_{2j}, $j = 0, \dots, s$, are given by (3.12) and (3.13), respectively. Therefore, (3.20) takes the form

$$\frac{d\varphi}{dy} = 1 - \frac{y^{2s}}{\sum_{j=0}^{s} q_j y^{2j}},$$

where

$$q_j = \binom{2s-j}{j} \left[\frac{(2s-2j)!}{(s-j)!} \right]^2, \qquad j = 0, \dots, s.$$

In the end we get

$$\text{(3.21)} \qquad \frac{d\varphi}{dy} = \frac{\sum_{r=0}^{s-1} q_r y^{2r}}{\sum_{r=0}^{s} q_r y^{2r}} = \frac{q_0 + q_1 y^2 + \dots + q_{s-1} y^{2s-2}}{q_0 + q_1 y^2 + \dots + q_{s-1} y^{2s-2} + y^{2s}}.$$

Hence, being $q_j > 0$, $j = 0, \dots, s$, $\varphi(y)$ is monotonically increasing in the domain $\widehat{y}_k(\widehat{\Phi})$. As a consequence, $\widehat{y}_k(\varphi)$ is also monotonically increasing in $(0, \pi)$. Finally, since

$$\frac{d^2\varphi}{dy^2} = \frac{\sum_{j=0}^{s} 2(j-s)\, q_j y^{2(s+j)-1}}{\left(\sum_{j=0}^{s} q_j y^{2j}\right)^2} \begin{cases} > 0 & \text{if } y < 0 \\ < 0 & \text{if } y > 0, \end{cases}$$

we obtain (3.11). ■

3.2. *A set of functions for parametrizing the roots of* (3.6-a).

For the complete determination of the root locus curve we need to extend the parametrization of the roots to the whole domain $(-\pi, \pi]$.

First, let us define the odd extensions of the functions $\{\widehat{y}_k(\varphi)\}$ to the interval $(-\pi, 0)$, that is

$$\check{y}_k(\varphi) := -\widehat{y}_k(-\varphi), \qquad \varphi \in (-\pi, 0) \qquad \text{for } k = 1, \dots, s.$$

By observing that $\mathcal{P}_{s,s}(-\mathrm{i}y) = -\mathcal{P}_{s,s}(\mathrm{i}y)$ for all $y \in \mathbb{R}$, it is immediate to prove the following result.

LEMMA 3.6. – *The set* $\{i\check{y}_1(\varphi), i\check{y}_2(\varphi), \ldots, i\check{y}_s(\varphi)\}$ *provides the s roots of equation* (3.6-a), *for* $\varphi \in (-\pi, 0)$. *Moreover, for every* $k \in \mathcal{K}$, $\check{y}_k(\varphi)$ *is smooth and monotonically increasing* (*in* $(-\pi, 0)$) *and fulfils the inequality*

$$\check{y}_k(\varphi)\,\frac{d^2\check{y}_k}{d\varphi^2}(\varphi) > 0 \qquad \forall \varphi \in (-\pi, 0)\,. \tag{3.22}$$

To terminate the parametrization we have left to consider the cases $\varphi = 0$ and $\varphi = \pi$. By Theorem 3.1 and Lemma 3.6, we have that for every $l \in \mathcal{K}$, $\hat{y}_l(\varphi)$ and $\check{y}_l(\varphi)$ are monotonically increasing in $(0, \pi)$ and $(-\pi, 0)$, respectively. Hence they have limit as $\varphi \to \pi^-$ and as $\varphi \to 0^-$, respectively. Therefore let

$$\lim_{\varphi \to \pi^-} \hat{y}_l(\varphi) = y_l^\pi, \qquad l = 1, \ldots, s\,,$$

$$\lim_{\varphi \to 0^-} \check{y}_l(\varphi) = y_l^0, \qquad l = 1, \ldots, s\,.$$

Concerning the numbers r and r' of finite roots of (3.6-a), corresponding to the values $\varphi = \pi$ and $\varphi = 0$, we have to distinguish the following two cases (see Remark 3.1).

(1) If s is even $r = s$ and $r' = s - 1$;

(2) if s is odd $r = s - 1$ and $r' = s$.

Next we are ready to state the following lemma.

LEMMA 3.7.

(i) *Let* $\varphi = \pi$ *and assume* y_i^π $(i \in \mathcal{K})$ *to be finite. Then* iy_i^π *is one of the r finite and distinct roots of* (3.6-a). *Viceversa, every finite root of* (3.6-a) *is the limit of one of the functions* $i\hat{y}_l(\cdot)$ $(l \in \mathcal{K})$ *as* $\varphi \to \pi^-$.

(ii) *Similarly, let* $\varphi = 0$ *and assume* y_j^0 $(j \in \mathcal{K})$ *to be finite. Then* iy_j^0 *is one of the* r' *finite and distinct roots of* (3.6-a). *Viceversa, every finite root of* (3.6-a) *is the limit of one of the functions* $i\check{y}_l(\cdot)$ $(l \in \mathcal{K})$ *as* $\varphi \to 0^-$.

PROOF. – Consider first the case (i), that is $\varphi = \pi$. By defining

$$\mathcal{F}_\varphi(z) : (\varphi, z) \to \frac{M_s(z)}{M_s(-z)} - \exp(i\varphi)\,, \tag{3.23}$$

the first statement, which asserts that iy_i^π is a root of $\mathcal{F}_\pi(z)$, that is of

$$\frac{M_s(z)}{M_s(-z)} - 1\,, \tag{3.24}$$

follows directly by the continuity of $\mathcal{T}_\varphi(z)$ in a neighbourhood of the imaginary axis.

Viceversa, for what concerns the second statement, let Ω be a compact set including inside all the r (pure imaginary) roots of (3.24), such that no zeros of $M_s(-z)$ lie within it. Then, the functions (3.23) are analytic in Ω and, by Lemma 3.3, different from zero for $z \in \partial\Omega$.

Now, because of the uniform convergence of $\{\mathcal{T}_\varphi(z)\}$ on every compact subset of Ω, we have from Rouché's theorem that each finite zero of (3.24) is the limit of a corresponding zero of $\mathcal{T}_\varphi(z)$ (as $\varphi \to \pi^-$). This proves the first part of the lemma; the proof of (ii) is analogous. ■

In conclusion, by virtue of Lemmas 3.4, 3.6 and 3.7, we are able to give a suitable parametrization of the set of functions (3.7), in the whole domain $(-\pi, \pi]$. Specifically, we set, for every $k \in \mathcal{K}$,

$$y_k(\varphi) := \begin{cases} \check{y}_k(\varphi), & \varphi \in (-\pi, 0), \\ y_k^0 & \varphi = 0, \qquad \text{(if } y_k^0 \text{ is finite)}, \\ \hat{y}_k(\varphi), & \varphi \in (0, \pi), \\ y_k^\pi, & \varphi = \pi, \qquad \text{(if } y_k^\pi \text{ is finite)}. \end{cases} \tag{3.25}$$

3.3. *An ordering relation for the functions* $y_k(\cdot)$.

LEMMA 3.8. – *Let* $\Phi = (-\pi, \pi]$ *and* $y_i(\Phi)$ *denote the image of* Φ *under* y_i (*with* $i = 1, \ldots, s$). *Then,*

(i) $y_l(\Phi) \cap y_k(\Phi) = \emptyset \quad \forall l, k = 1, \ldots, s, \ l \neq k$;

(ii) $\bigcup_{l=1}^{s} y_l(\Phi) = \mathbb{R}$.

PROOF. – Suppose that $y_l(\varphi') = y_k(\varphi'')$ for some $k \neq l$ and some $\varphi', \varphi'' \in \Phi$. By (3.6-a), this would imply $\exp(\mathrm{i}\varphi') = \exp(\mathrm{i}\varphi'')$ and consequently $\varphi' = \varphi''$, which would contradict Lemma 3.3. Hence (i) is proved. Furthermore, (ii) is an immediate consequence of the property

$$|\mathcal{P}_{s,s}(\mathrm{i}y)| = \left| \frac{M_s(\mathrm{i}y)}{M_s(-\mathrm{i}y)} \right| \equiv 1, \qquad \forall y \in \mathbb{R},$$

which implies that, for any given y, there exists a corresponding $\varphi \in \Phi$ such that (3.6-a) is fulfilled. ■

Summarizing the results stated through Lemmas 3.6, 3.7 and Theorem 3.1, we have that every branch y_i is *odd* and monotonically increasing in $(-\pi, 0)$

as well as in $(0, \pi)$. Hence we have, for every $i \in \mathcal{K}$,

$$y_i(\Phi) = (\nu_i, \sigma_i] \cup (-\sigma_i, -\nu_i],$$

where $\nu_i = min(|y_i^0|, |y_i^\pi|)$ and $\sigma_i = max(|y_i^0|, |y_i^\pi|)$. Furthermore, by virtue of the results stated by Lemma 3.8, we can conventionally label the numbers ν_i, σ_i in the following way:

$$\nu_1 < \sigma_1 = \nu_2 < \sigma_2 = \nu_3 < \sigma_3 \ldots < \sigma_{s-2} = \nu_{s-1} < \sigma_{s-1} = \nu_s < \sigma_s.$$

As a consequence we can order the roots (3.25) according to the following rule:

$$y_l(\cdot) \prec y_k(\cdot) \quad \text{if } \nu_l < \nu_k. \tag{3.26}$$

With this choice we have $\nu_1 = 0$ (by Lemma 3.8) and hence

$$y_1(0) = 0\,. \tag{3.27}$$

Moreover (see Remark 3.1) we have $\sigma_s = +\infty$. Hence, in both cases s odd and s even, y_s is the unbounded branch. More in detail,

$$\lim_{\varphi \to 0^+} y_s(\varphi) = -\infty\,, \qquad \lim_{\varphi \to 0^-} y_s(\varphi) = +\infty\,, \qquad \text{if } s \text{ is even}\,, \tag{3.28}$$

$$\lim_{\varphi \to \pi^-} y_s(\varphi) = +\infty\,, \qquad \lim_{\varphi \to -\pi^+} y_s(\varphi) = -\infty\,, \qquad \text{if } s \text{ is odd}\,. \tag{3.29}$$

3.4. *A compact representation for the root locus.*

In this section we consider the matter of representing the roots of (3.6-a) through a unique function in the extended domain $(-s\pi, s\pi]$, instead of s functions in $(-\pi, \pi]$ (which is done by means of (3.25)). To do this, simply consider the periodic extensions $\bar{y}_k(\varphi)$ (with period 2π) of the functions $y_k(\varphi)$ and define the function

$$\mathcal{Y}_s(\varphi) := \begin{cases} \bar{y}_1(\varphi), & \varphi \in (-\pi, \pi], \\ \bar{y}_2(\varphi), & \varphi \in (-2\pi, -\pi] \cup (\pi, 2\pi], \\ \vdots & \vdots \\ \vdots & \vdots \\ \bar{y}_{s-1}(\varphi), & \varphi \in (-(s-1)\pi, -(s-2)\pi] \cup ((s-2)\pi, (s-1)\pi], \\ \bar{y}_s(\varphi), & \varphi \in (-s\pi, -(s-1)\pi] \cup ((s-1)\pi, s\pi)\,. \end{cases}$$

Evidently, by the properties of the functions (3.25) (and of theirs periodic extensions), $\mathcal{Y}_s(\varphi)$ is odd. Moreover, by (3.28) and (3.29), $\bar{y}_s(\varphi)$ is upper-unbounded as $\varphi \to s\pi$ and thus

$$\mathcal{Y}_s(s\pi) := \lim_{\varphi \to (s\pi)^-} \mathcal{Y}_s(\varphi) = +\infty \,. \tag{3.30}$$

An illustrative example now follows.

EXAMPLE 3.1. – Consider Gaussian collocation at 2 points. In this case $\mathcal{K} = \{1, 2\}$ and

$$\begin{cases} \widehat{y}_1(\varphi) = \dfrac{\left(\sqrt{6}\sqrt{(7-\cos(\varphi))\sin(\varphi/2)^2} - 3\sin(\varphi)\right)}{2\sin(\varphi/2)^2}, \\ \widehat{y}_2(\varphi = \dfrac{-\left(\sqrt{6}\sqrt{(7-\cos(\varphi))\sin(\varphi/2)^2} + 3\sin(\varphi)\right)}{2\sin(\varphi/2)^2}, \end{cases} \qquad \varphi \in (0, \pi),$$

which are plotted in Figure 2 (left side), together with $\check{y}_1(\varphi)$ and $\check{y}_2(\varphi)$. In this case we get

$$\mathcal{Y}_2(\varphi) := \begin{cases} \widehat{y}_2(\varphi + 2\pi), & \varphi \in (-2\pi, -\pi), & y_2^{\pi}, & \varphi = -\pi, \\ \check{y}_1(\varphi), & \varphi \in (-\pi, 0), & y_1^0, & \varphi = 0, \\ \widehat{y}_1(\varphi), & \varphi \in (0, \pi), & y_1^{\pi}, & \varphi = \pi, \\ \check{y}_2(\varphi - 2\pi), & \varphi \in (\pi, 2\pi), \end{cases}$$

which is plotted in Figure 2 (right side). According to (3.30), it is easy to verify that

$$\mathcal{Y}_2(2\pi) := \lim_{\varphi \to (2\pi)^-} \mathcal{Y}_2(\varphi) = +\infty \,.$$

Concerning $\mathcal{Y}_s$, we are able to state the following result, whose importance is crucial to the final discussion.

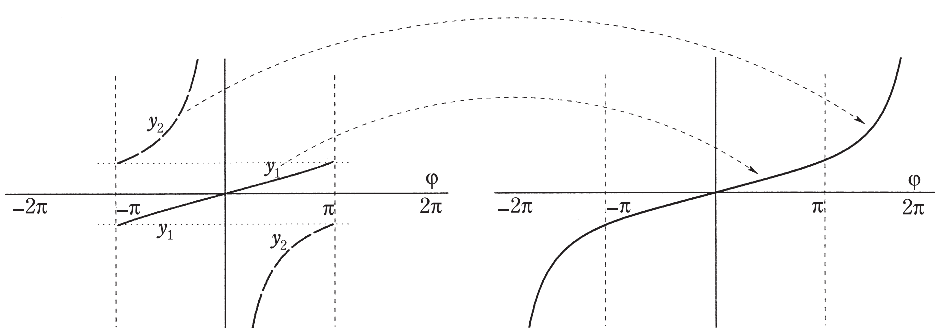

Figure 2. – Construction of $\mathcal{Y}_2$ by means of y_1 and y_2 (case $s = 2$).

LEMMA 3.9. – *The function* $\mathcal{Y}_s(\varphi):(-s\pi,\, s\pi]\to\mathbb{R}\cup\infty$ *is smooth, odd and monotonically increasing. Furthermore*

$$\mathcal{Y}_s(\varphi)>\varphi\,,\qquad \text{for } \varphi>0\,. \tag{3.31}$$

PROOF. – By Theorem 3.1 and Lemma 3.6, $\mathcal{Y}_s(\varphi)$ is continuous, smooth and monotonically increasing in every open interval $(k\pi,(k+1)\pi)$, $k=-s$, $-s+1,\dots,s-1$.

Then analyse, for $k=-s,\,-s+1,\dots,s-1$, the *critical* points $\varphi_k^\star=k\pi$. Consider equation (3.6-a), fix k and let $\varphi\to\varphi_k^\star$. On account of the periodic and continuous dependence of the right-hand side of equation (3.6-a) with respect to φ, the continuity of $\mathcal{Y}_s(\varphi)$ also at the points $\{\varphi_k^\star\}$ is easily obtained on the basis of the ordering relation (3.26), by making use of the same arguments used to prove Lemma 3.7. Moreover, the smoothness of $\mathcal{Y}_s(\varphi)$ at every $\varphi_k^\star$ is still obtained as a consequence of Implicit Function Theorem.

Finally, on account of the continuity, we immediately obtain the *global* monotonicity. This proves the first part of the Lemma.

Since $\mathcal{Y}_s(0)=0$ (by (3.27)), $\mathcal{Y}_s(\varphi)>0$ in $(0,\,s\pi]$. Moreover, by exploiting formula (3.21), we get $y_1'(0)=1$ and thus

$$\mathcal{Y}_s'(0)=1\,. \tag{3.32}$$

Therefore, property (3.31) is a direct consequence of the convexity of $\mathcal{Y}_s$ for $\varphi>0$, which is due to (3.11) and (3.22). ■

At this point, by making use of the function $\mathcal{Y}_s(\varphi)$, we are able to exploit equation (3.6-b) to rewrite $\overline{U}_m$ (see (3.5)) into the following compact form:

$$\overline{U}_m=\left\{(a,\,b)\in\mathbb{R}^2\,|\,\exists\varphi\in(-s\pi,\,s\pi]:(a+b\exp(-\mathrm{i}m\varphi))=\mathrm{i}m\,\mathcal{Y}_s(\varphi)\right\}, \tag{3.33}$$

which turns out to be very well suited to the stability analysis.

The case $\varphi=0$ is singular and needs to be separately considered. Doing this, by a general result concerning R-K methods [Gug97], we obtain the following result.

LEMMA 3.10. – *For the s-stage Gaussian collocation, applied to the test equation* (2.1), *the following relation holds*:

$$l^0\in\overline{U}_m,$$

l^0 *being the straight line defined by* (2.3).

By the way, observe that $l^0\cap\Sigma_\star=\emptyset$. Henceforth, in order to apply Lemma

3.2, we shall focus our attention on the set

$$V_m := \overline{U}_m \setminus l^0. \tag{3.24}$$

With routine algebraic manipulations, by (3.33) we obtain

$$V_m = \{(a_m(\varphi),\, b_m(\varphi));\ \varphi \in (0,\, s\pi]\},$$

where

$$\begin{cases} a_m(\varphi) = m\,\mathcal{Y}_s(\varphi)\,\cot(m\varphi), \\ b_m(\varphi) = -\dfrac{m\,\mathcal{Y}_s(\varphi)}{\sin(m\varphi)}. \end{cases} \tag{3.35}$$

Observe that V_m is parametrized in the interval $(0,\, s\pi]$. This is because $a_m(\varphi)$ and $b_m(\varphi)$ are even functions with respect to φ (as they should).

The analysis of formulæ (3.35) gives that V_m is a piecewise regular algebraic curve (as shown in Figure 3), with singularities at $\varphi = l\pi/m$, $l = 0, 1, \dots, sm$.

We introduce some final notations. First define the intervals

$$I_l = \left((l-1)\frac{\pi}{m},\, l\frac{\pi}{m}\right), \tag{3.36}$$

with $l = 1, \dots, sm$. Then we define, in the (a, b)-plane, the algebraic curves

$$\gamma_l = \{(a_m(\varphi),\, b_m(\varphi)) \,|\, \varphi \in I_l\}, \tag{3.37}$$

where, naturally, $\{\gamma_l\}$ depend on m and s.

With the exception of the case when $\varphi \to 0^+$ (where $(a_m(\varphi),\, b_m(\varphi)) \to (1,\, -1)$, as is common to any R-K process for DDEs [Gug97]), we get

$$\lim_{\varphi \to l(\pi/m)} |a_m(\varphi)| = +\infty, \qquad \lim_{\varphi \to l(\pi/m)} |b_m(\varphi)| = +\infty, \qquad l = 1, \dots, sm.$$

Finally we define the convex cones

$$\begin{cases} \Omega^- = \{(a,\, b) \in \mathbb{R}^2 \,|\, -b \geqslant |a|\}, \\ \Omega^+ = \{(a,\, b) \in \mathbb{R}^2 \,|\, b \geqslant |a|\}. \end{cases} \tag{3.38}$$

The following result gives a useful geometric characterization of the root locus curve.

LEMMA 3.11. – *The curves* γ_l, $l = 1, \ldots, sm$, *do not intersect each other at any finite point of the* (a, b)*-plane. Furthermore, the following relationships hold*:

$$\gamma_l \subset \Omega^- \quad \text{if } l \text{ is odd} \quad \textit{and} \quad \gamma_l \subset \Omega^+ \quad \text{if } l \text{ is even}. \tag{3.39}$$

PROOF. – Let $\varphi' \in I_{l'}$ and $\varphi'' \in I_{l''}$, with $l'' > l'$. If the curves $\gamma_{l'}$ and $\gamma_{l''}$ intersected, we should have

$$\begin{cases} a_m(\varphi') = a_m(\varphi''), \\ b_m(\varphi') = b_m(\varphi''), \end{cases} \tag{3.40}$$

for certain values φ', φ''. Since

$$\frac{a_m(\varphi)}{b_m(\varphi)} = -\cos(m\varphi), \tag{3.41}$$

(3.40) should imply

$$\varphi'' = \varphi' + (l'' - l')\frac{\pi}{m}.$$

However, by the monotonicity of $\mathcal{Y}_s(\varphi)$ (Lemma 3.9) it follows that (3.40) cannot hold. Finally, by (3.41), one readily gets (3.39). ■

An example for the root locus, relevant to the choice $s = 2$ and $m = 2$, is shown in Figure 3. Here, the numbers plotted inside the different connected regions refer to the numbers of roots of (2.7) with modulus larger than 1.

4. – The main result.

In this section we show that s-stage Gaussian collocation is stable.

THEOREM 4.1. – *The* s*-stage Gaussian collocation is* $\tau(0)$*-stable.*

PROOF. – According to Lemma 3.2, our goal is to show that $V_m \cap \Sigma_\star = \emptyset$ for all $m \geq 1$. By making use of (3.37), we write $V_m = \bigcup_{l=1}^{sm} \gamma_l$.

Now, by Lemma 3.11 we have that, if l is even, the curves γ_l lie in Ω^+ and hence do not intersect $\Sigma_\star$. Therefore, we can restrict our attention to the curves γ_l with odd index l, which lie in Ω^- (see (3.39)). Relevant to these curves, we can suitably express the condition of *non-intersection*, that is

$$\gamma_l \subset \Omega^- \setminus \Sigma_\star \quad \text{if } l \text{ is odd}. \tag{4.1}$$

In order to do this, we consider an algebraic reformulation of (4.1). To this purpose, it turns out to be convenient to represent the stability region $\Sigma_\star$ (Figure 1) as

$$\Sigma_\star = \overset{\triangle}{\Sigma} \cup \Sigma\,,$$

where $\overset{\triangle}{\Sigma} = \{(a, b) \in \mathbb{R}^2 \,|\, a + |b| < 0\}$ and $\Sigma = \Sigma_\star \setminus \overset{\triangle}{\Sigma} = \Sigma_\star \cap \Omega^-$.

Now examine $\gamma_\star$ (see (2.4)), which bounds Σ to the right. After some manipulations we obtain that the points belonging to the curve fulfil the relation

$$\sqrt{b^2 - a^2} = \arccos\left(-\frac{a}{b}\right),$$

where $\arccos : [-1, 1] \to [0, \pi]$.

By restricting our attention to the pairs (a, b) belonging to Ω^-, the property $(a, b) \in \Omega^- \setminus \Sigma$ is expressed by the inequality $\sqrt{b^2 - a^2} > \arccos(-a/b)$ or equivalently, with $\varrho = -a/b$, by

$$-b\sqrt{1 - \varrho^2} > \arccos(\varrho)\,. \tag{4.2}$$

On account of (4.2), with

$$\varrho_m(\varphi) = -\frac{a_m(\varphi)}{b_m(\varphi)} = \cos(m\varphi)\,,$$

condition (4.1) reads

$$b_m(\varphi)\sqrt{1 - \varrho_m(\varphi)^2} + \arccos(\varrho_m(\varphi)) < 0\,, \qquad \forall \varphi \in \left((l-1)\frac{\pi}{m},\, l\frac{\pi}{m}\right), \tag{4.3}$$

with $1 \leqslant l \leqslant sm$ and l odd. Since $\varrho \geqslant \arccos(\cos(\varrho))$ if $\varrho \geqslant 0$, inequality (4.3) is certainly fulfilled if

$$\mathcal{Y}_s(\varphi) > \varphi \qquad \forall \varphi \in (0, s\pi)\,. \tag{4.4}$$

Therefore, applying Lemma 3.9 concludes the proof. ∎

We make here some final considerations to have a better understanding of the stability set Σ_m. Firstly observe that

$$a_m\left(\frac{l\pi}{2m}\right) = 0\,, \qquad l = 1, \ldots, sm\,.$$

Thus, if we restrict our attention to the set $\{\gamma_{2l-1}\}$, $l \geqslant 1$, that is to the curves belonging to Ω^-, and look at (3.37), we get (by virtue of Lemma 3.9) the fol-

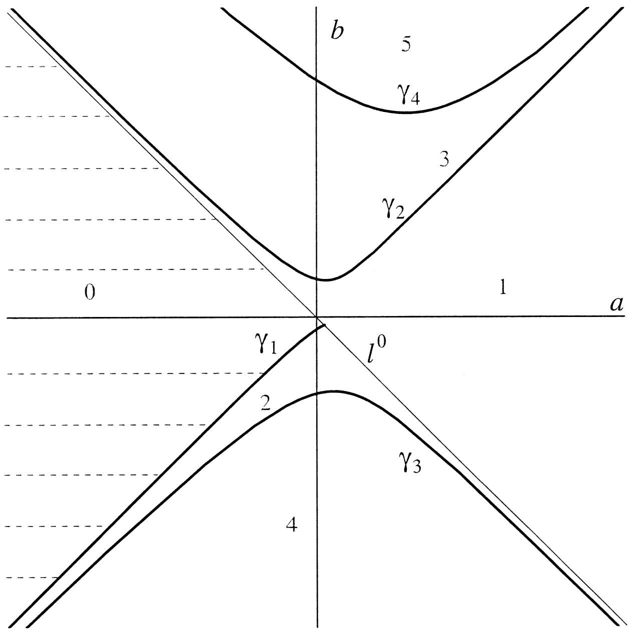

Figure 3. – Example of root locus for the Gaussian collocation method at 2 points ($s=2$ and $m=2$).

lowing relation:

$$b_m\left(\frac{(2l'-1)\,\pi}{2m}\right) > b_m\left(\frac{(2l''-1)\,\pi}{2m}\right), \qquad \text{if } l'' > l'.$$

This induces an useful ordering relation for the curves $\{\gamma_{2l-1}\}$, $l=1, 2, \ldots$.

As a consequence, we are able to determine the actual boundary of the numerical asymptotic stability regions, which is

$$\partial\Sigma_m = l^0 \cup \gamma_1 .$$

According to this, the dashed region in Figure 3, which refers to the case $s=2$, identifies Σ_2.

5. – Conclusions.

In this work we have provided an analytic proof of $\tau(0)$-stability of Gaussian collocation methods for DDEs. Although such property has also been proved by means of geometric properties of order stars [GH99], the approach considered here establishes an interesting link between stability and certain results for hypergeometric series (involved in the diagonal Padè approximations to the exponential).

REFERENCES

[Bar75] V. K. Barwell, *Special stability problems for functional differential equations*, BIT, **15** (1975), 130-135.

[BC63] R. Bellmann and K. L. Cooke, *Differential difference equations*, Academic Press, New York, 1963.

[Bel84] A. Bellen, *One-step collocation for delay differential equations*, J. Comput. Appl. Math., **10** (1984), 275-283.

[DKS86] K. Dekker, J. F. B. M. Kraaijevanger and M. N. Spijker, *The order of B-convergence of the Gaussian Runge-Kutta method*, Computing, **36** (1986), 35-41.

[GH99] N. Guglielmi and E. Hairer, *Order stars and stability for delay differential equations*, Numer. Math., **83**(3) (1999), 371-383.

[Gug97] N. Guglielmi, *On the asymptotic stability properties of Runge-Kutta methods for delay differential equations*, Numer. Math., **77**(4) (1997), 467-485.

[Hai82] E. Hairer, *Constructive characterization of A-stable approximations to* $\exp z$ *and its connection with algebraically stable Runge-Kutta methods*, Numer. Math, **39** (1982), 247-258.

[Hen74] P. Henrici, *Applied and Computational Complex Analysis*, vol. 1, Wiley, New York, 1974.

[HW96] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems*, Springer Series in Computational Mathematics, **14**, Springer-Verlag, Berlin, 2nd edition, 1996.

[IN91] A. Iserles and S. P. Nørsett, *Order Stars*, Chapman & Hall, London, 1991.

[Kua93] Y. Kuang, *Delay Differential Equations with Application in Population Dynamics*, Academic Press, Boston, 1993.

[Sla66] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, Cambridge, 1966.

[Zen85] M. Zennaro, *On the P-stability of one-step collocation for delay differential equations*, ISNM, **74** (1985), 334-343.

Dipartimento di Matematica Pura ed Applicata, Università dell'Aquila
via Vetoio (Coppito), I-67010 L'Aquila, Italy; e-mail: guglielm@univaq.it