Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 221–261.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_221_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 221-261

On the Canonical Ideal of a Set of Points.

MARTIN KREUZER

Sunto. – *Dato un insieme $\mathbb{X}$ di s punti nello spazio proiettivo, si costruisce un esplicito ideale canonico $\mathfrak{I}$ nel suo anello di coordinate R. Si descrivono le componenti omogenee di $\mathfrak{I}$ e la struttura della mappa di moltiplicazione $R_\sigma \otimes \mathfrak{I}_{\sigma+1} \to \mathfrak{I}_{2\sigma+1}$, dove $\sigma = \max\{i \mid H_{\mathbb{X}}(i) < s\}$. Tra le applicazioni ci sono varie caratterizzazioni di insiemi di punti coomologicamente uniformi, disuguaglianze nelle loro funzioni di Hilbert, il calcolo del primo modulo delle sizigie di $\mathfrak{I}$ in casi particolari, una generalizzazione della «trasformata di Gale» a trasformate canoniche di grado superiore e infine alcune osservazioni sui codici MDS.*

1. – Introduction.

Given a set of points $\mathbb{X} = \{P_1, \ldots, P_s\}$ in projective space $\mathbb{P}^d$ over an algebraically closed field K, we are interested in studying relations between the geometry of the configuration of the points and the algebraic structure of certain ideals and modules over the homogeneous coordinate ring $R = K[X_0, \ldots, X_d]/I_{\mathbb{X}}$ of $\mathbb{X}$. In three previous papers [6], [10], and [11], we saw that many geometric properties of $\mathbb{X}$ are encoded in the R-module structure of the *canonical module*

$$\omega_R \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-1)$$

of R, where we assume $\mathbb{X} \cap \mathcal{Z}(X_0) = \emptyset$ and let $x_0 = X_0 + I_{\mathbb{X}} \in R$.

The purpose of this paper is to refine those methods by noting that, for reduced 0-dimensional subschemes $\mathbb{X} \subseteq \mathbb{P}^d$ as above, there exists an ideal $\mathfrak{I}_{R/K[x_0]} \subseteq R$ which is – up to a shift in degrees – isomorphic to the canonical module. Such an ideal is called a *canonical ideal* of $\mathbb{X}$. In section 1 the ideal $\mathfrak{I}_{R/K[x_0]}$ is constructed in an analogous manner to the local case (cf. [9]), but with additional care taken to keep all maps and modules homogeneous and to make them completely explicit.

The canonical ideal of $\mathbb{X}$ constructed in this way depends on the choice of the element $x_0 \in R$, but in a very manageable fashion (cf. Prop. 1.8). It has the advantage of being contained in $R_{\geq \sigma_{\mathbb{X}}+1} = \bigoplus_{i \geq \sigma_{\mathbb{X}}+1} R_i$, where $\sigma_{\mathbb{X}} = \max\{i \in \mathbb{Z} \mid \dim_K(R_i) < s\}$. By [6], 1.13, this part of the ring R can be described precisely using the *separators* $f_1, \ldots, f_s \in R_{\sigma_{\mathbb{X}}+1}$ of $\mathbb{X}$: we have a K-basis for each homogeneous component R_i with $i \geqslant \sigma_{\mathbb{X}}+1$, and an explicit description of the multiplication of R in terms of those bases. This in turn allows us to completely describe the homogeneous components $\mathfrak{I}_{\sigma_{\mathbb{X}}+1}$ and $\mathfrak{I}_{2\sigma_{\mathbb{X}}+1}$ of $\mathfrak{I} = \mathfrak{I}_{R/K[x_0]}$ by constructing K-bases for them (cf. Cor. 1.10 and Cor. 1.11), and to compute the matrix of the multiplication map $R_{\sigma_{\mathbb{X}}} \otimes \mathfrak{I}_{\sigma_{\mathbb{X}}+1} \to \mathfrak{I}_{2\sigma_{\mathbb{X}}+1}$ (cf. Cor. 1.14).

In the remaining part of the paper we show several applications of the canonical ideal. The main application is the solution of a question about the Hilbert function of cohomologically uniform sets of points posed in [11]. This uniformity condition was introduced in [11] as an intermediate condition between 1-uniformity and $\Delta_{\mathbb{X}}$-uniformity. Here $\Delta_{\mathbb{X}} = H_{\mathbb{X}}(\sigma_{\mathbb{X}}+1) - H_{\mathbb{X}}(\sigma_{\mathbb{X}})$ is the last increase of the Hilbert function $H_{\mathbb{X}}(i) = \dim_K(R_i)$ of $\mathbb{X}$, and $\mathbb{X}$ is called n-uniform, if every subset $\mathbb{Y} \subseteq \mathbb{X}$ with $s-n \leqslant \#\mathbb{Y} \leqslant s$ satisfies $H_{\mathbb{Y}} = \min\{H_{\mathbb{X}}, \#\mathbb{Y}\}$. Cohomological uniformity is defined by the non-existence of a splitting $X = \mathbb{Y} \cup \mathbb{Y}'$ such that $\sum_{P_i \in \mathbb{Y}} K \cdot Lf_i \cap \sum_{P_j \in \mathbb{Y}'} K \cdot Lf_j = (0)$, where Lf_i is the image of f_i in $R/(x_0)$. We characterize cohomological uniformity in terms of the structure of the canonical ideal in the following way.

THEOREM 0.1. – *For a set of points $\mathbb{X} \subseteq \mathbb{P}^d$, the following conditions are equivalent.*

a) $\mathbb{X}$ is cohomologically uniform.

b) The multiplication map $R_{\sigma_{\mathbb{X}}} \otimes \mathfrak{I}_{\sigma_{\mathbb{X}}+1} \to \mathfrak{I}_{2\sigma_{\mathbb{X}}+1}$ is nondegenerate and surjective.

It is then an immediate consequence of [10], 2.6 and 3.1 that $\Delta_{\mathbb{X}}$-uniform points are cohomologically uniform and cohomologically uniform sets of points are 1-uniform.

At the end of section 2 we also relate cohomological uniformity in the case of $d+2 \leqslant s \leqslant \binom{d+2}{2}$ points to the condition that $\mathbb{X}$ does not *split linearly*, i.e. that there are no two linear subspaces $L_1, L_2 \subseteq \mathbb{P}^d$ such that $\mathbb{X} \subseteq L_1 \cup L_2$ and $L_1 \cap L_2 \neq \emptyset$ (cf. Prop. 2.7). We obtain another proof of the result of [2] which says in our situation that $\mathbb{X}$ does not split linearly if and only if the canonical ideal is generated by its elemets of degree two. An example shows that cohomological uniformity generalizes linear splitting in a suitable way to point sets with $\sigma_{\mathbb{X}} \geqslant 2$ (cf. Ex. 2.8).

The main result of section 3 is the following affirmative soultion of [11], Question, p. 248.

THEOREM 0.2. – *If* $\mathbb{X} \subseteq \mathbb{P}^d$ *is a cohomologically uniform set of points, then*

$$H_{\mathbb{X}}(n) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) \leq s - \Delta_{\mathbb{X}} + 1$$

for all $n \in \{0, \dots, \sigma_{\mathbb{X}}\}$.

In order to prove this theorem, we characterize cohomological uniformity of $\mathbb{X}$ by the existence of elements $r \in R_{\sigma_{\mathbb{X}}}$ and $\varphi \in \mathfrak{I}_{\sigma_{\mathbb{X}}+1}$ such that $\mathfrak{I}_{2\sigma_{\mathbb{X}}+1}$ is of the form

$$\mathfrak{I}_{2\sigma_{\mathbb{X}}+1} = R_{\sigma_{\mathbb{X}}} \varphi \oplus Kr\varphi_2 \oplus \dots \oplus Kr\varphi_{\Delta_{\mathbb{X}}}$$

for any K-basis $\{\varphi, \varphi_2, \dots, \varphi_{\Delta_{\mathbb{X}}}\}$ of $\mathfrak{I}_{\sigma_{\mathbb{X}}+1}$ (cf. Prop. 3.9). The core part of the proof is to show that one can in fact choose $r = l^{\sigma_{\mathbb{X}}}$ for a generic element $l \in R_1$ in this characterization (cf. Prop. 3.12). Then the desired inequalities follow easily (cf. Cor. 3.3). For a discussion of the meaning of Theorem 0.2 and for examples we refer the reader to [11].

In the fourth section we apply our knowledge of the canonical ideal in the case of $d + 1 < s < \binom{d+2}{2}$ cohomologically uniform points in order to give a description of its first syzygy module. Both for the syzygy module of $\mathfrak{I}$, considered as a $P = K[X_0, \dots, X_d]$-module, and of $\mathfrak{I}$, considered as an R-module, we provide explicit homogeneous systems of generators (cf. Lemma 4.2.b and Prop. 4.3), and we show how one can compute those elements effectively (cf. Rem. 4.7). In the case of $s = d + 3 \geq 6$ cohomologically uniform points in $\mathbb{P}^d$, we prove that the R-syzygy module of $\mathfrak{I}$ is generated by its homogeneous elements of lowest degree (cf. Prop. 4.4), and we relate this to the analogous result of [3] for the P-syzygy module of $\mathfrak{I}$ (cf. Rem. 4.5).

Finally, in section 5, we use the homogeneous components of the canonical ideal to generalize the Gale transform of a set of s points $\mathbb{X}$ (see for instance [5]). We show that the Gale transform can be defined explicitly by representing a K-basis of $\mathfrak{I}_{2\sigma}$ in terms of the separators, and that it consists of s distinct points, if and only if every subset of $s-2$ points of $\mathbb{X}$ spans $\mathbb{P}^d$ (cf. Prop. 5.2). More generally, we can use the homogeneous components $\mathfrak{I}_{2\sigma+1-i}$ for $i = 1, \dots, \sigma$ to define higher «canonical transforms» $\kappa_i(\mathbb{X})$ of $\mathbb{X}$. After describing those components explicitly (cf. Prop. 5.1), we formulate again the precise uniformity condition on $\mathbb{X}$ to ensure that $\kappa_i(\mathbb{X})$ consists of s distinct points. We end the paper by connecting the uniformity of $\mathbb{X}$ to the uniformity of $\kappa_\sigma(\mathbb{X})$ (cf. Prop. 5.10), and by interpreting this result in the language of Coding Theory (cf. Cor. 5.11).

Acknowledgements. Mayor parts of this paper were done while the author was enjoying the hospitality of the Departments of Mathematics of Queen's

University (Kingston, Canada) and Universitá di Genova (Italy). He wishes to express his deep gratitude to those institutions, and in particular to A. V. Geramita (Kingston) and L. Robbiano (Genova). He is also thankful to E. Kunz (Regensburg) for his continuing interest in the subject and for providing the author with the repeated opportunity to lecture about it in the Oberseminar.

1. – Construction of the canonical ideal.

After fixing notations, we describe how to construct the canonical ideal of the projective coordinate ring of a set of points. We follow the procedure outlined in [9], 2. Vortrag, with two major differences: we are dealing with a graded situation, and we want explicit descriptions of the homogeneous components of the canonical ideal.

Throughout this paper we work over an algebraically closed field K of arbitrary characteristic. Our central object of interest is a given set of points $\mathbb{X} = \{P_1, \ldots, P_s\}$ in $\mathbb{P}^d$, the d-dimensional projective space over K. The coordinate functions $\{X_0, \ldots, X_d\}$ of $\mathbb{P}^d$ are always chosen in such a manner that $\mathbb{X} \subseteq D_+(X_0)$, i.e. such that no point of $\mathbb{X}$ lies on the hyperplane $\mathcal{Z}(X_0)$. The projective coordinate ring of $\mathbb{X}$ in $\mathbb{P}^d$ is $R = K[X_0, \ldots, X_d]/I_{\mathbb{X}}$, where $I_{\mathbb{X}}$ denotes the homogeneous saturated ideal of $\mathbb{X}$.

Let us collect a few elementary observations. The ring $R = \bigoplus_{n \geq 0} R_n$ is a 1-dimensional reduced Cohen-Macaulay K-algebra, and $x_0 := X_0 + I_{\mathbb{X}} \in R_1$ is not a zerodivisor of R. The Hilbert function $H_{\mathbb{X}}: \mathbb{Z} \to \mathbb{N}$ $(n \mapsto \dim_K R_n)$ of $\mathbb{X}$ satisfies

$$\ldots = 0 = H_{\mathbb{X}}(-1) < 1 = H_{\mathbb{X}}(0) < \ldots < H_{\mathbb{X}}(\sigma_{\mathbb{X}}) < s = H_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1) = H_{\mathbb{X}}(\sigma_{\mathbb{X}} + 2) = \ldots$$

for some number $\sigma_{\mathbb{X}} \geq -1$. Its first difference function $\Delta H_{\mathbb{X}}(n) := H_{\mathbb{X}}(n) - H_{\mathbb{X}}(n-1)$ therefore has a nonzero value $\Delta H_{\mathbb{X}}(n) \neq 0$ if and only if $0 \leq n \leq \sigma_{\mathbb{X}} + 1$. We let $\Delta_{\mathbb{X}} := \Delta H_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1)$ and write simply $\sigma = \sigma_{\mathbb{X}}$ and $\Delta = \Delta_{\mathbb{X}}$, if no confusion can arise.

As in [6] and [10], for $i \in \{1, \ldots, s\}$ we denote by $f_i \in R_{\sigma+1}$ a *separator* of P_i in $\mathbb{X}$, i.e. a function such that $f_i(P_j) = 0$ for $j \neq i$ and $f_i(P_i) \neq 0$. Here we let $f(P) := f(1, p_1, \ldots, p_d)$, if $f \in R$ is a homogeneous element and $P = (1 : p_1 : \ldots : p_d) \in D_+(X_0)$ a closed point. We normalize $\{f_1, \ldots, f_s\}$ by requiring $f_i(P_i) = 1$ for $i = 1, \ldots, s$. Now [6], 1.13, says that $\{x_0^n f_1, \ldots, x_0^n f_s\}$ is a K-basis of $R_{\sigma+1+n}$ for each $n \geq 0$, and [6], 3.2.a, shows that the multiplication in R satisfies $rf_i = r(P_i)\, x_0^n f_i$ for $i = 1, \ldots, s$ and $r \in R_n$.

LEMMA 1.1. – *A homogeneous element $r \in R_n$ is not a zerodivisor of R if and only if $r(P_i) \neq 0$ for $i = 1, \ldots, s$.*

PROOF. – If $n \geq \sigma_{\mathbb{X}} + 1$, we have $r = r(P_1)\, x_0^{n-\sigma-1} f_1 + \ldots + r(P_s)\, x_0^{n-\sigma-1} f_s$, and if $n \leq \sigma_{\mathbb{X}}$, then $r x_0^{\sigma+1-n} = r(P_1)\, f_1 + \ldots + r(P_s)\, f_s$. In any case, $r(P_i) = 0$ implies $r f_i = 0$, i.e. r is a zerodivisor of R. The converse follows from $r f_i = r(P_i)\, x_0^n\, f_i$. ■

The image of f_i in $\overline{R} := R/(x_0)$ is denoted by Lf_i and is called the *leading form* of f_i ($i = 1, \ldots, s$). By [6], 2.13, we can renumber $\{P_1, \ldots, P_s\}$ in such a way that $\{Lf_1, \ldots, Lf_\Delta\}$ is a K-basis of $\overline{R}_{\sigma+1}$. Then we write

$$Lf_{\Delta+j} = \beta_{j1} Lf_1 + \ldots + \beta_{j\Delta} Lf_\Delta$$

for $j = 1, \ldots, s - \Delta$, and we form the matrix $\mathfrak{B} := (\beta_{ji})^{\mathrm{transp}}$.

LEMMA 1.2. – *a) For $i = 1, \ldots, \Delta$ we have $\beta_{1i} + \ldots + \beta_{s-\Delta i} = -1$.*

b) The elements $g_1, \ldots, g_{s-\Delta} \in R_\sigma$ which satisfy

$$x_0 g_i = f_{\Delta+i} - \beta_{i1} f_1 - \ldots - \beta_{i\Delta} f_\Delta$$

form a K-basis of R_σ.

PROOF. – Part *a*) follows, if we consider the equation $f_1 + \ldots + f_s = x_0^{\sigma+1}$ and pass to $\overline{R}$. Part *b*) follows from the observation that the elements $f_{\Delta+i} - \beta_{i1} f_1 - \ldots - \beta_{i\Delta} f_\Delta$ with $i = 1, \ldots, s - \Delta$ are linearly independent elements of $x_0 R_\sigma$. ■

Notice that $\mathfrak{B}$ depends on the numbering of $\{P_1, \ldots, P_s\}$ in an obvious way. The dependency of $\mathfrak{B}$ on the choice of x_0 is governed by the following rule.

LEMMA 1.3. – *Let $l \in R_1$ be a nonzerodivisor, and let β_{ji}^l be constructed as above, but using l instead of x_0. Then*

$$\beta_{ji}^l = \frac{l(P_{\Delta+j})^\sigma}{l(P_i)^\sigma} \cdot \beta_{ji}$$

for $i = 1, \ldots, \Delta$ and $j = 1, \ldots, s - \Delta$.

PROOF. – If $f_1^l, \ldots, f_s^l$ denote the separators w.r.t. l, then $f_1^l + \ldots + f_s^l = l^{\sigma+1}$ yields $f_i^l = l(P_i)^{\sigma+1} f_i$ for $i = 1, \ldots, s$. Now notice that $f_{\Delta+j}^l - (l(P_{\Delta+j})^\sigma / l(P_1)^\sigma)\, \beta_{j1} f_1^l - \ldots - (l(P_{\Delta+j})^\sigma / l(P_\Delta)^\sigma)\, \beta_{j\Delta} f_\Delta^l = l(P_{\Delta+j})^{\sigma+1} f_{\Delta+j} - l(P_{\Delta+j})^\sigma\, l(P_1)\, \beta_{j1} f_1 - \ldots - l(P_{\Delta+j})^\sigma\, l(P_\Delta)\, \beta_{j\Delta} f_\Delta = l(P_{\Delta+j})^\sigma\, l g_j$ is an element of lR_σ. ■

At this point we are ready to start with our construction of the canonical ideal. Recall that the integral closure $\widetilde{R}$ of R in its total ring of quotients can be described as follows (cf. [13] or [6]):

$$\widetilde{R} \cong R/\mathfrak{p}_1 \times \ldots \times R/\mathfrak{p}_s \cong K[T_1] \times \ldots \times K[T_s]$$

where $\mathfrak{p}_i$ is the homogeneous ideal of P_i in $\mathbb{X}$, and T_i is the image of x_0 in $R/\mathfrak{p}_i$. Using the above notations, the canonical map $\iota\colon R \hookrightarrow \widetilde{R}$ satisfies $\iota(r) = (r(P_1)T_1^n, \ldots, r(P_s)\,T_s^n)$ for $r \in R_n$. In particular, $\iota(x_0) = (T_1, \ldots, T_s)$ and $\iota(f_i) = (0, \ldots, 0, T_i^{\sigma+1}, 0, \ldots, 0)$. Since $\widetilde{R}$ is graded by $\deg T_i = 1$, the map ι is homogeneous of degree zero.

From this description we find that the full ring of quotients $Q(R)$ of R can be identified with $Q(R) \cong K(T_1) \times \ldots \times K(T_s)$. In the next step we compute $Q^h(R)$, the homogeneous ring of quotients of R, i.e. the localization of R w.r.t. the set of all homogeneous nonzerodivisors.

PROPOSITION 1.4. – *a) Using the above identifications, we have*

$$Q^h(R) \cong K[T_1, T_1^{-1}] \times \ldots \times K[T_s, T_s^{-1}].$$

Here an element $f/g \in Q^h(R)$ with $f \in R_m$ and a nonzerodivisor $g \in R_n$ is identified with the tuple

$$\left(\frac{f(P_1)}{g(P_1)}\, T_1^{m-n}, \ldots, \frac{f(P_s)}{g(P_s)}\, T_s^{m-n}\right).$$

b) We have $Q^h(R) \cong R_{x_0}$.

PROOF. – Let $f/g \in Q^h(R)$ be as in *a*). The image of f in $Q(R)$ is $(f(P_1)\,T_1^m, \ldots, f(P_s)\,T_s^m)$, and the image of g in $Q(R)$ is $(g(P_1)\,T_1^n, \ldots, g(P_s)\,T_s^n)$. Therefore f/g is identified in $Q(R)$ as claimed and is contained in $K[T_1, T_1^{-1}] \times \ldots \times K[T_s, T_s^{-1}]$.

Conversely, for any element $(g_1, \ldots, g_s) \in K[T_1, T_1^{-1}] \times \ldots \times K[T_s, T_s^{-1}]$ and any $n \gg 0$, we have $(T_1^n g_1, \ldots, T_s^n g_s) = \iota(f)$ for some $f \in R$, because $\dim_K R_m = s = \dim_K (K[T_1, T_1^{-1}] \times \ldots \times K[T_s, T_s^{-1}])_m$ for $m \gg 0$. Thus $(g_1, \ldots, g_s)$ is the image of $f/x_0^n \in Q^h(R)$, proving *a*) and *b*). ■

In what follows, we let $L_0 := K[x_0, x_0^{-1}]$ and $L := Q^h(R)$. From 1.4 *b*) we conclude that $L = L_0 \bigotimes_{K[x_0]} R$. Since R is a Cohen-Macaulay ring, it is a free $K[x_0]$-module of rank s. Therefore L is a free L_0-module of rank s, and there is an L_0-basis $\{e_1, \ldots, e_s\}$ of L which is identified with the standard basis of $K[T_1, T_1^{-1}] \times \ldots \times K[T_s, T_s^{-1}]$ under the isomorphism of 1.4 *a*). Let $\sigma_{L/L_0}\colon L \to L_0$ $(e_i \mapsto 1)$ be the *canonical trace map*.

PROPOSITION 1.5. – *The homomorphism* $\Sigma\colon L \to \underline{\mathrm{Hom}}_{L_0}(L, L_0)$ $(1 \mapsto \sigma_{L/L_0})$ *is an isomorphism of graded L-modules.*

PROOF. – Let $\varphi\colon L \to L_0$ be an L_0-linear map and $g_i(x_0, x_0^{-1}) := \varphi(e_i)$ for $i = 1, \ldots, s$. Then $\varphi = (g_1(T_1, T_1^{-1}), \ldots, g_s(T_s, T_s^{-1}))\cdot\sigma_{L/L_0}$ shows that Σ is surjective. The map Σ is also injective, since from $\psi = (g_1(T_1, T_1^{-1}), \ldots, g_s(T_s, T_s^{-1}))\cdot\sigma_{L/L_0} = 0$ we obtain $0 = \psi(e_i) = g_i(x_0, x_0^{-1})$ for $i = 1, \ldots, s$, and hence $(g_1(T_1, T_1^{-1}), \ldots, g_s(T_s, T_s^{-1})) = 0$. ■

DEFINITION. – Now we consider the following homomorphism of graded R-modules

$$\Phi\colon \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \hookrightarrow \underline{\mathrm{Hom}}_{L_0}(L, L_0) \xrightarrow{\Sigma^{-1}} L\,,$$

$$\phi \mapsto \varphi \otimes \mathrm{id}_{L_0}\,.$$

Its image is a homogeneous fractional R-ideal $\mathfrak{C}_{R/K[x_0]}$ of L which is called the *Dedekind complementary module* of R with respect to x_0.

It is not difficult to see (cf. [6], 3.1) that the Hilbert function of $\mathfrak{C} := \mathfrak{C}_{R/K[x_0]} \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$ is given by $H_{\mathfrak{C}}(n) = s - H_{\mathbb{X}}(-n-1)$ for all $n \in \mathbb{Z}$.

PROPOSITION 1.6. – (*Explicit description of* $\mathfrak{C}_{R/K[x_0]}$).
Let $n \geq 0$, *and let* $\varphi \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-\sigma-1+n}$. *For* $i = 1, \ldots, s$ *we write* $\varphi(f_i) = c_i x_0^n$ *with* $c_i \in K$. *Then*

$$\Phi(\varphi) = (c_1 T_1^{-\sigma-1+n}, \ldots, c_s T_s^{-\sigma-1+n}) \in L\,.$$

PROOF. – Since $\iota\colon R \hookrightarrow L$ satisfies $\iota(f_i) = (0, \ldots, 0, T_i^{\sigma+1}, 0, \ldots, 0)$, the map $\varphi \otimes \mathrm{id}_{L_0}\colon L \cong R \bigotimes_{K[x_0]} L_0 \to L_0$ is given by $(\varphi \otimes \mathrm{id}_{L_0})(x_0^{\sigma+1} e_i) = (\varphi \otimes \mathrm{id}_{L_0})((0, \ldots, 0, T_i^{\sigma+1}, 0, \ldots, 0)) = \varphi(f_i) = c_i x_0^n$. Therefore we have $\varphi \otimes \mathrm{id}_{L_0} = x_0^{-\sigma-1+n}(c_1, \ldots, c_s)\,\sigma_{L/L_0}$ in $\underline{\mathrm{Hom}}_{L_0}(L, L_0)$. Thus the claim follows from the description of Σ^{-1} given in the proof of 1.5. ■

COROLLARY 1.7. – *The set* $x_0^{2\sigma+2}\cdot\mathfrak{C}_{R/K[x_0]} \subseteq L$ *is an ideal of R.*

PROOF. – Clearly this set is an R-submodule of L. We shall show that it is contained in the image of R in L. Let $\varphi \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-\sigma-1+n}$ with

$n \geqslant 0$, and let $\varphi(f_i) = c_i x_0^n$ with $c_i \in K$ for $i = 1, \ldots, s$. Then we have

$$x_0^{2\sigma+2}\Phi(\varphi) = x_0^{2\sigma+2}\cdot(c_1 T_1^{-\sigma-1+n}, \ldots, c_s T_s^{-\sigma-1+n}) =$$

$$(c_1 T_1^{\sigma+1+n}, \ldots, c_s T_s^{\sigma+1+n}),$$

and this element is the image of $x_0^n(c_1 f_1 + \ldots + c_s f_s) \in R$ in L. ■

DEFINITION. – The ideal $\mathfrak{I}_{R/K[x_0]} := x_0^{2\sigma+2}\cdot\mathfrak{C}_{R/K[x_0]}$ of R is called the *canonical ideal* of $\mathbb{X}$ with respect to x_0. Since the choice of x_0 will remain fixed, we shall also denote this ideal by $\mathfrak{I} := \mathfrak{I}_{R/K[x_0]}$.

Later we shall show that, in general, $\sigma_{\mathbb{X}} + 2$ is the smallest number n such that $x_0^n\cdot\mathfrak{C}_{R/K[x_0]}$ is contained in R, but in special cases also smaller numbers n can suffice for this to be true. It is also clear that $\mathfrak{I}_{R/K[x_0]}$ depends on the choice of the linear nonzerodivisor x_0 of R. The next proposition makes this dependency explicit.

PROPOSITION 1.8. – *If $l \in R_1$ is a nonzerodivisor of R, then $x_0\cdot\mathfrak{C}_{R/K[x_0]} = l\cdot\mathfrak{C}_{R/K[l]}$ and $l^{2\sigma+1}\cdot\mathfrak{I}_{R/K[x_0]} = x_0^{2\sigma+1}\cdot\mathfrak{I}_{R/K[l]} \subseteq R$.*

In other words, for an element $f = x_0^n(c_1 f_1 + \ldots + c_s f_s)$ of $R_{\sigma+1+n}$, with $c_i \in K$ and $n \geqslant 0$, we have $f \in \mathfrak{I}_{R/K[l]}$ if and only if $x_0^n(l(P_1)^{-2\sigma-1}c_1 f_1 + \ldots + l(P_s)^{-2\sigma-1}c_s f_s) \in \mathfrak{I}_{R/K[x_0]}$.

PROOF. – In view of the definition, it suffices to prove the first claim. Because of symmetry reasons, we only show $x_0\cdot\mathfrak{C}_{R/K[x_0]} \subseteq l\cdot\mathfrak{C}_{R/K[l]}$. Let $\varphi\colon R \to K[x_0]$ be a homogeneous $K[x_0]$-linear map of degree $-\sigma_{\mathbb{X}} - 1 + n$ with $n \geqslant 0$, and let $\varphi(f_i) = c_i x_0^n$ with $c_i \in K$ for $i = 1, \ldots, s$. Since $\varphi(R_{\sigma-n}) \subseteq K[x_0]_{-1} = (0)$, we have $\lambda_1 c_1 + \ldots + \lambda_s c_s = 0$ whenever $\lambda_1, \ldots, \lambda_s \in K$ and $\lambda_1 f_1 + \ldots + \lambda_s f_s = x_0^{n+1}g$ for some $g \in R_{\sigma-n}$. By 1.6, we have $\Phi(x_0\varphi) = (c_1 T_1^{-\sigma+n}, \ldots, c_s T_s^{-\sigma+n})$ in L.

Now let U_i be the image of l in $R/\mathfrak{p}_i$, and consider the representation $L = K[U_1, U_1^{-1}] \times \ldots \times K[U_s, U_s^{-1}]$. Since $U_i = l(P_i)\,T_i$ for $i = 1, \ldots, s$, the element $\Phi(x_0\varphi)$ is given by $(c_1 l(P_1)^{\sigma-n}U_1^{-\sigma+n}, \ldots, c_s l(P_s)^{\sigma-n}U_s^{-\sigma+n})$ in this representation.

Next we let $\psi\colon R \to K[l, l^{-1}]$ be the homogeneous $K[l]$-linear map of degree $-\sigma_{\mathbb{X}} - 1 + n$ such that $\psi(f_i^{(l)}) = c_i l(P_i)^{\sigma-n} l^n$ for $i = 1, \ldots, s$. Here $f_1^{(l)}, \ldots, f_s^{(l)}$ are the normalized separators with respect to l. Using 1.6 again, we see that $(c_1 l(P_1)^{\sigma-n}U_1^{-\sigma+n}, \ldots, c_s l(P_s)^{\sigma-n}U_s^{-\sigma+n})$ is also the image of $l\psi$ in L.

Therefore it suffices to show $im\psi \subseteq K[l]$. For this we need to prove $\psi(R_{\sigma-n}) = 0$. Let $g \in R_{\sigma-n}$, and let $x_0^{n+1}g = \lambda_1 f_1 + \ldots + \lambda_s f_s$ with $\lambda_1, \ldots, \lambda_s \in K$. By the proof of 1.3 we know $f_i^{(l)} = l(P_i)^{\sigma+1}f_i$ for $i = 1, \ldots, s$. Therefore we have $l^{n+1}g = \lambda_1 l(P_1)^{n+1}f_1 + \ldots + \lambda_s l(P_s)^{n+1}f_s = \lambda_1 l(P_1)^{-\sigma+n}f_1^{(l)} +$

$\ldots + \lambda_s l(P_s)^{-\sigma+n} f_s^{(l)}$. Thus we obtain $l^{n+1}\psi(g) = \psi(l^{n+1}g) = \lambda_1 l(P_1)^{-\sigma+n} \cdot \psi(f_1^{(l)}) + \ldots + \lambda_s l(P_s)^{-\sigma+n}\psi(f_s^{(l)}) = l^n(\lambda_1 c_1 + \ldots + \lambda_s c_s) = 0$. Altogether we get $\psi(g) = 0$, as desired. ■

The explanation why $\mathfrak{J}_{R/K[x_0]}$ is important for the study of $\mathbb{X}$ and why it is called the canonical ideal of $\mathbb{X}$ is provided by the following proposition and its corollaries. Recall that the graded R-module $\omega_R = \underline{\mathrm{Hom}}_K(H^1_{\mathfrak{m}}(R), K) \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-1)$ is the *canonical module* of R and can be used to characterize many geometric properties of $\mathbb{X}$ (cf. [10]).

PROPOSITION 1.9. – *There are isomorphisms of graded R-modules*

$$\omega_R \cong \mathfrak{C}_{R/K[x_0]}(-1) \cong \mathfrak{J}_{R/K[x_0]}(2\sigma_{\mathbb{X}} + 1).$$

Here an element $\varphi \in (\omega_R)_{-\sigma+n} = \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-\sigma-1+n}$ with $n \geqslant 0$ is identified with the element $\varphi(f_1) f_1 + \ldots + \varphi(f_s) f_s$ of $\mathfrak{J}_{\sigma+1+n}$.

PROOF. – Since Φ is a monomorphism of graded R-modules and $\mathfrak{J} = x_0^{2\sigma+2}\mathfrak{C}_{R/K[x_0]} \cong \mathfrak{C}_{R/K[x_0]}(-2\sigma_{\mathbb{X}} - 2)$, the first claim is clear. For the second claim, let $\varphi \in (\omega_R)_{-\sigma+n}$ be given by $\varphi(f_i) = c_i x_0^n$, with $n \geqslant 0$ and $c_i \in K$ for $i = 1, \ldots, s$. Then multiplication by $x_0^{2\sigma+2}$ provides an isomorphism $\mathfrak{C}_{R/K[x_0]}(-1) \cong \mathfrak{C}_{R/K[x_0]}(2\sigma_{\mathbb{X}} + 1)$ which identifies $\Phi(\varphi) = (c_1 T_1^{-\sigma-1+n}, \ldots, c_s T_s^{-\sigma-1+n})$ with $x_0^n(c_1 f_1 + \ldots + c_s f_s) = \varphi(f_1) f_1 + \ldots + \varphi(f_s) f_s$. ■

As a first application, we can use the isomorphism of this proposition to give explicit descriptions of $\mathfrak{J}_{\sigma+1} \cong (\omega_R)_{-\sigma}$ and $\mathfrak{J}_{2\sigma+1} \cong (\omega_R)_0$. From what we mentioned earlier, we know the Hilbert funtion of $\mathfrak{J}$, namely

$$H_{\mathfrak{J}}(n) = s - H_{\mathbb{X}}(2\sigma_{\mathbb{X}} + 1 - n)$$

for all $n \in \mathbb{Z}$. Thus $\mathfrak{J}_{\sigma+1}$ and $\mathfrak{J}_{2\sigma+1}$ are the first and last nontrivial homogeneous components of $\mathfrak{J}$. In section 5 we shall also give a (somewhat less explicit) description of the remaining nontrivial homogeneous components of $\mathfrak{J}$.

COROLLARY 1.10. – *The elements $\pi_i := f_i + \beta_{1i} f_{\Delta+1} + \ldots + \beta_{s-\Delta i} f_s$ such that $1 \leqslant i \leqslant \Delta$ form a K-basis of $\mathfrak{J}_{\sigma+1}$.*

SINCE $\{Lf_1, \ldots, Lf_\Delta\}$ is a K-basis of $\overline{R}_{\sigma+1}$, the projections $\pi_1, \ldots, \pi_\Delta$ defined by $\pi_i(f_j) := \delta_{ij}$ for $j = 1, \ldots, \Delta$ form a K-basis of $(\omega_R)_{-\sigma} = \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-\sigma-1}$ (cf. [10], 1.5). For $j = 1, \ldots, s - \Delta$ this yields $\pi_i(f_{\Delta+j}) = \pi_i(\beta_{j1} f_1 + \ldots + \beta_{j\Delta} f_\Delta) = \beta_{ij}$. Now an application of 1.9 finishes the proof. ■

COROLLARY 1.11. – *The elements* $h_i := x_0^\sigma(f_{i+1} - f_i)$ *with* $i = 1, \ldots, s-1$ *form a* K*-basis of* $\mathfrak{J}_{2\sigma+1}$. *An element* $x_0^\sigma(c_1 f_1 + \ldots + c_s f_s)$ *of* $R_{2\sigma+1}$ *is contained in* $\mathfrak{J}_{2\sigma+1}$ *if and only if* $c_1 + \ldots + c_s = 0$.

PROOF. – An element φ of $(\omega_R)_0 \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-1}$ yields a K-linear map $\varphi|_{R_{\sigma+1}}: R_{\sigma+1} \to K[x_0]_\sigma$ with the property $(\varphi|_{R_{\sigma+1}})(x_0^{\sigma+1}) = x_0^{\sigma+1}\varphi(1) \in x_0^{\sigma+1}K[x_0]_{-1} = (0)$. Conversely, a K-linear map $\psi: R_{\sigma+1} \to K[x_0]_\sigma$ extends to a $K[x_0]$-linear homogeneous map $\overline{\psi}: R \to K[x_0]$ of degree -1 if and only if $\psi(x_0^{\sigma+1}) = 0$, because $\dim_K(\omega_R)_0 = s-1$ and $\overline{\psi}$ is uniquely determined by $\overline{\psi}|_{R_{\sigma+1}}$ (cf. [10], 1.4). Now $x_0^{\sigma+1} = f_1 + \ldots + f_s$ implies $\varphi(f_1) + \ldots + \varphi(f_s) = 0$ for $\varphi \in (\omega_R)_0$, and the claims of the corollary follow from Proposition 1.9. ■

Using Corollary 1.10, we can now give the promised examples which show that, in general, one has to multiply $\mathfrak{J}_{R/K[x_0]}$ by $x_0^{2\sigma+2}$ in order to get an ideal of R, but in special cases a lower power of x_0 may suffice.

EXAMPLE 1.12. – Let char $K \neq 2, 3$, and let $\mathbb{X} = \{(1:0:0), (1:1:0), (1:2:0)\} \subseteq \mathbb{P}^2$. Then $\mathbb{X}$ is a complete intersection of type $(3, 1)$, i.e. $I_{\mathbb{X}} = (X_2, X_1(X_1 - X_0)(X_1 - 2X_0))$, its Hilbert function is given by $\Delta H_{\mathbb{X}}$: $1\ 1\ 1\ 0 \ldots$, and $\sigma_{\mathbb{X}} = 1$. It is easy to compute $f_1 = (1/2)(x_1 - x_0)(x_1 - 2x_0)$, $f_2 = -x_1(x_1 - 2x_0)$, and $f_3 = (1/2)\, x_1(x_1 - x_0)$ (cf. [6], 1.15). Thus $Lf_2 = \beta_{11} Lf_1$ with $\beta_{11} = -2$, and $Lf_3 = \beta_{21} Lf_1$ with $\beta_{21} = 1$. By Corollary 1.10, the element $\pi_1 = f_1 - 2f_2 + f_3$ is a K-basis of $\mathfrak{J}_{\sigma+1}$. Since $\mathbb{X}$ is a complete intersection, $\{\pi_1\}$ is even an R-basis of $\mathfrak{J}$. Clearly, $\pi_1 \notin x_0 R_\sigma$, so that in this case $x_0^{2\sigma+1}\mathfrak{C}_{R/K[x_0]} \nsubseteq R$.

EXAMPLE 1.13. – Let $K := \mathbb{C}$, and let $\mathbb{X} := \{(1:0:0), (1:2:0), (1: 1 + \sqrt{3}\, i : 0)\} \subseteq \mathbb{P}^2$. Again $\mathbb{X}$ is a complete intersection of type $(3, 1)$, its Hilbert function is $\Delta H_{\mathbb{X}}$: $1\ 1\ 1\ 0 \ldots$, and $\sigma_{\mathbb{X}} = 1$. This time we find

$$f_1 = (1/(2 + 2\sqrt{3}\, i))(x_1 - 2x_0)(x_1 - x_0 - \sqrt{3}\, i x_0),$$

$$f_2 = (1/(2 - 2\sqrt{3}\, i)\, x_1(x_1 - x_0 - \sqrt{3}\, i x_0), \quad \text{and} \quad f_3 = -(1/4)\, x_1(x_1 - 2x_0).$$

Thus $Lf_2 = \beta_{11} Lf_1$ with $\beta_{11} = (1/2)(\sqrt{3}\, i - 1)$ and $Lf_3 = \beta_{21} Lf_1$ with $\beta_{21} = -(1/2)(\sqrt{3}\, i + 1)$. Therefore the R-basis $\pi_1 = f_1 + \beta_{11} f_2 + \beta_{21} f_3$ of $\mathfrak{J}_{R/K[x_0]}$ satisfies $L\pi_1 = Lf_1 + \beta_{11} Lf_2 + \beta_{21} Lf_3 = 0$, i.e. we have $\pi_1 \in x_0 R_\sigma$. This means that we could have taken $x_0^{2\sigma+1}\mathfrak{C}_{R/K[x_0]} = x_0^3\mathfrak{C}_{R/K[x_0]}$ and obtained an ideal of R.

Let us also show that $x_0^2\mathfrak{C}_{R/K[x_0]} \nsubseteq R$ here. This fractional ideal starts in degree zero, so that we want to show $\pi_1 \notin (x_0^2)$. But $\pi_1 = \lambda x_0^2 = \lambda f_1 + \lambda f_2 + \lambda f_3$ for some $\lambda \in K$ implies $\beta_{11} = \beta_{21} = 1$, which is not the case.

In [10], 2.6 and 3.1, the multiplication map $R_\sigma \otimes \mathfrak{J}_{\sigma+1} \to \mathfrak{J}_{2\sigma+1}$ has been used to characterize geometrical properties of $\mathbb{X}$. By applying 1.10 and 1.11, we shall now give an explicit matrix for this map.

COROLLARY 1.14. – *Let* $\{g_1, \ldots, g_{s-\Delta}\}$, $\{\pi_1, \ldots, \pi_\Delta\}$, *and* $\{h_1, \ldots, h_{s-1}\}$ *be as in* 1.2.b, 1.10, *and* 1.11, *resp. Then the multiplication map* $R_\sigma \otimes \mathfrak{I}_{\sigma+1} \to \mathfrak{I}_{2\sigma+1}$ *is given by*

$$g_j \cdot \pi_i = \begin{cases} \beta_{j1} h_{\Delta+j-1} & \text{for } i=1 \text{ and } j=1, \ldots, s-\Delta\,, \\ \beta_{ji} h_{\Delta+j-1} - \beta_{ji} h_{i-1} & \text{for } i=2, \ldots, \Delta \text{ and } j=1, \ldots, s-\Delta\,. \end{cases}$$

PROOF. – For $i \in \{1, \ldots, \Delta\}$ and $j \in \{1, \ldots, s-\Delta\}$ we have

$$x_0 g_j \pi_i = (f_{\Delta+j} - \beta_{j1} f_1 - \ldots - \beta_{j\Delta} f_\Delta) \cdot (f_1 + \beta_{\Delta+1i} f_{\Delta+1} + \ldots + \beta_{s-\Delta i} f_{s-\Delta}) = x_0^\sigma (\beta_{ji} f_{\Delta+j} - \beta_{ji} f_i)\,.$$

From this the claim follows immediately. ■

Finally, we remind the reader that in complete analogy with [9], 6.13, one can show the following proposition in our situation.

PROPOSITION 1.15. – *The ring* $R/\mathfrak{I}$ *is a* 0*-dimensional Gorenstein ring.*

2. – Canonical ideals of cohomologically uniform schemes.

In this section we shall apply our knowledge of the canonical ideal in order to study the property of cohomological uniformity introduced in [11]. Recall that a 0-dimensional scheme $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ is called *n-uniform* for some $n \geq 0$, if every subset $\mathbb{Y} \subseteq \mathbb{X}$ consisting of $s-n$ points has the same Hilbert function, namely $H_\mathbb{Y} = \min\{H_\mathbb{X}, s-n\}$. In [6], 1-uniform schemes were also called Cayley-Bacharach schemes.

Our main goals are to characterize cohomological uniformity in terms of the structure of the canonical ideal of $\mathbb{X}$ and to show that this property is intermediate between 1-uniformity and $\Delta_\mathbb{X}$-uniformity. Towards the end of the section we shall also give a concrete geometrical interpretation of cohomological uniformity in the case of $d+2 \leq s \leq \binom{d+2}{2}$ points with generic Hilbert function.

Let us start with the definition. We continue to use all notations and conventions of section 1.

DEFINITION. – We say that $\mathbb{X}$ *splits cohomologically*, if we can decompose $\mathbb{X} = \mathbb{Y} \cup \mathbb{Y}'$ such that $\mathbb{Y} \neq \emptyset$, $\mathbb{Y}' \neq \emptyset$, $\mathbb{Y} \cap \mathbb{Y}' = \emptyset$ and $\sum_{P_i \in \mathbb{Y}} K \cdot L f_i \cap \sum_{P_i \in \mathbb{Y}'} K \cdot L f_i = (0)$ in $\overline{R}_{\sigma+1}$.

If $\mathbb{X}$ does not split cohomologically, we say that $\mathbb{X}$ is *cohomologically uniform.*

The choice of this name is explained in [11], sec. 3. From our next proposition and Lemma 1.3 it follows that the definition is in fact independent of the choice of the linear nonzerodivisor $x_0 \in R_1$ (for another proof see [11]).

PROPOSITION 2.1. – *The following conditions are equivalent.*

a) $\mathbb{X}$ *splits cohomologically.*

b) We can renumber the points of $\{P_1, \ldots, P_\Delta\}$ *and of* $\{P_{\Delta+1}, \ldots, P_s\}$ *such that the matrix* $\mathfrak{B}$ *is of the form* $\mathfrak{B} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, *where each block may have zero rows or columns and is strictly smaller than* $\mathfrak{B}$.

PROOF. – Suppose condition *a)* holds. If $Lf_i = 0$ for some $i \in \{1, \ldots, s\}$, then one of the columns of $\mathfrak{B}$ is zero, i.e. after renumbering $\{P_{\Delta+1}, \ldots, P_s\}$ the matrix $\mathfrak{B}$ is of the form $\mathfrak{B} = (0 \;\; *)$. Therefore we shall assume now that all elements $Lf_1, \ldots, Lf_s$ are different from zero. Let $\mathbb{X} = \mathbb{Y} \cup \mathbb{Y}'$ be the decomposition according to the above definition. Renumber $\{P_1, \ldots, P_\Delta\}$ and $\{P_{\Delta+1}, \ldots, P_s\}$ such that we have $\mathbb{Y} = \{P_1, \ldots, P_\delta, P_{\Delta+1}, \ldots, P_{\Delta+\varepsilon}\}$ for some $0 \leqslant \delta \leqslant \Delta$ and $0 \leqslant \varepsilon \leqslant s - \Delta$.

Here we cannot have $\delta = 0$, since in that case $\overline{R}_{\sigma+1}$ agrees with $\sum_{P_i \in \mathbb{Y}'} K \cdot Lf_i$, and for all points $P_j \in \mathbb{Y}$ we have $Lf_j = 0$. In an analogous way we see that $\delta < \Delta$ holds. In case $\varepsilon = 0$ we have $Lf_j \in \sum_{P_i \in \mathbb{Y}'} K \cdot Lf_i$ for $\Delta + 1, \ldots, s$, so that $\mathfrak{B}$ is of the form $\mathfrak{B} = \begin{pmatrix} 0 \\ * \end{pmatrix}$. Similarly, if $\varepsilon = s - \Delta$, the matrix $\mathfrak{B}$ is of the form $\mathfrak{B} = \begin{pmatrix} * \\ 0 \end{pmatrix}$. Hence we can assume that $1 \leqslant \varepsilon < s - \Delta$.

Now we observe that for $j = 1, \ldots, \varepsilon$ it follows from $\beta_{j\delta+1} Lf_{\delta+1} + \ldots + \beta_{j\Delta} Lf_\Delta = Lf_{\Delta+j} - \beta_{j1} Lf_1 - \ldots - \beta_{j\delta} Lf_\delta \in \sum_{P_i \in \mathbb{Y}} K \cdot Lf_i \cap \sum_{P_i \in \mathbb{Y}'} K \cdot Lf_i$ that $\beta_{j\delta+1} = \ldots = \beta_{j\Delta} = 0$. Also, for $j = \varepsilon + 1, \ldots, s - \Delta$ it follows from

$$\beta_{j1} Lf_1 + \ldots + \beta_{j\delta} Lf_\delta = Lf_{\Delta+j} - \beta_{j\delta+1} Lf_{\delta+1} - \ldots - \beta_{j\Delta} Lf_\Delta \in \sum_{P_i \in \mathbb{Y}} K \cdot Lf_i \cap \sum_{P_i \in \mathbb{Y}'} K \cdot Lf_i$$

that $\beta_{j1} = \ldots = \beta_{j\delta} = 0$. Thus the matrix $\mathfrak{B}$ has the required shape.

Conversely, suppose that $\beta_{j1} = \ldots = \beta_{j\Delta} = 0$ for some $j \in \{1, \ldots, s - \Delta\}$. Then we can choose $\mathbb{Y} = \{P_j\}$ and $\mathbb{Y}' = \mathbb{X} \backslash \{P_j\}$, and we get the desired decomposition of $\overline{R}_{\sigma+1}$. On the other hand, if the matrix $\mathfrak{B}$ is of the form $\mathfrak{B} =$

$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ or $\mathfrak{B} = \begin{pmatrix} * \\ 0 \end{pmatrix}$ or $\mathfrak{B} = \begin{pmatrix} 0 \\ * \end{pmatrix}$, we let $\delta \in \{1, \ldots, \Delta - 1\}$ be the number of rows of the upper blocks and $\varepsilon \in \{1, \ldots, s - \Delta - 1\}$ the number of columns of the left-hand blocks. Then we can choose $\mathbb{Y} = \{P_1, \ldots, P_\delta, P_{\Delta+1}, \ldots, P_{\Delta+\varepsilon}\}$ and $\mathbb{Y}' = \{P_{\delta+1}, \ldots, P_\Delta, P_{\Delta+\varepsilon+1}, \ldots, P_s\}$, and we get the desired decomposition of $\overline{R}_{\sigma+1}$ again. ∎

This proposition implies that cohomologically uniform schemes have $Lf_i \neq 0$ for $i = 1, \ldots, s$, i.e. that they are 1-uniform (cf. [6], 2.6). Let us illustrate the phenomenon of cohomological splitting with an example. We note that condition 2.1.b) yields a computational way to check for this property.

EXAMPLE 2.2. – Consider two skew lines $L_1, L_2 \subseteq \mathbb{P}^3$, e.g. $L_1 = \mathcal{Z}(X_2, X_3)$ and $L_2 = \mathcal{Z}(X_3 - X_0, X_1)$. Let $\mathbb{X} = \{P_1, \ldots, P_6\} \subseteq \mathbb{P}^3$ with $\{P_1, P_3, P_4\} \subseteq L_1$ and $\{P_2, P_5, P_6\} \subseteq L_2$, e.g. $P_1 = (1{:}\ 0{:}\ 0{:}\ 0)$, $P_2 = (1{:}\ 0{:}\ 0{:}\ 1)$, $P_3 = (1{:}\ 1{:}\ 0{:}\ 0)$, $P_4 = (1{:}\ -1{:}\ 0{:}\ 0)$, $P_5 = (1{:}\ 0{:}\ 1{:}\ 1)$, and $P_6 = (1{:}\ 0{:}\ -1{:}\ 1)$. Here we assume char $K \neq 2$. Then $\mathbb{X}$ has Hilbert function $\Delta H_{\mathbb{X}}$: 1 3 2 0… and $\sigma_{\mathbb{X}} = 1$. We compute the separators of $\mathbb{X}$ and find $f_1 = x_0^2 - x_0 x_3 - x_1^2$, $f_2 = x_0 x_3 - x_2^2$, $f_3 = (1/2)\, x_0 x_1 + (1/2)\, x_1^2$, $f_4 = -(1/2)\, x_0 x_1 + (1/2)\, x_1^2$, $f_5 = (1/2)\, x_0 x_2 + (1/2)\, x_2^2$, and $f_6 = -(1/2)\, x_0 x_2 + (1/2)\, x_2^2$.

Thus $\{Lf_1, Lf_2\}$ is a K-basis of $\overline{R}_{\sigma+1}$, and we see that the matrix $\mathfrak{B}$ of $\mathbb{X}$ is given by $\begin{pmatrix} -1/2 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 \end{pmatrix}$. Using 2.1 we conclude that $\mathbb{X}$ splits cohomologically in the form $\mathbb{X} = \{P_1, P_3, P_4\} \cup \{P_2, P_5, P_6\}$.

The following lemma gives us a different coherence property of the matrix $\mathfrak{B}$ of a cohomologically uniform scheme.

LEMMA 2.3. – *Suppose* $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ *is a cohomologically uniform set of points, and* $\sim$ *is an equivalence relation on the set* $\{1, \ldots, s\}$ *with the property that* $i \sim j$ *whenever* $i \in \{1, \ldots, \Delta\}$, $j \in \{\Delta + 1, \ldots, s\}$, *and* $\beta_{j-\Delta i} \neq 0$. *Then* $1 \sim 2 \sim \ldots \sim s$.

PROOF. – W.l.o.g. let $\{\Delta + 1, \ldots, \Delta + t_1\}$ with $t_1 \geq 1$ be the set of those numbers j among $\{\Delta + 1, \ldots, s\}$ for which $\beta_{j-\Delta 1} \neq 0$. Here we have $t_1 \geq 1$ because of 1.2.a. By assumption we then get $1 \sim \Delta + 1 \sim \ldots \sim \Delta + t_1$, and for $j \in \{\Delta + t_1 + 1, \ldots, s\}$ we have $\beta_{j-\Delta 1} = 0$. Since $\mathbb{X}$ is cohomologically uniform, Proposition 2.1 yields a number $i \in \{2, \ldots, \Delta\}$ such that not all elements of $\{\beta_{1i}, \ldots, \beta_{t_1 i}\}$ are zero. W.l.o.g. let $i = 2$ and $\beta_{\nu_1 2} \neq 0$ with $\nu_1 \in \{1, \ldots, t_1\}$. Then we have $2 \sim \Delta + \nu_1 \sim 1$.

W.l.o.g. let $\{\Delta + t_1 + 1, \ldots, \Delta + t_2\}$ with $t_2 \geq t_1$ be the set of those numbers j among $\{\Delta + t_1 + 1, \ldots, s\}$ for which $\beta_{j-\Delta 2} \neq 0$. (Notice that we allow $t_2 = t_1$

and an empty set.) Thus we have $2 \sim \Delta + t_1 + 1 \sim \ldots \sim \Delta + t_2$ and $\beta_{j-\Delta\, 2} = 0$ for $j = \Delta + t_1 + 2, \ldots, s$. Since $\mathbb{X}$ is cohomologically uniform, Proposition 2.1 yields a number $i \in \{3, \ldots, \Delta\}$ such that not all elements of the set $\{\beta_{1i}, \ldots, \beta_{t_2 i}\}$ are zero. W.l.o.g. let $i = 3$ and $\beta_{\nu_2 3} \neq 0$ with $\nu_2 \in \{1, \ldots, t_2\}$. Then we have $3 \sim \Delta + \nu_2 \sim 2 \sim 1$.

Continuing in this manner we finally obtain $t_\Delta = s - \Delta$, since not all elements of $\{\beta_{s-\Delta\, 1}, \ldots, \beta_{s-\Delta\Delta}\}$ are zero, and we get $\Delta \sim \Delta - 1 \sim \ldots \sim 1$. By 2.1, each column of $\mathfrak{B}$ has a nonzero entry, so $1 \sim \ldots \sim \Delta \sim \Delta + 1 \sim \ldots \sim s$. ∎

Now we are ready to prove the main theorem of this section.

THEOREM 2.4. – *Let* $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ *be a set of points. The following conditions are equivalent.*

a) $\mathbb{X}$ *is cohomologically uniform.*

b) The multiplication map $\mu\colon R_\sigma \otimes \mathfrak{I}_{\sigma+1} \to \mathfrak{I}_{2\sigma+1}$ *is nondegenerate and surjective.*

c) The multiplication map $\tilde{\mu}\colon R_\sigma \otimes (\omega_R)_{-\sigma} \to (\omega_R)_0$ *is nondegenerate and surjective.*

PROOF. – «$a) \Rightarrow b)$». From Proposition 2.1 it follows that $\mathbb{X}$ is a Cayley-Bacharach scheme. In [10], 2.6 it was shown that the multiplication map $\tilde{\mu}$ is nondegenerate for Cayley-Bacharach schemes. By 1.9, also μ is nondegenerate.

Because of 1.11, we still need to show that the elements $h_i = x_0^\sigma(f_{i+1} - f_i)$ such that $1 \leqslant i \leqslant s - 1$ are in the image of μ. Define a relation $\sim$ on the set $\{1, \ldots, s\}$ by $i \sim j \Leftrightarrow x_0^\sigma(f_i - f_j) \in \operatorname{im}\mu$. Obviously $\sim$ is an equivalence relation. If $i \in \{1, \ldots, \Delta\}$ and $j \in \{\Delta + 1, \ldots, s\}$, then $g_{j-\Delta}\pi_i = \beta_{j-\Delta i} x_0^\sigma(f_j - f_i) \in \operatorname{im}\mu$ by 1.14. Hence if $\beta_{j-\Delta i} \neq 0$, then $i \sim j$. Thus we can apply the lemma and obtain $1 \sim \ldots \sim s$, i.e. $x_0^\sigma(f_j - f_i) \in \operatorname{im}\mu$ for $i, j \in \{1, \ldots, s\}$.

«$b) \Leftrightarrow a)$» Suppose that $\mathbb{X}$ splits cohomologically in the form $\mathbb{X} = \mathbb{Y} \cup \mathbb{Y}'$. Then $\Delta_{\mathbb{X}} \geqslant 2$, and we can renumber $\{P_1, \ldots, P_s\}$ such that $\mathbb{Y} = \{P_1, \ldots, P_\delta, P_{\Delta+1}, \ldots, P_{\Delta+\varepsilon}\}$ with $1 \leqslant \delta \leqslant \Delta - 1$ and $0 \leqslant \varepsilon \leqslant s - \Delta$. Since the image of μ is generated by the elements $g_{j-\Delta}\pi_i = \beta_{j-\Delta i} x_0^\sigma(f_j - f_i)$ with $i \in \{1, \ldots, \Delta\}$ and $j \in \{\Delta + 1, \ldots, s\}$, it is already generated by the subset of those elements for which $\beta_{j-\Delta i} \neq 0$. As $\mathbb{X}$ splits cohomologically, that subset is contained in the set

$$M := \{x_0^\sigma(f_j - f_i) \mid i = 1, \ldots, \delta \text{ and } j = \Delta + 1, \ldots, \Delta + \varepsilon\} \cup \{x_0^\sigma(f_j - f_i) \mid i = \delta + 1, \ldots, \Delta \text{ and } j = \Delta + \varepsilon + 1, \ldots, s\}.$$

Now let $x = x_0^\sigma(c_1 f_1 + \ldots + c_s f_s)$ with $c_1, \ldots, c_s \in K$ be an arbitrary element of $\operatorname{im}\mu$. We express x with the generators from M and get

$$x = \sum_{i=1}^{\delta} \sum_{j=\Delta+1}^{\Delta+\varepsilon} \zeta_{ij} x_0^\sigma (f_j - f_i) + \sum_{i=\delta+1}^{\Delta} \sum_{j=\Delta+\varepsilon+1}^{s} \zeta_{ij} x_0^\sigma (f_j - f_i)$$

with $\zeta_{ij} \in K$. By comparing the two representations of x, we find

$$c_i = \begin{cases} -\zeta_{i\Delta+1} - \ldots - \zeta_{i\Delta+\varepsilon} & \text{for } i = 1, \ldots, \delta\,, \\ -\zeta_{i\Delta+\varepsilon+1} - \ldots - \zeta_{is} & \text{for } i = \delta+1, \ldots, \Delta\,, \\ \zeta_{1i} + \ldots + \zeta_{\delta i} & \text{for } i = \Delta+1, \ldots, \Delta+\varepsilon\,, \\ \zeta_{\delta+1 i} + \ldots + \zeta_{\Delta i} & \text{for } i = \Delta+\varepsilon+1, \ldots, s\,. \end{cases}$$

Thus we conclude $\sum_{P_i \in \mathbb{Y}} c_i = 0$ and $\sum_{P_i \in \mathbb{Y}'} c_i = 0$. In particular, the element $x_0^\sigma(f_\Delta - f_1)$ is not in the image of μ. This contradicts the hypothesis.

«$b) \Leftrightarrow c)$» is a consequence of 1.9. ■

Using [10], 3.2, this theorem implies that $\Delta_{\mathbb{X}}$-uniform schemes are cohomologically uniform. In view of [6], example 3.11, it was asked in [11] whether cohomological uniformity is the right condition to show the inequalities $H_{\mathbb{X}}(n) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) \leq s - \Delta_{\mathbb{X}} + 1$ for $n = 1, \ldots, \sigma_{\mathbb{X}}$ for the Hilbert function of $\mathbb{X}$. This is the topic of our next section.

But before we want to specialize the situation for a moment and consider schemes $\mathbb{X}$ consisting of «few» points in $\mathbb{P}^d$. More precisely, let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ consist of $d + 2 \leq s \leq \binom{d+2}{2}$ points, and suppose that $\mathbb{X}$ is nondegenerate, i.e. it is not contained in a hyperplane, and that $\sigma_{\mathbb{X}} = 1$. Then $\mathbb{X}$ has generic Hilbert function, i.e. the Hilbert function of $\mathbb{X}$ is $H_{\mathbb{X}}$: $1\ d+1\ s\ s \ldots$, and we have $\Delta_{\mathbb{X}} = s - d - 1$. In this situation [10], 3.2 and 4.2 yield that some conditions of uniformity coincide.

REMARK 2.5. – The following conditions are equivalent.

a) $\mathbb{X}$ is in linearly general position, i.e. any subscheme $\mathbb{Y} \subseteq \mathbb{X}$ consisting of $d+1$ points has Hilbert function $H_{\mathbb{Y}}$: $1\ d+1\ d+1 \ldots$.

b) $\mathbb{X}$ is $\Delta_{\mathbb{X}}$-uniform.

c) The multiplication map $\mu: R_1 \otimes \mathfrak{I}_2 \to \mathfrak{I}_3$ is biinjective, i.e. $\mu(r \otimes \varphi) = 0$ implies $r = 0$ or $\varphi = 0$.

For $i = 1, \ldots, s$, let us write $P_i = (1: p_{i1}: \ldots: p_{id})$ with $p_{ij} \in K$. Because of 1.14 we know how the matrix of the above multiplication map μ depends on the coefficients of the matrix $\mathfrak{B} = (\beta_{ij})$. In our situation, this matrix $\mathfrak{B}$ has a particularly simple description.

LEMMA 2.6. – *If $\mathbb{X}$ is in linearly general position, and if we choose the coordinate system suitably, the matrix $\mathfrak{B} = (\beta_{ij})$ is given by*

$$\mathfrak{B} = \begin{bmatrix} -1 + p_{11} + \ldots + p_{1d} & -p_{11} & \ldots & -p_{1d} \\ \vdots & \vdots & & \vdots \\ -1 + p_{\Delta 1} + \ldots + p_{\Delta d} & -p_{\Delta 1} & \ldots & -p_{\Delta d} \end{bmatrix}$$

PROOF. – Recall that we have numbered $P_1, \ldots, P_s$ such that $\{Lf_1, \ldots, Lf_\Delta\}$ is a K-basis of $\overline{R}_{\sigma+1}$. Since $\mathbb{X}$ is in linearly general position, $P_{\Delta+1}, \ldots, P_s$ span $\mathbb{P}^d$. Thus we may change the coordinate system such that $P_{\Delta+1} = P_{s-d} = (1\colon 0\colon \ldots\colon 0)$, $P_{s-d+1} = (1\colon 1\colon 0\colon \ldots\colon 0)$, $\ldots$, $P_s = (1\colon 0\colon \ldots\colon 0\colon 1)$. Both $\{x_0 - x_1 - \ldots - x_d, x_1, \ldots, x_d\}$ and $\{g_1, \ldots, g_{d+1}\}$ are K-bases of R_1 (cf. 1.2.b). They attain the same values $(x_0 - x_1 - \ldots - x_d)(P_{s-d+i}) = \delta_{i0} = g_1(P_{s-d+i})$ and $x_j(P_{s-d+i}) = \delta_{ij} = g_{j+1}(P_{s-d+i})$ for $i = 0, \ldots, d$ and $j = 1, \ldots, d$ at the points $P_{s-d}, \ldots, P_s$. Since those points span $\mathbb{P}^d$, we have $g_1 = x_0 - x_1 - \ldots - x_d$, $g_2 = x_1, \ldots, g_{d+1} = x_d$, and the claim follows from $\beta_{ij} = -g_i(P_j)$ for $i = 1, \ldots, d+1$ and $j = 1, \ldots, \Delta$. ■

The following notion will be used to explain the geometrical meaning of cohomological splitting in the present situation.

DEFINITION. – We say that a reduced 0-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^d$ *splits linearly*, if there exist linear subspaces $L_1, L_2 \subseteq \mathbb{P}^d$ such that $\mathbb{X} \subseteq L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$.

For example, the scheme of 2.2 splits linearly. Like 2.5, the next proposition shows that several notions which are distinct in general, coincide in the case of «few» points. The equivalence of conditions a) and d) follows also from [2], 1.2 and 1.5.

PROPOSITION 2.7. – *Let $\mathbb{X} \subseteq \mathbb{P}^d$ be a nondegenerate, reduced, 0-dimensional subscheme consisting of $d+2 \leqslant s \leqslant \binom{d+2}{2}$ points with generic Hilbert function. Then the following conditions are equivalent.*

a) *$\mathbb{X}$ does not split linearly.*

b) *$\mathbb{X}$ is cohomologically uniform.*

c) *The multiplication map $\mu\colon R_1 \otimes \mathfrak{J}_2 \to \mathfrak{J}_3$ is nondegenerate and surjective.*

d) *$\mathbb{X}$ is pure, i.e. its canonical ideal $\mathfrak{J}$ is generated by the elements of $\mathfrak{J}_2$* (cf. [11], *section* 4).

PROOF. – «$a) \Leftrightarrow b)$». The scheme $\mathbb{X}$ has the Cayley-Bacharach property w.r.t hypersurfaces of degree $\sigma_{\mathbb{X}} = 1$, because a hyperplane L_1 containing exactly $s-1$ points of $\mathbb{X}$ and the 0-dimensional linear space L_2 consisting of the remaining point would constitute a linear splitting. Thus $\mathbb{X}$ is 1-uniform.

Suppose $\mathbb{X}$ splits cohomologically, and we have renumbered $\{P_1, \ldots, P_s\}$ such that the second part of 2.1.b holds. Then we have $g_j(P_i) = 0$ for $j \in \{1, \ldots, \varepsilon\}$ and $i \in \{\delta+1, \ldots, \Delta\} \cup \{\Delta+\varepsilon+1, \ldots, s\}$. Hence the linear subspace $L_1 := \mathcal{Z}(g_1, \ldots, g_\varepsilon)$ of $\mathbb{P}^d$ contains $\{P_{\delta+1}, \ldots, P_\Delta\} \cup \{P_{\Delta+\varepsilon+1}, \ldots, P_s\}$. Analogously it follows that the linear subspace $L_2 = \mathcal{Z}(g_{\varepsilon+1}, \ldots, g_{s-\Delta})$ of $\mathbb{P}^d$ contains $\{P_1, \ldots, P_\delta\} \cup \{P_{\Delta+1}, \ldots, P_{\Delta+\varepsilon}\}$. Altogether, $L_1 \cup L_2$ contains $\mathbb{X}$ and $L_1 \cap L_2 = \mathcal{Z}(g_1, \ldots, g_{s-\Delta}) = \mathcal{Z}(\bigoplus_{n \geq 1} R_n) = \emptyset$, contradicting $a)$.

The equivalence «$b) \Leftrightarrow c)$» is a special case of 2.4. For «$c) \Leftrightarrow d)$» we note that $\mathfrak{I}_3$ does not contain a minimal generator of $\mathfrak{I}$, that we have $\dim_K \mathfrak{I}_3 = s-1$, and that $\dim_K \mathfrak{I}_4 = s$. In view of this and 1.11, it suffices to find one element $r = x_0^2(c_1 f_1 + \ldots + c_s f_s)$ in $\mathfrak{I}_4$ such that $r \in R_2 \mathfrak{I}_2$ and $c_1 + \ldots + c_s \neq 0$. For this we can take any nonzero element $r' = c_1' f_1 + \ldots + c_s' f_s \in \mathfrak{I}_2$ such that $c_i' \neq 0$ for some $i \in \{1, \ldots, s\}$ and multiply it by $f_i \in R_2$.

As «$d) \Leftrightarrow c)$» is trivially true, we are left with «$b) \Leftrightarrow a)$». Suppose there are linear subspaces $L_1, L_2 \in \mathbb{P}^d$ such that $\mathbb{X} \subseteq L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$. W.l.o.g. we may assume that $\dim L_1 + \dim L_2 = d-1$. We let $\varepsilon := d - \dim L_1 = \dim L_2 + 1 \in \{1, \ldots, d\}$. By a linear change of coordinates, we may also assume $L_1 = \mathcal{Z}(X_0, \ldots, X_{\varepsilon-1})$ and $L_2 = \mathcal{Z}(X_\varepsilon, \ldots, X_d)$. Notice that here we may loose the property that $x_0 \in R_1$ is not a zerodivisor. If we denote the number of points of $\mathbb{X} \cap L_1$ by $r \in \{1, \ldots, s-1\}$, we can renumber $\{P_1, \ldots, P_s\}$ such that $\mathbb{X} \cap L_1 = \{P_1, \ldots, P_r\}$ and $\mathbb{X} \cap L_2 = \{P_{r+1}, \ldots, P_s\}$. Furthermore, we choose a nonzerodivisor $l \in R_1$ of R, we construct the normalized separators $f_1^l, \ldots, f_s^l$ w.r.t l, and for $i = 1, \ldots, d$ we let x_i be the image of X_i in R_1.

Because of $\mathbb{X} \subseteq L_1 \cup L_2$, we have $x_i x_j = 0$ for $i \in \{0, \ldots, \varepsilon-1\}$ and $j \in \{\varepsilon, \ldots, d\}$. Thus f_i^l has a decomposition $f_i^l = f_i' + f_i''$ such that $f_i' \in K[x_0, \ldots, x_{\varepsilon-1}] \subseteq R$ and $f_i'' \in K[x_\varepsilon, \ldots, x_d] \subseteq R$. For every $i \in \{1, \ldots, s\}$, the definitions of L_1 and L_2 imply $f_i'(P_j) = 0$ for $j \in \{1, \ldots, r\}$ and $f_i''(P_j) = 0$ for $j \in \{r+1, \ldots, s\}$. Since also $f_i'(P_j) = f_i^l(P_j) = 0$ for $i = 1, \ldots, r$ and $j = r+1, \ldots, s$, we obtain $f_i' = 0$ for $i = 1, \ldots, r$. Analogously, we have $f_i'' = 0$ for $i = r+1, \ldots, s$. Altogether we have shown that $f_i^l \in K[x_\varepsilon, \ldots, x_d]$ for $i = 1, \ldots, r$ and $f_i^l \in K[x_0, \ldots, x_{\varepsilon-1}]$ for $i = r+1, \ldots, s$.

Now we can conclude that $l \cdot (Kx_\varepsilon + \ldots + Kx_d) \subseteq Kf_1^l \oplus \ldots \oplus Kf_r^l$ and $l \cdot (Kx_0 + \ldots + Kx_{\varepsilon-1}) \subseteq Kf_{r+1}^l \oplus \ldots \oplus Kf_s^l$. Hence $c_1 f_1^l + \ldots + c_s f_s^l \in l \cdot R_1$ for some $c_1, \ldots, c_s \in K$ implies $c_1 f_1^l + \ldots + c_r f_r^l \in lR_1$ and $c_{r+1} f_{r+1}^l + \ldots + c_s f_s^l \in lR_1$. Thus we find the splitting $(R/(l))_2 = \sum_{i=1}^{r} K \cdot Lf_i^l \oplus \sum_{i=r+1}^{s} K \cdot Lf_i^l$, i.e. $\mathbb{X}$ splits cohomologically. ■

We end this section with an example which shows that not every cohomologically split scheme splits linearly. The second part of this example also demonstrates that cohomological splitting or uniformity may depend on quite subtle geometrical properties of the configuration of the points of $\mathbb{X}$. This example suggests us to ask if for every cohomologically split scheme $\mathbb{X} = \mathbb{Y} \cup \mathbb{Y}'$ there exist disjoint varieties V_1, $V_2 \subseteq \mathbb{P}^d$ of «small» degree such that $\mathbb{Y} \subseteq V_1$ and $\mathbb{Y}' \subseteq V_2$.

EXAMPLE 2.8. – *a*) Let C_1, $C_2 \subseteq \mathbb{P}^3_{\mathbb{C}}$ be two skew twisted cubics, and let $\mathbb{X}$ consist of 20 points on C_1 and 20 points on C_2. For instance, we can take $C_1 = \{(u^3 \colon u^2 v \colon uv^2 \colon v^3) \mid (u \colon v) \in \mathbb{P}^1_{\mathbb{C}}\}$ and $C_2 = \{(2uv^2 \colon u^3 \colon v^3 \colon u^2 v) \mid (u \colon v) \in \mathbb{P}^1_{\mathbb{C}}\}$, and then choose the points corresponding to $(u \colon v) = (1 \colon i)$ for $i = 1, \ldots, 20$ on both C_1 and C_2. The Hilbert function of $\mathbb{X}$ is $H_{\mathbb{X}}$: 1 4 10 20 26 32 38 40 40 $\ldots$, so that $\sigma_{\mathbb{X}} = 6$ and $\Delta_{\mathbb{X}} = 2$. The matrix $\mathfrak{B}$ is of the form $\begin{pmatrix} * \ldots * & 0 \ldots 0 \\ 0 \ldots 0 & * \ldots * \end{pmatrix}$, and therefore $\mathbb{X}$ splits in the form $\mathbb{X} = (\mathbb{X} \cap C_1) \cup (\mathbb{X} \cap C_2)$. Clearly $\mathbb{X}$ does not split linearly.

We note the inequality $H_{\mathbb{X}}(3) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - 3) = 40 > 39 = s - \Delta_{\mathbb{X}} + 1$, a phenomenon which will be explored more deeply in the next section.

b) If we replace C_2 by $C_3 = \{(u^3 \colon uv^2 \colon v^3 \colon 2u^2 v) \mid (u \colon v) \in \mathbb{P}^1_{\mathbb{C}}\}$, then $C_1 \cap C_3 = \{(1 \colon 0 \colon 0 \colon 0)\}$, and the analogously defined scheme $\mathbb{X}$ has $H_{\mathbb{X}}$: 1 4 10 19 25 31 37 40 $\ldots$. Since its matrix $\mathfrak{B}$ does not split and has nonzero columns, $\mathbb{X}$ is cohomologically uniform. We note that the inequalities $H_{\mathbb{X}}(n) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) \leqslant s - \Delta_{\mathbb{X}} + 1$ hold in this case for $n = 0, \ldots, \sigma_{\mathbb{X}}$.

The calculations for this example were done using the program COP for computations with zerodimensional schemes (cf. [1]).

3. – Hilbert functions of cohomologically uniform schemes.

As in the previous sections, we let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ be a set of points with canonical ideal $\mathfrak{J} \subseteq R$. We have seen that cohomological uniformity of $\mathbb{X}$ is an intermediate property between 1-unformity and $\Delta_{\mathbb{X}}$-uniformity. In [11] it was shown that $\Delta_{\mathbb{X}}$-uniformity implies certain inequalities for the Hilbert function of $\mathbb{X}$ which do not hold in general for 1-uniform schemes (cf. [6], 3.10 and 3.11). Our goal in this section is to confirm the view expressed in [11], section 3, that cohomological uniformity is the correct condition which implies those inequalities.

THEOREM 3.1. – *Let* $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ *be a cohomologically uniform* *set of points, and suppose that* $\operatorname{char}(K) \notin \{2, \ldots, \sigma_{\mathbb{X}}\}$. *Then we have*

$$H_{\mathbb{X}}(n) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) \leqslant s - \Delta_{\mathbb{X}} + 1$$

for all $n \in \{0, \ldots, \sigma_{\mathbb{X}}\}$.

Notice that if $\mathbb{X}$ consists of only one point, then $\sigma_{\mathbb{X}} = -1$ and the theorem does not claim anything. Similarly, the theorem is trivial in case $\sigma_{\mathbb{X}} = 0$. Therefore we shall assume $s \geqslant 3$ and $\sigma_{\mathbb{X}} \geqslant 1$ for the rest of this section. For proof of Theorem 3.1 we can assume that $\mathbb{X} \subseteq \mathbb{P}^d$ is nondegenerate, since the Hilbert function of $\mathbb{X}$ does not change, if we replace $\mathbb{P}^d$ by the linear span of $\mathbb{X}$. This proof relies on a detailed understanding of how those inequalities result from structural properties of the canonical ideal of $\mathbb{X}$. The next proposition and its corollary provide us with the basic link.

PROPOSITION 3.2. – *Let* $r \in R_\sigma$ *and* $\varphi \in \mathfrak{I}_{\sigma+1}$. *The following conditions are equivalent.*

a) There exist elements $\varphi_2, \ldots, \varphi_\Delta \in \mathfrak{I}_{\sigma+1}$ *such that*

$$\mathfrak{I}_{2\sigma+1} = R_\sigma \varphi \oplus Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta \,.$$

b) If $\varphi_2, \ldots, \varphi_\Delta \in \mathfrak{I}_{\sigma+1}$ *are such that* $\{\varphi, \varphi_2, \ldots, \varphi_\Delta\}$ *is a K-basis of* $\mathfrak{I}_{\sigma+1}$, *then*

$$\mathfrak{I}_{2\sigma+1} = R_\sigma \varphi \oplus Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta \,.$$

c) We have $\varphi \neq 0$, *and if* $r' \in R_\sigma$, $\varphi' \in \mathfrak{I}_{\sigma+1}$ *are such that* $r'\varphi = r\varphi'$, *then* $r' = \lambda r$ *and* $\varphi' = \lambda\varphi$ *for some* $\lambda \in K$.

PROOF. – First we show that a) implies c). From $s - 1 = \dim_K \mathfrak{I}_{2\sigma+1} > \Delta_{\mathbb{X}} - 1 \geqslant \dim_K (Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta)$ we conclude that $\varphi \neq 0$. Now suppose that $r\varphi' = r'\varphi$ for some $r' \in R_\sigma$, $\varphi' \in \mathfrak{I}_{\sigma+1}$. From the hypothesis it is clear that $\{\varphi, \varphi_2, \ldots, \varphi_\Delta\}$ are K-linearly independent, and hence form a K-basis of $\mathfrak{I}_{\sigma+1}$. Thus we can write $\varphi' = \lambda_1\varphi + \lambda_2\varphi_2 + \ldots + \lambda_\Delta\varphi_\Delta$ with $\lambda_1, \ldots, \lambda_\Delta \in K$, and we obtain $r(\lambda_2\varphi_2 + \ldots + \lambda_\Delta\varphi_\Delta) = (r' - \lambda_1 r)\varphi \in R_\sigma\varphi \cap (Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta) = (0)$. Therefore $\lambda_2 = \ldots = \lambda_\Delta = 0$ and $\varphi' = \lambda_1\varphi$. Since the hypothesis also implies $\dim_K(R_\sigma\varphi) = s - \Delta$, we get $r' - \lambda_1 r = 0$, i.e. $r' = \lambda_1 r$.

As «$b) \Leftrightarrow a)$» is trivially true, we are left with proving «$c) \Leftrightarrow b)$». We claim that $\dim_K (R_\sigma\varphi) = s - \Delta$, i.e. that no element $r' \in R_\sigma \setminus \{0\}$ annihilates φ. Otherwise we could choose $\varphi' = 0$, and $r'\varphi = 0 = r\varphi'$ would imply $\varphi' = \lambda\varphi$, $r' = \lambda r$ for $\lambda = 0$, contradicting $r' \neq 0$. We also claim that we may assume $r \neq 0$. Otherwise $r'\varphi = 0 \Rightarrow r' = 0$ and [10], 2.6, imply that $\mathbb{X}$ is 1-uniform with $\Delta_{\mathbb{X}} = 1$, so that the claim is obviously true.

Suppose now that $\varphi' = \lambda_2\varphi_2 + \ldots + \lambda_\Delta\varphi_\Delta$ with $\lambda_2, \ldots, \lambda_\Delta \in K$ satisfies

$r\varphi' = 0$. Then $r\varphi' = r'\varphi$ for $r' = 0$, and thus $r' = \lambda r$, $\varphi' = \lambda\varphi$ with $\lambda = 0$, i.e. we have $\varphi' = 0$. This shows $\dim_K(Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta) = \Delta_{\mathbb{X}} - 1$. Finally, it is clear from the hypothesis that $(R_\sigma \varphi) \cap (Kr\varphi_2 \oplus \ldots \oplus Kr\varphi_\Delta) = (0)$, and the claim follows by adding up dimensions. ∎

Note that if $\mathbb{X}$ is 1-uniform and $\Delta_{\mathbb{X}} = 1$, then condition *c*) holds for $r = 0$ and every $\varphi \in \mathfrak{I}_{\sigma+1} \setminus \{0\}$. The existance of $r \in R_\sigma$ and $\varphi \in \mathfrak{I}_{\sigma+1}$ such that 3.2. *a*)-*c*) hold has strong consequences.

COROLLARY 3.3. – *a*) *If there exists an element $l \in R_1$ such that the equivalent conditions of* 3.2 *are satisfied for $r = l^\sigma$ and some $\varphi \in \mathfrak{I}_{\sigma+1}$, then*

$$H_{\mathbb{X}}(n) + H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) \leqslant s - \Delta_{\mathbb{X}} + 1$$

for all $n \in \{0, \ldots, \sigma_{\mathbb{X}}\}$.

b) *If there exist $r \in R_\sigma$ and $\varphi \in \mathfrak{I}_{\sigma+1}$ such that the equivalent conditions of* 3.2 *are satisfied, then $\mathbb{X}$ is cohomologically uniform.*

PROOF. – To see *a*), we choose $\varphi_2, \ldots, \varphi_\Delta \in \mathfrak{I}_{\sigma+1}$ as in 3.2 *b*). Like in the proof of 3.2, we find $Ann_R(\varphi) \cap R_\sigma = (0)$. Then we observe that $l^n(\lambda_2\varphi_2 + \ldots + \lambda_\Delta\varphi_\Delta) \in R_n\varphi$ for $n \in \{0, \ldots, \sigma\}$ and $\lambda_2, \ldots, \lambda_\Delta \in K$ implies $l^\sigma(\lambda_2\varphi_2 + \ldots + \lambda_\Delta\varphi_\Delta) \in R_\sigma\varphi$, and therefore $\lambda_2 = \ldots = \lambda_\Delta = 0$. Thus $s - H_{\mathbb{X}}(\sigma_{\mathbb{X}} - n) = \dim_K \mathfrak{I}_{\sigma+1+n} \geqslant \dim_K(R_n\varphi) + \Delta_{\mathbb{X}} - 1 = H_{\mathbb{X}}(n) + \Delta_{\mathbb{X}} - 1$.

Now we show claim *b*). From $\mathrm{Ann}_R(\varphi) \cap R_\sigma = (0)$ and [10], 2.6, it follows that $\mathbb{X}$ is 1-uniform. A look at Theorem 2.4 finishes the proof. ∎

This corollary reduces the proof of Theorem 3.1 to showing that for every cohomologically uniform scheme $\mathbb{X}$ there exist $l \in R_1$ and $\varphi \in \mathfrak{I}_{\sigma+1}$ such that 3.2 *a*)-*c*) hold for $r = l^\sigma$ and φ. This question will now be reduced to a linear algebra problem.

Recall that we have explicit bases $\{g_1, \ldots, g_{s-\Delta}\}$ of R_σ (cf. 1.2 *b*)) and $\{\pi_1, \ldots, \pi_\Delta\}$ of $\mathfrak{I}_{\sigma+1}$ (cf. 1.10).

PROPOSITION 3.4. – *Let $r \in R_\sigma$ and $\varphi \in \mathfrak{I}_{\sigma+1}$. Write $r = a_1 g_1 + \ldots + a_{s-\Delta} g_{s-\Delta}$ with $a_1, \ldots, a_{s-\Delta} \in K$ and $\varphi = c_1\pi_1 + \ldots + c_\Delta\pi_\Delta$ with $c_1, \ldots, c_\Delta \in K$. Let $b_i = a_1\beta_{1i} + \ldots + a_{s-\Delta}\beta_{s-\Delta i}$ for $i = 1, \ldots, \Delta$ and $d_j = c_1\beta_{j1} + \ldots + c_\Delta\beta_{j\Delta}$ for $j =$*

$1, \ldots, s-\Delta$. *Define the matrix*

$$M(r, \varphi) = \begin{bmatrix} -b_1 & & 0 & c_1\beta_{11} & \cdots & c_1\beta_{s-\Delta\, 1} \\ & \ddots & & \vdots & & \vdots \\ 0 & & -b_\Delta & c_\Delta\beta_{1\,\Delta} & \cdots & c_\Delta\beta_{s-\Delta\,\Delta} \\ a_1\beta_{11} & \cdots & a_1\beta_{1\,\Delta} & -d_1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ a_{s-\Delta}\beta_{s-\Delta\,1} & \cdots & a_{s-\Delta}\beta_{s-\Delta\,\Delta} & 0 & & -d_{s-\Delta} \end{bmatrix}.$$

Then conditions 3.2 *a*)-*c*) *are equivalent with* $\operatorname{rk} M(r, \varphi) = s-1$.

PROOF. – Using 1.14, we calculate

$$r\pi_i = a_1 g_1 \pi_i + \ldots + a_{s-\Delta} g_{s-\Delta} \pi_i =$$

$$a_1 x_0^\sigma(\beta_{1i} f_{\Delta+1} - \beta_{1i} f_i) + \ldots + a_{s-\Delta} x_0^\sigma(\beta_{s-\Delta i} f_s - \beta_{s-\Delta i} f_i) =$$

$$x_0^\sigma(-b_i f_i + a_1\beta_{1i} f_{\Delta+1} + \ldots + a_{s-\Delta}\beta_{s-\Delta i} f_s)$$

for $i = 1, \ldots, \Delta$ and

$$g_j\varphi = c_1 g_j \pi_1 + \ldots + c_\Delta g_j \pi_\Delta = c_1 x_0^\sigma(\beta_{j1} f_{\Delta+j} - \beta_{j1} f_1) + \ldots + c_\Delta x_0^\sigma(\beta_{j\Delta} f_{\Delta+j} - \beta_{j\Delta} f_\Delta) =$$

$$x_0^\sigma(-c_1\beta_{j1} f_1 - \ldots - c_\Delta \beta_{j\Delta} f_\Delta + d_j f_{\Delta+j})$$

for $j = 1, \ldots, s-\Delta$. Therefore the columns of the matrix

$$\begin{bmatrix} T_1^{2\sigma+1} & & 0 \\ & \ddots & \\ 0 & & T_s^{2\sigma+1} \end{bmatrix} \cdot M(r, \varphi)$$

represent the images of $r\pi_1, \ldots, r\pi_\Delta, -g_1\varphi, \ldots, -g_{s-\Delta}\varphi$ under the canonical injection $\iota\colon R \hookrightarrow \widetilde{R} \cong K[T_1] \times \ldots \times K[T_s]$ of section 1. Thus $\operatorname{rk} M(r, \varphi) = s-1$ means that $r\mathfrak{I}_{\sigma+1} + R_\sigma\varphi$ is $(s-1)$-dimensional. This is equivalent to $\dim_K((r\mathfrak{I}_{\sigma+1}) \cap (R_\sigma\varphi)) = 1$ and hence to conditions 3.2 *a*)-*c*). ■

PROPOSITION 3.5. – *Let* $r = a_1 g_1 + \ldots + a_{s-\Delta} g_{s-\Delta} \in R_\sigma$ *with* $a_1, \ldots, a_{s-\Delta} \in K$. *The following conditions are equivalent.*

a) *There exists an element* $\varphi \in \mathfrak{I}_{\sigma+1}$ *such that conditions* 3.2 *a*)-*c*) *hold.*

b) The matrix

$$\mathfrak{M}(r)=\begin{pmatrix} -b_1 & & 0 & a_1\beta_{11} & \dots & a_{s-\Delta}\beta_{s-\Delta} \\ & \ddots & & \vdots & & \vdots \\ 0 & & -b_\Delta & a_1\beta_{1\Delta} & \dots & a_{s-\Delta}\beta_{s-\Delta\,\Delta} \\ \beta_{11}Y_1 & \dots & \beta_{1\Delta}Y_\Delta & -D_1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \beta_{s-\Delta\,1}Y_1 & \dots & \beta_{s-\Delta\,\Delta}Y_\Delta & 0 & & -D_{s-\Delta} \end{pmatrix}$$

has rank $s-1$.

Here $b_i=a_1\beta_{1i}+\ldots+a_{s-\Delta}\beta_{s-\Delta i}$ *for* $i=1,\ldots,\Delta$, *the symbols* $Y_1,\ldots,Y_\Delta$ *denote independent variables over* K, *and* $D_j=\beta_{j1}Y_1+\ldots+\beta_{j\Delta}Y_\Delta$ *for* $j=1,\ldots,s-\Delta$.

PROOF. – This follows from the preceding proposition, since $\mathfrak{M}(r)$ has rank $s-1$ if and only if one of its specializations $M(r,\varphi)^{\text{transp}}$ has rank $s-1$. ∎

Now we can prove Theorem 3.1 in some special cases.

COROLLARY 3.6. – *a) If in the matrix* $\mathfrak{B}$ *of* $\mathbb{X}$ *there is a row* $(\beta_{1i},\ldots,\beta_{s-\Delta i})$, *all of whose entries are nonzero, and if* $l\in R_1$ *is a nonzerodivisor, then* $\operatorname{rk}\mathfrak{M}(l^\sigma)=s-1$ *and the Hilbert function of* $\mathbb{X}$ *satisfies the inequalities of* 3.1.

b) If $\mathbb{X}$ *is* $\Delta_{\mathbb{X}}$*-uniform, then* $\operatorname{rk}\mathfrak{M}(l^\sigma)=s-1$ *for all nonzerodivisors* $l\in R_1$.

PROOF. – To show *a*), we may assume that $\beta_{11}\neq 0,\ldots\beta_{s-\Delta\,1}\neq 0$. Then we cancel the first row and the first column of $\mathfrak{M}(l^\sigma)$. The coefficient of $Y_1^{s-\Delta}$ in the determinant of the resulting matrix is

$$(-l(P_{\Delta+1})^\sigma\beta_{11})\ldots(-l(P_s)^\sigma\beta_{s-\Delta\,1})\neq 0\,.$$

For the proof of claim *b*) we recall that $g_j\pi_i=\beta_{ji}(h_{\Delta+j-1}-\delta_{i1}h_{i-1})$ by 1.14. Thus [10], 3.2 implies that all entries of $\mathfrak{B}$ are nonzero and we can apply *a*). ∎

In view of 3.6 *a*) it is natural to ask if $\operatorname{rk}\mathfrak{M}(l^\sigma)=s-1$ holds for all nonzerodivisors $l\in R_1$. Our next example demonstrates that this is not the case and that we can only hope to show $\operatorname{rk}\mathfrak{M}(l^\sigma)=s-1$ for generic $l\in R_1$.

EXAMPLE 3.7. – Let $\mathbb{X}\subseteq\mathbb{P}^3$ consist of the following six points: $P_1=$

$(1:1:1:0)$, $P_2=(1:2:0:1)$, $P_3=(1:0:0:0)$, $P_4=(1:1:0:0)$, $P_5=(1:0:1:0)$, and $P_6=(1:0:0:1)$. We calculate $H_{\mathbb{X}}$: $1\ 4\ 6\ 6\ldots$, so that $\sigma_{\mathbb{X}}=1$ and $\Delta_{\mathbb{X}}=2$. We also calculate that the matrix $\mathfrak{B}$ of $\mathbb{X}$ is given by

$$\mathfrak{B} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & -2 & 0 & -1 \end{bmatrix}$$

Since the first two columns have nonzero entries, and since no column is zero, we know from 2.1 that $\mathbb{X}$ is cohomologically uniform. But we may check that

$$\operatorname{rk}\mathfrak{M}(x_0^\sigma) = \operatorname{rk}\begin{bmatrix} 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & -2 & 0 & -1 \\ Y_1 & 2Y_2 & -Y_1-2Y_2 & 0 & 0 & 0 \\ -Y_1 & -2Y_2 & 0 & Y_1+2Y_2 & 0 & 0 \\ -Y_1 & 0 & 0 & 0 & Y_1 & 0 \\ 0 & -Y_2 & 0 & 0 & 0 & Y_2 \end{bmatrix} = 4 = s-2\,.$$

Next we shall show that the converse of Corollary 3.3.*b* holds. This will allow us to prove Theorem 3.1 in even more cases.

In the sequel $Y_1, \ldots, Y_\Delta$ and $Z_1, \ldots, Z_{s-\Delta}$ will denote sets of independent variables over K, and we let $B_i = \beta_{1i}Z_1 + \ldots + \beta_{s-\Delta i}Z_{s-\Delta}$ for $i=1, \ldots, \Delta$ as well as $D_j = \beta_{j1}Y_1 + \ldots + \beta_{j\Delta}Y_\Delta$ for $j=1, \ldots, s-\Delta$.

LEMMA 3.8. – *Let* $1 \leqslant u < \Delta_{\mathbb{X}}$, *let* $1 \leqslant v < s-\Delta_{\mathbb{X}}$, *and let* $M_{u,v}$ *be the matrix*

$$M_{u,v} = \begin{bmatrix} -B_1 & & 0 & \beta_{11}Z_1 & \ldots & \beta_{v1}Z_v \\ & \ddots & & \vdots & & \vdots \\ 0 & & -B_u & \beta_{1u}Z_1 & \ldots & \beta_{vu}Z_v \\ \beta_{11}Y_1 & \ldots & \beta_{1u}Y_u & -D_1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \beta_{v1}Y_1 & \ldots & \beta_{vu}Y_u & 0 & & -D_v \end{bmatrix}.$$

Then $\det M_{u,v}=0$ *implies that* $\mathbb{X}$ *splits cohomologically.*

PROOF. – We proceed by induction on $u+v$. In case $u+v=2$, we are considering the matrix $M_{1,1} = \begin{pmatrix} -B_1 & \beta_{11}Z_1 \\ \beta_{11}Y_1 & -D_1 \end{pmatrix}$. Since $\det M_{1,1}=0$, none of the variables $Y_2, \dots, Y_\Delta$, $Z_2, \dots, Z_{s-\Delta}$ may occur in B_1 or D_1. Hence $\beta_{12} = \dots = \beta_{1\Delta} = 0$ and $\beta_{21} = \dots = \beta_{s-\Delta\,1} = 0$, i.e. $\mathfrak{B}$ splits.

In order to prove the induction step, we grade the polynomial ring $K[Y_1, \dots, Y_\Delta, Z_1, \dots, Z_{s-\Delta}]$ by $\deg Y_1 = \dots = \deg Y_\Delta = \deg Z_1 = \dots = \deg Z_v = 0$ and $\deg Z_{v+1} = \dots = \deg Z_{s-\Delta} = 1$. We expand $\det M_{u,v}$ and look at its leading form. Note that $Z_{v+1}, \dots, Z_{s-\Delta}$ only occur in $B_1, \dots, B_u$, and renumber $P_1, \dots, P_u$ such that $\{1, \dots, u'\}$ with $1 \leqslant u' \leqslant u$ are precisely those indices i among $\{1, \dots, u\}$ for which $\beta_{v+1\,i} = \dots = \beta_{s-\Delta\,i} = 0$.

Here we must have $u' \geqslant 1$, since $u' = 0$ would imply that the leading form of $\det M_{u,v}$ is $\pm(\beta_{v+1\,1} Z_{v+1} + \dots + \beta_{s-\Delta\,1} Z_{s-\Delta}) \dots (\beta_{v+1\,u} Z_{v+1} + \dots + \beta_{s-\Delta\,u} Z_{s-\Delta}) \cdot D_1 \dots D_v \neq 0$, contradicting $\det M_{u,v} = 0$. If $u' \geqslant 1$, the leading form of $\det M_{u,v}$ is $\pm(\beta_{v+1\,u'+1} Z_{v+1} + \dots + \beta_{s-\Delta\,u'+1} Z_{s-\Delta}) \dots (\beta_{v+1\,u} Z_{v+1} + \dots + \beta_{s-\Delta\,u} Z_{s-\Delta}) \cdot \det M_{u',v}$. Thus if $u' < u$, the claim follows from the induction hypothesis.

We are left with the case $u' = u$, i.e. with the case that $\beta_{v+1\,i} = \dots = \beta_{s-\Delta\,i} = 0$ for $i = 1, \dots, u$. In this case we grade the polynomial ring $K[Y_1, \dots, Y_\Delta, Z_1, \dots, Z_{s-\Delta}]$ by $\deg Y_1 = \dots = \deg Y_u = \deg Z_1 = \dots = \deg Z_{s-\Delta} = 0$ and $\deg Y_{u+1} = \dots = \deg Y_\Delta = 1$. Again we consider the leading form of $\det M_{u,v}$ w.r.t. this grading. Note that $Y_{u+1}, \dots, Y_\Delta$ only occur in $D_1, \dots, D_v$, and renumber $P_{\Delta+1}, \dots, P_{\Delta+v}$ in such a way that $\{1, \dots, v'\}$ with $1 \leqslant v' \leqslant v$ are precisely those indices j among $\{1, \dots, v\}$ for which $\beta_{j\,u+1} = \dots = \beta_{j\Delta} = 0$.

Here we have $v' \geqslant 1$, since $v' = 0$ would imply that the leading form of $\det M_{u,v}$ is $\pm(\beta_{1\,u+1} Y_{u+1} + \dots + \beta_{1\Delta} Y_\Delta) \dots (\beta_{v\,u+1} Y_{u+1} + \dots + \beta_{v\Delta} Y_\Delta) \cdot B_1 \dots B_u \neq 0$, contradicting $\det M_{u,v} = 0$. If $v' \geqslant 1$, the leading form of $\det M_{u,v}$ is $\pm(\beta_{v'+1\,u+1} Y_{u+1} + \dots + \beta_{v'+1\,\Delta} Y_\Delta) \dots (\beta_{v\,u+1} Y_{u+1} + \dots + \beta_{v\Delta} Y_\Delta) \cdot \det M_{u,v'}$. Thus if $v' < v$, the claim follows from the induction hypothesis.

Finally we are left with the case $u' = u$ and $v' = v$, i.e. with the case that $\beta_{v+1\,i} = \dots = \beta_{s-\Delta\,i} = 0$ for $i = 1, \dots, u$ and $\beta_{j\,u+1} = \dots = \beta_{j\Delta} = 0$ for $j = 1, \dots, v$. Obviously, in this case the matrix $\mathfrak{B}$ of $\mathbb{X}$ splits, too. ■

PROPOSITION 3.9. – *Let $\mathbb{X} \subseteq \mathbb{P}^d$ be a reduced, 0-dimensional subscheme. The following conditions are equivalent.*

a) $\mathbb{X}$ is cohomologically uniform.

b) There are elements $r \in R_\sigma$, $\varphi \in \mathfrak{I}_{\sigma+1}$ such that conditions 3.2 a)-c) hold.

PROOF. – Because of 3.3 *b*), it suffices to show «*a*) ⇔ *b*)». By 3.5, we have to find an element $r = a_1 g_1 + \dots + a_{s-\Delta} g_{s-\Delta} \in R_\sigma$ such that $\operatorname{rk} \mathfrak{M}(r) = s - 1$. Thus we have to show that the matrix

$$\mathfrak{M} = \begin{pmatrix} -B_1 & & 0 & \beta_{11} Z_1 & \dots & \beta_{s-\Delta\,1} Z_{s-\Delta} \\ & \ddots & & \vdots & & \vdots \\ 0 & & -B_\Delta & \beta_{1\Delta} Z_1 & \dots & \beta_{s-\Delta\,\Delta} Z_{s-\Delta} \\ \beta_{11} Y_1 & \dots & \beta_{1\Delta} Y_\Delta & -D_1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \beta_{s-\Delta\,1} Y_1 & \dots & \beta_{s-\Delta\,\Delta} Y_\Delta & 0 & & -D_{s-\Delta} \end{pmatrix}$$

has rank $s-1$. Let $\mathfrak{M}'$ be the matrix which is obtained from $\mathfrak{M}$ by deleting the last row and the last column. We shall show that $\mathbb{X}$ splits cohomologically, if $\det \mathfrak{M}' = 0$.

Notice that $Z_{s-\Delta}$ appears only in the entries $-B_1, \dots, -B_\Delta$ of $\mathfrak{M}'$. W.l.o.g. let $u \in \{1, \dots, \Delta\}$ be such that $\beta_{s-\Delta\,1} = \dots = \beta_{s-\Delta\,u} = 0$ and $\beta_{s-\Delta\,u+1} \neq 0, \dots, \beta_{s-\Delta\,\Delta} \neq 0$. Here $u = \Delta$ is impossible because of Prop. 2.1. The leading coefficient of $\det \mathfrak{M}'$ w.r.t. $Z_{s-\Delta}$ is $(-\beta_{s-\Delta\,u+1}) \dots (-\beta_{s-\Delta\,\Delta}) \cdot \det M_{u,\,s-\Delta-1}$. Hence $\det M_{u,\,s-\Delta-1} = 0$, and an application of the lemma finishes the proof. ■

Thus we have found yet another characterization of cohomological uniformity. It allows us to determine the generic rank of $\mathfrak{M}(l^\sigma)$ in case $\sigma_{\mathbb{X}} = 1$. Here and later we use the phrase «for generic $l \in R_1$» to mean that there exists a Zariski-open subset U of the affine space $\mathbb{A}(R_1)$ such that the claimed property holds whenever l corresponds to a closed point of U.

COROLLARY 3.10. – *Let $X \subseteq \mathbb{P}^d$ be a reduced 0-dimensional subscheme with $\sigma_{\mathbb{X}} = 1$. For generic $l \in R_1$ we have* $\operatorname{rk} \mathfrak{M}(l) = s-1$.

PROOF. – It suffices to apply the proposition and to note that the condition $\operatorname{rk} \mathfrak{M}(l) = s-1$ defines an open subset of $\mathbb{A}(R_1)$. ■

Our next lemma and proposition constitute the heart of the proof of Theorem 3.1.

LEMMA 3.11. – *Let $\mathbb{X} \subseteq \mathbb{P}^d$ be a nondegenerate, reduced, 0-dimensional subscheme with $\sigma_{\mathbb{X}} \geq 2$, suppose that* $\operatorname{char}(K) \notin \{2, \dots, \sigma_{\mathbb{X}}\}$, *and suppose there are $i \in \{1, \dots, \Delta\}$, $j \in \{1, \dots, s-\Delta\}$, and pairwise distinct elements $\nu_1, \dots, \nu_j \in \{1, \dots, s-\Delta\}$ such that $l(P_{\Delta+\nu_1})^\sigma \beta_{\nu_1 i} + \dots + l(P_{\Delta+\nu_j})^\sigma \beta_{\nu_j i} = 0$ for generic $l \in R_1$. Then we have $\beta_{\nu_1 i} = \dots = \beta_{\nu_j i} = 0$.*

PROOF. – Let $\mathbb{P}^N = \mathbb{P}(A_\sigma)$ with $N = \binom{d+\sigma}{d}$ be the projective space associated with $A_\sigma = K[X_0, \ldots, X_d]_\sigma$. If we use the lexicographically ordered set of monomials $\{X_0^\sigma, X_0^{\sigma-1}X_1, \ldots, X_d^\sigma\}$ of degree σ as a K-basis of A_σ, and if $L = \lambda_0 X_0 + \ldots + \lambda_d X_d \in A_1$ with $\lambda_0, \ldots, \lambda_d \in K$ is a linear form, then $L^\sigma \in A_\sigma$ is given in this basis by the coordinate tuple $(\lambda_0^\sigma, \lambda_0^{\sigma-1}\lambda_1, \ldots, \lambda_d^\sigma)$, where the coefficient of $\lambda_0^{\nu_0} \ldots \lambda_d^{\nu_d}$ is the multinomial coefficient $\binom{\sigma}{\nu_0, \ldots, \nu_d}$. Since $\operatorname{char}(K) \notin \{2, \ldots, \sigma\}$, all of those coefficients are nonzero. In the lexicographically ordered basis $\left\{ \binom{\sigma}{\nu_0, \ldots, \nu_d} X_0^{\nu_0} \ldots X_d^{\nu_d} \,\middle|\, \nu_0 + \ldots + \nu_d = \sigma \right\}$ of A_σ, the element L^σ is then given by the coordinate tuple $(\lambda_0^\sigma, \lambda_0^{\sigma-1}\lambda_1, \ldots, \lambda_d^\sigma)$. The set of all elements of the form L^σ such that $L \in A_1$ is therefore precisely the set of closed points of a Veronese variety V, namely of the σ^{th} Veronese embedding of $\mathbb{P}^d$ in $\mathbb{P}^N$.

Now we consider the linear subspace $\Lambda = \mathbb{P}((I_{\mathbb{X}})_\sigma)$ of $\mathbb{P}^N$. It has codimension $codim(\Lambda, \mathbb{P}^N) = s - \Delta$. Since $I_{\mathbb{X}}$ is a radical ideal, it follows from $L^\sigma \in (I_{\mathbb{X}})_\sigma$ for some $L \in A_1$ that $L \in (I_{\mathbb{X}})_1$, i.e. that $L = 0$, since $\mathbb{X}$ is nondegenerate. Altogether we conclude $\Lambda \cap V = \emptyset$. Next we let $\pi \colon \mathbb{P}^N \setminus \Lambda \to \mathbb{P}^{s-\Delta-1} = \mathbb{P}(R_\sigma)$ be the morphism given by $\langle F \rangle \mapsto \langle F + I_{\mathbb{X}} \rangle$ for $F \in A_\sigma \setminus (I_{\mathbb{X}})_\sigma$, i.e. we let π be the projection from $\mathbb{P}^N$ with center Λ. Since $\Lambda \cap V = \emptyset$, the image $\overline{V} = \pi(V) \subseteq \mathbb{P}^{s-\Delta-1}$ is a closed subscheme (cf. [4], 14.2), and since V is well-known to be irreducible and nondegenerate, the same is true for $\overline{V}$. Thus the set of all closed points of $\mathbb{P}(R_\sigma)$ of the form l^σ with $l \in R_1$ is not contained in a hyperplane of $\mathbb{P}(R_\sigma)$.

If we look at the coordinate system of $\mathbb{P}(R_\sigma)$ which corresponds to the K-basis $\{g_1, \ldots, g_{s-\Delta}\}$ of R_σ, we can use the equation $l^\sigma = l(P_{\Delta+1})^\sigma g_1 + \ldots + l(P_s)^\sigma g_{s-\Delta}$ for $l \in R_1$ to conclude that the closed point $\langle l^\sigma \rangle \in \mathbb{P}(R_\sigma)$ is in that coordinate system given by the tuple $(l(P_{\Delta+1})^\sigma : \ldots : l(P_s)^\sigma)$. By the hypothesis, it is for generically chosen $l \in R_1$ contained in the variety $H = \mathcal{Z}(\beta_{\nu_1 i} X_{\nu_1} + \ldots + \beta_{\nu_j i} X_{\nu_j}) \subseteq \mathbb{P}(R_\sigma)$. Since $\overline{V}$ is nondegenerate, the variety H cannot be a hyperplane, i.e. we have to have $H = \mathbb{P}(R_\sigma)$ and $\beta_{\nu_1 i} = \ldots = \beta_{\nu_j i} = 0$. ■

PROPOSITION 3.12. – *If $\mathbb{X} \subseteq \mathbb{P}^d$ is a nondegenerate, reduced, 0-dimensional, cohomologically uniform scheme, and if* $\operatorname{char}(K) \notin \{2, \ldots, \sigma_{\mathbb{X}}\}$, *then* $\operatorname{rk} \mathfrak{M}(l^\sigma) = s - 1$ *for generic* $l \in R_1$.

PROOF. – Because of 3.10 it suffices to consider the case $\sigma_{\mathbb{X}} \geq 2$. We renumber the points $P_{\Delta+1}, \ldots, P_s$ such that $\beta_{11} \neq 0, \ldots, \beta_{t_1 1} \neq 0$, $\beta_{t_1+1\,1} = \ldots =$

$\beta_{s-\varDelta\,1}=0$ for some $1\leqslant t_1\leqslant s-\varDelta$. Note that $t_1\geqslant 1$, since $\beta_{11}+\ldots+\beta_{s-\varDelta\,1}=-1$ (cf. 1.2.a). Next we renumber $P_2,\ldots,P_\varDelta$ such that one of $\beta_{12},\ldots,\beta_{t_12}$ is not zero. This is possible, because $\mathfrak{B}$ does not split. Then we renumber $P_{\varDelta+t_1+1},\ldots,P_s$ such that $\beta_{t_1+1\,2}\neq 0$, ..., $\beta_{t_22}\neq 0$, $\beta_{t_2+1\,2}=\ldots=\beta_{s-\varDelta\,2}=0$ for some $t_1\leqslant t_2\leqslant s-\varDelta$, and we renumber $P_3,\ldots,P_\varDelta$ such that one of $\beta_{13},\ldots,\beta_{t_23}$ is not zero. Continuing this way, we find numbers $1\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_{\varDelta-1}\leqslant s-\varDelta$ such that the matrix $\mathfrak{B}$ of $\mathbb{X}$ looks as follows:

$$\mathfrak{B}=\begin{bmatrix}
\beta_{11} & \cdots & \beta_{t_11} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\beta_{12} & \cdots & \beta_{t_12} & \beta_{t_1+1\,2} & \cdots & \beta_{t_22} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & & \vdots & \beta_{t_2+1\,3} & \cdots & \beta_{t_33} & \cdots & \vdots & & \vdots\\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & & 0 & \cdots & 0\\
\beta_{1\varDelta} & \cdots & \beta_{t_1\varDelta} & \beta_{t_1+1\,\varDelta} & \cdots & \beta_{t_2\varDelta} & \beta_{t_2+1\,\varDelta} & \cdots & \beta_{t_3\varDelta} & \cdots & \beta_{t_{\varDelta-1}+1\,\varDelta} & \cdots & \beta_{s-\varDelta\,\varDelta}
\end{bmatrix}.$$

Notice that $t_\varDelta=s-\varDelta$, because no column of $\mathfrak{B}$ is zero by 2.1. For $j=1,\ldots,s-\varDelta$ we let $\lambda_j=l(P_{\varDelta+j})^\sigma$. In addition, we have arranged this matrix such that $\gamma_2=\lambda_1\beta_{12}+\ldots+\lambda_{t_1}\beta_{t_12}\neq 0$, ..., $\gamma_\varDelta=\lambda_1\beta_{1\varDelta}+\ldots+\lambda_{t_{\varDelta-1}}\beta_{t_{\varDelta-1}\varDelta}\neq 0$ for generic $\ell\in R_1$.

In the matrix $\mathfrak{M}(l^\sigma)$ we add columns $2,\ldots,\varDelta+t_1$ to column 1 and obtain the following matrix

$$\mathfrak{M}_0=$$

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & \lambda_1\beta_{11} & \cdots & \lambda_{t_1}\beta_{t_11} & 0 & \cdots & 0\\
\delta_2 & -b_2 & & 0 & \lambda_1\beta_{12} & \cdots & \lambda_{t_1}\beta_{t_12} & \lambda_{t_1+1}\beta_{t_1+1\,2} & \cdots & \lambda_{s-\varDelta}\beta_{s-\varDelta\,2}\\
\vdots & & \ddots & & \vdots & & \vdots & \vdots & & \vdots\\
\delta_\varDelta & 0 & & -b_\varDelta & \lambda_1\beta_{1\varDelta} & \cdots & \lambda_{t_1}\beta_{t_1\varDelta} & \lambda_{t_1+1}\beta_{t_1+1\,\varDelta} & \cdots & \lambda_{s-\varDelta}\beta_{s-\varDelta\,\varDelta}\\
0 & \beta_{12}Y_2 & \cdots & \beta_{1\varDelta}Y_\varDelta & -D_1 & & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \ddots & & \vdots & & \vdots\\
0 & \beta_{t_12}Y_2 & \cdots & \beta_{t_1\varDelta}Y_\varDelta & 0 & & -D_{t_1} & 0 & \cdots & 0\\
D_{t_1+1} & \beta_{t_1+1\,2}Y_2 & \cdots & \beta_{t_1+1\,\varDelta}Y_\varDelta & 0 & \cdots & 0 & -D_{t_1+1} & & 0\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\
D_{s-\varDelta} & \beta_{s-\varDelta\,2}Y_2 & \cdots & \beta_{s-\varDelta\,\varDelta}Y_\varDelta & 0 & \cdots & 0 & 0 & & -D_{s-\varDelta}
\end{bmatrix}.$$

If $\varDelta_{\mathbb{X}}=1$, we have $t_1=s-\varDelta$, and the claim $\operatorname{rk}\mathfrak{M}(l^\sigma)=\operatorname{rk}\mathfrak{M}_0=s-1$ follows from $(-D_1)\ldots(-D_{s-\varDelta})\neq 0$. Therefore we shall assume $\varDelta_{\mathbb{X}}\geqslant 2$ from now on.

Let the matrix $\mathfrak{M}_1$ be obtained from $\mathfrak{M}_0$ by deleting its first row and its first column. We shall show $\operatorname{rk}\mathfrak{M}(l^\sigma)=\operatorname{rk}\mathfrak{M}_0=s-1$ by proving $\det\mathfrak{M}_1\neq 0$. Notice that Y_1 appears only in the entries $-D_1, \ldots, -D_{t_1}$ of $\mathfrak{M}_1$. Let the matrix $\mathfrak{M}_2$ be obtained form $\mathfrak{M}_1$ by deleting the columns and rows containing those entries. Then the coefficient of $Y_1^{t_1}$ in $\det \mathfrak{M}_1$ is $(-\beta_{11})\ldots(-\beta_{t_1 1})\cdot\det\mathfrak{M}_2$. Hence $\det\mathfrak{M}_2 \neq 0$ implies $\det\mathfrak{M}_1 \neq 0$.

More generally, for each $i \in \{2, \ldots, \Delta\}$ we define a submatrix $\mathfrak{M}_i$ of $\mathfrak{M}_1$ by

$$\mathfrak{M}_i = \begin{pmatrix} -b_i & & 0 & \lambda_{t_{i-1}+1}\beta_{t_{i-1}+1\,i} & \cdots & \lambda_{t_i}\beta_{t_i i} & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & & -b_\Delta & \lambda_{t_{i-1}+1}\beta_{t_{i-1}+1\,\Delta} & \cdots & \lambda_{t_i}\beta_{t_i\Delta} & \lambda_{t_i+1}\beta_{t_i+1\,\Delta} & \cdots & \lambda_{s-\Delta}\beta_{s-\Delta\,\Delta} \\ \beta_{t_{i-1}+1\,i}Y_i & \cdots & \beta_{t_{i-1}+1\,\Delta}Y_\Delta & -D_{t_{i-1}+1} & & & 0 & \cdots & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \beta_{t_i i}Y_i & \cdots & \beta_{t_i\Delta}Y_\Delta & 0 & & -D_{t_i} & 0 & \cdots & 0 \\ 0 & \cdots & \beta_{t_i+1\,\Delta}Y_\Delta & 0 & \cdots & 0 & -D_{t_i+1} & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & \beta_{s-\Delta\,\Delta}Y_\Delta & 0 & \cdots & 0 & 0 & & -D_{s-\Delta} \end{pmatrix}.$$

Using downward induction on $i \in \{2, \ldots, \Delta\}$, we shall show that if $\det\mathfrak{M}_i = 0$, then the matrix $\mathfrak{B}$ splits. For cohomologically uniform schemes $\mathbb{X}$, this proves $\det\mathfrak{M}_2 \neq 0$.

We begin the induction by looking at the case $i = \Delta$. If $\det\mathfrak{M}_\Delta = 0$, then $t_{\Delta-1} < s - \Delta$, since $t_{\Delta-1} = s - \Delta$ implies $\det\mathfrak{M}_\Delta = -b_\Delta \neq 0$ for generic $\ell \in R_1$. Starting from column 2, we add all columns of $\mathfrak{M}_\Delta$ to column 1, and we obtain the matrix

$$\mathfrak{M}'_\Delta = \begin{pmatrix} -\gamma_\Delta & \lambda_{t_{\Delta-1}+1}\beta_{t_{\Delta-1}+1\,\Delta} & \cdots & \lambda_{s-\Delta}\beta_{s-\Delta\,\Delta} \\ 0 & -D_{t_{\Delta-1}+1} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & -D_{s-\Delta} \end{pmatrix}$$

which has $\det\mathfrak{M}'_\Delta = \det\mathfrak{M}_\Delta \neq 0$, since $-\gamma_\Delta = -\lambda_1\beta_{1\Delta} - \ldots - \lambda_{t_{\Delta-1}}\beta_{t_{\Delta-1}\,\Delta} \neq 0$ for generic $\ell \in R_1$.

Finally we prove the induction step. We assume $i < \Delta$ and $\det\mathfrak{M}_i = 0$. If $t_{i-1} = t_i$, then the only nonzero entry in the first row of $\mathfrak{M}_i$ is $-b_i$, and we have $\det\mathfrak{M}_i = (-b_i)\cdot\det\mathfrak{M}_{i+1}$. In this case the claim follows from the induction hypothesis, because $b_i \neq 0$ for generic $\ell \in R_1$. Thus we may also assume $t_{i-1} < t_i$. Then we add columns $2, \ldots, \Delta - i + 1 + t_i - t_{i-1}$ of $\mathfrak{M}_i$ to column 1, and we obtain the following matrix

$$\mathfrak{M}_i' =$$

$$\left[\begin{array}{cccccccccc}
-\gamma_i & 0 & \dots & 0 & \lambda_{t_{i-1}+1}\beta_{t_{i-1}+1\, i} & \dots & \lambda_{t_i}\beta_{t_i i} & 0 & \dots & 0 \\
\eta_{i+1} & -b_{i+1} & & 0 & \lambda_{t_{i-1}+1}\beta_{t_{i-1}+1\, i+1} & \dots & \lambda_{t_i}\beta_{t_i\, i+1} & * & \dots & * \\
\vdots & & \ddots & & \vdots & & \vdots & \vdots & & \vdots \\
\eta_\Delta & 0 & & -b_\Delta & \lambda_{t_{i-1}+1}\beta_{t_{i-1}+1\, \Delta} & \dots & \lambda_{t_i}\beta_{t_i \Delta} & * & \dots & * \\
0 & \beta_{t_{i-1}+1\, i+1}Y_{i+1} & \dots & \beta_{t_i\, \Delta}Y_\Delta & -D_{t_{i-1}+1} & & 0 & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
0 & \beta_{t_i\, i+1}Y_{i+1} & \dots & \beta_{t_i\, \Delta}Y_\Delta & 0 & & -D_{t_i} & 0 & \dots & 0 \\
D_{t_i+1} & \beta_{t_i+1\, i+1}Y_{i+1} & \dots & \beta_{t_i+1\, \Delta}Y_\Delta & 0 & \dots & 0 & -D_{t_i+1} & & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\
D_{s-\Delta} & \beta_{s-\Delta\, i+1}Y_{i+1} & \dots & \beta_{s-\Delta\, \Delta}Y_\Delta & 0 & \dots & 0 & 0 & & -D_{s-\Delta}
\end{array}\right].$$

Here the elements « $*$ » have to be replaced by the corresponding entries of $\mathfrak{M}_i$. Notice that the only entries of $\mathfrak{M}_i'$ containing Y_i are $-D_{t_{i-1}+1}, \dots, -D_{t_i}$. Let the matrix $\mathfrak{M}_i\ll$ be obtained form $\mathfrak{M}_i'$ by deleting the rows and columns containing those entries. Then the coefficient of $Y_i^{t_i - t_{i-1}}$ in $\det \mathfrak{M}_i'$ is $(-\beta_{t_{i-1}+1\ i}) \dots (-\beta_{t_i i}) \cdot \det \mathfrak{M}_i\ll$, and $\det \mathfrak{M}_i = \det \mathfrak{M}_i' = 0$ implies $\det \mathfrak{M}_i'' = 0$. We observe that $-\gamma_i = -\lambda_1 \beta_{1i} - \dots - \lambda_{t_{i-1}} \beta_{t_{i-1}\ i} \neq 0$ is the only nonzero entry in the first row of $\mathfrak{M}_i''$, and that $\det \mathfrak{M}_i'' = (-\gamma_i) \cdot \det \mathfrak{M}_{i+1}$. Thus the claim follows from the induction hypothesis. ■

In view of 3.3 *a*), 3.5, and 3.12, the proof of Theorem 3.1 is now complete.

4. – The syzygy module of the canonical ideal.

In this section we give a description of the first syzygy module of the canonical ideal in the case of $d+1 < s < \binom{d+2}{2}$ points $\mathbb{X} = \{P_1, \dots, P_s\}$ in $\mathbb{P}^d$. We shall assume that $\mathbb{X}$ is nondegenerate, cohomologically uniform and satisfies $\sigma_{\mathbb{X}} = 1$. We use the notations introduced in the previous sections. Our first goal is to find an explicit homogeneous system of generators of the first syzygy module of $\mathfrak{I}$.

The Hilbert function of $\mathfrak{I}$ is $H_{\mathfrak{I}}$: $0\ 0\ s-d-1\ s-1\ s\ s \dots$, and a K-basis of $\mathfrak{I}_2$ is given by $\{\pi_1, \dots, \pi_\Delta\}$, where $\Delta = s-d-1$. Since $\mathbb{X}$ is cohomologically uniform, Prop. 2.7 implies that $\{\pi_1, \dots, \pi_\Delta\}$ is a minimal homogeneous system of generators of $\mathfrak{I}$. Now we choose a generic element $\varphi = c_1 f_1 + \dots + c_s f_s \in \mathfrak{I}_2$ with $c_1, \dots, c_s \in K$. By using the representation $\varphi = c_1 \pi_1 + \dots + c_\Delta \pi_\Delta$ and

Prop. 2.1, we see that we may assume $c_1, \ldots, c_s \neq 0$. After a generic change of coordinates, we may also assume by Prop. 3.9 that

$$\mathfrak{I}_3 = R_1 \varphi \oplus K x_0 \pi_2 \oplus \ldots \oplus K x_0 \pi_\Delta .$$

Therefore there are unique elements $l_{ij} \in R_1$ and $\lambda_{ij}^{(2)}, \ldots, \lambda_{ij}^{(\Delta)} \in K$ such that

$$x_i \pi_j = l_{ij} \varphi + \lambda_{ij}^{(2)} x_0 \pi_2 + \ldots + \lambda_{ij}^{(\Delta)} x_0 \pi_\Delta$$

for $i = 1, \ldots, d$ and $j = 2, \ldots, \Delta$. Since $\{\varphi, \pi_2, \ldots, \pi_\Delta\}$ is a minimal homogeneous system of generators of $\mathfrak{I}$, we can use those relations to describe the R-module

$$S_R := \mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta) = \{(r_1, \ldots, r_\Delta) \in R^\Delta \mid r_1 \varphi + r_2 \pi_2 + \ldots + r_\Delta \pi_\Delta = 0\}$$

explicitly.

REMARKS 4.1. – *a*) From the exact sequence of graded R-modules

$$0 \to \mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta) \to R(-2)^\Delta \xrightarrow{\varepsilon} \mathfrak{I} \to 0$$

with $\varepsilon(e_1) = \varphi$ and $\varepsilon(e_i) = \pi_i$ for $i = 2, \ldots, \Delta$, we obtain H_{S_R}: $0\ 0\ 0\ (\Delta - 1)\, d\ (\Delta - 1)\, s (\Delta - 1)\, s \ldots$.

b) The elements

$$\sigma_{ij} = (l_{ij}, \lambda_{ij}^{(2)} x_0, \ldots, \lambda_{ij}^{(j-1)} x_0, \lambda_{ij}^{(j)} x_0 - x_i, \lambda_{ij}^{(j+1)} x_0, \ldots, \lambda_{ij}^{(\Delta)} x_0) \in R(-2)^\Delta$$

such that $1 \leqslant i \leqslant d$ and $2 \leqslant j \leqslant \Delta$ form a K-basis of $(S_R)_3$, since they are K-linearly independent and $H_{S_R}(3) = (\Delta - 1)\, d$.

c) Let $P = K[X_0, \ldots, X_d]$ and

$$S_P = \mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta) = \{(F_1, \ldots, F_\Delta) \in P^\Delta \mid F_1 \varphi + F_2 \pi_2 + \ldots + F_\Delta \pi_\Delta = 0\}.$$

From the exact sequence of graded P-modules

$$0 \to \mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta) \to P(-2)^\Delta \to \mathfrak{I} \to 0$$

we obtain an exact sequence of graded R-modules

$$0 \to \mathrm{Tor}_P^1(\mathfrak{I}, R) \to \mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta) \otimes R \to \mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta) \to 0 .$$

In particular, we get $H_{S_P}(3) = (\Delta - 1)\, d = H_{S_R}(3)$. Therefore any set of preimages

$$\Sigma_{ij} = (L_{ij}, \lambda_{ij}^{(2)} X_0, \ldots, \lambda_{ij}^{(j)} X_0 - X_i, \lambda_{ij}^{(\Delta)} X_0) \in P(-2)^\Delta$$

of the elements $\sigma_{ij} \in R(-2)^\Delta$ forms a K-basis of $(S_P)_3$.

d) From the short exact sequence $0 \to I_{\mathbb{X}} \to P \to R \to 0$ we get a long exact sequence of graded P-modules

$$0 \to \mathrm{Tor}_P^1(\mathfrak{I}, R) \to I_{\mathbb{X}} \underset{P}{\otimes} \mathfrak{I} \to \mathfrak{I} \xrightarrow{\mathrm{id}} \mathfrak{I} \to 0$$

from which we conclude that $\mathrm{Tor}_P^1(\mathfrak{I}, R) \cong I_{\mathbb{X}} \underset{P}{\otimes} R \underset{R}{\otimes} \mathfrak{I} \cong (I_{\mathbb{X}}/I_{\mathbb{X}}^2) \underset{R}{\otimes} \mathfrak{I}$. Thus the second exact sequence of *c*) translates in degree four into a short exact sequence of K-vector spaces

$$0 \to (I_{\mathbb{X}})_2 \otimes \mathfrak{I}_2 \xrightarrow{\iota} \mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta)_4 \to \mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta)_4 \to 0$$

where ι is given by $\iota(F_1 \otimes \varphi + F_2 \otimes \pi_2 + \ldots + F_\Delta \otimes \pi_\Delta) = (F_1, \ldots, F_\Delta)$ for $F_1, \ldots, F_\Delta \in (I_{\mathbb{X}})_2$.

Next we determine how big a part of $\mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta)_4$ is generated by the elements Σ_{ij}.

LEMMA 4.2. – *a*) *The set* $\{\Sigma_{ij} \mid 1 \leqslant i \leqslant d,\, 2 \leqslant j \leqslant \Delta\}$ *generates a subspace of* $\mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta)_4$ *which is at least* $(\Delta - 1)((1/2)\, d^2 + (3/2)\, d)$*-dimensional.*

b) *The P-module* $\mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta)$ *is generated by the elements*

$$\{\Sigma_{ij} \mid 1 \leqslant i \leqslant d,\, 2 \leqslant j \leqslant \Delta\} \cup \{(F, 0, \ldots, 0) \mid F \in (I_{\mathbb{X}})_2\} \cup \{\Theta_j \mid 2 \leqslant j \leqslant \Delta\}$$

where $\Theta_j = (Q_j, 0, \ldots, 0, -X_0^2, 0, \ldots, 0)$ *and Q_j is a preimage in P_2 of the unique element* $q_j \in R_2$ *such that* $x_0^2 \pi_j = q_j \varphi$.

PROOF. – «*a*)» Let $\leqslant_\sigma$ be the lexicographical term ordering on P, where $X_0 <_\sigma X_1 <_\sigma \ldots <_\sigma X_d$. On $P(-2)^\Delta$ we define a module term ordering $\leqslant_\tau$ in the following way. If $t_1, t_2 \in P$ are power products of variables, and if $i, j \in \{1, \ldots, \Delta\}$, then we let $t_1 e_i \leqslant_\tau t_2 e_j$ if and only if one of four conditions is satisfied:

1) $i = j = 1$ and $t_1 \leqslant_\sigma t_2$,

2) $i = 1$ and $j > 1$,

3) $i > 1$, $j > 1$, and $t_1 <_\sigma t_2$,

4) $1 < i \leqslant j$ and $t_1 = t_2$.

It is easy to check that $\leqslant_\tau$ is in fact a module term ordering (cf. [4], p. 324) and that the elements Σ_{ij} have leading terms $\mathrm{Lt}_\tau(\Sigma_{ij}) = (0, \ldots, 0, -X_i, 0, \ldots, 0)$. Now it is a standard fact that, for $U = \langle\{\Sigma_{ij} \mid 1 \leqslant i \leqslant d,\, 2 \leqslant j \leqslant \Delta\}\rangle \subseteq P(-2)^\Delta$, we have

$$\dim_K(U_4) = \dim_K(\mathrm{Lt}_\tau(U))_4 \geqslant \dim_K \langle\{\mathrm{Lt}_\tau(\Sigma_{ij}) \mid 1 \leqslant i \leqslant d,\, 2 \leqslant j \leqslant \Delta\}\rangle_4$$

(cf. [4], 15.26). The last number is obviously $(\Delta - 1)((1/2)\, d^2 + (3/2)\, d)$.

«b)» By subtracting suitable multiples of the syzygies Σ_{ij}, every P-syzygy of $(\varphi, \pi_2, \ldots, \pi_\Delta)$ can be reduced to one of the form $(F, \gamma_2 X_0^2, \ldots, \gamma_\Delta X_0^2)$ with $F \in P_2$ and $\gamma_2, \ldots, \gamma_\Delta \in K$. Then one can subtract $\sum_{j=2}^{\Delta} \gamma_j \Theta_j$. The result is a syzygy of the form $(G, 0, \ldots, 0)$ with $G \in P_2$. Since $\mathrm{Ann}_R(\varphi) = (0)$, this implies $G \in (I_{\mathbb{X}})_2$. It is well-known that it follows from $\sigma_{\mathbb{X}} = 1 = \alpha_{\mathbb{X}} - 1$ that $\mathrm{Syz}_P^1(\varphi, \pi_2, \ldots, \pi_\Delta)$ is generated by its homogeneous elements of degrees three and four (cf. e.g. [10], 5.1). ∎

PROPOSITION 4.3. – *For $j = 2, \ldots, \Delta$ let $\theta_j = (q_j, 0, \ldots, 0, -x_0^2, 0, \ldots, 0) \in R(-2)^\Delta$ be the image of Θ_j (cf. 4.2 b)). The graded R-module $\mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta)$ is generated by the homogeneous elements $\{\sigma_{ij} \mid 1 \leqslant i \leqslant d, 2 \leqslant j \leqslant \Delta\}$ of degree three and the homogeneous elements $\{\theta_j \mid 2 \leqslant j \leqslant \Delta\}$ of degree four.*

PROOF. – The claim is a consequence of Remark 4.1.*c* and Lemma 4.2.*b*. ∎

Notice that the elements σ_{ij} are minimal generators of $\mathrm{Syz}_R^1(\varphi, \pi_2, \ldots, \pi_\Delta)$, whereas it is not clear how many of the elements θ_j are actually needed. In the case $\Delta = 2$, i.e. for $s = d + 3$ points in $\mathbb{P}^d$, we can in fact do without the elements θ_j.

PROPOSITION 4.4. – *Let $d \geqslant 3$, and let $\mathbb{X} \subseteq \mathbb{P}^d$ be a set of $s = d + 3$ cohomologically uniform points with generic Hilbert function $\Delta H_{\mathbb{X}}$: 1 d 2. Then the R-module $\mathrm{Syz}_R^1(\varphi, \pi_2)$ is generated by its homogeneous elements of degree three.*

PROOF. – We choose the coordinate system and the element $\varphi \in \mathfrak{I}_2$ as at the beginning of this section. For $1 \leqslant i \leqslant d$ we write $x_i \pi_2 = l_i \varphi + \lambda_i x_0 \pi_2$ with $l_i \in R_1$ and $\lambda_i \in K$, and we let $\sigma_i = (l_i, \lambda_i x_0 - x_i) \in R(-2)^2$. Furthermore, we let $q \in R_2$ be the unique element such that $x_0^2 \pi_2 = q\varphi$. By Prop. 4.3, we need to show that $(-q, x_0^2) \in \sum_{i=1}^{d} P_1 \sigma_i$. Since $\mathrm{Ann}_R(\varphi) = (0)$, this amounts to showing that x_0^2 is an element of the ideal $J = (\lambda_1 x_0 - x_1, \ldots, \lambda_d x_0 - x_d)$ of R.

As a first step, we prove that not all elements $l_1, \ldots, l_d \in R_1$ can be zero. Otherwise $(\lambda_i x_0 - x_i)\pi_2 = 0$ for $i = 1, \ldots, d$ and $\pi_2 = f_2 + \beta_{12} f_3 + \ldots + \beta_{d+1\,2} f_{d+3}$ imply that $(\lambda_1 x_0 - x_1, \ldots, \lambda_d x_0 - x_d)$ equals $\mathfrak{p}_2$, the ideal of P_2 in $\mathbb{X}$. Since $\mathbb{X}$ does not split cohomologically, Lemma 2.3 yields $\beta_{j2} \neq 0$ for some $j \in \{1, \ldots, d+1\}$. Since not all elements of $\mathfrak{p}_2$ vanish at P_{j+2}, there exists a $k \in \{1, \ldots, d+1\}$ such that $(\lambda_k x_0 - x_k)(P_{j+2}) \neq 0$ Now $(\lambda_k x_0 - x_k)\,\pi_2(P_{j+2}) = (\lambda_k x_0 - x_k)(P_{j+2})\beta_{j2} = 0$ implies $(\lambda_k x_0 - x_k)(P_{j+2}) = 0$, a contradiction.

Let $Q = (1: \lambda_1: \ldots: \lambda_d) \in \mathbb{P}^d$. In the next step, we prove $Q \notin \mathbb{X}$. Suppose $Q = P_1$, and let $\mathfrak{p}_1 \subseteq R$ be the ideal of P_1 in $\mathbb{X}$. If we use the equation $l_i \varphi + (\lambda_i x_0 - x_i)\pi_2 = 0$, as well as $\lambda_i x_0 - x_i \in J = \mathfrak{p}_1$ and $\pi_2 = f_2 + \beta_{12} f_3 + \ldots + \beta_{d+1\,2} f_{d+3} \in \mathfrak{p}_1$, we obtain $l_i \varphi + \mathfrak{p}_1^2 = 0$ in $R/\mathfrak{p}_1^2$. Since the residue class of $\varphi \notin \mathfrak{p}_1$ is not a zero divisor in that ring, we arrive at $l_i \in \mathfrak{p}_1^2$, which means $l_i = 0$, because $l_i \in R_1$ and $\mathfrak{p}_1^2$ is generated in degree two. But the first step shows that $l_i = 0$ cannot hold simultaneously for $i = 1, \ldots, d$.

Next we suppose $Q \in \{P_2, \ldots, P_s\}$. Then the equation $l_i \varphi + (\lambda_i x_0 - x_i)\pi_2 = 0$ shows that l_i vanishes at P_1 and Q, i.e. there are elements $c_{ij} \in K$ such that $l_i = \sum_{j=1}^{d} c_{ij}(\lambda_j x_0 - x_j)$ for $i = 1, \ldots, d$. Furthermore, by passing to the ring R/J^2 and arguing as before, we see that $\pi_2(Q) \neq 0$, i.e. Q has to be P_2 or one of the points P_j such that $3 \leq j \leq d+3$ and $\beta_{j-2\,2} \neq 0$. Using the representation of $l_i \in J$, we get equations $\sum_{j=1}^{d} c_{ij} l_j \varphi + l_i \pi_2 = 0$. Again the conditions $l_i \in \mathfrak{p}_1$, $\pi_2 \in \mathfrak{p}_1$, and $\varphi \notin \mathfrak{p}_1$ can only be satisfied if $\sum_{j=1}^{d} c_{ij} l_j = 0$ and $l_i \pi_2 = 0$. Now $l_i \varphi + (\lambda_i x_0 - x_i)\pi_2 = 0$ implies that $l_i(P_\nu) = 0$ for all $\nu \in \{1, \ldots, d+3\}$ such that $\pi_2(P_\nu) = 0$, while $l_i \pi_2 = 0$ means $l_i(P_\nu) = 0$ for all $\nu \in \{1, \ldots, d+3\}$ such that $\pi_2(P_\nu) \neq 0$. Thus we get $l_i = 0$ for $i = 1, \ldots, d$, contradicting the first step.

Altogether we have shown that $Q \notin \mathbb{X}$, and we can consider the Hilbert function of $\mathbb{Y} = \mathbb{X} \cup \{Q\}$. We claim that it is given by $\Delta H_{\mathbb{Y}}$: $1\ d\ 3$. The only other possibility is $\Delta H_{\mathbb{Y}}$: $1\ d\ 2\ 1$. In that case the canonical Ideal $\mathfrak{J}_{\mathbb{Y}}$ of $\mathbb{Y}$ starts in degree three and satisfies $H_{\mathfrak{J}_{\mathbb{Y}}}(3) = 1$, $H_{\mathfrak{J}_{\mathbb{Y}}}(4) = 3$. Thus a nonzero element of $(\mathfrak{J}_{\mathbb{Y}})_3$ has a nontrivial annihilator, i.e. $\mathbb{Y}$ is not a Cayley-Bacharach scheme. Therefore there exists an $i \in \{1, \ldots, s\}$ such that $\Delta H_{\mathbb{Y} \setminus \{P_i\}}$: $1\ d\ 1\ 1$ and $\Delta H_{\mathbb{X} \setminus \{P_i\}}$: $1\ d\ 1$. Now [7], 5.2 shows that Q and three points of $\mathbb{X}$ lie on a line in $\mathbb{P}^d$. Since the other d points of $\mathbb{X}$ span at most a $(d-1)$-dimensional linear space, we get that $\mathbb{X}$ splits linearly, in contradiction with our hypothesis.

Thus we have $H_{\mathbb{Y}}$: $1\ d+1\ d+4\ d+4\ \ldots$, and the formula $H_{R/J}(i) = H_{\mathbb{X}}(i) + H_{\{Q\}}(i) - H_{\mathbb{Y}}(i)$ proves $H_{R/J}(2) = 0$, so that $x_0^2 \in J$, as was to be shown. ∎

It is instructive to compare the preceding result to a similar one for the P-module $Syz_P^1(\varphi, \pi_2)$ which follows from [3].

REMARK 4.5. – Let $d \geq 3$, and let $\mathbb{X} \subseteq \mathbb{P}^d$ be a set of $s = d+3$ points. Then the following conditions are equivalent.

a) The set $\mathbb{X}$ is cohomologically uniform, has generic Hilbert function $\Delta H_{\mathbb{X}}$: $1\ d\ 2$, and $\mathrm{Syz}^1_P(\varphi, \pi_2)$ is minimally generated by $(\Delta-1)\,d$ homogeneous elements of degree three and $d-1$ homogeneous elements of degree four. (The number $d-1$ is the minimal possible one, here.)

b) No $d+2$ points of $\mathbb{X}$ are on a hyperplane and no d points of $\mathbb{X}$ are on a linear subspace of $\mathbb{P}^d$ of dimension $d-2$.

The implication «*a*)-*b*)» follows from 2.7 and [3], 4.3. Conversely, [3], 1.3 shows that $\mathbb{X}$ has generic Hilbert function, [3], 4.1 and our Prop. 2.7 imply that $\mathbb{X}$ is cohomologically uniform, and [3], 4.3 yields the remaining claim.

It is an elementary exercise to verify directly that condition 4.5 *b*) implies that $\mathbb{X}$ does not split linearly. Our next example shows that it is in fact a stronger hypothesis.

EXAMPLE 4.6. – Let $\mathbb{X} \subseteq \mathbb{P}^3$ consist of three points on a line and three generically chosen points. Then we have $\Delta H_{\mathbb{X}}$: $1\ 3\ 2$, and $\mathbb{X}$ is cohomologically uniform. But $\mathrm{Syz}^1_P(\varphi, \pi_2)$ needs three minimal generators in degree four besides its three minimal generators of degree three. The module $\mathrm{Syz}^1_R(\varphi, \pi_2)$, however, is minimally generated in degree three by Prop. 4.4.

To conclude the discussion of $\mathrm{Syz}^1_R(\varphi, \pi_2, \ldots, \pi_\Delta)$, we show how the elements σ_{ij} and θ_j can be computed effectively in terms of the constants β_{ij}, the separators $f_1, \ldots, f_s \in R_2$, and the coordinates of the points. We note that the numbers β_{ij} and the separators $f_1, \ldots, f_s \in R_2$ can be found as a by-product of the Buchberger-Möller Algorithm (cf. [12], sect. 1.2).

REMARK 4.7. – Let us write $x_0 l_{ij} = l_{ij}^{(1)} f_1 + \ldots + l_{ij}^{(s)} f_s$ with $l_{ij}^{(1)}, \ldots, l_{ij}^{(s)} \in K$. Recall that we have $P_i = (1 : p_{i1} : \ldots : p_{id})$ for $i = 1, \ldots, s$ and $\varphi = c_1 f_1 + \ldots + c_s f_s$ with $c_{\Delta+k} = c_1 \beta_{k1} + \ldots + c_\Delta \beta_{k\Delta}$ for $k = 1, \ldots, s-\Delta$. For $i = 1, \ldots, d$ and $j = 2, \ldots, \Delta$ we express both sides of $x_i \pi_j = l_{ij}\varphi + \lambda_{ij}^{(2)} x_0 \pi_2 + \ldots + \lambda_{ij}^{(\Delta)} x_0 \pi_\Delta$ in the K-basis $\{x_0 f_1, \ldots, x_0 f_s\}$ of R_3 (cf. [6], 1.13). We get

$$x_i \pi_j = x_0^{-1}(p_{1i} f_1 + \ldots + p_{si} f_s)(f_j + \beta_{1j} f_{\Delta+1} + \ldots + \beta_{s-\Delta\, j} f_s) =$$

$$p_{ji} x_0 f_j + p_{\Delta+1\, i} \beta_{1j} x_0 f_{\Delta+1} + \ldots + p_{si} \beta_{s-\Delta\, j} x_0 f_s$$

as well as

$$l_{ij}\varphi + \lambda_{ij}^{(2)} x_0 \pi_2 + \ldots + \lambda_{ij}^{(\Delta)} x_0 \pi_\Delta = l_{ij}^{(1)} c_1 x_0 f_1 + \ldots + l_{ij}^{(s)} c_s x_0 f_s +$$

$$(\lambda_{ij}^{(2)} x_0 f_2 + \lambda_{ij}^{(2)} \beta_{12} x_0 f_{\Delta+1} + \ldots + \lambda_{ij}^{(2)} \beta_{s-\Delta\, 2} x_0 f_s) + \ldots +$$

$$(\lambda_{ij}^{(\Delta)} x_0 f_\Delta + \lambda_{ij}^{(\Delta)} \beta_{1\Delta} x_0 f_{\Delta+1} + \ldots + \lambda_{ij}^{(\Delta)} \beta_{s-\Delta\, \Delta} x_0 f_s) =$$

$$l_{ij}^{(1)} c_1 x_0 f_1 + (l_{ij}^{(2)} c_2 + \lambda_{ij}^{(2)}) x_0 f_2 + \ldots + (l_{ij}^{(\Delta)} c_\Delta + \lambda_{ij}^{(\Delta)}) x_0 f_\Delta + (l_{ij}^{(\Delta)+1} c_{\Delta+1} + \lambda_{ij}^{(2)} \beta_{12} + \ldots + \lambda_{ij}^{(\Delta)} \beta_{1\Delta}) x_0 f_{\Delta+1} + \ldots + (l_{ij}^{(s)} c_s + \lambda_{ij}^{(2)} \beta_{s-\Delta\, 2} + \ldots + \lambda_{ij}^{(\Delta)} \beta_{s-\Delta\, \Delta}) x_0 f_s .$$

Comparing coefficients yields $l_{ij}^{(1)} = 0$, $\lambda_{ij}^{(k)} = -l_{ij}^{(k)} c_j$ for $k \in \{2, \ldots, \Delta\} \setminus \{j\}$ and $\lambda_{ij}^{(j)} = p_{ji} - l_{ij}^{(j)} c_j$. Substituting this into the remaining equations yields

$$(*) \qquad (p_{\Delta+k\, i} - p_{ji}) \beta_{kj} = l_{ij}^{(\Delta)+k} c_{\Delta+k} - l_{ij}^{(2)} c_2 \beta_{k2} - \ldots - l_{ij}^{(\Delta)} c_\Delta \beta_{k\Delta}$$

for $k = 1, \ldots, s - \Delta$. Hence the elements $\lambda_{ij}^{(2)}, \ldots, \lambda_{ij}^{(\Delta)}$ are uniquely determined by $l_{ij}^{(2)}, \ldots, l_{ij}^{(s)}$. The condition $l_{ij}^{(1)} = 0$ means that the hyperplane $\mathcal{Z}(l_{ij})$ contains P_1, and that we can find $m_1, \ldots, m_d \in K$ such that $l_{ij} = m_1(x_1 - p_{11} x_0) + \ldots + m_d(x_d - p_{1d} x_0)$. Therefore we have $l_{ij}^{(k)} = m_1(p_{k1} - p_{11}) + \ldots + m_d(p_{kd} - p_{1d})$ for $k = 2, \ldots, s$. Now equations $(*)$ yield $s - \Delta = d + 1$ linear equations $(**)$ for $m_1, \ldots, m_d$. But those equations are not linearly independent, because adding them up gives $\sum_{k=1}^{s-\Delta} p_{\Delta+k\, i} \beta_{kj} + p_{ji} = \sum_{k=1}^{s-\Delta} l_{ij}^{(\Delta)+k} c_{\Delta+k} + l_{ij}^{(2)} c_2 + \ldots + l_{ij}^{(\Delta)} c_\Delta = \sum_{k=1}^{s} (x_i \pi_j)(P_k) + \sum_{k=1}^{s} l_{ij}^{(k)} c_k = 0$ in view of $x_i \pi_j$, $l_{ij} \varphi \in \mathfrak{J}_3$ and 1.11.

Altogether, it follows from 3.9 and the above that the system of linear equations

$$\begin{aligned}(**) \qquad (p_{\Delta+k\, i} - p_{ji}) \beta_{jk} &= [m_1(p_{\Delta+k\, 1} - p_{11}) + \ldots + m_d(p_{\Delta+k\, d} - p_{1d})] c_{\Delta+k} - \\ &[m_1(p_{21} - p_{11}) + \ldots + m_d(p_{2d} - p_{1d})] c_2 \beta_{k2} - \ldots - \\ &[m_1(p_{\Delta 1} - p_{11}) + \ldots + m_d(p_{\Delta d} - p_{1d})] c_\Delta \beta_{k\Delta} = \\ &(p_{\Delta+k\, 1} c_{\Delta+k} - p_{11} c_1 \beta_{k1} - \ldots - p_{\Delta 1} c_\Delta \beta_{k\Delta}) m_1 + \ldots + \\ &(p_{\Delta+k\, d} c_{\Delta+k} - p_{1d} c_1 \beta_{k1} - \ldots - p_{\Delta d} c_\Delta \beta_{k\Delta}) m_d\end{aligned}$$

for $k = 1, \ldots, s - \Delta$ uniquely determines $m_1, \ldots, m_d \in K$, and hence $l_{ij} \in R_1$ as well as $\lambda_{ij}^{(2)}, \ldots, \lambda_{ij}^{(\Delta)} \in K$. Thus it suffices to solve the linear system of equations $(**)$ in order to compute the syzygies σ_{ij}.

Furthermore, if we want to find for $j \in \{2, \ldots, \Delta\}$ the unique element $q_j \in R_2$ such that $x_0^2 \pi_2 = q_j \varphi$, we write $q_j = \gamma_{j1} f_1 + \ldots \gamma_{js} f_s$ with $\gamma_{j1}, \ldots, \gamma_{js} \in K$, and we compare coefficients in

$$x_0^2 \pi_j = x_0^2 (f_j + \beta_{1j} f_{\Delta+1} + \ldots + \beta_{d+1\, j} f_s) = q_j \varphi = x_0^2 (\gamma_{j1} c_1 f_1 + \ldots + \gamma_{js} c_s f_s).$$

Thus $\gamma_{jk} = (1/c_j) \delta_{jk}$ for $k = 1, \ldots, \Delta$ and $\gamma_{jk} = (1/c_k) \beta_{k-\Delta\, j}$ for $k = \Delta + 1, \ldots, s$. This effectively computes the syzygies $\theta_j = (q_j, -x_0^2)$ in terms of the elements β_{ij} and the separators $f_1, \ldots, f_s$.

5. – Canonical transforms.

The canonical ideal can also be used to generalize the Gale transform of a set of points. In [5], it was shown that this transform may be defined using the elements of $\mathfrak{I}_{2\sigma}$. In this section we shall describe similar «canonical tansforms» based on each of the homogeneous components $\mathfrak{I}_{\sigma+1}, \ldots, \mathfrak{I}_{2\sigma}$. We start by characterizing those components as explicitly as we can. The notations and assumptions are the same as in section 1. In particular, we let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ be a set of points with homogeneous coordinate ring R and canonical ideal $\mathfrak{I} \subseteq R$.

PROPOSITION 5.1. – *For all $i \in \{0, \ldots, \sigma\}$, the homogeneous component $\mathfrak{I}_{\sigma+1+i}$ of the canonical ideal consists precisely of those elements $\varphi = c_1 x_0^i f_1 + \ldots + c_s x_0^i f_s \in R_{\sigma+1+i}$ such that $c_1, \ldots, c_s \in K$ and $r(P_1)c_1 + \ldots + r(P_s)c_s = 0$ for all $r \in R_{\sigma-i}$.*

In other words, if we apply the monomorphisms $\iota_j \colon R_j \to K^s$ given by $r \mapsto (r(P_1), \ldots, r(P_s))$ for $j \geq 0$, the homogeneous component $\mathfrak{I}_{\sigma+1+i}$ of the canonical ideal corresponds to the orthogonal space of $R_{\sigma-i}$ with respect to the standard pairing.

PROOF. – If we consider $\varphi \in \mathfrak{I}_{\sigma+1+i} \cong (\omega_R)_{-\sigma-i}$ as a homogeneous $K[x_0]$-linear map $\varphi \colon R \to K[x_0]$ of degree $-\sigma-1+i$, we have $\varphi(R_{\sigma-i}) \subseteq K[x_0]_{-1} = (0)$. For every element $r \in R_{\sigma-i}$, this implies $\varphi(x_0^{i+1} r) = x_0^{i+1}\varphi(r) = 0$. Since we have $x_0^{i+1} r = r(P_1) f_1 + \ldots + r(P_s) f_s$, we get $r(P_1)\varphi(f_1) + \ldots + r(P_s)\varphi(f_s) = 0$. Now Proposition 1.9 yields $\varphi(f_j) = c_j$ for $j = 1, \ldots, s$, and therefore $r(P_1)c_1 + \ldots + r(P_s)c_s = 0$.

The reverse implication follows from the fact that the orthogonal space of $\iota_{\sigma-i}(R_{\sigma-i})$ has dimension $s - \dim_K(R_{\sigma-i}) = \dim_K(\mathfrak{I}_{\sigma+1+i})$. ■

The next proposition provides the basis for our definition of the Gale transform which differs slightly from the one in [5].

PROPOSITION 5.2. – *Suppose $\mathbb{X} \subseteq \mathbb{P}^d$ has the property that any subset of $s-2$ points of $\mathbb{X}$ spans $\mathbb{P}^d$. We choose a K-basis $\{t_1, \ldots, t_{s-d-1}\}$ of $\mathfrak{I}_{2\sigma}$ and write $t_i = \gamma_{i1} x_0^{\sigma-1} f_1 + \ldots + \gamma_{is} x_0^{\sigma-1} f_s$ with $\gamma_{i1}, \ldots, \gamma_{is} \in K$ for $i = 1, \ldots, s-d-1$. Then any two columns of the matrix $\Gamma = (\gamma_{ij})$ are K-linearly independent.*

PROOF. – We assume that there are indices $\mu, \nu \in \{1, \ldots, s\}$ and an element $\lambda \in K \setminus \{0\}$ such that $\mu \neq \nu$ and $\gamma_{i\mu} = \lambda \gamma_{i\nu}$ for $i = 1, \ldots, s-d-1$. Let us denote the homogeneous coordinate ring of the subscheme $\mathbb{Y} = \mathbb{X} \setminus \{P_\nu\}$ of $\mathbb{X}$ by $R_{\mathbb{Y}} = K[X_0, \ldots, X_d]/I_{\mathbb{Y}}$, and its canonical ideal by $\mathfrak{I}_{\mathbb{Y}} \subseteq R_{\mathbb{Y}}$. Since $\mathbb{Y}$ spans

$\mathbb{P}^d$, we have

$$\dim_K(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}} = (s-1) - d - 1 \,.$$

We note that $\sigma_{\mathbb{X}} - 1 \leqslant \sigma_{\mathbb{Y}} \leqslant \sigma_{\mathbb{X}}$ and $\omega_{R_{\mathbb{Y}}} \cong \{\varphi \in \omega_R \mid \overline{I_{\mathbb{Y}}} \cdot \varphi = 0\}$, where $\overline{I_{\mathbb{Y}}}$ is the image of $I_{\mathbb{Y}}$ in R (cf. [10], 1.3 d)). Now we distinguish two cases.

Case 1: $\sigma_{\mathbb{Y}} = \sigma_{\mathbb{X}}$. In this case we have $(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}} \cong \{c_1 x_0^{\sigma-1} f_1 + \ldots + c_s x_0^{\sigma-1} f_s \in \mathfrak{I}_{2\sigma} \mid c_\nu = 0\}$, because $\overline{I_{\mathbb{Y}}}$ is given by $\overline{I_{\mathbb{Y}}} = (f_\nu)^{\text{sat}}$ and $(\overline{I_{\mathbb{Y}}})_{2\sigma} = Kx_0^{\sigma-1} f_\nu$. Since $(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}} \mathfrak{I}_{2\sigma}$, we may assume $\gamma_{1\nu} \neq 0$. Then we find

$$(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}} = K \cdot (\gamma_{1\nu} t_2 - \gamma_{2\nu} t_1) \oplus \ldots \oplus K \cdot (\gamma_{i\nu} t_{s-d-1} - \gamma_{s-d-1\,\nu} t_1)$$

and all elements of $(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}}$ are of the form $\tilde{c}_1 x_0^{\sigma-1} f_1 + \ldots + \tilde{c}_s x_0^{\sigma-1} f_s$ with $\tilde{c}_1, \ldots, \tilde{c}_s \in K$ and $\tilde{c}_\mu = \tilde{c}_\nu = 0$.

Case 2: $\sigma_{\mathbb{Y}} = \sigma_{\mathbb{X}} - 1$. Again we have $(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}} \cong (\omega_{R_{\mathbb{Y}}})_{-1} \subseteq (\omega_R)_{-1} \cong \mathfrak{I}_{2\sigma_{\mathbb{X}}}$, and the same argument as above shows that all elements $\tilde{c}_1 x_0^{\sigma-1} f_1 + \ldots + \tilde{c}_s x_0^{\sigma-1} f_s \in (\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}}$ satisfy $\tilde{c}_\mu = \tilde{c}_\nu = 0$.

Im both cases all elements of $(\mathfrak{I}_{\mathbb{Y}})_{2\sigma_{\mathbb{Y}}}$ are also contained in the canonical ideal of $\mathbb{Y}' = \mathbb{Y} \backslash \{P_\mu\}$. Then $\dim_K(\mathfrak{I}_{\mathbb{Y}'})_{2\sigma} = (s-2) - (d-1) - 1$ contradicts the assumption that the points of $\mathbb{Y}'$ span $\mathbb{P}^d$. ■

DEFINITION 5.3. – Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^d$ be a set of points with the property that any subset consisting of $s-2$ points of $\mathbb{X}$ spans $\mathbb{P}^d$, and let Γ be the matrix defined in Proposition 5.2. Then the set of points $\kappa(\mathbb{X}) \subseteq \mathbb{P}^{s-d-2}$ defined by the columns of the matrix Γ is called the *Gale transform* (or the *associated set of points*) of $\mathbb{X}$.

By Proposition 5.2, the set $\kappa(\mathbb{X})$ consists of s distinct points of $\mathbb{P}^{s-d-1}$. It is clear that $\kappa(\mathbb{X})$ does not depend on the choice of the linear nonzerodivisor $x_0 \in R$ (see the proof of 1.3), and that it changes by a coordinate transformation of $\mathbb{P}^{s-d-1}$, if we choose a different basis for $\mathfrak{I}_{2\sigma}$ in Proposition 5.2. In [5], Gale transforms were defined assuming only that subsets of $s-1$ points of $\mathbb{X}$ span $\mathbb{P}^d$. Out next example shows that in this case $\kappa(\mathbb{X})$ does not necessarily consist of s distinct points in $\mathbb{P}^{s-d-1}$.

EXAMPLE 5.4. – Let K be a field of characterstic $\operatorname{char}(K) \neq 2$, and let $\mathbb{X} \subseteq \mathbb{P}^3$ consist of the following six points: $P_1 = (1{:}\ 1{:}\ 0{:}\ 0)$, $P_2 = (1{:}\ 0{:}\ 1{:}\ 0)$, $P_3 = (1{:}\ -1{:}\ 0{:}\ 0)$, $P_4 = (1{:}\ 0{:}\ -1{:}\ 0)$, $P_5 = (1{:}\ 0{:}\ 0{:}\ 1)$, and $P_6 = (1{:}\ 0{:}\ 0{:}\ -1)$. Then the Hilbert function of $\mathbb{X}$ is $H_{\mathbb{X}}$: 1 4 6 6 …, and we have $\sigma_{\mathbb{X}} = 1$ as well as $\Delta_{\mathbb{X}} = 2$. A basis of $\overline{R}_{\sigma+1}$ is given by Lf_1 and Lf_2, and we have $Lf_3 = Lf_1$, $Lf_4 = Lf_2$, $Lf_5 = Lf_6 = -Lf_1 - Lf_2$. Thus the homogeneous component $\mathfrak{I}_2 = \mathfrak{I}_{\sigma+1} = \mathfrak{I}_{2\sigma}$ has the K-basis $\{\pi_1, \pi_2\}$, where $\pi_1 = f_1 + f_3 - f_5 - f_6$ and $\pi_2 = f_2 + f_4 - f_5 - f_6$.

It is clear that any five points of $\mathbb{X}$ span $\mathbb{P}^3$. But the Gale transform $\kappa(\mathbb{X}) \subseteq \mathbb{P}^1$ consists only of the three points $(1:0)$, $(0:1)$, and $(1:1)$.

If $\sigma_{\mathbb{X}} = 1$, as in the previous example, the description of $\kappa(\mathbb{X})$ can be simplified considerably.

REMARK 5.5. – Let $\mathbb{X} \subseteq \mathbb{P}^d$ be a set of s points such that $\sigma_{\mathbb{X}} = 1$ and such that any $s-2$ points of $\mathbb{X}$ span $\mathbb{P}^d$. We form the matrix

$$\widetilde{\mathfrak{B}} = \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & \mathfrak{B} \\ 0 & & 1 & \end{array}\right)$$

Then, by Corollary 1.10, the Gale transform of $\mathbb{X}$ is the set of points $\kappa(\mathbb{X}) \subseteq \mathbb{P}^{\Delta-1}$ defined by the columns of the matrix $\widetilde{\mathfrak{B}}$.

Next we want to generalize the Gale transform by using other components of the canonical ideal as well. Again we are interested in the correct conditions under which those constructions yield sets of s distinct points. To be able to formulate those conditions, we introduce the following notion of uniformity.

DEFINITION 5.6. – Let $1 \leqslant i \leqslant s-1$ and $1 \leqslant j \leqslant \sigma_{\mathbb{X}}$. We say that $\mathbb{X}$ is (i, j)-uniform, if every subscheme $\mathbb{Y} \subseteq \mathbb{X}$ consisting of $\deg \mathbb{Y} = s - i$ points satisfies $H_{\mathbb{Y}}(j) = H_{\mathbb{X}}(j)$.

This notion generalizes most uniformity conditions for 0-dimensional schemes considered in [10] and elsewhere. Recall that $\mathbb{X}$ is called i-uniform, if every subset $\mathbb{Y} \subseteq \mathbb{X}$ of degree $s - i \leqslant \deg \mathbb{Y} \leqslant s$ satisfies $H_{\mathbb{Y}} = \min\{H_{\mathbb{X}}, \deg \mathbb{Y}\}$.

REMARK 5.7. – *a*) A set of points $\mathbb{X} \subseteq \mathbb{P}^d$ is i-uniform, if and only if $\mathbb{X}$ is (i, σ)-uniform. In particular, $\mathbb{X}$ is a Cayley-Bacharach scheme, if and only if $\mathbb{X}$ is $(1, \sigma)$-uniform, and $\mathbb{X}$ is in uniform position, if and only if $\mathbb{X}$ is $(s-1, \sigma)$-uniform.

b) If $\mathbb{X} \subseteq \mathbb{P}^d$ is nondegenerate, then $\mathbb{X}$ is in linearly general position (i.e. any $d+1$ points of $\mathbb{X}$ span $\mathbb{P}^d$), if and only if $\mathbb{X}$ is $(s-d-1, 1)$-uniform. More generally, $\mathbb{X}$ is in lineraly general position of i^{th} order (cf. [10]), if and only if $\mathbb{X}$ is $\left(s - \binom{d+i}{d}, i\right)$-uniform.

PROPOSITION 5.8. – *Let* $\mathbb{X} \subseteq \mathbb{P}^d$ *be a set of* s *points, let* $1 \leqslant i \leqslant \sigma$, *let* $r = s - H_{\mathbb{X}}(i)$, *and let* $\{t_1, \ldots, t_r\}$ *be a* K-*basis of* $\mathfrak{I}_{2\sigma+1-i}$. *For* $j = 1, \ldots, r$

we write $t_j = \gamma_{j1} x_0^{\sigma - i} f_1 + \ldots + \gamma_{js} x_0^{\sigma - i} f_s$ *with* $\gamma_{j1}, \ldots, \gamma_{js} \in K$, *and we form the matrix* $\Gamma_i = (\gamma_{jk})$. *Then the following conditions are equivalent.*

a) $\mathbb{X}$ *is* $(2, i)$*-uniform.*

b) The columns of the matrix Γ_i *are pairwise K-linearly independent.*

PROOF. – Suppose *b)* holds. Let $1 \leqslant j < k \leqslant s$. Clearly, condition *b)* does not depend on the choice of the basis $\{t_1, \ldots, t_r\}$ of $\mathfrak{J}_{2\sigma + 1 - i}$. If we choose it suitably, the j^{th} and k^{th} columns of Γ_i are given by $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$, respectively. Then we get

$$(*) \qquad \dim_K \{c_1 x_0^{\sigma - i} f_1 + \ldots + c_s x_0^{\sigma - i} f_s \in \mathfrak{J}_{2\sigma + 1 - i} \mid c_1, \ldots, c_s \in K,\ c_j = c_k = 0\} = \dim_K \mathfrak{J}_{2\sigma + 1 - i} - 2 = r - 2\,.$$

This implies $\dim_K (\omega_{R_{\mathbb{Y}}})_{-i} = \dim_K (\omega_R)_{-i} - 2$ for all subschemes $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg \mathbb{Y} = s - 2$ and canonical module $\omega_{R_{\mathbb{Y}}}$. Since we have

$$H_{\mathbb{Y}}(i) = \deg \mathbb{Y} - \dim_K (\omega_{R_{\mathbb{Y}}})_{-i} = (s - 2) - (\dim_K (\omega_R)_{-i} - 2) = H_{\mathbb{X}}(i)\,,$$

we see that $\mathbb{X}$ is $(2, i)$-uniform.

Conversely, given *a)*, the same calculation shows that equation $(*)$ holds for all $1 \leqslant j < k \leqslant s$, i.e. that columns j and k of Γ_i are K-linearly independent. ■

DEFINITION 5.9. – Let $\mathbb{X} \subseteq \mathbb{P}^d$ be a $(2, i)$-uniform set of s points, let $i \in \{1, \ldots, \sigma\}$, let $r = s - H_{\mathbb{X}}(i)$, and let Γ_i be the matrix defined in the previous proposition. Then the set of s distinct points $\kappa_i(\mathbb{X}) \subseteq \mathbb{P}^r$ defined by the columns of Γ_i is called the i^{th} *canonical transform* of $\mathbb{X}$.

It is clear that $\kappa_1(\mathbb{X}) = \kappa(\mathbb{X})$ is the Gale transform of $\mathbb{X}$, and $\kappa_i(\mathbb{X})$ is determined up to a coordinate transformation in $\mathbb{P}^r$. It seems to be a largely unexplored question how properties of the embedding $\mathbb{X} \subseteq \mathbb{P}^s$ are related to properties of $\kappa_i(\mathbb{X}) \subseteq \mathbb{P}^r$. Our next proposition gives a small result in this direction.

PROPOSITION 5.10. – *Let* $\mathbb{X} \subseteq \mathbb{P}^d$ *be a set of s points. Then* $\mathbb{X}$ *is* $\Delta_{\mathbb{X}}$*-uniform (or* $(\Delta_{\mathbb{X}}, \sigma_{\mathbb{X}})$*-uniform in the terminology of Definition* 5.6*), if and only if* $\kappa_\sigma(\mathbb{X}) \subseteq \mathbb{P}^{\Delta - 1}$ *is in linearly general position (i.e. if and only if any subset of* $\Delta_{\mathbb{X}}$ *points of* $\kappa_\sigma(\mathbb{X})$ *spans* $\mathbb{P}^{\Delta - 1}$)

PROOF. – By Corollary 1.10, the set $\kappa_\sigma(\mathbb{X})$ is given by the columns of the matrix $\widetilde{\mathfrak{B}}$ defined in Remark 5.5. Those columns are the coordinate vectors of $Lf_1, \dots, Lf_s$ in the K-basis $\{Lf_1, \dots, Lf_\Delta\}$ of $\overline{R}_{\sigma+1}$. Now the claim follows from [10], 3.4. ∎

We end this section (and this paper) by pointing out some connections between the material presented above and Coding Theory. Let $p > 0$ be a prime number, $e > 0$, $q = p^e$, and $\mathbb{F}_q$ the field with q elements. In Coding Theory, a linear subspace $C \subseteq \mathbb{F}_q$ is called a *linear code*. The number s is called the *length* of C, the number $d(C) = \dim_{\mathbb{F}_q}(C)$ is called the *dimension* of C, and the minimal number $m(C)$ of nonzero components of a nonzero vector of C is called the *minimal distance* of C. With those notations, C is also called an $[s, d(C)]_q$-code. It satisfies the *Singleton bound* $m(C) \leqslant s - d(C) + 1$, and if it achieves equality there, it is called an *MDS-code* («maximum distance separable»).

In [8], J. P. Hansen described the following way to associate linear codes to a set of $\mathbb{F}_q$-rational points $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^d(\mathbb{F}_q) \cap D_+(X_0) \subseteq \mathbb{P}^d_{\mathbb{F}_q}$. If $R = \mathbb{F}_q[X_0, \dots, X_d]/I_{\mathbb{X}}$ is the homogeneous coordinate ring of $\mathbb{X}$ and $1 \leqslant r \leqslant \sigma_{\mathbb{X}}$, then the image of the map

$$\begin{aligned} \Phi_r \colon R_r &\to \mathbb{F}_q^s \\ f &\mapsto (f(P_1), \dots, f(P_s)) \end{aligned}$$

is called the r^{th} *associated Reed-Muller code* of $\mathbb{X}$ and denoted by $C_r(\mathbb{X}) = \operatorname{im} \Phi_r$. In case $r = 1$, the code $C(\mathbb{X}) = \operatorname{im} \Phi_1$ is also called the *associated linear code* of $\mathbb{X}$. In [8], Prop. 6 and Thm. 8, the uniformity of $\mathbb{X}$ and invariants of $C_r(\mathbb{X})$ were interrelated as follows.

Proposition 5.11. – *For all $1 \leqslant r \leqslant \sigma_{\mathbb{X}}$ and $i \geqslant 1$, we have $m(C_r(\mathbb{X})) \geqslant s - i + 1$, if and only if $\mathbb{X}$ is $(s - i, r)$-uniform. In particular, $C_r(\mathbb{X})$ is an MDS-code, if and only if $\mathbb{X}$ is $(s - H_{\mathbb{X}}(r))$-uniform.*

If we specialize to the case $r = \sigma_{\mathbb{X}}$, we see that $C_\sigma(\mathbb{X})$ is the linear code generated by the vectors $(-\beta_{j1}, \dots, -\beta_{j\Delta}, 0, \dots, 0, 1, 0, \dots, 0)$ for $j = 1, \dots, s - \Delta$. Thus $C_\sigma(\mathbb{X})$ is precisely the kernel of the linear map with matrix $\widetilde{\mathfrak{B}}$ (defined as in 5.5). In Coding Theory, one says that $\widetilde{\mathfrak{B}}$ is a *parity check matrix* for $C_\sigma(\mathbb{X})$. From Propositions 5.10 and 5.11 we get the following relation between the linear code $C_\sigma(\mathbb{X})$ and the canonical transform $\kappa_\sigma(\mathbb{X})$.

Corollary 5.12. – *For a set of $\mathbb{F}_q$-rational points $\mathbb{X} \subseteq \mathbb{P}^d_{\mathbb{F}_q}$ as above, the following conditions are equivalent.*

a) The linear code $C_\sigma(\mathbb{X})$ is an MDS-code.

b) The set of points $\mathbb{X}$ is $\Delta_{\mathbb{X}}$-uniform.

c) The σ^{th} canonical transform $\kappa_\sigma(\mathbb{X})$ is in linearly general position.

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Pervenuta in Redazione
il 12 ottobre 1998