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Global Existence and Regularity of Solutions
for Complex Ginzburg-Landau Equations

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Sunto. – Si considerano equazioni di Ginzburg-Landau complesse del tipo \( u_t - \alpha \Delta u + P(|u|^2)u = 0 \) in \( \mathbb{R}^N \) dove \( P \) è polinomio di grado \( K \) a coefficienti complessi e \( \alpha \) è un numero complesso con parte reale positiva \( \Re \alpha \). Nell'ipotesi che la parte reale del coefficiente del termine di grado massimo \( P \) sia positiva, si dimostra l'esistenza e la regolarità di una soluzione globale nel caso \( |\alpha| < C\Re \alpha \), dove \( C \) dipende da \( K \) e \( N \).

1. – Introduction.

Let \( K \) be an integer, \( K \geq 1 \), \( \alpha \) and \( \mu_j, j \in \{0, \ldots, K\} \), complex numbers with \( \Re \alpha > 0 \), and \( \Re \mu_K > 0 \). We consider the initial value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \alpha \Delta u + \sum_{j=0}^{K} \mu_j |u|^{2j} u &= 0, \quad x \in \mathbb{R}^N, \ t > 0, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{align*}
\]

Without loss of generality, we suppose that \( \Re \mu_0 > 0 \). For example, when \( K = 1 \), we obtain the well-known cubic Ginzburg-Landau equation, and when \( K = 2 \), the equation given by Fauve-Thual in [3] as a model of localized structures generated by subcritical instabilities. In [1], Doering, Gibbon and Levermore have considered a system of the same form but with periodic boundary conditions:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Ru + (1 + iv) \Delta u - (1 + i\mu) |u|^{2\sigma} u, \quad x \in T^N, \ t > 0, \\
u(0, x) &= u_0(x), \quad x \in T^N.
\end{align*}
\]

They obtained existence of global-weak solutions in all dimensions and for all \( \sigma > 0 \) and parameter values \( R \), \( \nu \) and \( \mu \). Under certain assumptions, they also obtained global strong solutions. But their proofs use essentially the boundedness of the domain \( T^N \). The case of the whole space is considered...
in [4], [5] by Ginibre and Velo, for the system
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \gamma_1 u + (a + i\gamma_2) Du - (b + i\gamma_3) g(|u|^2) u, \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\]
with \(a > 0, b > 0\) and \(g \geq 0\) satisfying
\[
x^\sigma \leq g(x) \leq C(1 + x^\sigma)
\]
for some \(\sigma (0 < \sigma < \infty)\), some \(C \geq 1\) and all \(x \geq 0\). They obtained existence and uniqueness of solutions globally defined in time with initial data corresponding to the spaces \(L^p\) for \(p \geq 2\) or \(H^1 \cap L^{2^* + 2}\). They also studied the case where the nonlinear term is of the form \((b + i\gamma_3) f(u)\) with \(f\) belonging to \(C^1(\mathbb{C}, \mathbb{C})\) and obtain local existence of solutions for initial data belonging to \(L^p, p \geq 2\).

In this article, under assumptions on \(a\), we obtain for (1.1) existence and regularity of strong global solutions when \(u_0\) belongs to the space \(W^{2-\frac{2}{q}, q}\) with \(q > 1 + N/2\) (so the results are different from [4], [5]) and we deduce existence of global weak-solutions when \(u_0\) belongs to \(L^p, p \geq 2\) or \(H^1 \cap L^p\) with \(p > 2\). The methods are different from [4], [5] and do not use a priori estimates obtained by multiplying the first equation of (1.1) by \(\Delta u\).

We use the notations:
\[
L^p = L^p(\mathbb{R}^N, \mathbb{C}), \\
W^{s, p} = W^{s, p}(\mathbb{R}^N, \mathbb{C}).
\]

Our purpose is to prove the following results:

**Theorem 1.** – Assume that \(u_0\) belongs to \(W^{2-\frac{2}{q}, q}\) with \(q > 1 + N/2\) and that \(a\) verifies
\[
|a| < \frac{(2K + 1) q - 2 - 2K}{(2K + 1) q - 2} \Re \alpha
\]
and \(\Re \mu_K > 0\). Then (1.1) has a unique global solution \(u\) belonging to \(C([0, t], W^{2-\frac{2}{q}, q})\) for all \(t > 0\).

**Theorem 2.** – Assume that \(u_0\) belongs to \(H^1 \cap L^p\) with \(p = 2 + 2\sigma (\sigma > 0)\) and that \(a\) satisfies the condition
\[
|a| < \min \left( \frac{(2K + 1) N + 2}{(2K + 1) N - 2}, \frac{1 + \sigma}{\sigma} \right) \Re \alpha
\]
and \( \Re \mu_K > 0 \). Then (1.1) possesses at least a global weak-solution \( u \) of the form \( u = v + w \) such that\[ v \in C([0, + \infty), H^1), \quad w \in C([0, + \infty), W^{2-2/q, q}) \] and \( w(0, \cdot) = 0 \), with\[ q = 1 + \frac{2\sigma + 1}{2K + 1}. \]

In particular if \( \sigma \geq K \) and\[ |a| < \frac{(2K + 1) N + 2}{(2K + 1) N - 2} \Re a, \]

\( u \) belongs to \( C([0, + \infty), H^1) \).

Section 2 is devoted to some results on the \( L^p \) regularity of solutions of linear equations analogous to (1.1). In section 3, we prove the local existence of solutions when \( u_0 \in L^\infty \). In section 4, we provide estimates on this local solution; then we prove, under assumptions on \( \alpha \), that when \( u_0 \) belongs to \( W^{2-2/q, q} \), with \( q > 1 + N/2 \), the solution is global in time. Then we pass to the limit to cover the case where \( u_0 \) belongs to \( L^p \) or \( H^1 \cap L^p \) with \( p = 2 + 2\sigma \) (\( \sigma > 0 \)).

Remark 1. – The same results hold if we consider problem (1.1) in a bounded regular domain of \( \mathbb{R}^N \) and add a Dirichlet or Neumann boundary condition.

In this article, we denote:\[ F(u) = \sum_{j=0}^{K} \mu_j |u|^{2j}. \]

2. – \( L^p \) regularity.

Consider the Cauchy problem:\[
\begin{cases}
\frac{\partial u}{\partial t} - \alpha \Delta u + \mu u = f, & x \in \mathbb{R}^N, \ t > 0, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\]

We assume that \( \alpha \) and \( \mu \) are complex numbers with positive real parts \( \alpha_r \) and \( \mu_r \), that \( p \) belongs to \( (1, + \infty) \), \( f \) is given in \( L^p(\mathbb{R}_+, L^p) \) and \( u_0 \) in \( W^{2-2/p, p} \). We are interested in the \( L^p \) regularity of solutions of (2.1), we will prove that
(2.1) has a unique solution \( u \in W^{1,p}(\mathbb{R}_+; L^p) \cap L^p(\mathbb{R}_+; W^{2,p}) \) and that \( u, \Delta u \) and \( \partial u/\partial t \) depend continuously on \( f \).

To obtain this result, we will use the imaginary powers of the operators appearing in (2.1), according to an idea of Prüss and Sohr [9]. We refer to the book of Triebel [12] for a definition of the imaginary powers of an operator.

Let us recall some definitions. Let \( A \) be a closed linear operator in \( L^p(\mathbb{R}_+; L^p) \), with dense domain \( D(A) \); \( N(A) \) and \( R(A) \) denote the kernel and the range of \( A \), \( \varrho(A) \) and \( \sigma(A) \) the resolvent set and the spectrum of \( A \). Finally, \( B(L^p) \) is the space of bounded linear operators in \( L^p(\mathbb{R}_+; L^p) \).

**Definition 2** [9]. – Let \( \theta \) belong to \([0, \pi)\). A closed linear densely defined operator \( A \) in \( L^p(\mathbb{R}_+; L^p) \) belongs to the class \( BIP(L^p, \theta) \), if it satisfies:

(H1) The set \((- \infty, 0)\) is included in \( \varrho(A) \), the kernel \( N(A) \) is reduced to \( 0 \), the range \( R(A) \) is dense in \( L^p(\mathbb{R}_+; L^p) \), and, there exists a \( M \geq 1 \), such that

\[
|(t + A)^{-1}| \leq M/t \quad \text{for all} \quad t > 0.
\]

(H2) For all \( s \in \mathbb{R} \), \( A^{is} \) belongs to \( B(L^p) \), and there exists a \( K_0 \) such that

\[
|A^{is}| \leq K_0 \exp(\theta |s|).
\]

**Definition 3.** – Let \( A, B \) two linear operators. We say that \( A, B \) are resolvent commuting if for all \( \lambda \) (respectively \( \nu \)) in the resolvent set \( \varrho(A) \) (respectively \( \varrho(B) \)) \( (\lambda - A)^{-1}(\nu - B)^{-1} = (\nu - B)^{-1}(\lambda - A)^{-1} \).

Let us quote the main result of [9]:

**Theorem 4** [9]. – We are given \( k \geq 2 \) elements \( A_i \) in \( BIP(L^p, \theta_i) \), such that, for each pair \( i \neq j \), \( A_i \) and \( A_j \) are resolvent commuting and satisfy \( \theta_i + \theta_j < \pi \). Let \( \theta = \max \theta_i \) and assume that there is only one \( i \) with \( \theta = \theta_i \).

Then the operator \( A \) defined by

\[
D(A) = \bigcap_{1 \leq i \leq k} D(A_i), \quad A = \sum_{i=1}^{k} A_i,
\]

is closed and belongs to the class \( BIP(L^p, \theta) \). Moreover, there is a constant \( C > 0 \) such that

\[
\sum_{i=1}^{k} |A_i x| \leq C |Ax|, \quad \forall x \in D(A).
\]

In particular, \( N(A) = 0 \) and \( R(A) \) is dense in \( L^p(\mathbb{R}_+; L^p) \).
In the following sequence of lemmas, we show that the operators appearing in (2.1) belong to the class $BIP(L^p, \theta)$ and we characterize the relevant $\theta$.

**Lemma 5.** Define $A_1$ and $B_1$ respectively by

\[
D(A_1) = W^1_{0, p}(\mathbb{R}_+; L^p), \quad A_1 = \partial_t^2, \\
D(B_1) = L^p(\mathbb{R}_+; W^{2,p}), \quad B_1 = -\Delta.
\]

Then for all $\varepsilon > 0$:

\[A_1 \in BIP(L^p, \pi/2 + \varepsilon) \text{ and } B_1 \in BIP(L^p, \varepsilon).\]

**Proof.** The result for $A_1$ is due to [2] [9]. In the scalar case the result for $B_1$ is due to [11] [10], the vector generalization is straightforward. 

**Lemma 6.** Let $\beta$ be a complex number of positive real part $\beta_+$. The operator $I_\beta$ is the multiplication by $\beta$ in $L^p(\mathbb{R}_+; L^p)$. Then

\[I_\beta \in BIP(L^p, |\text{Arg } \beta|),\]

where Arg is the principal determination of the argument.

**Proof.** It suffices to prove (2.3). Let $s \in \mathbb{R}$, then we have

\[|\beta^{is}| = |\exp(is \text{ Log } |\beta| - s \text{ Arg } \beta)| = \exp(-s \text{ Arg } \beta),\]

thus

\[|\beta^{is}| \leq \exp(|s| |\text{Arg } \beta|).\]

**Lemma 7.** Define an operator $B_a$ by

\[D(B_a) = L^p(\mathbb{R}_+; W^{2,p}), \quad B_a = -a\Delta,\]

then for all $\varepsilon > 0$

\[B_a \in BIP(L^p, |\text{Arg } a| + \varepsilon).\]

**Proof.** Remark that $B_a = I_a B_1$, then thanks to the corollary 3 of [9],

\[B_a \in BIP(L^p, \theta_{B_1} + \theta_{I_a}).\]

Now, we can prove the following theorem:

**Theorem 8.** Let $\alpha, \mu$ be complex numbers such that $\alpha_+ > 0, \mu_+ > 0$; let $u_0$ belong to $W^{2-2/p, p}$. Then for all $f \in L(\mathbb{R}_+; L^p), 1 < p < \infty$, the Cauchy problem (2.1) has a unique solution $u \in W^{1,p}(\mathbb{R}_+; L^p) \cap L^p(\mathbb{R}_+; W^{2,p})$. Moreover
there exists a constant $C > 0$, such that

$$\left(2.4\right) \quad \left| \frac{\partial u}{\partial t} \right|_{L^p(R^+; L^p)} + \left| u \right|_{L^p(R^+; w^{2,p})} \leq C \left( \left| f \right|_{L^p(R^+; L^p)} + \left| u_0 \right|_{w^{2-2\mu,p}} \right).$$

**Proof.** – Consider the problem

$$\left(2.5\right) \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = f, & x \in \mathbb{R}^N, \ t > 0, \\ v(0, x) = u_0, & x \in \mathbb{R}^N. \end{cases}$$

where $u_0$ is as in the statement of Theorem 8. It is a well known fact (see for example [7]) that (2.5) possesses a unique solution $v$, which belongs to $W^{1,p}(R^+; L^p) \cap L^p(R^+; W^{2,p})$. Moreover there exists a constant $C_1$ such that $v$ verifies

$$\left(2.6\right) \quad \left| \frac{\partial v}{\partial t} \right|_{L^p(R^+; L^p)} + \left| v \right|_{L^p(R^+; w^{2,p})} \leq C_1 \left( \left| f \right|_{L^p(R^+; L^p)} + \left| u_0 \right|_{w^{2-2\mu,p}} \right).$$

Define

$$f_v = f - \frac{\partial v}{\partial t} + \alpha \Delta v - \mu v.$$ 

We observe that $f_v$ belongs to $L^p(R^+; L^p)$. The function $w = u - v$ is solution of

$$\left(2.7\right) \quad \begin{cases} \frac{\partial w}{\partial t} - \alpha \Delta w + \mu w = f_v, & x \in \mathbb{R}^N, \ t > 0, \\ w(0, x) = 0, & x \in \mathbb{R}^N. \end{cases}$$

Let $A$ be the operator defined by

$$D(A) = W^{1,p}_0(R^+; L^p) \cap L^p(R^+; W^{2,p}), \quad A = A_1 + B_\alpha + I_\mu,$$

we can rewrite the problem (2.7) under the form

$$Aw = f_v.$$ 

Thanks to Lemma 5, 7 for $\varepsilon$ sufficiently small, the hypotheses of Theorem 4 are satisfied; therefore $A$ is invertible in $D(A)$ and there exists a constant $C > 0$ such that

$$\left(2.8\right) \quad \left| \frac{\partial w}{\partial t} \right|_{L^p(R^+; L^p)} + \left| w \right|_{L^p(R^+; w^{2,p})} \leq C \left| f_v \right|_{L^p(R^+; L^p)}.$$
Now $u = w + v$ and (2.4) comes from (2.6) and (2.8). This concludes the proof of Theorem 8. □

Remark 9. – The result of Theorem 8 can be also obtained using the results of Hieber and Prüss [6].

3. – Local existence.

In this section, we prove the following existence result:

Lemma 10. – Let $u_0 \in L^\infty$; then there exists a positive number $T_0$, depending only on $|u_0|_{L^\infty}$ and $F$, such that (1.1) has at least a solution $u \in L^\infty(0, T_0; L^\infty)$. Moreover $u$ is infinitely differentiable over $(0, T_0) \times \mathbb{R}^N$.

Proof. – Let $G$ be the Green function corresponding to the linear initial value problem

$$
\begin{align*}
\frac{\partial u}{\partial t} - \alpha \Delta u &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \\
\quad u(0, x) &= u_0(x), \quad x \in \mathbb{R}^N.
\end{align*}
$$

It is given explicitly by

$$
G(t, x) = (4\pi\alpha t)^{-N/2} \exp \left( - |x|^2 / 4\pi\alpha t \right),
$$

where the fractional powers are defined as principal determination when necessary.

Let $\tau$ and $\varrho$ be positive real numbers and $B(u_0, \tau, \varrho)$ be the ball in $L^\infty(0, \tau; L^\infty)$ of center $u_0$ and radius $\varrho$. Define an application $\mathcal{C}$ from $L^\infty(0, \tau; L^\infty)$ into itself by

$$
(\mathcal{C}u)(t, \cdot) = G(t, \cdot) * u_0 - \int_0^t G(t - s, \cdot) F(u(\cdot, s)) \, ds.
$$

For $u \in B(u_0, \tau, \varrho)$ and $t \leq \tau$; we have

$$
| (\mathcal{C}u)(t, \cdot) - u_0(\cdot) |_{L^\infty} \leq |u_0|_{L^\infty} + |G(t, \cdot) * u_0|_{L^\infty}
$$

$$
+ \int_0^t |G(t - s, \cdot) * F(u(\cdot, s))|_{L^\infty} \, ds,
$$
and by Young’s inequality, we obtain
\[
\|(\mathcal{G}u)(t, \cdot) - u_0(\cdot)\|_{L^p} \leq (1 + |G(t, \cdot)|_{L^p})|u_0|_{L^p}
\]
\[
+ \int_0^t |G(t - s, \cdot)|_{L^p} |F(u(\cdot, s))|_{L^p} \, ds ,
\]
but
\[
|G(t, \cdot)|_{L^p} = (4\pi|a|t)^{-N/2} \int_{\mathbb{R}^N} \exp \left( -|x|^2 / 4\pi|a|^2 t \right) dx
\]
\[
= (4\pi|a|t)^{-N/2} \int_{\mathbb{R}^N} \exp \left( -\alpha_r |x|^2 / 4\pi|a|^2 t \right) dx
\]
\[
= |a|^{-N/2} \int_{\mathbb{R}^N} \exp \left( -\alpha_r |y|^2 / |a|^2 \right) dy
\]
\[
= k_N .
\]
Thus, we have
\[
(3.1) \quad \|(\mathcal{G}u)(t, \cdot) - u_0(\cdot)\|_{L^p} \leq (1 + k_N) |u_0|_{L^p} + k_N \int_0^t |F(u(\cdot, s))|_{L^p} \, ds .
\]
The function $F$ is Lipschitz continuous on the set
\[
V_{a, 0} = \{v \in \mathbb{C}; \ |v - a| \leq q \},
\]
with Lipschitz constant $L(a, q)$; set
\[
\lambda = \text{ess sup}_{x \in \mathbb{R}^N} \{L(u_0(x), q)\} < + \infty .
\]
We deduce from (3.1) that
\[
\|(\mathcal{G}u)(t, \cdot) - u_0(\cdot)\|_{L^p} \leq (1 + k_N) |u_0|_{L^p} + k_N \tau (\lambda q + |F(u_0)|_{L^p}) .
\]
Choose $q > 2(1 + k_N) |u_0|_{L^p}$ and $\tau$ sufficiently small such that
\[
\|(\mathcal{G}u)(t, \cdot) - u_0(\cdot)\|_{L^p} \leq q ;
\]
we deduce that
\[
\mathcal{G}(B(u_0, \tau, q)) \subset B(u_0, \tau, q) .
\]
For \( u \) and \( v \) in \( B(u_0, t, r) \) and \( t \leq \tau \), we have
\[
|\mathcal{C}(u)(t, \cdot) - \mathcal{C}(v)(t, \cdot)|_{L^r} \leq k_N \lambda \int_0^t |u(\cdot, s) - v(\cdot, s)|_{L^r} \, ds
\]
\[
\leq k_N \lambda \tau |u - v|_{L^r(0, \tau; L^r)}.
\]
Thus
\[
|\mathcal{C}u - \mathcal{C}v|_{L^r(0, \tau; L^r)} \leq k_N \lambda \tau |u - v|_{L^r(0, \tau; L^r)},
\]
and therefore
\[
|\mathcal{C}u - \mathcal{C}v|_{L^r(0, \tau; L^r)} \leq k_0 |u - v|_{L^r(0, \tau; L^r)},
\]
with \( k_0 < 1 \). We can deduce that \( \mathcal{C} \) is a contraction from \( B(u_0, \tau, \rho) \) to itself. By Banach’s fixed point theorem, we conclude that \( \mathcal{C} \) has a fixed point in \( B(u_0, \tau, \rho) \), which is a solution of (1.1). The proof that \( u \) is infinitely differentiable over \( (0, T) \times \mathbb{R}^N \) is identical to the proof of Proposition 2.1 of [8], to which the reader is referred. This concludes the proof of Lemma 10.

\[\text{4. – Global estimates and global existence.}\]

Let \( s \in (0, +\infty) \), in this section, we denote \( K = 2K + 2, \ s = 2s + 2, \) and \( m = 2K + 2s + 2 \).

Theorem 11. – Assume that \( u_0 \) belongs to \( L^\infty \cap L^2 \). Let \( u \) be the local solution obtained at Lemma 10 and \( T \) less than the maximal existence time. Then there exists two positive constants \( C_K, C_\lambda \) such that for all \( t \in [0, T] \),
\[
|u(t, \cdot)|_{L^r} \leq \exp(C_K t) |u_0|_{L^r}
\]
and
\[
|u|_{L^\infty(0, \tau; L^r)} \leq \frac{\exp(2C_K T)}{2C'_K} |u_0|^2_{L^2}
\]
and for \( \alpha \) such that \( |\alpha| < (1 + s) \alpha/s \), we have
\[
|u(t, \cdot)|_{L^\alpha} \leq \exp(C_K t) |u_0|_{L^\alpha}
\]
and
\[
|u|_{L^\infty(0, \tau; L^m)} \leq \frac{\exp(2(s + 1) C_K T)}{2(s + 1) C'_K} |u_0|^s_s.
\]
PROOF. – Multiplying the first equation of (1.1) by $|u|^{2s} \overline{u}$ and integrating the result by parts in space, we obtain

$$\int_{\mathbb{R}^N} \frac{\partial u}{\partial t} |u|^{2s} \overline{u} \, dx + \alpha \int_{\mathbb{R}^N} \nabla u \nabla (|u|^{2s} \overline{u}) \, dx + \int_{\mathbb{R}^N} F(u) |u|^{2s} \overline{u} \, dx = 0.\tag{4.5}$$

An elementary calculation shows that

$$\alpha \int_{\mathbb{R}^N} \nabla u \nabla (|u|^{2s} \overline{u}) \, dx = (s+1) \alpha \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s} \, dx +$$

$$+ s\alpha \sum_{k=1}^n \int_{\mathbb{R}^N} \left( \frac{\partial u}{\partial x_k} \right)^2 |u|^{2s-2} \, dx.$$

Let $Q$ be the polynominal

$$Q(y) = \sum_{j=0}^K \Re \mu_j y^{2j} \quad \text{for all } y \in \mathbb{R}.$$

We take the real part of (4.5) and we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^{2s+2} \, dx + (s+1)^2 \alpha \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s} \, dx +$$

$$(s+1) \int_{\mathbb{R}^N} Q(|u|)|u|^{2s+2} \, dx = -s(s+1) \Re \left( \alpha \sum_{k=1}^n \int_{\mathbb{R}^N} \left( \frac{\partial u}{\partial x_k} \right)^2 |u|^{2s-2} \, dx \right).$$

Since $\Re \mu_K > 0$, there exists two positive constants $C_K, C'_K$ such that

$$Q(y) \equiv C'_K y^{2K} - C_K \quad \text{for all } y \in \mathbb{R}.$$

It follows from (4.6) that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^{2s+2} \, dx +$$

$$(s+1)^2 \alpha \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s} \, dx + C_K (s+1) \int_{\mathbb{R}^N} |u|^{2K+2s+2} \, dx \leq$$

$$C_K (s+1) \int_{\mathbb{R}^N} |u|^{2s+2} \, dx - s(s+1) \Re \left( \alpha \sum_{k=1}^n \int_{\mathbb{R}^N} \left( \frac{\partial u}{\partial x_k} \right)^2 |u|^{2s-2} \, dx \right).$$
When $s = 0$, (4.7) gives

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 \, dx \leq C_K \int_{\mathbb{R}^N} |u|^2 \, dx.$$  

Hence, by integrating, we find (4.1). We return to (4.7) with $s = 0$, which we now integrate between 0 and $T$; this yields

$$C_K \int_0^T \int_{\mathbb{R}^N} |u|^{2K+2} \, ds \, dx \leq C_K \int_0^T \int_{\mathbb{R}^N} |u|^2 \, ds \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |u_0|^2 \, dx.$$  

It follows from (4.1) that

$$(4.8) \quad \int_0^T \int_{\mathbb{R}^N} |u|^2 \, ds \, dx \leq \frac{1}{2C_K} \left( \exp(2C_K T) - 1 \right) \int_{\mathbb{R}^N} |u_0|^2 \, dx,$$  

and we deduce that

$$C_K \int_0^T \int_{\mathbb{R}^N} |u|^{2K+2} \, dx \leq \frac{\exp(2C_K T)}{2} \int_{\mathbb{R}^N} |u_0|^2 \, dx,$$  

i.e. (4.2). When $s$ is different from 0, we notice that

$$-\Re \left( \alpha \sum_{k=1}^n \int_{\mathbb{R}^N} \left( \frac{\partial u}{\partial x_k} \frac{\bar{u}}{u} \right)^2 |u|^{2s-2} \, dx \right) \leq |\alpha| \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s} \, dx,$$  

and we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^{2s+2} \, dx + (s+1)^2 \alpha' \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s} \, dx +$$  

$$C_K' (s+1) \int_{\mathbb{R}^N} |u|^{2K+2s+2} \, dx \leq C_K (s+1) \int_{\mathbb{R}^N} |u|^{2s+2} \, dx,$$  

where

$$\alpha' = \alpha - \frac{|\alpha| s}{s+1}.$$  

If we suppose

$$|\alpha| < \frac{1+s}{s} \alpha_r,$$
then \( \alpha' \) is positive and by the above argument we obtain (4.3), (4.4). This concludes the proof of Theorem 11.

In the next two theorems, we suppose now that \( u_0 \) belongs to \( W^{2 - 2/q, q} \), with the condition

\[
q > 1 + \frac{N}{2}.
\]

In this case, the space \( W^{2 - 2/q, q} \) is included in \( L^\infty \), and the local existence of a solution \( u \) is also a consequence of Lemma 10.

Define

\[
H(u) = F(u) - \mu_0 u = \sum_{j=1}^{K} \mu_j |u|^{2j}u.
\]

**Theorem 12.** Assume that \( u_0 \) belongs to \( W^{2 - 2/q, q} \) with \( q > 1 + N/2 \) and that \( \alpha \) verifies

\[
|\alpha| < \frac{(2K + 1) q - 2K}{(2K + 1) q - 2K - 2} \Re \alpha
\]

and \( \Re \mu_K > 0 \). Let \( u \) be the local solution obtained at Lemma 10 and \( T \) less than the maximal existence time. Then we have

\[
H(u) \in L^{q}(0, T; L^q).
\]

**Proof.** Since

\[
q > 1 + \frac{N}{2} > 1 + \frac{1}{2K+1},
\]

there exists \( s > 0 \) such that

\[
q = 1 + \frac{2s + 1}{2K+1}.
\]

Moreover, since

\[
2s + 2 > \frac{2s}{2K+1} + 2 > 1 + \frac{2s + 1}{2K+1},
\]

the initial condition \( u_0 \) belongs to \( L^{2s+2} \) and we deduce from (4.4) that \( u \) be-
longs to $L^m(0, T; L^m)$ provided that

\begin{equation}
|\alpha| < \frac{1+s}{s} \Re \alpha.
\end{equation}

Since

$$s = (2K+1) \frac{q}{2} - K - 1,$$

we notice that the condition (4.11) is exactly (4.9). Let us show now that $H(u)$ belongs to $L^q(0, T; L^q)$, we already have

\begin{equation}
\int_0^T \int_{\mathbb{R}^N} (|u|^{2K+1})^q \leq \int_0^T \int_{\mathbb{R}^N} |u|^m < +\infty.
\end{equation}

On the other hand, for all $j, j \in \{1, \ldots, K-1\}$, we have

\begin{equation}
q < 3q \leq (2j+1) q < m.
\end{equation}

If $q \leq 2$, it follows from (4.8), (4.12) and by interpolation that for all $j, j \in \{1, \ldots, K-1\}$,

$$|u|^{2j} u \in L^q(0, T; L^q).$$

If $q > 2$, we deduce from (4.13) that is sufficient to see that $u$ belongs to $L^q(0, T; L^q)$ under the assumptions (4.9). But if $q > 2$, there exists $s_0 > 0$ such that $q = 2s_0 + 2$ and we deduce (4.3) that $u$ belongs to $L^q(0, T; L^q)$ provided that

$$|\alpha| < \frac{1+s_0}{s_0} \Re \alpha.$$

Finally, there remains to show that

\begin{equation}
\frac{1+s}{s} < \frac{1+s_0}{s_0}.
\end{equation}

But since

$$s_0 = \frac{q}{2} - 1 \quad \text{and} \quad s = (2K+1) \frac{q}{2} - K - 1,$$

we deduce that $s > s_0$ and as a consequence (4.14). This concludes the proof of Theorem 12.

\textbf{Theorem 13.} – Assume that $u_0$ belongs to $W^{2-2/q, q}$ with $q > 1 + N/2$ and
that \( \alpha \) verifies
\[
|\alpha| < \frac{(2K + 1) q - 2K}{(2K + 1) q - 2K - 2} \Re \alpha
\]
and \( \Re \mu_K > 0 \). Then (1.1) has a unique global solution \( u \) belonging to
\[
C([0, t], W^{2-2/q, q})
\]
for all \( t > 0 \).

**Proof.** – Let \( T_m \) be the maximum existence time. It follows from Theorem 12 that \( H(u) \in L^q(0, T_m; L^q) \) and we infer from Theorem 8 that
\[
u \in W^{1, q}(0, T_m; L^q) \cap L^q(0, T_m; W^{2, q}).
\]
The Sobolev embedding theorem gives
\[
u \in C([0, T_m], W^{2-2/q, q}).
\]
Therefore \( T_m \) is not the maximal existence time, which proves that \( u \) is a global solution of (1.1).

In order to obtain the uniqueness, we consider two solutions \( u \) and \( v \) of (1.1), the function \( w = u - v \) verifies
\[
\frac{\partial w}{\partial t} - \alpha \Delta w + F(u) - F(v) = 0.
\]
Multiplying (4.15) by \( \overline{w} \) and integrating by parts in space, this yields
\[
\int_{R^N} \frac{\partial w}{\partial t} \overline{w} dx + \alpha \int_{R^N} \left| \nabla w \right|^2 dx + \int_{R^N} (F(u) - F(v)) \overline{w} dx = 0.
\]
Since \( u \) and \( v \) belong to \( C([0, T_m], L^\infty) \), there exists a positive constant \( C \) such that over \([0, T_m]\)
\[
|F(u) - F(v)|_{L^2} \leq C|u - v|_{L^2}.
\]
It follows from (4.16) that
\[
\frac{1}{2} \frac{d}{dt} \int_{R^N} |w|^2 dx + \alpha \int_{R^N} \left| \nabla w \right|^2 dx \leq C \int_{R^N} |w|^2 dx,
\]
so that Gronwall’s inequality gives the desired result. This completes the proof of Theorem 13. ■

Denote \( C^\infty_0 \) the space of functions infinitely differentiable with compact support; a consequence of Theorem 13 is the following:
Corollary 14. – Assume that $u_0$ belongs to $C^\infty_0$ and that $\alpha$ verifies
\[ |\alpha| < \frac{(2K + 1) N + 2}{(2K + 1) N - 2} \Re \alpha \]
and $\Re \mu_K > 0$. Then (1.1) has a unique global solution $u$ belonging to $C([0, t], L^\infty)$ for all $t > 0$.

Proof. – We just need to prove that $u$ belongs to $C([0, t], L^\infty)$ for all $t > 0$.

Let $\varepsilon > 0$ and define
\[ r(\varepsilon) = \frac{(2K + 1)(1 + N/2 + \varepsilon) - 2K}{(2K + 1)(1 + N/2 + \varepsilon) - 2K - 2}. \]

$u_0$ belongs to $W^{2-2/q, q}$ with $q = 1 + N/2 + \varepsilon$ and since
\[ \lim_{\varepsilon \to 0} r(\varepsilon) = \frac{(2K + 1) N + 2}{(2K + 1) N - 2} \]
and
\[ r(\varepsilon) < \frac{(2K + 1) N + 2}{(2K + 1) N - 2}, \]
we can choose $\varepsilon$ sufficiently small such that
\[ |\alpha| < r(\varepsilon) \Re \alpha. \]

Thanks to Theorem 13, $u$ belongs to $C([0, t], L^\infty)$ for all $t > 0$. ■

We now define notion of global weak-solution of (1.1):

Definition 15. – A function $u$ is a global weak solution of (1.1) if it belongs to
\[ L^\infty_{\text{loc}}(\mathbb{R}_+ : L^2) \cap L^2_{\text{loc}}(\mathbb{R}_+ : H^1) \cap L^{2K+2}_{\text{loc}}(\mathbb{R}_+ : L^{2K+2}), \]
and for any $\psi \in C^\infty_0([0, + \infty) \times \mathbb{R}^N)$, it verifies
\[ \int_{\mathbb{R}_+ \times \mathbb{R}^N} \left(-u \frac{\partial \psi}{\partial t} + \alpha \nabla u \nabla \psi + F(u) \psi\right) ds dx = \int_{\mathbb{R}^N} u_0(x) \psi(\cdot, x) dx. \]

Theorem 16. – Assume that $\alpha$ satisfies the condition
\[ |\alpha| < \frac{(2K + 1) N + 2}{(2K + 1) N - 2} \Re \alpha \]
and \( \Re \mu_K > 0 \). Then for all \( u_0 \in L^2 \), (1.1) possesses at least a global weak-solution.

**Proof.** – Let \( t > 0 \), introduce a sequence \( (u_{0m}) \) in \( C_0^\infty \) such that \( (u_{0m}) \) tends to \( u_0 \) in \( L^2 \) and let \( u_m \) be a solution of (1.1) over \( (0, t) \) with initial condition \( (u_{0m}) \). It follows from (4.1) that the sequence \( (u_m) \) remains in a bounded set of \( L^\infty (0, t; L^2) \), from (4.2) that the sequence \( (u_m) \) remains in a bounded set of \( L^{2K+2} (0, t; L^{2K+2}) \). With the help of (4.7), we can see that the sequence \( (u_m) \) remains in a bounded set of \( L^2 (0, t; W^{1,2}) \).

These estimates ensure the existence of an element \( u \) and a subsequence still denoted \( (u_m) \) such that as \( m \) tends to infinity, \( (u_m) \) tends weakly to \( u \) in \( L^2 (0, t; W^{1,2}) \), weakly to \( u \) in \( L^{2K+2} (0, t; L^{2K+2}) \) and tends weakly-star to \( u \) in \( L^\infty (0, t; L^2) \). We also introduce \( v_m \) and \( w_m \) solutions of the following problems:

\[
\begin{aligned}
\frac{\partial v_m}{\partial t} - \alpha \Delta v_m + \mu_0 v_m &= 0, \quad x \in \mathbb{R}^N, \quad t \in (0, t), \\
v_m(0, x) &= u_{0m}(x), \quad x \in \mathbb{R}^N
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial w_m}{\partial t} - \alpha \Delta w_m + \mu_0 w_m + H(u_m) &= 0, \quad x \in \mathbb{R}^N, \quad t \in (0, t), \\
w_m(0, x) &= 0, \quad x \in \mathbb{R}^N
\end{aligned}
\]

We recall that \( H = F - \mu_0 \text{Id} \); it is clear that \( u_m = v_m + w_m \). The sequence \( (v_m) \) converges to \( v \), solution of the problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} - \alpha \Delta v + \mu_0 v &= 0, \quad x \in \mathbb{R}^N, \quad t \in (0, t), \\
v(0, x) &= u_0(x), \quad x \in \mathbb{R}^N
\end{aligned}
\]

in \( C([0, t], L^2) \cap L^2 (0, t; H^1) \).

It follows from (4.1) that the term \( H(u_m) \) is bounded in the space \( L^q (0, t; L^q) \) with \( q = (2K+2)/(2K+1) \) and we infer from Theorem 8 that the sequence \( (w_m) \) is bounded in \( W^{1,q} (0, t; L^q) \cap L^q (0, t; W^{2,q}) \). This estimate ensures the existence of an element \( w \) and a subsequence still denoted \( (w_m) \) such that \( (w_m) \) tends weakly to \( w \) in \( W^{1,q} (0, t; L^q) \cap L^q (0, t; W^{2,q}) \). We can now deduce that the subsequence \( (u_m) \) converges to \( u = v + w \) in \( L^1_{\text{loc}} (0, t; L^q_{\text{loc}}) \) and almost everywhere. As a consequence, the sequence \( \langle H(u_m) \rangle \) tends to \( H(u) \) in \( L^1 (0, t; L^q_{\text{loc}}) \) and for all \( \psi \) in \( C^\infty_0 ((0, +\infty) \times \mathbb{R}^n) \), as \( m \) tends to infinity

\[
(4.17) \quad \langle H(u_m), \psi \rangle \text{ tends to } \langle H(u), \psi \rangle.
\]
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It is also clear that when \( m \) tends to infinity
\[
(4.18) \quad \left\langle \frac{\partial u_m}{\partial t} - \alpha \Delta u_m + \mu_0 u_m, \psi \right\rangle \text{ tends to } \left\langle \frac{\partial u}{\partial t} - \alpha \Delta u + \mu_0 u, \psi \right\rangle.
\]

It follows from (4.17) and (4.18) that \( u \) satisfies
\[
\frac{\partial u}{\partial t} - \alpha \Delta u + F(u) = 0,
\]
in the distribution sense.

Since \( v \in C([0, t), L^2) \), \( w \in C([0, t), W^{2, -2q, q}) \) and \( u = v + w \), the initial condition \( u(0, x) = u_0(x), x \in \mathbb{R}^n \) make sense. This concludes the proof of Theorem 16.

Using the previous decomposition, estimates (4.3), (4.4) and Theorem 8, we can also prove the following results:

**Theorem 17.** – Assume that \( u_0 \) belongs to \( L^2 \cap L^p \), with \( p = 2 + 2 \sigma \ (\sigma > 0) \) and that \( a \) satisfies the condition
\[
|\alpha| < \min \left( \frac{(2K + 1) N + 2}{(2K + 1) N - 2}, \frac{1 + \sigma}{\sigma} \right) \Re \alpha
\]
and \( \Re \mu_K > 0 \). Then (1.1) possesses at least a global weak-solution of the form \( u = v + w \) with
\[
v \in C([0, + \infty), L^p) \quad \text{and} \quad w \in C([0, + \infty), W^{2, -2q, q}) \quad \text{and} \quad w(0, \cdot) = 0,
\]
with
\[
q = 1 + \frac{2\sigma + 1}{2K + 1}.
\]
In particular if \( p \geq KN \),
\[
u \in C([0, + \infty), L^p).
\]

**Remark 18.** – In fact, the assumption \( u_0 \) belongs to \( L^2 \) is not really necessary if we change the definition of a weak-global solution.

**Theorem 19.** – Assume that \( u_0 \) belongs to \( H^1 \cap L^p \) with \( p = 2 + 2s \ (\sigma > 0) \) and that \( a \) satisfies the condition
\[
|\alpha| < \min \left( \frac{(2K + 1) N + 2}{(2K + 1) N - 2}, \frac{1 + \sigma}{\sigma} \right) \Re \alpha
\]
and \( \Re \mu_K > 0 \). Then (1.1) possesses at least a global weak-solution \( u \) of the form \( u = v + w \) with

\[
v \in C([0, + \infty), H^1), \quad w \in C([0, + \infty), W^{2-2/q, q}) \text{ and } w(0, \cdot) = 0,
\]

with

\[
q = 1 + \frac{2\sigma + 1}{2K + 1}.
\]

In particular if \( \sigma \geq K \) and

\[
|\alpha| < \frac{(2K + 1) N + 2}{(2K + 1) N - 2} \Re \alpha,
\]

\( u \) belongs to \( C([0, + \infty), H^1) \).

**Remark 20.** – In the second part of this Theorem, we can take \( u_0 \) in \( H^1 \) for \( N \leq 2 \) and for \( N = 3 \) if \( K \leq 2 \), which covers the case of the Fauve-Thual equation.

**References**


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