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Stéphane Descombes, Mohand Moussaoui

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Global Existence and Regularity of Solutions for Complex Ginzburg-Landau Equations

STÉPHANE DESCOMBES - MOHAND MOUSSAOUI

Sunto. – *Si considerano equazioni di Ginzburg-Landau complesse del tipo* $u_t - \alpha \Delta u + P(|u|^2)u = 0$ *in* $\mathbb{R}^N$ *dove P è polinomio di grado K a coefficienti complessi e* α *è un numero complesso con parte reale positiva* $\Re\alpha$. *Nell'ipotesi che la parte reale del coefficiente del termine di grado massimo P sia positiva, si dimostra l'esistenza e la regolarità di una soluzione globale nel caso* $|\alpha| < C\Re\alpha$, *dove C dipende da K e N.*

1. – Introduction.

Let K be an integer, $K \geq 1$, α and $\mu_j, j \in \{0, \ldots, K\}$, complex numbers with $\Re\alpha > 0$, and $\Re\mu_K > 0$. We consider the initial value problem

$$(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \alpha \Delta u + \displaystyle\sum_{j=0}^{K} \mu_j |u|^{2j} u = 0 , & x \in \mathbb{R}^N , \ t > 0 , \\ u(0, x) = u_0(x) , & x \in \mathbb{R}^N . \end{cases}$$

Without loss of generality, we suppose that $\Re\mu_0 > 0$. For example, when $K = 1$, we obtain the well-known cubic Ginzburg-Landau equation, and when $K = 2$, the equation given by Fauve-Thual in [3] as a model of localized structures generated by subcritical instabilities. In [1], Doering, Gibbon and Levermore have considered a system of the same form but with periodic boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Ru + (1 + i\nu)\, \Delta u - (1 + i\mu)\, |u|^{2\sigma} u , & x \in \mathbb{T}^N , \ t > 0 , \\ u(0, x) = u_0(x) , & x \in \mathbb{T}^N . \end{cases}$$

They obtained existence of global-weak solutions in all dimensions and for all $\sigma > 0$ and parameter values R, ν and μ. Under certain assumptions, they also obtained global strong solutions. But their proofs use essentially the boundedness of the domain $\mathbb{T}^N$. The case of the whole space is considered in [4], [5] by Ginibre and Velo, for the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma_1 u + (a + i\gamma_2)\, \Delta u - (b + i\gamma_3)\, g(|u|^2) u\,, & x \in \mathbb{R}^N\,,\ t > 0\,, \\ u(0, x) = u_0(x)\,, & x \in \mathbb{R}^N\,, \end{cases}$$

with $a > 0$, $b > 0$ and $g \geqslant 0$ satisfying

$$x^{\sigma} \leqslant g(x) \leqslant C(1 + x^{\sigma})$$

for some σ ($0 < \sigma < \infty$), some $C \geqslant 1$ and all $x \geqslant 0$. They obtained existence and uniqueness of solutions globally defined in time with initial data corresponding to the spaces $\boldsymbol{L}^p$ for $p \geqslant 2$ or $\boldsymbol{H}^1 \cap \boldsymbol{L}^{2\sigma+2}$. They also studied the case where the nonlinear term is of the form $(b + i\gamma_3)\, f(u)$ with f belonging to $C^1(\mathbb{C}, \mathbb{C})$ and obtain local existence of solutions for initial data belonging to $\boldsymbol{L}^p$, $p \geqslant 2$.

In this article, under assumptions on α, we obtain for (1.1) existence and regularity of strong global solutions when u_0 belongs to the space $\boldsymbol{W}^{2-2/q,\,q}$ with $q > 1 + N/2$ (so the results are different from [4], [5]) and we deduce existence of global weak-solutions when u_0 belongs to $\boldsymbol{L}^p$, $p \geqslant 2$ or $\boldsymbol{H}^1 \cap \boldsymbol{L}^p$ with $p > 2$. The methods are different from [4], [5] and do not use a priori estimates obtained by multiplying the first equation of (1.1) by $\Delta \overline{u}$.

We use the notations:

$$\boldsymbol{L}^p = L^p(\mathbb{R}^N, \mathbb{C})\,,$$

$$\boldsymbol{W}^{s,\,p} = W^{s,\,p}(\mathbb{R}^N, \mathbb{C})\,.$$

Our purpose is to prove the following results:

THEOREM 1. – *Assume that u_0 belongs to $\boldsymbol{W}^{2-2/q,\,q}$ with $q > 1 + N/2$ and that α verifies*

$$|\alpha| < \frac{(2K+1)\, q - 2K}{(2K+1)\, q - 2K - 2}\, \Re\alpha$$

and $\Re\mu_K > 0$. Then (1.1) *has a unique global solution u belonging to*

$$C([0, \boldsymbol{t}], \boldsymbol{W}^{2-2/q,\,q})$$

for all $\boldsymbol{t} > 0$.

THEOREM 2. – *Assume that u_0 belongs to $\boldsymbol{H}^1 \cap \boldsymbol{L}^p$ with $p = 2 + 2\sigma$ ($\sigma > 0$) and that α satisfies the condition*

$$|\alpha| < \min\left(\frac{(2K+1)\, N + 2}{(2K+1)\, N - 2}\,,\ \frac{1+\sigma}{\sigma} \right) \Re\alpha$$

and $\Re\mu_K > 0$. *Then* (1.1) *possesses at least a global weak-solution* u *of the form* $u = v + w$ *such that*

$$v \in C([0, +\infty), \boldsymbol{H}^1), \qquad w \in C([0, +\infty), \boldsymbol{W}^{2-2/q, q}) \text{ and } w(0, \cdot) = 0,$$

with

$$q = 1 + \frac{2\sigma + 1}{2K + 1}.$$

In particular if $\sigma \geq K$ *and*

$$|a| < \frac{(2K+1)N + 2}{(2K+1)N - 2} \Re\alpha,$$

u *belongs to* $C([0, +\infty), \boldsymbol{H}^1)$.

Section 2 is devoted to some results on the L^p regularity of solutions of linear equations analogous to (1.1). In section 3, we prove the local existence of solutions when $u_0 \in \boldsymbol{L}^\infty$. In section 4, we provide estimates on this local solution; then we prove, under assumptions on α, that when u_0 belongs to $\boldsymbol{W}^{2-2/q, q}$, with $q > 1 + N/2$, the solution is global in time. Then we pass to the limit to cover the case where u_0 belongs to $\boldsymbol{L}^p$ or $\boldsymbol{H}^1 \cap \boldsymbol{L}^p$ with $p = 2 + 2\sigma$ ($\sigma > 0$).

REMARK 1. – *The same results hold if we consider problem* (1.1) *in a bounded regular domain of* $\mathbb{R}^N$ *and add a Dirichlet or Neumann boundary condition.*

In this article, we denote:

$$F(u) = \sum_{j=0}^{K} \mu_j |u|^{2j} u.$$

2. – L^p regularity.

Consider the Cauchy problem:

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \alpha \Delta u + \mu u = f, & x \in \mathbb{R}^N, \ t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

We assume that α and μ are complex numbers with positive real parts α_r and μ_r, that p belongs to $(1, +\infty)$, f is given in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$ and u_0 in $\boldsymbol{W}^{2-2/p,p}$. We are interested in the L^p regularity of solutions of (2.1), we will prove that

(2.1) has a unique solution $u \in W^{1,p}(\mathbb{R}_+; \boldsymbol{L}^p) \cap L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})$ and that u, Δu and $\partial u/\partial t$ depend continuously on f.

To obtain this result, we will use the imaginary powers of the operators appearing in (2.1), according to an idea of Prüss and Sohr [9]. We refer to the book of Triebel [12] for a definition of the imaginary powers of an operator. Let us recall some definitions. Let A be a closed linear operator in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$, with dense domain $D(A)$; $N(A)$ and $R(A)$ denote the kernel and the range of A, $\varrho(A)$ and $\sigma(A)$ the resolvent set and the spectrum of A. Finally, $B(L^p)$ is the space of bounded linear operators in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$.

DEFINITION 2 [9]. – *Let θ belong to $[0, \pi)$. A closed linear densely defined operator A in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$ belongs to the class $BIP(L^p, \theta)$, if it satisfies:*

(H1) *The set $(-\infty, 0)$ is included in $\varrho(A)$, the kernel $N(A)$ is reduced to 0, the range $R(A)$ is dense in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$, and, there exists a $M \geq 1$, such that*

$$|(t+A)^{-1}| \leq M/t \quad \text{for all } t > 0 . \tag{2.2}$$

(H2) *For all $s \in \mathbb{R}$, A^{is} belongs to $B(L^p)$, and there exists a K_0 such that*

$$|A^{is}| \leq K_0 \exp(\theta |s|). \tag{2.3}$$

DEFINITION 3. – *Let A, B two linear operators. We say that A, B are resolvent commuting if for all λ (respectively ν) in the resolvent set $\varrho(A)$ (respectively $\varrho(B)$) $(\lambda - A)^{-1}(\nu - B)^{-1} = (\nu - B)^{-1}(\lambda - A)^{-1}$.*

Let us quote the main result of [9]:

THEOREM 4 [9]. – *We are given $k \geq 2$ elements A_i in $BIP(L^p, \theta_i)$, such that, for each pair $i \neq j$, A_i and A_j are resolvent commuting and satisfy $\theta_i + \theta_j < \pi$. Let $\theta = \max \theta_i$ and assume that there is only one i with $\theta = \theta_i$.*

Then the operator A defined by

$$D(A) = \bigcap_1^k D(A_i), \qquad A = \sum_{i=1}^k A_i ,$$

is closed and belongs to the class $BIP(L^p, \theta)$. Moreover, there is a constant $C > 0$ such that

$$\sum_{i=1}^k |A_i x| \leq C |Ax| , \qquad \forall x \in D(A).$$

In particular, $N(A) = 0$ and $R(A)$ is dense in $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$.

In the following sequence of lemmas, we show that the operators appearing in (2.1) belong to the class $BIP(L^p, \theta)$ and we characterize the relevant θ.

LEMMA 5. – *Define* A_1 *and* B_1 *respectively by*

$$D(A_1) = W_0^{1,p}(\mathbb{R}_+; \boldsymbol{L}^p), \quad A_1 = \partial/\partial t,$$
$$D(B_1) = L^P(\mathbb{R}_+; \boldsymbol{W}^{2,p}), \quad B_1 = -\Delta.$$

Then for all $\varepsilon > 0$:

$$A_1 \in BIP(L^p, \pi/2 + \varepsilon) \text{ and } B_1 \in BIP(L^p, \varepsilon).$$

PROOF. – The result for A_1 is due to [2] [9]. In the scalar case the result for B_1 is due to [11] [10], the vector generalization is straightforward. ■

LEMMA 6. – *Let* β *be a complex number of positive real part* β_r. *The operator* I_β *is the multiplication by* β *in* $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$. *Then*

$$I_\beta \in BIP(L^p, |\operatorname{Arg}\beta|),$$

where Arg *is the principal determination of the argument.*

PROOF. – It suffices to prove (2.3). Let $s \in \mathbb{R}$, then we have

$$|\beta^{is}| = |\exp(is \operatorname{Log}|\beta| - s \operatorname{Arg}\beta)| = \exp(-s \operatorname{Arg}\beta),$$

thus

$$|\beta^{is}| \leqslant \exp(|s|\,|\operatorname{Arg}\beta|). \quad ■$$

LEMMA 7. – *Define an operator* B_α *by*

$$D(B_\alpha) = L^P(\mathbb{R}_+; \boldsymbol{W}^{2,p}), \quad B_\alpha = -\alpha\Delta,$$

then for all $\varepsilon > 0$

$$B_\alpha \in BIP(L^p, |\operatorname{Arg}\alpha| + \varepsilon).$$

PROOF. – Remark that $B_\alpha = I_\alpha B_1$, then thanks to the corollary 3 of [9],

$$B_\alpha \in BIP(L^p, \theta_{B_1} + \theta_{I_\alpha}). \quad ■$$

Now, we can prove the following theorem:

THEOREM 8. – *Let* α, μ *be complex numbers such that* $\alpha_r > 0, \mu_r > 0$; *let* u_0 *belong to* $\boldsymbol{W}^{2-2/p,p}$. *Then for all* $f \in L(\mathbb{R}_+; \boldsymbol{L}^p)$, $1 < p < \infty$, *the Cauchy problem* (2.1) *has a unique solution* $u \in W^{1,p}(\mathbb{R}_+; \boldsymbol{L}^p) \cap L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})$. *Moreover* *there exists a constant* $C > 0$, *such that*

$$\left| \frac{\partial u}{\partial t} \right|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} + |u|_{L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})} \leq C(|f|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} + |u_0|_{\boldsymbol{W}^{2-2/p, p}}) . \tag{2.4}$$

PROOF. – Consider the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = f, & x \in \mathbb{R}^N, \ t > 0, \\ v(0, x) = u_0, & x \in \mathbb{R}^N. \end{cases} \tag{2.5}$$

where u_0 is as in the statement of Theorem 8. It is a well known fact (see for example [7]) that (2.5) possesses a unique solution v, which belongs to $W^{1,p}(\mathbb{R}_+; \boldsymbol{L}^p) \cap L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})$. Moreover there exists a constant C_1 such that v verifies

$$\left| \frac{\partial v}{\partial t} \right|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} + |v|_{L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})} \leq C_1(|f|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} + |u_0|_{\boldsymbol{W}^{2-2/p, p}}) . \tag{2.6}$$

Define

$$f_v = f - \frac{\partial v}{\partial t} + \alpha \Delta v - \mu v .$$

We observe that f_v belongs to $L^p(\mathbb{R}_+; \boldsymbol{L}^p)$. The function $w = u - v$ is solution of

$$\begin{cases} \dfrac{\partial w}{\partial t} - \alpha \Delta w + \mu w = f_v, & x \in \mathbb{R}^N, \ t > 0, \\ w(0, x) = 0, & x \in \mathbb{R}^N. \end{cases} \tag{2.7}$$

Let A be the operator defined by

$$D(A) = W_0^{1,p}(\mathbb{R}_+; \boldsymbol{L}^p) \cap L^P(\mathbb{R}_+; \boldsymbol{W}^{2,p}), \qquad A = A_1 + B_\alpha + I_\mu ,$$

we can rewrite the problem (2.7) under the form

$$Aw = f_v .$$

Thanks to Lemma 5, 7 for ε sufficiently small, the hypotheses of Theorem 4 are satisfied; therefore A is invertible in $D(A)$ and there exists a constant $C > 0$ such that

$$\left| \frac{\partial w}{\partial t} \right|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} + |w|_{L^p(\mathbb{R}_+; \boldsymbol{W}^{2,p})} \leq C|f_v|_{L^p(\mathbb{R}_+; \boldsymbol{L}^p)} . \tag{2.8}$$

Now $u = w + v$ and (2.4) comes from (2.6) and (2.8). This concludes the proof of Theorem 8. ■

REMARK 9. – *The result of Theorem* 8 *can be also obtained using the results of Hieber and Prüss* [6].

3. – Local existence.

In this section, we prove the following existence result:

LEMMA 10. – *Let* $u_0 \in \boldsymbol{L}^\infty$; *then there exists a positive number* T_0, *depending only on* $|u_0|_{\boldsymbol{L}^\infty}$ *and* F, *such that* (1.1) *has at least a solution* $u \in L^\infty(0, T_0; \boldsymbol{L}^\infty)$. *Moreover* u *is infinitely differentiable over* $(0, T_0) \times \mathbb{R}^N$.

PROOF. – Let G be the Green function corresponding to the linear initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha \Delta u = 0\,, & x \in \mathbb{R}^N,\ t > 0\,, \\ u(0, x) = u_0(x)\,, & x \in \mathbb{R}^N. \end{cases}$$

It is given explicitly by

$$G(t, x) = (4\pi\alpha t)^{-N/2} \exp\left(-|x|^2/4\pi\alpha t\right),$$

where the fractional powers are defined as principal determination when necessary.

Let τ and ϱ be positive real numbers and $B(u_0, \tau, \varrho)$ be the ball in $L^\infty(0, \tau; \boldsymbol{L}^\infty)$ of center u_0 and radius ϱ. Define an application $\mathcal{T}$ from $L^\infty(0, \tau; \boldsymbol{L}^\infty)$ into itself by

$$(\mathcal{T}u)(t, \cdot) = G(t, \cdot) \star u_0 - \int_0^t G(t-s, \cdot) F(u(\cdot, s))\, ds\,.$$

For $u \in B(u_0, \tau, \varrho)$ and $t \leqslant \tau$; we have

$$|(\mathcal{T}u)(t, \cdot) - u_0(\cdot)|_{\boldsymbol{L}^\infty} \leqslant |u_0|_{\boldsymbol{L}^\infty} + |G(t, \cdot) \star u_0|_{\boldsymbol{L}^\infty} + \int_0^t |G(t-s, \cdot) \star F(u(\cdot, s))|_{\boldsymbol{L}^\infty}\, ds\,,$$

and by Young's inequality, we obtain

$$|(\mathcal{C}u)(t,\cdot)-u_0(\cdot)|_{L^\infty} \leqslant (1+|G(t,\cdot)|_{L^1})|u_0|_{L^\infty} + \int_0^t |G(t-s,\cdot)|_{L^1}|F(u(\cdot,s))|_{L^\infty}\,ds\,,$$

but

$$\begin{aligned} |G(t,\cdot)|_{L^1} &= |(4\pi\alpha t)^{-N/2}|\int_{\mathbb{R}^N} |\exp(-|x|^2/4\pi\alpha t)|\,dx \\ &= (4\pi|\alpha|t)^{-N/2}\int_{\mathbb{R}^N}\exp(-\alpha_r|x|^2/4\pi|\alpha|^2 t)\,dx \\ &= |\alpha|^{-N/2}\int_{\mathbb{R}^N}\exp(-\alpha_r|y|^2/|\alpha|^2)\,dy \\ &= k_N\,. \end{aligned}$$

Thus, we have

$$(3.1)\qquad |(\mathcal{C}u)(t,\cdot)-u_0(\cdot)|_{L^\infty} \leqslant (1+k_N)|u_0|_{L^\infty} + k_N\int_0^t |F(u(\cdot,s))|_{L^\infty}\,ds\,.$$

The function F is Lipschitz continuous on the set

$$V_{a,\varrho} = \{v\in\mathbb{C};\ |v-a|\leqslant\varrho\}\,,$$

with Lipschitz constant $L(a,\varrho)$; set

$$\lambda = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N}\{L(u_0(x),\varrho)\} < +\infty\,.$$

We deduce from (3.1) that

$$|(\mathcal{C}u)(t,\cdot)-u_0(\cdot)|_{L^\infty} \leqslant (1+k_N)|u_0|_{L^\infty} + k_N\tau(\lambda\varrho + |F(u_0)|_{L^\infty})\,.$$

Choose $\varrho > 2(1+k_N)|u_0|_{L^\infty}$ and τ sufficiently small such that

$$|(\mathcal{C}u)(t,\cdot)-u_0(\cdot)|_{L^\infty} \leqslant \varrho\,;$$

we deduce that

$$\mathcal{C}(B(u_0,\tau,\varrho))\subset B(u_0,\tau,\varrho)\,.$$

For u and v in $B(u_0, \tau, \varrho)$ and $t \leqslant \tau$, we have

$$|(\mathcal{C}u)(t, \cdot) - (\mathcal{C}v)(t, \cdot)|_{\boldsymbol{L}^\infty} \leqslant k_N \lambda \int_0^t |u(\cdot, s) - v(\cdot, s)|_{\boldsymbol{L}^\infty}\, ds$$
$$\leqslant k_N \lambda \tau |u - v|_{L^\infty(0, t; \boldsymbol{L}^\infty)}\,.$$

Thus

$$|\mathcal{C}u - \mathcal{C}v|_{L^\infty(0, t; \boldsymbol{L}^\infty)} \leqslant k_N \lambda \tau |u - v|_{L^\infty(0, t; \boldsymbol{L}^\infty)}\,,$$

and therefore

$$|\mathcal{C}u - \mathcal{C}v|_{L^\infty(0, t; \boldsymbol{L}^\infty)} \leqslant k_0\, |u - v|_{L^\infty(0, t; \boldsymbol{L}^\infty)}\,,$$

with $k_0 < 1$. We can deduce that $\mathcal{C}$ is a contraction from $B(u_0, \tau, \varrho)$ to itself. By Banach's fixed point theorem, we conclude that $\mathcal{C}$ has a fixed point in $B(u_0, \tau, \varrho)$, which is a solution of (1.1). The proof that u is infinitely differentiable over $(0, T) \times \mathbb{R}^N$ is identical to the proof of Proposition 2.1 of [8], to which the reader is referred. This concludes the proof of Lemma 10. ■

4. – Global estimates and global existence.

Let $s \in (0, +\infty)$, in this section, we denote $\boldsymbol{K} = 2K + 2$, $\boldsymbol{s} = 2s + 2$, and $\boldsymbol{m} = 2K + 2s + 2$.

THEOREM 11. – *Assume that u_0 belongs to $\boldsymbol{L}^\infty \cap \boldsymbol{L}^2$. Let u be the local solution obtained at Lemma* 10 *and T less than the maximal existence time. Then there exists two positive constants C_K, C'_K such that for all $t \in [0, T]$,*

$$|u(t, \cdot)|_{\boldsymbol{L}^2} \leqslant \exp(C_K t)\, |u_0|_{\boldsymbol{L}^2} \tag{4.1}$$

and

$$|u|^{\boldsymbol{K}}_{L^{\boldsymbol{K}}(0, T; \boldsymbol{L}^{\boldsymbol{K}})} \leqslant \frac{\exp(2C_K T)}{2C'_K}\, |u_0|^2_{\boldsymbol{L}^2} \tag{4.2}$$

and for α such that $|\alpha| < (1 + s)\, \alpha_r / s$, we have

$$|u(t, \cdot)|_{\boldsymbol{L}^{\boldsymbol{s}}} \leqslant \exp(C_K t)\, |u_0|_{\boldsymbol{L}^{\boldsymbol{s}}} \tag{4.3}$$

and

$$|u|^{\boldsymbol{m}}_{L^{\boldsymbol{m}}(0, T; \boldsymbol{L}^{\boldsymbol{m}})} \leqslant \frac{\exp(2(s+1)\, C_K T)}{2(s+1)\, C'_K}\, |u_0|^{\boldsymbol{s}}_{\boldsymbol{s}}\,. \tag{4.4}$$

Proof. – Multiplying the first equation of (1.1) by $|u|^{2s}\overline{u}$ and integrating the result by parts in space, we obtain

$$\int_{\mathbb{R}^N} \frac{\partial u}{\partial t}\,|u|^{2s}\overline{u}\,dx + \alpha\int_{\mathbb{R}^N} \nabla u\,\nabla(|u|^{2s}\overline{u})\,dx + \int_{\mathbb{R}^N} F(u)\,|u|^{2s}\overline{u}\,dx = 0\,. \tag{4.5}$$

An elementary calculation shows that

$$\alpha\int_{\mathbb{R}^N} \nabla u\,\nabla(|u|^{2s}\overline{u})\,dx = (s+1)\,\alpha\int_{\mathbb{R}^N} |\nabla u|^2\,|u|^{2s}\,dx$$

$$+ s\alpha\sum_{k=1}^{n}\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial x_k}\,\overline{u}\right)^2 |u|^{2s-2}\,dx.$$

Let Q be the polynominal

$$Q(y) = \sum_{j=0}^{K} \Re\mu_j\,y^{2j} \qquad \text{for all } y\in\mathbb{R}\,.$$

We take the real part of (4.5) and we obtain

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^N} |u|^{2s+2}\,dx + (s+1)^2\alpha_r\int_{\mathbb{R}^N} |\nabla u|^2\,|u|^{2s}\,dx + \tag{4.6}$$

$$(s+1)\int_{\mathbb{R}^N} Q(|u|)|u|^{2s+2}\,dx = -\,s(s+1)\,\Re\left(\alpha\sum_{k=1}^{n}\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial x_k}\,\overline{u}\right)^2 |u|^{2s-2}\,dx\right).$$

Since $\Re\mu_K > 0$, there exists two positive constants C_K, C'_K such that

$$Q(y) \geqslant C'_K\,y^{2K} - C_K \qquad \text{for all } y\in\mathbb{R}\,.$$

It follows from (4.6) that

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^N} |u|^{2s+2}\,dx + \tag{4.7}$$

$$(s+1)^2\alpha_r\int_{\mathbb{R}^N} |\nabla u|^2\,|u|^{2s}\,dx + C'_K(s+1)\int_{\mathbb{R}^N} |u|^{2K+2s+2}\,dx \leqslant$$

$$C_K(s+1)\int_{\mathbb{R}^N} |u|^{2s+2}\,dx - s(s+1)\,\Re\left(\alpha\sum_{k=1}^{n}\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial x_k}\,\overline{u}\right)^2 |u|^{2s-2}\,dx\right).$$

When $s = 0$, (4.7) gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2\, dx \leqslant C_K \int_{\mathbb{R}^N} |u|^2\, dx \, .$$

Hence, by integrating, we find (4.1). We return to (4.7) with $s = 0$, which we now integrate between 0 and T; this yields

$$C_K' \int_0^T \int_{\mathbb{R}^N} |u|^{2K+2}\, ds\, dx \leqslant C_K \int_0^T \int_{\mathbb{R}^N} |u|^2\, ds\, dx + \frac{1}{2}\int_{\mathbb{R}^N} |u_0|^2\, dx \, .$$

It follows from (4.1) that

$$\int_0^T \int_{\mathbb{R}^N} |u|^2\, ds\, dx \leqslant \frac{1}{2C_K}\left(\exp(2C_K T) - 1\right)\int_{\mathbb{R}^N} |u_0|^2\, dx \, , \tag{4.8}$$

and we deduce that

$$C_K' \int_0^T \int_{\mathbb{R}^N} |u|^{2K+2}\, dx \leqslant \frac{\exp(2C_K T)}{2}\int_{\mathbb{R}^N} |u_0|^2\, dx \, ,$$

i.e. (4.2). When s is different from 0, we notice that

$$-\Re\left(\alpha \sum_{k=1}^{n}\int_{\mathbb{R}^N}\left(\frac{\partial u}{\partial x_k}\,\overline{u}\right)^2 |u|^{2s-2}\, dx\right) \leqslant |\alpha| \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s}\, dx \, ,$$

and we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^{2s+2}\, dx + (s+1)^2 \alpha' \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2s}\, dx +$$

$$C_K'(s+1)\int_{\mathbb{R}^N} |u|^{2K+2s+2}\, dx \leqslant C_K(s+1)\int_{\mathbb{R}^N} |u|^{2s+2}\, dx \, ,$$

where

$$\alpha' = \alpha_r - \frac{|\alpha|\, s}{s+1} \, .$$

If we suppose

$$|\alpha| < \frac{1+s}{s}\,\alpha_r \, ,$$

then α' is positive and by the above argument we obtain (4.3), (4.4). This concludes the proof of Theorem 11. ■

In the next two theorems, we suppose now that u_0 belongs to $\boldsymbol{W}^{2-2/q,\,q}$, with the condition

$$q > 1 + \frac{N}{2}\,.$$

In this case, the space $\boldsymbol{W}^{2-2/q,\,q}$ is included in $\boldsymbol{L}^\infty$, and the local existence of a solution u is also a consequence of Lemma 10.

Define

$$H(u) = F(u) - \mu_0 u = \sum_{j=1}^{K} \mu_j \, |u|^{2j} u\,.$$

THEOREM 12. – *Assume that u_0 belongs to $\boldsymbol{W}^{2-2/q,\,q}$ with $q > 1 + N/2$ and that α verifies*

$$|\alpha| < \frac{(2K+1)\,q - 2K}{(2K+1)\,q - 2K - 2}\,\Re\alpha \tag{4.9}$$

and $\Re\mu_K > 0$. Let u be the local solution obtained at Lemma 10 and T less than the maximal existence time. Then we have

$$H(u) \in L^q(0,\,T;\,\boldsymbol{L}^q)\,. \tag{4.10}$$

PROOF. – Since

$$q > 1 + \frac{N}{2} > 1 + \frac{1}{2K+1}\,,$$

there exists $s > 0$ such that

$$q = 1 + \frac{2s+1}{2K+1}\,.$$

Moreover, since

$$2s + 2 > \frac{2s}{2K+1} + 2 > 1 + \frac{2s+1}{2K+1}\,,$$

the initial condition u_0 belongs to $\boldsymbol{L}^{2s+2}$ and we deduce from (4.4) that u belongs to $L^{m}(0, T; \boldsymbol{L}^{m})$ provided that

$$|\alpha| < \frac{1+s}{s} \Re\alpha . \tag{4.11}$$

Since

$$s = (2K+1)\frac{q}{2} - K - 1 ,$$

we notice that the condition (4.11) is exactly (4.9). Let us show now that $H(u)$ belongs to $L^{q}(0, T; \boldsymbol{L}^{q})$, we already have

$$\int_0^T \int_{\mathbb{R}^N} (|u|^{2K+1})^q \leq \int_0^T \int_{\mathbb{R}^N} |u|^{m} < +\infty . \tag{4.12}$$

On the other hand, for all j, $j \in \{1, \dots, K-1\}$, we have

$$q < 3q \leq (2j+1)\, q < \boldsymbol{m} . \tag{4.13}$$

If $q \leq 2$, it follows from (4.8), (4.12) and by interpolation that for all j, $j \in \{1, \dots, K-1\}$,

$$|u|^{2j} u \in L^{q}(0, T; \boldsymbol{L}^{q}) .$$

If $q > 2$, we deduce from (4.13) that is sufficient to see that u belongs to $L^{q}(0, T; \boldsymbol{L}^{q})$ under the assumptions (4.9). But if $q > 2$, there exists $s_0 > 0$ such that $q = 2s_0 + 2$ and we deduce (4.3) that u belongs to $L^{q}(0, T; \boldsymbol{L}^{q})$ provided that

$$|\alpha| < \frac{1+s_0}{s_0} \Re\alpha .$$

Finally, there remains to show that

$$\frac{1+s}{s} < \frac{1+s_0}{s_0} . \tag{4.14}$$

But since

$$s_0 = \frac{q}{2} - 1 \quad \text{and} \quad s = (2K+1)\frac{q}{2} - K - 1 ,$$

we deduce that $s > s_0$ and as a consequence (4.14). This concludes the proof of Theorem 12. ■

THEOREM 13. – *Assume that* u_0 *belongs to* $\boldsymbol{W}^{2-2/q,\, q}$ *with* $q > 1 + N/2$ *and* *that α verifies*

$$|\alpha| < \frac{(2K+1)\,q-2K}{(2K+1)\,q-2K-2}\,\Re\alpha$$

and $\Re\mu_K > 0$. *Then* (1.1) *has a unique global solution u belonging to*

$$C([0,\boldsymbol{t}],\boldsymbol{W}^{2-2/q,\,q})$$

for all $\boldsymbol{t} > 0$.

PROOF. – Let T_m be the maximum existence time. It follows from Theorem 12 that $H(u) \in L^q(0, T_m; \boldsymbol{L}^q)$ and we infer from Theorem 8 that

$$u \in W^{1,\,q}(0, T_m; \boldsymbol{L}^q) \cap L^q(0, T_m; \boldsymbol{W}^{2,\,q}).$$

The Sobolev embedding theorem gives

$$u \in C([0, T_m], \boldsymbol{W}^{2-2q/q,\,q}).$$

Therefore T_m is not the maximal existence time, which proves that u is a global solution of (1.1).

In order to obtain the uniqueness, we consider two solutions u and v of (1.1), the function $w = u - v$ verifies

$$\frac{\partial w}{\partial t} - \alpha\Delta w + F(u) - F(v) = 0\,. \tag{4.15}$$

Multiplying (4.15) by $\overline{w}$ and integrating by parts in space, this yields

$$\int_{\mathbb{R}^N} \frac{\partial w}{\partial t}\overline{w}\,dx + \alpha\int_{\mathbb{R}^N} |\nabla w|^2\,dx + \int_{\mathbb{R}^N} (F(u)-F(v))\,\overline{w}\,dx = 0\,. \tag{4.16}$$

Since u and v belong to $C([0, T_m], \boldsymbol{L}^\infty)$, there exists a positive constant C such that over $[0, T_m]$

$$|F(u) - F(v)|_{L^2} \leq C|u-v|_{L^2}.$$

It follows from (4.16) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |w|^2\,dx + \alpha_r\int_{\mathbb{R}^N} |\nabla w|^2\,dx \leq C\int_{\mathbb{R}^N} |w|^2\,dx\,,$$

so that Gronwall's inequality gives the desired result. This completes the proof of Theorem 13. ■

Denote $\boldsymbol{C}_0^\infty$ the space of functions infinitely differentiable with compact support; a consequence of Theorem 13 is the following:

COROLLARY 14. – *Assume that* u_0 *belongs to* $\boldsymbol{C}_0^\infty$ *and that* α *verifies*

$$|\alpha| < \frac{(2K+1)\,N+2}{(2K+1)\,N-2}\,\Re\alpha$$

and $\Re\mu_K > 0$. *Then* (1.1) *has a unique global solution* u *belonging to* $C([0,\boldsymbol{t}],\boldsymbol{L}^\infty)$ *for all* $\boldsymbol{t}>0$.

PROOF. – We just need to prove that u belongs to $C([0,\boldsymbol{t}],\boldsymbol{L}^\infty)$ for all $\boldsymbol{t}>0$. Let $\varepsilon>0$ and define

$$r(\varepsilon) = \frac{(2K+1)(1+N/2+\varepsilon)-2K}{(2K+1)(1+N/2+\varepsilon)-2K-2}\,.$$

u_0 belongs to $\boldsymbol{W}^{2-2/q,\,q}$ with $q = 1+N/2+\varepsilon$ and since

$$\lim_{\varepsilon\to 0} r(\varepsilon) = \frac{(2K+1)\,N+2}{(2K+1)\,N-2}$$

and

$$r(\varepsilon) < \frac{(2K+1)\,N+2}{(2K+1)\,N-2}\,,$$

we can choose ε sufficiently small such that

$$|\alpha| < r(\varepsilon)\,\Re\alpha\,.$$

Thanks to Theorem 13, u belongs to $C([0,t],\boldsymbol{L}^\infty)$ for all $t>0$. ■

We now define notion of global weak-solution of (1.1):

DEFINITION 15. – *A function* u *is a global weak solution of* (1.1) *if it belongs to*

$$L^\infty_{\mathrm{loc}}(\mathbb{R}_+ : \boldsymbol{L}^2) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+ : \boldsymbol{H}^1) \cap L^{2K+2}_{\mathrm{loc}}(\mathbb{R}_+ ; \boldsymbol{L}^{2K+2})\,,$$

and for any $\psi \in \boldsymbol{C}_0^\infty([0,+\infty)\times\mathbb{R}^N)$, *it verifies*

$$\int_{\mathbb{R}_+\times\mathbb{R}^N} \left(-u\,\frac{\partial\psi}{\partial t} + \alpha\,\nabla u\,\nabla\psi + F(u)\,\psi\right) ds\,dx = \int_{\mathbb{R}^N} u_0(x)\,\psi(\cdot,x)\,dx\,.$$

THEOREM 16. – *Assume that* α *satisfies the condition*

$$|\alpha| < \frac{(2K+1)\,N+2}{(2K+1)\,N-2}\,\Re\alpha$$

and $\Re\mu_K > 0$. *Then for all* $u_0 \in \boldsymbol{L}^2$, (1.1) *possesses at least a global weak-solution.*

PROOF. – Let $\boldsymbol{t} > 0$, introduce a sequence (u_{0m}) in $\boldsymbol{C}_0^\infty$ such that (u_{0m}) tends to u_0 in $\boldsymbol{L}^2$ and let u_m be a solution of (1.1) over $(0, \boldsymbol{t})$ with initial condition (u_{0m}). It follows from (4.1) that the sequence (u_m) remains in a bounded set of $L^\infty(0, \boldsymbol{t}; \boldsymbol{L}^2)$, from (4.2) that the sequence (u_m) remains in a bounded set of $L^{2K+2}(0, \boldsymbol{t}; \boldsymbol{L}^{2K+2})$. With the help of (4.7), we can see that the sequence (u_m) remains in a bounded set of $L^2(0, \boldsymbol{t}; \boldsymbol{W}^{1,2})$.

These estimates ensure the existence of an element u and a subsequence still denoted (u_m) such that as m tends to infinity, (u_m) tends weakly to u in $L^2(0, \boldsymbol{t}; \boldsymbol{W}^{1,2})$, weakly to u in $L^{2K+2}(0, \boldsymbol{t}; \boldsymbol{L}^{2K+2})$ and tends weakly-star to u in $L^\infty(0, \boldsymbol{t}; \boldsymbol{L}^2)$ We also introduce v_m and w_m solutions of the following problems:

$$\begin{cases} \dfrac{\partial v_m}{\partial t} - \alpha \Delta v_m + \mu_0 v_m = 0\,, & x \in \mathbb{R}^N,\ \ t \in (0, \boldsymbol{t})\,, \\ v_m(0, x) = u_{0m}(x)\,, & x \in \mathbb{R}^N \end{cases}$$

and

$$\begin{cases} \dfrac{\partial w_m}{\partial t} - \alpha \Delta w_m + \mu_0 w_m + H(u_m) = 0\,, & x \in \mathbb{R}^N,\ \ t \in (0, \boldsymbol{t})\,, \\ w_m(0, x) = 0\,, & x \in \mathbb{R}^N. \end{cases}$$

We recall that $H = F - \mu_0 Id$; it is clear that $u_m = v_m + w_m$. The sequence (v_m) converges to v, solution of the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \alpha \Delta v + \mu_0 v = 0\,, & x \in \mathbb{R}^N,\ \ t \in (0, \boldsymbol{t})\,, \\ v(0, x) = u_0(x)\,, & x \in \mathbb{R}^N, \end{cases}$$

in $C([0, \boldsymbol{t}), \boldsymbol{L}^2) \cap L^2(0, \boldsymbol{t}; \boldsymbol{H}^1)$.

It follows from (4.1) that the term $H(u_m)$ is bounded in the space $L^q(0, \boldsymbol{t}, \boldsymbol{L}^q)$ with $q = (2K+2)/(2K+1)$ and we infer from Theorem 8 that the sequence (w_m) is bounded in $W^{1,q}(0, \boldsymbol{t}; \boldsymbol{L}^q) \cap L^q(0, \boldsymbol{t}; \boldsymbol{W}^{2,q})$. This estimate ensures the existence of an element w and a subsequence still denoted (w_m) such that (w_m) tends weakly to w in $W^{1,q}(0, \boldsymbol{t}; \boldsymbol{L}^q) \cap L^q(0, \boldsymbol{t}; \boldsymbol{W}^{2,q})$. We can now deduce that the subsequence (u_m) converges to $u = v + w$ in $L^1_{\mathrm{loc}}(0, \boldsymbol{t}; \boldsymbol{L}^1_{\mathrm{loc}})$ and almost everywhere. As a consequence, the sequence $(H(u_m))$ tends to $H(u)$ in $L^1(0, \boldsymbol{t}; \boldsymbol{L}^1_{\mathrm{loc}})$ and for all ψ in $\boldsymbol{C}_0^\infty((0, +\infty) \times \mathbb{R}^n)$, as m tends to infinity

$$\langle H(u_m), \psi \rangle \text{ tends to } \langle H(u), \psi \rangle. \tag{4.17}$$

It is also clear that when m tends to infinity

$$(4.18) \qquad \left\langle \frac{\partial u_m}{\partial t} - \alpha \Delta u_m + \mu_0 u_m, \psi \right\rangle \text{ tends to } \left\langle \frac{\partial u}{\partial t} - \alpha \Delta u + \mu_0 u, \psi \right\rangle.$$

It follows from (4.17) and (4.18) that u satisfies

$$\frac{\partial u}{\partial t} - \alpha \Delta u + F(u) = 0 ,$$

in the distribution sense.

Since $v \in C([0, t), L^2)$, $w \in C([0, t), W^{2-2/q, q})$ and $u = v + w$, the initial condition $u(0, x) = u_0(x)$, $x \in \mathbb{R}^n$ make sense. This concludes the proof of Theorem 16. ■

Using the previous decomposition, estimates (4.3), (4.4) and Theorem 8, we can also prove the following results:

THEOREM 17. – *Assume that* u_0 *belongs to* $L^2 \cap L^p$, *with* $p = 2 + 2\sigma$ $(\sigma > 0)$ *and that* α *satisfies the condition*

$$|\alpha| < \min\left(\frac{(2K+1)N+2}{(2K+1)N-2}, \frac{1+\sigma}{\sigma}\right) \Re\alpha$$

and $\Re\mu_K > 0$. *Then* (1.1) *possesses at least a global weak-solution of the form* $u = v + w$ *with*

$$v \in C([0, +\infty), L^p), \qquad w \in C([0, +\infty), W^{2-2/q, q}) \text{ and } w(0, \cdot) = 0 ,$$

with

$$q = 1 + \frac{2\sigma + 1}{2K + 1} .$$

In particular if $p \geq KN$,

$$u \in C([0, +\infty), L^p) .$$

REMARK 18. – *In fact, the assumption* u_0 *belongs to* L^2 *is not really necessary if we change the definition of a weak-global solution.*

THEOREM 19. – *Assume that* u_0 *belongs to* $H^1 \cap L^p$ *with* $p = 2 + 2s$ $(\sigma > 0)$ *and that* α *satisfies the condition*

$$|\alpha| < \min\left(\frac{(2K+1)N+2}{(2K+1)N-2}, \frac{1+\sigma}{\sigma}\right) \Re\alpha$$

and $\Re\mu_K > 0$. *Then* (1.1) *possesses at least a global weak-solution* u *of the form* $u = v + w$ *with*

$$v \in C([0, +\infty), \boldsymbol{H}^1), \qquad w \in C([0, +\infty), \boldsymbol{W}^{2-2/q, q}) \text{ and } w(0, \cdot) = 0,$$

with

$$q = 1 + \frac{2\sigma + 1}{2K + 1}.$$

In particular if $\sigma \geq K$ *and*

$$|\alpha| < \frac{(2K+1)N + 2}{(2K+1)N - 2} \Re\alpha,$$

u *belongs to* $C([0, +\infty), \boldsymbol{H}^1)$.

REMARK 20. – *In the second part of this Theorem, we can take* u_0 *in* $\boldsymbol{H}^1$ *for* $N \leq 2$ *and for* $N = 3$ *if* $K \leq 2$, *which covers the case of the Fauve-Thual equation.*

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S. Descombes: Université Lyon 1, Analyse Numérique
Bât. 101, 69622 Villeurbanne Cedex, France

M. Moussaoui: Ecole Centrale de Lyon, Département Math-Info.
BP 163, 69131 Ecully Cedex, France

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