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Proximal Set-Open Topologies (*).

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Sunto. – *Introduciamo una nuova classe di topologie in spazi di funzioni derivanti da prossimità sul rango, che denotiamo sinteticamente PSOTs, acronimo di* proximal set-open topologies. *Le PSOTs sono una naturale generalizzazione delle classiche topologie di tipo set-open quando l'ordinaria inclusione viene sostituita con l'inclusione stretta associata ad una prossimità. Molte e note topologie di tipo set-open connesse a speciali networks sono esempi di PSOTs. Ogni PSOT è contraibile ad un sottospazio chiuso che é copia omeomorfa del rango. Prossimità distinte determinano in generale PSOTs distinte. Una PSOT indotta da un prossimità di Efremovic δ e da un network chiuso ed ereditariamente chiuso α coincide con la topologia della convergenza uniforme sugli elementi di α generata dalla uniformità minimale compatibile con δ. Quando specializziamo il network e la prossimità otteniamo una PSOT che ammette una struttura di gruppo topologico e verifica un teorema di tipo Arens. Infine diamo semplici condizioni necessarie e sufficienti per la metrizzabilità.*

Introduction.

Let X and Y be T_1 topological spaces. Let α be a closed network on X (see [MN$_1$]). We assume, without any loss of generality, that α is closed under finite unions and finite intersections and includes all singletons. Set-open topologies on function spaces have been widely studied. A typical one is $C_\alpha = C_\alpha(X, Y)$ which has a subbase:

$$\{[A, B]: A \in \alpha, B \text{ open in } Y\},$$

where:

$$[A, B] = \{f \in C(X, Y): f(A) \subset B\}. \tag{1}$$

When α consists of all finite (compact) sets, C_α is called the *point-open* (resp. *compact-open*) topology. To generalize the compact-open topology to real-valued noncontinuous functions,to balance the disadvantage A is compact but $f(A)$ is not compact, McCoy and Ntantu ([MN$_2$]) considered a modification

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of (1) by introducing as subbase the collection of all sets

(2) $$[A, B] = \{f \in Y^X : \overline{f(A)} \subset B\},$$

where $A \in \alpha$ and B is open in $\mathbb{R}$. Kundu and Raha ([KR]) also study the same topology but on $C(X, \mathbb{R})$ when members of α are *bounded*. A set E is *bounded or relatively pseudocompact* iff for each $f \in C(X, \mathbb{R})$ $f(E)$ is bounded in the euclidean metric.

In this paper, we consider new function space topologies deriving from proximity on the range space which generalize the above topology as well. Let δ be a compatible LO-proximity on Y. For $A \in \alpha$ and $B \subset Y$, we use the notation:

(3) $$[A : B]_\delta = \{f \in Y^X : f(A) \ll B\},$$

where:

$$A \ll B \quad \text{iff} \quad A \not\delta B^c$$

and B^c denotes the complement of B. The topology $C_{\alpha, \delta}$ generated by:

$$\{[A : B]_\delta : A \in \alpha, B \text{ open in } Y\}$$

is called the *proximal set-open topology* [PSOT] induced on Y^X by α and δ. Note that, in general, Y has several different proximities which induce different PSOTs. We now list how several of the known set-open topologies are special cases of PSOTs. When α consists of all *finite* subsets of X and δ is any compatible LO-proximity on Y, then relative PSOT is the *point-open topology* $C_p = C_p(X, Y)$. When α consists of all *compact* subsets of X and δ is any compatible R (resp. EF)-proximity on regular (resp. completely regular) space Y, we get the *compact-open topology* $C_k = C_k(X, Y)$. When α is the family of all nonempty subsets of X and δ is a compatible EF-proximity on Y, we get the topology of *Leader convergence* or *convergence in proximity* ([L]) $C_l = C_l(X, Y)$.

When $\delta = \delta_0$ the finest compatible LO-proximity on Y, we get the generalized set-open topology (2) of McCoy-Ntantu, C_{α, δ_0}. Again when $Y = \mathbb{R}$, but δ is a compatible EF-proximity on $\mathbb{R}$, and α consists of all bounded subsets of X, then we get the *bounded-open* topology of Buchwalter $C_b = C_b(X, Y)$. (See [B], [KR]).

The choice of α as a network implies that any PSOT is finer than the point-open topology, so that any PSOT contains a closed retract which is homeomorphic copy of the range space Y. Thus any continuous invariant property and any topological property which is hereditary or closed hereditary of a PSOT reflects on Y. We will prove also that some PSOTs absorb properties from Y. When δ is an EF-proximity, i.e. Y is uniformizable in the Weil sense, then

$C_{\alpha,\delta}$ is uniformizable too. More precisely, when α is hereditarily closed, $C_{\alpha,\delta}$ agrees with the topology of uniform convergence on members of α w.r.t. the minimal uniformity compatible with δ.

After pointing out properties of a general PSOT we dedicate our interest to PSOTs deriving from R-proximities. For that we state relevant properties of R and EF proximities. Then we focus our attention on general networks and compare PSOTs generated by a same network but different proximities. A PSOT can have good separation properties also when the proximity is LO but nicely related to the network. We go on comparing PSOTs with set-open topologies, uniform topologies, graph topology, Whitney and Krikorian topologies. Finally generalizing the concept of *boundedness* in the more general ones of Y-total boundedness and Y-compactness we fix definitively our attention on C_{α,δ_0}, where α is done by Y-totally bounded closed subsets of X. We show that C_{α,δ_0} satisfies an Arens-type theorem. Moreover it admits a topological group structure when the range space Y is a topological group for which the right and left uniformities agree. In conclusion we give necessary and sufficient conditions for C_{α,δ_0} to be metrizable.

1. – Preliminaries and generalities.

For definitions, notations and terminology we refer to [NW]. We only remark that briefly speaking a Lodato proximity, LO-proximity in short, δ, on a set X is a *symmetric nearness* between pairs of non-empty subsets of X which induces a topology on X with the property: *two sets are near iff their closures are near*. Any T_1 topological space has a finest compatible LO-proximity, δ_0, called the Wallmann proximity, defined by

$$A \not\delta_0 B \quad \text{iff} \quad \overline{A} \cap \overline{B} = \emptyset \tag{4}$$

and a weakest one, δ_f, defined by:

$$A \not\delta_f B \quad \text{iff} \quad \overline{A} \cap \overline{B} = \emptyset \text{ and } A \text{ or } B \text{ is finite}. \tag{5}$$

Another interesting compatible LO-proximity δ_1, called the Alexandroff proximity, is defined by:

$$A \not\delta_1 B \quad \text{iff} \quad \overline{A} \cap \overline{B} = \emptyset \text{ and } \overline{A} \text{ or } \overline{B} \text{ is compact}. \tag{6}$$

A regular proximity, R-proximity in short, is a LO-proximity with the further property:

$$x \not\delta A \Rightarrow \exists E \neq \emptyset \quad \text{such that} \quad x \not\delta X - E \text{ and } A \not\delta E. \tag{7}$$

Any regular proximity induces a regular topology and any regular space has compatible regular proximities. An Efremovic proximity, EF-proximity in short, is a LO-proximity with the following additional property:

(8) $$A\not\delta B \Rightarrow \exists E \neq \emptyset \quad \text{such that } A\not\delta X - E \text{ and } B\not\delta E.$$

An EF-proximity induces a completely regular topology and any completely regular space admits compatible EF-proximities. There exists for any completely regular space X a finest compatible EF-proximity δ_F defined by: $A\not\delta_F B$ iff there exists a continuous function $f: X \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$. The finest EF-proximity δ_F coincides with the finest compatible LO-proximity δ_0 iff the space X is normal. When a space X is Tychonoff any of its T_2 compactifications $\gamma(X)$ induces a compatible EF-proximity δ defined by:

(9) $$A\delta B \Leftrightarrow \mathrm{Cl}_{\gamma(X)}(A) \cap \mathrm{Cl}_{\gamma(X)}(B) \neq \emptyset .$$

Any diagonal uniformity μ induces a natural proximity $\delta = \delta(\mu)$ defined by:

(10) $$A\delta B \quad \text{iff} \quad U[A] \cap B \neq \emptyset \quad \text{for each } U \in \mu .$$

If μ is a Weil uniformity, then $\delta(\mu)$ is an EF-proximity. In the metric case two sets are near in the natural proximity iff their distance is zero. We remind further that there exists just one totally bounded uniformity, which is also the minimal one,compatible with a fixed EF-proximity. More we remark that it is enough agreeable to work with regular proximities, since every regular proximity derives from a diagonal uniformity which has the same properties of a Weil uniformity except for triangle inequality satisfied only pointwise, see [W].

A function f between two proximal spaces $(X, \delta), (Y, \delta')$ is *proximally continuous* iff two sets which are δ-near admit δ'-near images. Any uniformly continuous function is proximally continuous. In the metric case uniform continuity and proximal continuity coincide.

PROPOSITION 1.1. – *If δ is an* R-*proximity, then any two disjoint closed sets one of them is compact are far.*

PROOF. – If a point x is far from a set B, then in R-proximity there is a nhbd of x which is far from B. So if A is compact and B is a closed set disjoint from it, then a standard argument shows that A is far from B. ■

THEOREM 1.2. – *Let X be Tychonoff. If two closed sets A and B are far w.r.t. any* EF-*proximity, then necessarily one of them must be compact.*

PROOF. – If both A and B were not compact, then both had to contain nets $\{a_\lambda\}, \{b_\mu\}$ which don't accumulate in X. Taken an arbitrary T_2-compactification of X, $\gamma(X)$, the nets $\{a_\lambda\}, \{b_\mu\}$ must accumulate somewhere in it. Let a and

b be two points in $\gamma(X)$ towards $\{a_\lambda\}, \{b_\mu\}$ accumulate respectively. Collapse the two points a and b to just one point, so obtaining a new T_2-compactification of X which induces an EF-proximity in which A and B are near since their closures in it intersect. A contradiction. ■

COROLLARY 1.3. – *When X is Tychonoff,* $\inf\{\delta : \delta \text{ is EF}\} = \inf\{\delta : \delta \text{ is R}\} = \delta_1$, *where δ_1 is defined by* (6).

PROOF. – It is very simple to deduce from Proposition 1.1 and (7), (8) that when δ_1 is R then it is EF too. The agreement between δ_1 and infima comes from Theorem 1.2. ■

THEOREM 1.4. – *If X is Tychonoff ,the proximity δ_1, see* (6), *is an* R (*equivalently an* EF)*-proximity iff the space X is locally compact.*

PROOF. – It comes out from Corollary1.3 and the following result: A Tychonoff space X is locally compact iff the lattice of its T_2-compactifications admits minimum. See (9). ■

2. – PSOTs.

A collection α of subsets of a topological space X is a *network* iff for any point $x \in X$ and any open set V with $x \in V$ there exists a member $A \in \alpha$ which contains the point x and is contained in the open set V. Given a network α on X and a LO-proximity δ on Y , the *proximal set-open topology* on Y^X, briefly PSOT, induced by α and δ has as subbasic elements the sets introduced by (3). If δ is understood, we drop δ writing instead of $[A: B]_\delta$ more simply $[A: B]$. When Y satisfies higher separation axioms, we'll require δ to satisfy stronger proximity axioms. Mostly we deal with continuous functions.

As in [MN_1], we have:

LEMMA 2.5.

$$\text{(i)}\ [A: B_1 \cap B_2] = [A: B_1] \cap [A: B_2].$$

$$\text{(ii)}\ [A_1 \cup A_2: B] = [A_1: B] \cap [A_2: B].$$

LEMMA 2.6. – *If δ is a* LO*-proximity and α is a network containing all the singletons, then* $C_p = C_{p,\delta} \subset C_{\alpha,\delta}$.

PROOF. – It is trivial since a point x is δ-near to a set A iff x belongs to the closure of A. ■

When Y is Hausdorff it follows very easily from Lemma 2.6 that by identifying any point $y \in Y$ with the constant function from X to Y, c_y, naturally determined from it, Y embeds in Y^X equipped with any PSOT like a closed subspace $e(Y)$. More any evaluation function is a retraction of Y^X onto $e(Y)$.

COROLLARY 2.7. – *If* δ *is an* R*-proximity and* α *a network containig all compact sets, then* $C_k = C_{k,\delta} \subset C_{\alpha,\delta}$.

PROOF. – It follows immediately from Proposition 1.1. ■

The following example will prove that we cannot remove in Corollary 2.7 δ to be R.

EXAMPLE 1. – $C_k \neq C_{k,\delta_f}$.

Let $X = Y = Q$ be equipped with the natural topology, k the network of all compact sets. More let Y have the coarsest LO-proximity introduced in (5). Further let f be the identity map, $A = \{1/n: n \in N^+\} \cup \{0\}$ and V a bounded open set containing the compact set A. Then $f \in [A, V] \in C_k$ but no basic open set in the PSOT, C_{k,δ_f}, can be contained in [A, V]. Suppose $f \in \cap \{[A_i: V_i]_{\delta_f}: i = 1, 2, \ldots, n\}$, A_i finite, V_i bounded till k, $1 \leqslant k \leqslant n$; more, from k on, A_i not finite, consequently V_i with finite complement. Observe that there exists a point y in $\cap \{V_i: i = k+1, \ldots, n\}$ but not in V. Pick up $1/n$ in A but outside $\cup \{A_i: 1, \ldots, k\}$ and choose a closed rational interval I centered in $1/n$ and irrational radius which excludes all points of $\cup \{A_i: i = 1, \ldots, k\}$. Glueing the constant function c_y on I to the identity map on $Q - I$, we construct a continuous function g which is in $\cap \{[A_i: V_i]_{\delta_f}: i = 1, 2, \ldots, n\}$ but not in $[A, V]$. ■

COROLLARY 2.8. – *If* Y *is the real line* $\mathbb{R}$, δ *is an* R*-proximity on* $\mathbb{R}$ *and* α *is a network containing all bounded sets of* X*, then* $C_b = C_{b,\delta} \subset C_{\alpha,\delta}$.

PROOF. – It is enough to observe that if A is bounded $\overline{f(A)}$ is compact in $\mathbb{R}$. ■

Again in this case too, the inclusion is strict.

PROPOSITION 2.9. – *Let* δ *be an* EF*-proximity compatible with* $\mathbb{R}$*. If* X *is normal and* α *contains an unbounded subset, then* $C_{\alpha,\delta} \not\subset C_b$.

PROOF. – Suppose $A \in \alpha$ is unbounded and $f \in [A: V]_\delta$ where V is a proper open set in $\mathbb{R}$. Suppose $f \in \bigcap_{i=1}^{n} [A_i: V_i]_{\delta_0} \subset [A: V]_\delta$ where each A_i is bounded. We claim that $A \subset \bigcup_{i=1}^{n} A_i$ and so A is bounded. For if not, there is an $a \in A - \bigcup_{i=1}^{n} A_i$. Set

$$g/A_i = f \quad \text{and} \quad g(a) \in \mathbb{R} - V.$$

Since $g\colon \bigcup_{i=1}^{n} A_i \cup \{a\} \to \mathbb{R}$ is continuous, by Tietze's extension theorem there is a continuous extension $\bar{g}\colon X \to \mathbb{R}$. Then $\bar{g} \in \bigcap_{i=1}^{n} [A_i\colon V_i]_{\delta_0}$ but $\bar{g} \notin [A\colon V]_\delta$. A contradiction. ■

PROPOSITION 2.10. – *If δ and δ' are* LO-*proximities, $\delta < \delta'$ and further δ is* EF, *then $C_{\alpha,\delta} \subset C_{\alpha,\delta'}$, for any network α. In particular, if δ is* EF, *then $C_{\alpha,\delta} \subset C_{\alpha,\delta_0}$.*

PROOF. – Suppose $f \in [A\colon B]_\delta$, that is $f(A) \ll_\delta B$. Since δ is EF, there is an open $E \subset Y$ such that $f(A) \ll_\delta E \ll_\delta B$. Thus $f \in [A\colon E]_{\delta'} \subset [A\colon B]_\delta$ and so $C_{\alpha,\delta} \subset C_{\alpha,\delta'}$ ■

We remark that it may happen $\delta < \delta'$ and anyway get $C_{\alpha,\delta} = C_{\alpha,\delta'}$. An example of this situation is when each $A \in \alpha$ is compact and δ, δ' are both R-proximities. Later on we will be able to illustrate completely the relationship.

PROPOSITION 2.11. – *Let X, Y be Tychonoff spaces and δ_1 the Alexandroff proximity on Y given in* (6). *If δ is an* R-*proximity on Y and α an arbitrary network on X, then $C_{\alpha,\delta_1} \subset C_{\alpha,\delta}$ and when X is locally compact*

$$C_{\alpha,\delta_1} = \inf\{C_{\alpha,\delta}\colon \delta \text{ is R}\} = \inf\{C_{\alpha,\delta}\colon \delta \text{ is EF}\}.$$

PROOF. – It follows from Proposition 1.1, Theorem 1.2, Corollary 1.3 and Theorem 1.4. ■

For each EF-proximity δ on Y and any closed network α on X which includes all compact sets a first comparison can be summarized in the following picture:

$$C_p \subset C_k \subset C_{\alpha,\delta} \subset C_{\alpha,\delta_0} \subset C_{w,\delta_0},$$

$$C_p \subset C_k \subset C_{\alpha,\delta} \subset C_\alpha \subset C_w,$$

$$C_p \subset C_k \subset C_{\alpha,\delta} \subset C_{w,\delta} \subset C_{w,\delta_0}.$$

Here w denotes the family of all closed sets in X. ■

Now to add some further results we focus our attention on general networks. Let α, β be networks. We say that α *refines* β iff any element in α can be covered by a finite union of elements in β; α *approximates* β iff any element

A in α and any open set V containing A there is an element of β which sits between A and V.

PROPOSITION 2.12. – *If δ is* LO, *Y contains a non trivial path and $C_{\alpha,\delta} \subset C_{\beta,\delta}$, then α refines β.*

PROOF. – Analogous to Proposition 1-1.1 in [MN_1]. ■

PROPOSITION 2.13. – *If δ is EF, β is hereditarily closed network and α refines β, or β is an arbitrary closed network approximated from α then $C_{\alpha,\delta} \subset C_{\beta,\delta}$.*

PROOF. – Firstly suppose β is hereditarily closed network and α refines β. Let $f \in [A: V]_\delta$. Then there exists in μ^*, the unique totally bounded uniformity compatible with δ, a diagonal nhbd U such that $U[f(A)] \subset V$. Pick up again in μ^* an open diagonal nhbd W for which $WoW \subset U$ and a closed one $D \subset W$. Choose $a_1, a_2, \ldots, a_m \in A$ so that $f(A) \subset D[f(a_1)] \cup \ldots \cup D[f(a_m)]$. On the other hand $A \subset B_1 \cup \ldots \cup B_n$, $B_i \in \beta$, $i = 1, \ldots, n$. Call N the set of all pairs (i, j) which determine a non empty intersection $B_{ij} = B_i \cap f^{-1}(D[f(a_j)])$. These last ones all stay in β. Then $f \in \cap \{[B_{ij}: W[f(A)]]_\delta : (i, j) \in N\}$ and more any other function g in it is contained in $[A: V]_\delta$. Secondly, if α approximates β between A and $f^{-1}(W[f(A)])$ sits a member $B \in \beta$ such that $f \in [B: W[f(A)]]_\delta \subset [A: V]_\delta$. ■

COROLLARY 2.14. – *Let X, Y be Tychonoff spaces and δ* EF *on Y. Under the conditions of the previous propositions it follows*:

(i) *X is compact iff $C_k = C_{w,\delta}$ iff $C_k = C_w$.*
(ii) *Each compact set in X is finite iff $C_p = C_k$.*
(iii) *α is a finite network iff $C_p = C_{\alpha,\delta_0}$ iff $C_p = C_{\alpha,\delta}$.*
(iv) *α is a compact network iff $C_k = C_{\alpha,\delta_0}$ iff $C_k = C_{\alpha,\delta}$.*
(v) *X is finite iff $C_p = C_w$ iff $C_p = C_{w,\delta}$.*

Here again w denotes the family of all closed sets. ■

Let α be a network on X, $(X, \delta), (Y, \delta')$ LO-proximity spaces. A function $f: X \to Y$ is called *α-p-continuous* iff for

$$A, B \in \alpha, f(A) \not\delta' f(B) \quad \text{implies} \quad A \not\delta B .$$

Obviously:

(11) $p-$continuity $\Rightarrow$ $\alpha - p -$continuity $\Rightarrow$ continuity $\Rightarrow$ continuity on α .

The following is a generalization of Leader's result.

THEOREM 2.15. – *Suppose* $f_\lambda: (X, \delta) \to (Y, \delta')$ *is a net of* α-*p*-*continuous functions converging to* f *in* $C_{\alpha, \delta'}$; *more* δ' *is* EF, *then* f *is* α-*p*-*continuous.*

PROOF. – Suppose $A, B \in \alpha$, $f(A) \not\delta' f(B)$. Then there is an $E \subset Y$ such that $f(A) \not\delta' E$, $E^c \not\delta' f(B)$. Since $f_\lambda \to f$ in $C_{\alpha, \delta'}$, eventually $f_\lambda(A) \not\delta' E$ and $E^c \not\delta' f_\lambda(B)$. So eventually $A \not\delta f_\lambda^{-1}(E)$ and $B \not\delta f_\lambda^{-1}(E^c)$. Then $A \not\delta B$ and f is α-*p*-continuous. ■

COROLLARY 2.16.

(*a*) $C(X, Y)$ *is closed in* Y^X *equipped with any* PSOT *deriving from* $\alpha = CL(X)$ *and an* R-*proximity.*

(*b*) *Convergence in* $C_{\alpha, \delta}$ *preserves* α-*continuity when* δ *is an* R-*proximity and* α *is hereditarily closed; in particular continuity when* $\alpha = CL(X)$. ■

We are now able to give examples of strict inclusions relatively to Proposition 2.10. Let $X = Y = \mathbb{R}$ with the usual topology. Let δ_1 be the *EF*-proximity induced by the Alexandroff one-point compactification, δ_2 the *EF*-proximity induced by the two-points compactification and δ_3 the usual metric proximity. The finest *EF*-proximity on $\mathbb{R}$ is δ_0. We now give examples to show that $C_{\alpha, \delta_1}, \subsetneq C_{\alpha, \delta_2} \subsetneq C_{\alpha, \delta_3} \subsetneq C_{\alpha, \delta_0}$, where $\alpha = CL(X)$, following two different schemes of demonstrantion.

All inclusions follow from Proposition 2.10 since $\delta_1 < \delta_2 < \delta_3 < \delta_0$ and each of them is EF.

EXAMPLE 2. – $C_{\alpha, \delta_1} \subsetneq C_{\alpha, \delta_2}$.

Remind that δ_1, δ_2-*p*-continuity for bounded functions is simply equivalent to continuous extendability to the Alexandroff one-point compactification and to the two-points compactification of $\mathbb{R}$ respectively.

Let $f \in C(\mathbb{R}, \mathbb{R})$ be the function whose graph consists of segments joining $(2n, -2n)$ to $(2n+1, 2n+1)$, $(2n+1, 2n+1)$ to $(2n+2, -2n-2)$ for $n \in \mathbb{N} \cup \{0\}$ and their symmetric ones w.r.t. *y*-axis. Consider $A = \{2n+1: n \in \mathbb{N} \cup \{0\}\}$, $B = \{2n: n \in \mathbb{N} \cup \{0\}\}$. Here $A \delta_2 B$ but $f(A) \not\delta_2 f(B)$. So f, which is *p*-continuous from $(\mathbb{R}, \delta_1)$ in itself, is *not* *p*-continuous from $(\mathbb{R}, \delta_2)$ in itself. By (11) the set $\mathcal{F}$ of all δ_2-*p*-continuous functions is closed in C_{α, δ_2}. We show that $\mathcal{F}$ is *not* closed in C_{α,δ_1}, by constructing a sequence of functions in $\mathcal{F}$ which converges to f in C_{α,δ_1}. Set $f_n\colon \mathbb{R}\to\mathbb{R}$ by:

when n is odd

$$f_n(x) = \begin{cases} f(x), & \text{if } x\in[-n, n], \\ n, & \text{otherwise}. \end{cases}$$

when n is even

$$f_n(x) = \begin{cases} f(x), & \text{if } x\in[-n, n], \\ -n, & \text{otherwise}. \end{cases}$$

Then $f_n \in \mathcal{F}$ and we show that f_n converges to f in C_α, δ_1.

Let $f\in [C\colon V]_{\delta_1}$ where C is closed and V is open in $\mathbb{R}$. It follows that $\overline{f(C)} \cap X - V = \emptyset$ and $\overline{f(C)}$ or $X - V$ is bounded. If $\overline{f(C)}$ is bounded, C is bounded too. Put $|\sup C| \leqslant n_0 \in N$ Then $f_n(C) = f(C)$ for any $n > n_0$. So $f_n \in [C\colon V]_{\delta_1}$ for any $n > n_0$. If, on the other hand, $X - V$ is bounded and $|\sup(X-V)| \leqslant n_0 \in N$, then $\overline{f_n(C)} \cap X - V = \emptyset$ for each $n > n_0$ and again $f_n \in [C\colon V]_{\delta_1}$ when $n > n_0$.

Example 3. – $C_{\alpha,\delta_2}, \subsetneq C_{\alpha,\delta_3}$.

Let now $\mathcal{F}$ denote the set of all p-continuous functions from $(\mathbb{R}, \delta_3)$ in itself. Then $\mathcal{F}$ is closed in C_{α,δ_3}. We show instead, by using the same argument of the previous example, that $\mathcal{F}$ is *not* closed in C_{α,δ_2}. It is well known that the square function f, $f(x) = x^2$, doesn't belong to $\mathcal{F}$. Set for each $n \in N$:

$$f_n(x) = \begin{cases} x^2, & \text{if } x\in[-n, n], \\ n^2, & \text{otherwise}. \end{cases}$$

Suppose $f\in[A\colon V]_{\delta_2}$. Since $f(A)$ is bounded below by zero, it is clear that $\overline{f(A)} \cap X - V = \emptyset$ and $\overline{f(A)}$ or $X - V$ must be bounded above. Firstly, $\overline{f(A)}$ is bounded above implies A is bounded and there exists $n_0 \in \mathbb{N}$ such that for each $a\in A$, $|a| < n_0$. For $n > n_0$, $f_n(A) = f(A)$ and so $\overline{f_n(A)} \cap X - V = \emptyset$. Secondly, $X - V$ is bounded above, and $|\sup(X-V)| \leqslant n_1 \in \mathbb{N}$, implies for $n \geqslant n_1$, $\overline{f_n(A)} \subset \overline{f(A\cap[-n, n])} \cup \{n^2\}$ and $n^2 \notin X - V$. So again $\overline{f_n(A)} \cap X - V = \emptyset$. Thus $f_n \to f$ in C_{α,δ_2} but clearly f_n doesn't converge to f in C_{α,δ_3}.

Example 4. – $C_{\alpha,\delta_3}, \subsetneq C_{\alpha,\delta_0}$.

Suppose f is the identity map on $\mathbb{R}$, $A = \cup\{[n - 1/2^n, n + 1/2^n]\colon n \geqslant 2\}$, $V = \cup\{]n - 1/2^{n-1}, n + 1/2^{n-1}[\colon n\geqslant 2\}$. Then $f\in[A\colon V]_{\delta_0} \in C_{\alpha,\delta_0}$ but no $\cap\{[A_i\colon V_i]_{\delta_3}\colon i = 1, \ldots, n\} \in C_{\alpha,\delta_3}$ containing f can be contained in it. There will be $\varepsilon > 0$ such that the ε-collar of any A_i is contained in V_i. Perturbing f by $\varepsilon/2$ we determine a continuous function $g = f - \varepsilon/2$ which is in $\cap\{[A_i\colon V_i]_{\delta_3}\colon i = 1, \ldots, n\}$ but not in $[A\colon V]_{\delta_0}$, since $g(n - 1/2^n)$ is outside V when $3/2^n < \varepsilon/2$. Thus $C_{\alpha,\delta_3} \subsetneq C_{\alpha,\delta_0}$. ■

We now give compactness conditions in a PSOT when δ is EF on Y.

Since $C_p \subset C_{\alpha,\delta}$, $C_{\alpha,\delta}$ is *proper*. A compact subset $\mathcal{F} \subset \mathcal{C}_{\alpha,\delta}$ must be closed and pointwise bounded, ([MN$_1$]). Since $C_{\alpha,\delta}$ is conjoining, it is weakly conjoining and so it is hyper-Ascoli. Hence a compact subset $\mathcal{F} \subset \mathcal{C}_{\alpha,\delta}$ is closed, pointwise bounded and evenly continuous.

Suppose $\mathcal{F} \subset \mathcal{C}_{\alpha,\delta}$ is closed, pointwise bounded and evenly continuous. Then $\mathcal{F}$ is equicontinuous w.r.t. the unique totally bounded uniformity μ^* compatible with δ ([MN$_1$]. It follows:

THEOREM 2.17. – *Let X be pseudocompact and Y metrizable, more δ EF on Y. Then $\mathcal{F} \subset C_{\alpha,\delta}$ is compact if and only if F is closed, pointwise bounded and evenly continuous (or equicontinuous w.r.t. μ^*).*

3. – Comparison.

In the previous paragraph we compared PSOTs among themselves varying the network or proximity or both. Now we go on with comparison of PSOTs with set-open topologies, uniform topologies, graph topology, Whitney and Krikorian topologies.

We start with set-open topologies:

PROPOSITION 3.18. – *If δ ia an EF-proximity, α is a closed and hereditarily closed network, then $C_{\alpha,\delta} \subset C_\alpha$.*

PROOF. – Suppose $f \in [A: V]_\delta$. That means there exists a diagonal nhbd U in the minimal uniformity μ^* compatible with δ with $U[f(A)] \subset V$. Pick in μ^* an open symmetric diagonal nhbd W such that $WoW \subset U$ and more, always in μ^* a closed diagonal nhbd D contained in W. Since μ^* is totally bounded for some $a_1, a_2, \ldots, a_n$ in A

$$f(A) \subset D[f(a_1)] \cup D[f(a_2)] \cup \ldots \cup D[f(a_n)] \subset D[f(A)] \subset W[f(A)].$$

For each $i = 1, \ldots, n$, $A_i = A \cap f^{-1}(D[f(a_i)]$ is in α. Then

$$f \in \{\cap A_i, W[f(a_i)]: i = 1, \ldots, n\},$$

which is easily seen to be a basic element in C_α contained in $[A: V]_\delta$. ■

But usually the inclusion is strict.

EXAMPLE 5. – $C_{\alpha,\delta} \neq C_\alpha$.

Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. Y is supposed to be equipped with the natural proximity, α the network of all closed subsets of $\mathbb{R}^2$. The first projection π_x of $\mathbb{R}^2$ onto $\mathbb{R}$ is in $[C, \mathbb{R}^+] \in C_\alpha$, where $C = \{(x, y): y = 1/x, x > 0\}$. But no basic open set in $C_{\alpha,\delta}$ which contains π_x can be contained in $[C, \mathbb{R}^+]$. When $\pi_x \in \cap\{[A_i: V_i]_\delta: i = 1, \ldots, n\}$, where any A_i is closed in $\mathbb{R}^2$ and V_i is open in $\mathbb{R}$ we can find $\varepsilon > 0$ with the ε-collar of any $\pi_x(A_i)$ contained in V_i. Consider the composition g of π_x with the continuous function $f:(x, y) \in \mathbb{R}^2 \to (x - \varepsilon/2, y) \in \mathbb{R}^2$. Then $g \in \cap\{[A_i: V_i]_\delta: i = 1, \ldots, n\}$, but $g \notin [C, \mathbb{R}^+]$. ■

We continue with uniform topologies.

When Y is completely regular, μ is a compatible Weil uniformity on Y and α is a network on X, $C_{\alpha,\mu}$ denotes the topology of uniform convergence on members of α. $C_{\alpha,\mu}$ is uniformized by the Weil uniformity which admits as subbase the collection

$$\{\widehat{U}[A]: U \in \mu, A \in \alpha\},$$

where:

$$\widehat{U}[A] = \{(f, g) \in C(X, Y): (f(x), g(x)) \in U, \quad \forall x \in A\}.$$

THEOREM 3.19. – *If δ is* EF, *μ is any diagonal uniformity compatible with δ, α an arbitrary network on X, then $C_{\alpha,\delta} \subset C_{\alpha,\mu}$.*

PROOF. – Suppose $f \in [A: V]_\delta$. Then $f(A) \ll_\delta V$. Then there exists an $U \in \mu$ such that $U[f(A)] \subset V$. It is then easy to show that $\widehat{U}(A)(f) \subset [A: V]_\delta$. ■

THEOREM 3.20. – *If δ is* EF *and α is a closed, hereditarily closed network, then $C_{\alpha,\delta} = C_{\alpha,\mu^*}$.*

PROOF. – We have to show that if a net $f_\lambda \to f$ in $C_{\alpha,\delta}$, then it converges in C_{α,μ^*} too. Consider $U, W, D \in \mu^*$, W is open symmetric and $WoW \subset U$, more D closed and contained in W. Pick up $a_1, \ldots, a_n \in A$ such that:

$$f(A) \subset D[f(a_1)] \cup \ldots \cup D[f(a_n)].$$

Set $A_i = A \cap f^{-1}(D[f(a_i)])$. Then $f \in \cap\{[A_i: W[f(a_i)]_\delta: i = 1, \ldots, n\}$. So for some λ_0, f_λ is in it if $\lambda \geq \lambda_0$. It follows that, when $\lambda \geq \lambda_0, f_\lambda \in \widehat{U}[A](f)$. For each $a \in A$ and $\lambda \geq \lambda_0$, $f_\lambda(a) \in W[f(a_i)]$ for some $i = 1, \ldots, n$ but $f(a) \in W[f(a_i)]$ as well. Thus $(f_\lambda(a), f(a)) \in WoW \subset U$. ■

Theorem 3.20 jointly a result in [N] gives us the opportunity of introducing a special function space for which any two distinct EF-proximities on the range space induce different PSOTs.

EXAMPLE 6. – $\delta \lneqq \delta'$ and $C_{\alpha,\delta}, \subsetneqq C_{\alpha,\delta'}$.

Suppose X and Y are both non empty, X is equipped with the discrete topology and Y is completely regular.More $|X| \geq |Y|$, i.e there exists a function g from X *onto* Y obviously continuous. If $\delta \lneqq \delta'$ are two EF-proximities on Y then $\mu^* \lneqq \mu^{*\prime}$, where μ^*, $\mu^{*\prime}$ are the totally bounded uniformities compatible with them respectively. Thus there exists in $\mu^{*\prime}$ a diagonal nhbd U such that for any diagonal nhbd W in μ^* it is possible to find two points x_W, y_W in Y with $(x_W, y_W) \in W$ but outside U. Define for any $W \in \mu^*$ the function f_W: $Y \to Y$ in the following way:

$$f_W(y) = y, \quad \text{when} \quad y \neq y_W \quad \text{and} \quad f_W(y_W) = x_W.$$

By discreteness of X any $g_W = f_W o g$ is continuous. It is easy to show that the net $\{g_W\}$ converges to g in $C_{\alpha,\delta}$ but not in $C_{\alpha,\delta'}$ where W runs in μ^* ordered by the reverse inclusion. ■

Let C_γ denote the *graph topology* [Na]. Following summarizes our knowledge.

THEOREM 3.21. – *Let* $\alpha = CL(X)$, *and* δ EF *on* Y. *Then*

$$C_{\alpha,\delta} \subset C_\gamma.$$

More, X is countably compact iff $C_{\alpha,\delta} = C_\gamma$. ■

Let (Y, ϱ) be a metric space. For each $f \in C(X, Y)$ and $\phi \in C(X, \mathbb{R}^+)$ define:

$$B_\varrho(f, \phi) = \{g \in C(X, Y) \colon \varrho(f(x), g(x)) < \phi(x), \forall x \in X\}.$$

The family $\{B_\varrho(f, \phi) \colon f \in C(X, Y), \phi \in C(X, \mathbb{R}^+)\}$ is a base for a topology $C_W(X, Y)$, called the *fine* (Whitney or Morse) topology. From [MN_1] we get:

THEOREM 3.22. – *If* δ *is* EF *on* Y, α *is a closed network on* X, *then*

$$C_{\alpha,\delta} \subset C_W.$$

Moreover, X is pseudocompact if and only if $C_{\alpha,\delta} = C_W$.

See also [DHHM], [DDR]. ■

The Krikorian topology C_{k_r} introduced by Krikorian [K] and also studied in [DMN], [DHHM], has a base consisting of sets of the form

$$\mathcal{U}((A_\alpha), (V_\beta), \varphi) = \{f \in C(X, Y) \colon f(A_\alpha) \subset V_{\varphi(\alpha)}\}$$

where (A_α) is a locally finite closed cover of X, (V_β) is an open cover of Y and φ maps α's into β's.

If we suppose that X and Y are Tychonoff and δ is EF on Y, from [DHHM], with no pseudocompactness condition on X, we get:

THEOREM 3.23. – *The following are equivalent*:

(i) *X countably compact, Y first countable and paracompact implies* $C_{k_r} \subset C_{\alpha, \delta}$.

(ii) *Y Tychonoff with a countable base at a nonisolated point and* $C_{k_r} \subset C_{\alpha, \delta}$ *implies X is countably compact* [DHHM].

(iii) $C_{\alpha, \delta} \subset C_{k_r}$. ■

Some more results are contained in:

THEOREM 3.24. – *The following are equivalent*:

(*a*) *X is countably compact.*

(*b*) *Y first countable paracompact space and δ* EF *on Y implies*

$$C_{\alpha, \delta} = C_{k_r}. \quad ■$$

4. – Separation axioms.

We have seen that $C_p \subset C_{\alpha, \delta}$ for all compatible proximities δ on Y. Consequently, if Y is T_1 or T_2, then so is $C_{\alpha, \delta}$. We now give a sufficient condition for $C_{\alpha, \delta}$ to be regular. Let α be a network on X and δ a compatible LO-proximity on Y. Then δ is said to be *regular w.r.t.* α iff for each $A \in \alpha$, $f \in C(X, Y)$ and V open in Y, $f(A) \ll V$ implies there is an open set U in Y such that

$$f(A) \ll U \subset \overline{U} \ll V\,.$$

Naturally any EF-proximity is regular w.r.t. all networks while any R-proximity is regular w.r.t. any compact network.

THEOREM 4.25. – *If a compatible* LO-*proximity δ on Y is regular w.r.t. α, then $C_{\alpha, \delta}$ is regular.*

PROOF. – Suppose $A \in \alpha$, V open in Y and $f \in [A\colon V]_\delta$. Then $f(A) \ll V$. Hence there is an open set $U \subset Y$ such that $f(A) \ll U \subset \overline{U} \ll V$. So:

$$f \in [A\colon U]_\delta \subset \mathrm{Cl}\,[A\colon U]_\delta \subset [A, \overline{U}] \subset [A\colon V]_\delta$$

equivalently C_α, δ is regular. If $g \notin [A, \overline{U}]$, then there exists an $a \in A$ such that $g(a) \in \overline{U}^c$, or equivalently $g \in [a, \overline{U}^c]$, but $[a, \overline{U}^c] \cap [A\colon U]_\delta = \phi$. ■

THEOREM 4.26. – *If* δ *is a compatible* EF-*proximity on* Y *and* α *any network on* X, *then* $C_{\alpha, \delta}$ *is Tychonoff.*

PROOF. – If $f \in [A: V]_\delta$, then $f(A) \ll V$. There exists a proximally continuous function $\psi: Y \to [0, 1]$ such that $\psi(f(A)) = 0$, $\psi(V^c) = 1$. For $h \in C(X, Y)$, we define $\phi(h) = \sup \{\psi(h(a)) : a \in A\}$. Clearly $\phi(f) = 0$. If $g \notin [A: V]_\delta$, then $g(A)\, \delta V^c$. Since ψ is p-continuous, $\psi(g(A))\, \delta \{1\}$ and so $\phi(g) = 1$. Thus $\phi([A: V]_\delta^c) = 1$. We now show that ϕ is continuous. Suppose $\phi(h) = r \in [0, 1]$. For each $\varepsilon > 0$, there is an $a \in A$ such that $\psi(h(a) \in [0, 1] \cap (r - \varepsilon, r + \varepsilon) = U$. So $h(a) \in W = \psi^{-1}(U)$ which is open in Y. Thus $h \in [a, W]$ and $h \in [A: W']_\delta$ since $h(A) \ll W' = \psi^{-1}([0, 1] \cap [0, r + \varepsilon))$. So $h \in [a, W] \cap [A: W']_\delta$ which is open in $C_{\alpha, \delta}$ and if $g \in [a, W] \cap [A: W']_\delta$, then $\phi(g) \in W$. Hence ϕ is continuous. ■

5. – C_{α, δ_0}.

The concept of boundedness or relatively pseudocompactness can be naturally generalized in two different ways.

Let X and Y be topological spaces. A subset A of X is said to be *Y-compact* provided $f(A)$ for any $f \in C(X, Y)$ is relatively compact, i.e. $\overline{f(A)}$ is compact. If Y is uniformized by a uniformity μ, a subset A of X is said *Y-totally bounded* when $f(A)$ is totally bounded for any $f \in C(X, Y)$. Trivially when $Y = \mathbb{R}$ the three notions coincide. More when $Y = \mathbb{R}$ and X is realcompact boundedness is just compactness. In T_4 boundedness agrees with pseudocompactness which in that case is countable compactness. Of course Y-compactness implies Y-total boundedness but generally the three notions are distinct. Y-compactness flats in boundedness when Y contains a closed uniformly isomorphic copy of $\mathbb{R}$ with the standard uniformity.

Now suppose Y to be regular.

PROPOSITION 5.27. – *If* α *is a Y-compact network in X and* δ *is a regular proximity on Y, then* $C_{\alpha, \delta} = C_{\alpha, \delta_0}$.

PROOF. – Note that two disjoint closed sets one of which is compact are δ-far, see Proposition 1.1. So when A is Y-compact and V is open in Y, then $[A: V]_\delta = [A: V]_{\delta_0}$. ■

More, when Y is completely regular:

PROPOSITION 5.28. – *Let* δ *be an EF-proximity on Y and* μ *a uniformity compatible with* δ. *If the network* α *is hereditarily closed and Y-totally bounded, then* $C_{\alpha, \mu} = C_{\alpha, \delta}$.

PROOF. – From Theorem 3.20 it is enough to show that $C_{\alpha,\mu} \subset C_{\alpha,\delta}$. Consider a function f, a diagonal nhbd $U \in \mu$ and $A \in \alpha$. Pick $W \in \mu$ open symmetric and with $WoW \subset U$, more $D \in \mu$ closed and $DoD \subset W$. Since $f(A)$ is totally bounded, for some $a_1, \ldots, a_n \in A$ it follows

$$f(A) \subset D[f(a_1)] \cup \ldots \cup D[f(a_n)].$$

Set $A_i = A \cap f^{-1}(D[f(a_i)])$, $i = 1, \ldots, n$. Then $\cap [A_i \colon W[f(a_i)]]_\delta \in C_{\alpha,\delta}$ contains f and it is contained in $\widehat{U}[A](f)$. ■

The following theorem synthezizes the previous results:

GENERALIZED ARENS THEOREM 5.29. – *If Y is completely regular, μ is any diagonal uniformity consistent with Y and α is a hereditarily closed, Y-compact network in X, then $C_{\alpha,\mu} = C_{\alpha,\delta_0}$.* ■

When X is completely regular, Y is metric and α is the network of all Y-totally bounded subsets of X, C_{α,δ_0} can be an interesting object attached to X and Y. Suppose the continuous functions from X to Y generate a unifomity μ consistent with X, which characterizes, as it is well known, as the weakest one for which any of them is uniformly continuous. Any subset A of the uniform completion $\widehat{X}$ of X w.r.t. μ which is relatively compact has a trace on X which naturally is μ-totally bounded. On the other hand the collection α of all those traces, that is of all subsets of X whose closure in $\widehat{X}$ is compact, is a hereditarily closed network which is stable under finite unions. Thus the topology of uniform convergence on the members of α coincides with C_{α,δ_0}. We obtain particular cases by putting $Y = \mathbb{R}$ or $Y = \mathbb{R}^I$, where $\mathbb{R}^I$ is the set of all continuous functions from $I = [0, 1]$ to the reals equipped with the supmetric. In the former case $\widehat{X}$ is the Nachbin-Hewitt realcompactification of X; in the latter one $\widehat{X}$ is the completion of the Tukey-Shirota uniformity of X.

6. – Topological group structure.

When Y is a topological group, $C(X, Y)$ can be equipped with a natural algebraic structure. We give conditions for C_{α,δ_0} to be a topological group and then a homogeneous space. Any topological group can be uniformized by two natural diagonal uniformities, the left uniformity and the right one. Generally they are distinct, but they coincide when the group is abelian or compact and then, of course, when it is a (subgroup of a) product of an abelian group and a compact one.

THEOREM 6.30. – *If Y is a multiplicative topological group for which the left uniformity agrees with the right one and α is a hereditarily closed Y-*

compact network in X, then C_{α,δ_0} is a topological group w.r.t. the usual product.

PROOF. – When the left uniformity of a topological group agrees with the right one the passage to inverse in Y is uniformly continuous w.r.t. the left uniformity on both sides.This implies that for any nhbd U of the unity e, it is possible to find two nhbds of e, V, W, such that

$$VoV \subset U \quad \text{and} \quad VxW \subset Ux \quad \text{for each } x \in Y.$$

So choosing $x' \in Vx$ and $y' \in Wy$, since naturally $VxWy \subset Uxy$, then $x'y' \in Uxy$. Thus the product

$$(Y, \mathit{left}) \times (Y, \mathit{left}) \to (Y, \mathit{left})$$

is uniformly continuous too. Now it is possible to show that

$$(f, g) \in C_{\alpha,\mathit{left}} \times C_{\alpha,\mathit{left}} \to fg \in C_{\alpha,\mathit{left}}$$

is continuous. Fixed any nhbd U of the unity e and any member of $A \in \alpha$, after choosing V and W as above, it is easy to show that

$$\text{if } h \in \widehat{V}[A](f),\ k \in \widehat{W}[A](g), \qquad \text{then } hk \in \widehat{U}[A](fg).$$

Continuity of the passage to inverse in $C_{\alpha,\mathit{left}}$ derives in a same way from uniform continuity of the passage to inverse in Y. The final result follows from the agreement of $C_{\alpha,\mathit{left}}$ with C_{α,δ_0} due to generalized Arens theorem 5.29. ■

7. – Metrization theorem.

Let X be completely regular, (Y, d) a bounded metric space containing an arc, and α a Y-totally bounded network in X closed under finite unions and hereditarily closed.

THEOREM 7.31. – *The following properties are equivalent*:

(1) *C_{α,δ_0} is metrizable.*

(2) *C_{α,δ_0} is first countable.*

(3) *α contains a countable subcollection $\beta = \{A_1, \dots, A_n, \dots\}$ such that any $A \in \alpha$ is contained in some A_n.*

PROOF. – (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3). Let be $\gamma(0)$ the origin of an arc in Y and $c_\gamma(0)$ the constant function determined from it. Put $V = Y - \{\gamma(1)\}$. Then, for each $A \in \alpha$, $c_{\gamma(0)} \in [A : V]_{\delta_0}$. But $c_{\gamma(0)}$ has a countable family of basic nhbds $\{U_n : n \in \mathbb{N}\}$ of the type $U_n = \{\cap [A_i^n, V_i^n]_{\delta_0} : i \in F_n\}$. So for some integer n

$$c_{\gamma(0)} \in \cap\{[A_i^n, V_i^n]_{\delta_0} : i \in F_n\} \subset [A, V]_{\delta_0}.$$

Observe $\gamma(0) \in \cap\{V_i^n : i \in F_n\}$. The set A must be contained in $\{\cup A_i^n : i \in F_n\}$. If it wasn't so, we should find $a \in A$, $a \notin \{\cup A_i^n : i \in F_n\}$ and by complete regularity of X a continuous function $f: X \to [0, 1]$ such that $f(a) = 1$ and $f(\cup A_i^n) = 0$. The composition $\gamma \circ f: X \to Y$ is in $\cap\{[A_i^n V_i^n]_{\delta_0} : i \in F_n\}$, infact $\gamma \circ f(A_i^n) = \gamma(0) \in V_i^n$. But since $\gamma \circ f(a) = \gamma(1) \notin V$, $\gamma \circ f \notin [A, V]_{\delta_0}$. A contradiction. The subcollection β we look for is done from all finite unions of "A_i^n", where i runs F_n and n runs the integers.

(3) $\Rightarrow$ (1). From Prop. 5.28 C_{α, δ_0} agrees with the topology of uniform convergence on members of α and then it is just the topology of uniform convergence on members of β which is metrized, as well-known, by:

$$\varrho(f, g) = \sum_n \frac{1}{2^n} \varrho_n(f, g)$$

where ϱ_n is the pseudometric defined by:

$$\varrho_n(f, g) = \sup\{d(f(a), g(a)) : a \in A_n\}. \qquad \blacksquare$$

REFERENCES

[AD] R. Arens - J. Dugundji, *Topologies for function spaces*, Pacific J. Math., **1** (1951), 5-31.

[B] H. Buchwalter, *Parties bornées d'un éspaces topologique complétement regulier*, Sém. Choquet, 9^e année, n. **14** (1970).

[DDR] A. Di Concilio - G. Di Maio - A. Russo, *Su alcune topologie degli spazi di funzioni, con particolare riguardo alla topologia di Whitney*, Rend. Accad. Scienze Fis. Mat. Napoli, **49** (1982), 9-16.

[DHHM] G. Di Maio - L. Hola - D. Holy - R. A. McCoy, *Topologies on the space of continuous functions* (to appear).

[DMN] G. Di Maio - S. A. Naimpally, *Proximal graph topologies*, Q & A in Gen. Top., **10** (1992), 97-125.

[H] J. Hansard, *Function space topologies*, Pacific J. Math., **35** (1970), 381-388.

[Ha] D. Harris, *Regular-closed spaces and proximities*, Pacific J. Math., **34** (1970), 675-685.

[K] N. Krikorian, *A note concerning the fine topology on function spaces*, Compositio Math., **21** (1969), 343-348.

[KR] S. Kundu - A. B. Raha, *The bounded-open topology and its relatives*, Rend. Ist. Mat. Univ. Trieste, **27** (1995), 61-77.

[L] S. LEADER, *On completion of proximity spaces by local clusters*, Fund. Math., **48** (1960), 201-216.

[MN_1] R. A. MCCOY - I. NTANTU, *Topological properties of spaces of continuous functions*, Lecture Notes in Math., Springer-Verlag (1988).

[MN_2] R. A. MCCOY - I. NTANTU, *Completeness properties of function spaces*, Top. and Appl., **22** (1986), 191-206.

[N] L. NACHMANN, *On a conjecture of Leader*, Fund. Math., **65** (1969), 153-155.

[Na] S. A. NAIMPALLY, *Graph topology for function spaces*, Trans. Amer. Math. Soc., **123** (1966), 267-272.

[NW] S. A. NAIMPALLY - B. D. WARRACK, *Proximity spaces*, Cambridge University Press (1970).

[W] J. WILLIAMS, *Locally uniform spaces*, Trans. Amer. Math. Soc., **168** (1972), 435-469.

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