Bollettino Unione Matematica Italiana

Nickolay Tzvetkov

Remark on the null-condition for the nonlinear wave equation

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 135–145.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_135_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 135-145

Remark on the Null-Condition for the Nonlinear Wave Equation.

NICKOLAY TZVETKOV (*)

Sunto. – *Dimostriamo l'esistenza della soluzione globale per un sistema di equazioni delle onde con nonlinearità quadratica dipendente dalle variabili spazio-tempo. Come in* [3] *la tecnica è basata sulla trasformazione di Penrose.*

1. – Introduction.

In this note we shall consider the following system of nonlinear wave equations in Minkowski space-time $\boldsymbol{R}^{1+n}$:

$$(\partial_t^2 - \Delta)\, u^I = F^I(t, x, u, Du), \qquad I = 1, \ldots, N, \tag{1}$$

with initial data:

$$u(0, x) = f_0(x), \qquad u_t(0, x) = f_1(x). \tag{2}$$

Here $x \in \boldsymbol{R}^n$, Δ is the Laplace operator on $\boldsymbol{R}^n$ and $u = (u_1, \ldots, u_N)$ is $\boldsymbol{R}^N$-valued function. D denotes the space-time gradient $D = (\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$. The nonlinear terms F^I are supposed to be in the form:

$$F^I = \sum_{J, K} \Gamma^I_{JK}(u) B^I_{JK}(t, x, Du^J, Du^K), \tag{3}$$

where Γ^I_{JK} are smooth functions near to the origin of $\boldsymbol{R}^N$. B^I_{JK} are quadratic forms which are supposed to satisfy the null-condition. Similarly to [3] we introduce the following notion of the null-condition.

DEFINITION 1. – *The quadratic form $Q(t, x, Du, Dv)$ satisfies the null-condition if:*

(i) *$Q(t, x, Du, Dv)$ is homogeneous of order zero with respect to* (t, x).

(ii) *$Q(t, x, Du, Dv)$ is homogeneous of order one with respect to Du and Dv and a bilinear form of Du and Dv.*

(*) The author was partially supported by Contract MM-516 with the Bulgarian Ministry of Education, Science and Technology.

(iii) $Q(t, x, -t, x, -t, x) = 0$ *whenever* (t, x) *belongs to the null cone, i.e.* $t^2 - x^2 = 0$

REMARK. – We can replace assumption (iii) in the above definition with a stronger one:

$$\text{(iii)} \quad \Omega^{-1} Q(\sin T, \sin R, -\sin T, \sin R, -\sin T, \sin R) \in C^\infty(\boldsymbol{R} \times S^n),$$

where S^n is the n-dimensional sphere, parametrized by $(\sin R \cdot \omega, \cos R)$, $\omega \in S^{n-1}$ and Ω is the conformal factor of Penrose compactification map (cf. (4) below).

EXAMPLES. – It is easy to see that any of the following quadratic forms satisfy the null-condition:

$$Q_1(Du^J, Du^K) = \partial_t u^J \partial_t u^K - \sum_{i=1}^{n} \partial_{x_i} u^J \partial_{x_i} u^K,$$

$$Q_2(Du^J, Du^K) = \partial_{x_i} u^J \partial_{x_j} u^K - \partial_{x_j} u^J \partial_{x_i} u^K,$$

$$Q_3(Du^J, Du^K) = \partial_t u^J \partial_{x_j} u^K - \partial_{x_j} u^J \partial_t u^K,$$

$$Q_4(x, Du^J, Du^K) = \frac{x_i x_j}{|x|^2} \partial_t u^J \partial_t u^K - \partial_{x_i} u^J \partial_{x_j} u^K,$$

$$Q_5(x, Du^J, Du^K) = \frac{x_i x_j}{|x|^2} \partial_{x_k} u^J \partial_{x_l} u^K - \frac{x_k x_l}{|x|^2} \partial_{x_i} u^J \partial_{x_j} u^K,$$

$$Q_6(x, Du^J, Du^K) = \frac{x_i}{|x|} \partial_t u^J \partial_{x_j} u^K - \frac{x_j}{|x|} \partial_{x_i} u^J \partial_t u^K,$$

where $1 \leq i, j, k, l \leq n$.

When $n \geq 4$ one can prove global existence for the Cauchy problem (1)-(2) for general smooth nonlinear terms and small initial data only by using the energy estimate and scaled versions of the usual Sobolev inequalities (cf. [6]). This approach fails in the case $n \leq 3$ because logarithmic singularities appear in the energy inequality. Global existence for nonlinear terms of type Q_1, Q_2, Q_3 in the case $n = 3$ is proved by Klainerman [5]. The approach is based on the invariance of the null forms Q_1, Q_2, Q_3 under the actions of the Poincaré group. More precisely one can obtain decay inequalities for the nonlinear terms which give better decay of the nonlinear terms in the energy inequality than direct applying of the scaled version of Sobolev inequality. Another approach is using the conformal invariance of D'Alembertian (cf. [3]). By means of using the Penrose map one can reduce the global Cauchy problem (1)-(2) to a local one. The main difficulties are the singularities which appear in the nonlinear terms applying the Penrose map. The role of the null-condition is to cancel these singularities. In both papers [3] and [5] are considered nonlinearities which do not depend on the space-time variables. If we consider a system of wave and Klein-Gordon equations then the conformal invariance fails and one is not able to use the Penrose map. The approach using the invariant vector fields meets an essential difficulty since the usage of the radial vector field is not convenient. However, in [4] a stronger version of the null-condition leading to global existence for nonlinear systems of wave and Klein-Gordon equations is introduced.

In this paper we prove global existence for nonlinearities with variable coefficients. More precisely, when $n=2, 3$ we prove global existence for the Cauchy problem (1)-(2) with small initial data and nonlinearities in the form (3). If we go back to the examples we see that in fact the forms Q_1, Q_2, Q_3 could be obtained from Q_4, Q_5, Q_6 respectively. If we consider Q_4 for $i=j$ then after summing from 1 up to n we obtain Q_1. For $i=k$ and $j=l$ the term Q_5 is almost Q_2. For $i=j$ the term Q_6 is almost Q_3. Therefore we can obtain the null-forms with constant coefficients as a particular case of these with nonconstant ones. Our approach is also based on the Penrose map. We split the derivatives in the standard framework into radial and angular components. We show that the angular components cancel easily the singularity. Then we compute the radial components and cancel the singularities due to the properties of the nonlinearities satisfying the null-condition. Using the energy estimate and Sobolev inequalities with standard local existence arguments we obtain local solution of the transformed problem. To this solution corresponds global solution to the original problem (1)-(2).

The initial data (f_0, f_1) are small with respect to suitable weighted Sobolev norms. More precisely we shall suppose:

$$(f_0, f_1) \in H^{s, s-1}(\boldsymbol{R}^n) \times H^{s-1, s-2}(\boldsymbol{R}^n),$$

where the norm in $H^{s, \delta}$ is:

$$|f|^2_{s, \delta} = \sum_{j=1}^{s} |(1+|\cdot|^2)^{\delta+j} \nabla^j f|^2_{L^2(R^n)}.$$

If $\delta > s-n$ then the Penrose map transforms $H^{s, \delta}(\boldsymbol{R}^n)$ into $H^s(S^n)$ (cf. [1], p. 396).

We have the following Theorem:

THEOREM 1. – *Let* $n=2, 3$. *We suppose the initial data* (2) *are such that*:

$$|f_0|_{s, s-1} + |f_1|_{s-1, s-2} \leq \varepsilon,$$

for $s > 1 + n/2$ *and the quadratic forms* $B^I_{JK}(t, x, Du^I, Du^J)$ *satisfy the null-*

condition. In the case when $n=2$ we also suppose that Γ^I_{JK} are zero near to the origin.

Then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (1)-(2) *admits global solution with the following decay property*:

$$|u(t, x)| < \frac{c}{(1+t+|x|)^{(n-1)/2}(1+|t-|x||)^{(n-1)/2}} .$$

REMARK. – The assumption for $n=2$ about Γ^I_{JK} is essential. In this case the logarithmic singularities in the energy estimate applying the scaled version of Sobolev inequality appear when considering cubic nonlinearities.

2. – Proof of the Theorem.

We shall consider the Penrose map (cf. [7]):

$$P: \boldsymbol{R}^{1+n} \mapsto \boldsymbol{E}^{1+n}$$

which maps conformally $\boldsymbol{R}^{1+n}$ with the flat metric $\eta = dt^2 - dx^2$ into a bounded diamond-like region of the Einstein cylinder $\boldsymbol{E}^{1+n}$, where $\boldsymbol{E}^{1+n} = (\boldsymbol{R}\times S^n, g)$, $g = dT^2 - d\omega_n^2$, T stands for the time in $\boldsymbol{E}^{1+n}$ and ω_n^2 is the line-element of S^n.

If (t, r, ω), $\omega \in S^{n-1}$ are the polar coordinates of $\boldsymbol{R}^{1+n}$ and $(T, \sin R\cdot\omega, \cos R)$ are the local coordinates of $\boldsymbol{E}^{1+n}$ the Penrose map is given by:

$$\begin{aligned} &P:(t, r, \omega) \mapsto (T, R, \omega),\\ &T = \arctan(t+r) + \arctan(t-r),\\ &R = \arctan(t+r) - \arctan(t-r),\\ &\omega = \omega . \end{aligned}$$

We have $0 \leqslant R < \pi$ and $-\pi < T \pm R < \pi$, which determines the image $P(\boldsymbol{R}^{1+n})$ of the Penrose map. On $P(\boldsymbol{R}^{1+n})$ we can write the inverse of P as:

$$t = \frac{\sin T}{\Omega}, \qquad r = \frac{\sin R}{\Omega}, \qquad \omega = \omega ,$$

where:

$$\Omega = \cos T + \cos R = \frac{2}{(1+(t+r)^2)^{1/2}(1+(t-r)^2)^{1/2}} \tag{4}$$

We also have $g = \Omega^2\eta$. Therefore using the conformal invariance of

D'Alembertian and according to the general conformal theory (cf. [1], p. 260 for example) if we pose $u = \Omega^{(n-1)/2} v$ then we have:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^{n-1}}\right) u =$$

$$\Omega^{(n+3)/2}\left(\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial R^2} - \frac{n-1}{\tan R}\frac{\partial}{\partial R} + \frac{(n-1)^2}{4} - \frac{1}{\sin^2 R}\Delta_{S^{n-1}}\right) v =$$

$$\Omega^{(n+3)/2}\left(\frac{\partial^2}{\partial T^2} - \Delta_{S^n} + \frac{(n-1)^2}{4}\right) v$$

where Δ_{S^n} is Laplace-Beltrami operator on S^n. The term $(n-1)^2/4$ corresponds to the scalar curvature of $\boldsymbol{E}^{1+n}$ (cf. [1], p. 260 for example). We have the following parametrization of S^n:

$$Y_k = \sin R \cdot \omega_k, \qquad k = 1, \ldots, n,$$

$$Y_{n+1} = \cos R,$$

where $\omega = (\omega_1, \ldots, \omega_n) \in S^{n-1}$.

We introduce the generators of the group $SO(n)$:

$$Z_{ij} = Y_i \frac{\partial}{\partial Y_j} - Y_j \frac{\partial}{\partial Y_i}, \qquad 1 \leqslant i, j \leqslant n+1.$$

Then the Sobolev norms of S^n are defined by:

$$|u|_{H^k(S^n)} = |u|_{L^2(S^n)} + \sum_{l=1}^{k} \sum_{1 \leqslant i_m < j_m \leqslant n+1} |Z_{i_1 j_1} \ldots Z_{i_l j_l} u|_{L^2(S^n)}.$$

The following representation of Laplace-Beltrami operator holds:

$$\Delta_{S^n} = \sum_{1 \leqslant i < j \leqslant n+1} Z_{ij}^2.$$

In the next Proposition we state the energy estimate:

PROPOSITION 1. – *If*:

$$\left(\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{4}\right) v = G,$$

then we have:

$$|v_T(T,\,.)|_{L^2(S^n)} + |v(T,\,.)|_{H^1(S^n)} \leqslant$$

$$\text{const}\left(|v_T(0,\,.)|_{L^2(S^n)} + |v(0,\,.)|_{H^1(S^n)} + \int_0^T |G(s,\,.)|_{L^2(S^n)}\,ds \right).$$

In order to cancel the singularities of the angular derivatives we need the next Proposition:

PROPOSITION 2. – *The following decomposition of the vector fields* $\partial/\partial x_k$ *of angular and radial component holds*:

$$\frac{\partial}{\partial x_k} = \omega_k \frac{\partial}{\partial r} + \Omega \sum_{j=1}^{n} \frac{\omega_j}{\sin R} \cdot Z_{jk}, \qquad k = 1, \ldots, n\,.$$

PROOF. – We have that:

$$\Omega \sum_{j=1}^{n} \frac{\omega_j}{\sin R} \cdot Z_{jk} = \sum_{j=1}^{n} \left(\omega_j^2 \frac{\partial}{\partial x_k} - \omega_j \omega_k \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_k} - \omega_k \frac{\partial}{\partial r}\,.$$

This completes the proof of Proposition 2.

In the next Proposition we express the derivatives in $\boldsymbol{R}^{n+1}$ in the terms of derivation on $\boldsymbol{E}^{n+1}$.

PROPOSITION 3. – *If* $u = \Omega^{(n-1)/2} v$ *then we have*:

$$\frac{\partial u}{\partial x_k} = \frac{x_k}{r} \frac{\partial u}{\partial r} + \Omega^{(n+1)/2} \sum_{j=1}^{n} \frac{\omega_j}{\sin R} \cdot Z_{jk} v\,, \tag{5}$$

$$\frac{\partial u}{\partial t} = -\frac{t}{r} \frac{\partial u}{\partial r} + \frac{1}{\sin R} \cdot \Omega^{(n+1)/2} \left(-\frac{n-1}{2} \sin T \sin R \cdot v + \cos T \sin R \cdot \frac{\partial v}{\partial T} + \sin T \cos R \cdot \frac{\partial v}{\partial R} \right). \tag{6}$$

PROOF. – In order to prove (5) we use that Ω does not depend on the angular variables and Proposition 2. Using the definition of the Penrose map we obtain easily:

$$\frac{\partial}{\partial t} = (1+\cos T\cos R)\frac{\partial}{\partial T} - \sin T\sin R\frac{\partial}{\partial R}\,,$$

$$\frac{\partial}{\partial r} = -\sin T\sin R\frac{\partial}{\partial T} + (1+\cos T\cos R)\frac{\partial}{\partial R}\,.$$

If $u = \Omega^{(n-1)/2}v$ then the above relations yield

$$\frac{\partial u}{\partial t} = \Omega^{(n+1)/2}\cos T\frac{\partial v}{\partial T} + \Omega^{(n-1)/2}\left(-\frac{n-1}{2}\sin T\cos R\cdot v + \sin^2 T\cdot\frac{\partial v}{\partial T} - \sin T\sin R\cdot\frac{\partial v}{\partial R}\right),$$

$$\frac{\partial u}{\partial r} = \Omega^{(n+1)/2}\cos T\frac{\partial v}{\partial R} + \Omega^{(n-1)/2}\left(-\frac{n-1}{2}\cos T\sin R\cdot v - \sin T\sin R\cdot\frac{\partial v}{\partial T} + \sin^2 T\cdot\frac{\partial v}{\partial R}\right).$$

Now it is sufficient to compute $tu_r + ru_t$ to arrive at (6). This completes the proof of Proposition 3.

If we consider again the null-forms $Q_1, \ldots, Q_6$ using Proposition 3 we obtain:

$$Q_1(Du^J, Du^K) = \partial_t u^J\partial_t u^K - \partial_r u^J\partial_r u^K + \Omega^n\widetilde{Q}_1,$$

$$Q_2(Du^J, Du^K) = \Omega^n\widetilde{Q}_2\,,$$

$$Q_3(Du^J, Du^K) = \omega_j(\partial_t u^J\partial_r u^K - \partial_r u^J\partial_t u^K) + \Omega^n\widetilde{Q}_3,$$

$$Q_4(x, Du^J, Du^K) = \omega_i\omega_j(\partial_t u^J\partial_t u^K - \partial_r u^J\partial_r u^K) + \Omega^n\widetilde{Q}_4,$$

$$Q_5(x, Du^J, Du^K) = \Omega^n\widetilde{Q}_5,$$

$$Q_6(x, Du^J, Du^K) = \omega_i\omega_j(\partial_t u^J\partial_r u^K - \partial_r u^J\partial_t u^K) + \Omega^n\widetilde{Q}_6,$$

where $\widetilde{Q}_1, \ldots, \widetilde{Q}_6$ are smooth functions over $\boldsymbol{R} \times S^n \times \boldsymbol{R}^{N(n(n-1)/2+1)}$. Hence we have to examine the following forms:

$$P_1(Du^J, Du^K) = \partial_t u^J \partial_r u^K - \partial_r u^J \partial_t u^K,$$

$$P_2(Du^J, Du^K) = \partial_t u^J \partial_t u^K - \partial_r u^J \partial_r u^K.$$

A straightforward computation shows that the next presentation of P_1 and P_2 holds:

$$P_1(Du^J, Du^K) = \Omega^n \Big\{ \Omega(\partial_T v^J \partial_R v^K - \partial_T v^K \partial_R v^J) + \frac{n-1}{2} \sin T(\partial_R v^J v^K - v^J \partial_R v^K) - \frac{n-1}{2} \sin R(\partial_T v^J v^K - v^J \partial_T v^K) \Big\},$$

$$P_2(Du^J, Du^K) = \Omega^n \Big\{ \Omega(\partial_T v^J \partial_T v^K - \partial_R v^J \partial_R v^K) + \frac{n-1}{2} \sin R(\partial_R v^J v^K - v^J \partial_R v^K) - \frac{n-1}{2} \sin T(\partial_T v^J v^K - v^J \partial_T v^K) \Big\}.$$

Using the above relations and:

$$\frac{\partial}{\partial R} = \sum_{j=1}^{n} \omega_j Z_{(n+1)j}$$

we obtain:

$$Q_j = \Omega^n \widehat{Q}_j,$$

for $j = 1, \ldots, 6$, where $\widehat{Q}_j(T, R, \omega, \partial_T v, Z_{ij} v)$ are smooth functions over $\boldsymbol{R} \times S^n \times \boldsymbol{R}^{N(n(n-1)/2+1)}$. Hence the Cauchy problem (1)-(2) is transformed into the following:

$$\left(\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{4} \right) v^I = \widehat{F}^I(T, R, \omega, v, \partial_T v, Z_{ij} v), \quad I = 1, \ldots, N, \tag{7}$$

with initial data:

$$v(0, R, \omega) = \tilde{f}_0, \qquad v_t(0, R, \omega) = \tilde{f}_1, \tag{8}$$

where $u = \Omega^{(n-1)/2} v$. $\widehat{F}^I$ are smooth functions over $\boldsymbol{R} \times S^n \times B \times \boldsymbol{R}^{N(n(n-1)/2+1)}$, B is an open set containing the origin of $\boldsymbol{R}^N$ where Γ^I_{JK} are smooth, $\tilde{f}_0, \tilde{f}_1$ (defined over S^n) are the images of f_0, f_1 after the Penrose compactification map (cf. [3] for details). When $n = 2$ an additional factor $\Omega^{(n-1)/2}$ appears in the right hand side of the equation which is due to the assumption for Γ^I_{JK} in this case. Hence we reduce the problem of finding global solutions of (1)-(2) to the problem of finding solutions of (7)-(8) in the time interval $[0, \pi]$.

In the case of general nonlinearity using Proposition 3 we obtain the following decomposition of B^I_{JK}:

$$B^I_{JK}(t, x, Du^J, Du^K) = B^I_{JK}\left(t, x, -\frac{t}{r}\frac{\partial u^J}{\partial r}, \frac{x}{r}\frac{\partial u^J}{\partial r}, -\frac{t}{r}\frac{\partial u^K}{\partial r}, \frac{x}{r}\frac{\partial u^K}{\partial r}\right) +$$

$$\Omega^n C^I_{JK}(T, R, \omega, \partial_T v, Z_{ij} v),$$

where C^I_{JK} are smooth functions. Next we use that B^I_{JK} satisfy the null-condition and obtain:

$$B^I_{JK}\left(t, x, -\frac{t}{r}\frac{\partial u^J}{\partial r}, \frac{x}{r}\frac{\partial u^J}{\partial r}, -\frac{t}{r}\frac{\partial u^K}{\partial r}, \frac{x}{r}\frac{\partial u^K}{\partial r}\right) =$$

$$\frac{1}{r^2}\frac{\partial u^J}{\partial r}\cdot\frac{\partial u^K}{\partial r}\cdot Q(t, x, -t, x, -t, x) =$$

$$\frac{\Omega}{\sin^2 R}\frac{\partial u^J}{\partial r}\cdot\frac{\partial u^K}{\partial r}\cdot Q(\sin T, \sin R, -\sin T, \sin R, -\sin T, \sin R) =$$

$$\Omega^n D^I_{JK}(T, R, \omega, \partial_T v, Z_{ij} v),$$

where D^I_{JK} are smooth functions. Hence we have to solve locally a problem of type (7)-(8).

Further we set:

$$E_{s,a}(T) := \left\{ v \colon e_s(v, T) := \sum_{j=0}^{s} \sup_{0 \leq t \leq T} \left| \frac{\partial^j}{\partial T^j} v(t, .) \right|_{H^{s-j}(S^n)} \leq a \right\}.$$

For v, R^N-valued function we consider the map:

$$M \colon v \mapsto w,$$

where $w = Mv$ is the solution of the linear equation:

$$\left(\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{4}\right) w^I = \widehat{F}^I(T, R, \omega, v, \partial_T v, Z_{ij} v), \qquad I = 1, \dots, N,$$

with initial data:

$$w(0, R, \omega) = \tilde{f}_0, \qquad w_t(0, R, \omega) = \tilde{f}_1.$$

We have the following relations:

$$\left[\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{4}, Z_{ij}\right] = 0, \qquad \left[\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{4}, \partial_T\right] = 0,$$

where [., .] is the usual commutator. Therefore using Proposition 1 and Sobolev embedding we can obtain for $s > n/2 + 1$:

$$e_s(Mv, T) \leq c(|\tilde{f}_0|_{H^s(S^n)} + |\tilde{f}_1|_{H^{s-1}(S^n)} + T(e_s(v, T))^2).$$

We consider the iteration $v_{n+1} = Mv_n$, where $v_0 = 0$. Hence we have:

$$e_s(v_{n+1}, T) \leq c\delta + cT(e_s(v_n, T))^2,$$

where $\delta = |\tilde{f}_0|_{H^s(S^n)} + |\tilde{f}_1|_{H^{s-1}(S^n)}$. If we choose T and δ such that: $c\delta + c \cdot Ta^2 \leq a$ then we obtain $v_{n+1} \in E_{s,a}(T)$, provided $v_n \in E_{s,a}(T)$. In a similar way we can obtain that v_n is a Cauchy sequence in $E_{s,a}(T)$. Hence v_n converge to the unique solution of:

$$\left(\partial_T^2 - \Delta_{S^n} + \frac{(n-1)^2}{2}\right) v^I = \widehat{F}^I(T, R, \omega, v, \partial_T v, Z_{ij} v), \qquad I = 1, \ldots, N,$$

with initial data:

$$v(0, R, \omega) = \tilde{f}_0, \qquad v_t(0, R, \omega) = \tilde{f}_1,$$

for the time $[0, T)$. For verifying $c\delta + c \cdot Ta^2 \leq a$ it is sufficient to choose T and δ such that $T\delta \leq 1/4c^2$. Therefore if δ is small enough then T is greater than π. Due to [3] (cf. also [1], p. 396), we can estimate the weighted Sobolev norms on $\boldsymbol{R}^n$ with the usual Sobolev norms on S^n. More precisely we have:

$$|f_0|_{s, s-1} + |f_1|_{s-1, s-2} \leq c\delta.$$

Hence if we take the initial data (2) of the original problem small enough then we can obtain solution of (7)-(8) for time greater than π. To this solution will correspond global solution of the Cauchy problem (1)-(2). The decay property follows from the relation $u = \Omega^{(n-1)/2} v$ and the fact $v \in L^\infty$ which is due to the Sobolev embedding.

This completes the proof of the Theorem.

REFERENCES

[1] Y. CHOQUET-BRUHAT - CÉCILE DE WITT-MORETTE, *Analysis, Manifolds and Physics*, Part 2, North-Holland (1989).
[2] Y. CHOQUET-BRUHAT - D. CHRISTODOULOU, *Existence of global solutions of the Yang-Mils, Higgs and spinor field equations in* $3+1$ *dimensions*, Ann. E.N.S., 4ème Série, **14** (1981), 481-500.
[3] D. CHRISTODOULOU, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure. Appl. Math., **39** (1986), 267-282.
[4] V. GEORGIEV, *Global solution of the system of wave and Klein-Gordon equations*, Math. Z., **203** (1990), 683-698.
[5] S. KLAINERMAN, *The null condition and global existence to nonlinear wave equation*, Lectures in Apll. Math., **23** (1986), 293-326.
[6] S. KLAINERMAN, *Remarks on the global Sobolev inequalities in the Minkovski space* R^{n+1}, Comm. Pure Appl. Math., **40** (1987), 111-117.
[7] R. PENROSE, *Conformal treatment of infinity in relativity, groups and topology*, B. De Witt and C. De Witt (eds.), Gordon and Breach (1963).

Analyse numérique et EDP, Université de Paris-Sud, Bât. 425,
91405 Orsay Cedex, France
Institute of Mathematics, Section of Mathematical Physics
Bulgarian Academy of Sciences, Acad. G. Bonchev str. bl 8
1113 Sofia, Bulgaria

Pervenuta in Redazione
il 24 aprile 1998