Bollettino Unione Matematica Italiana

Giuseppe Melfi

On 5-tuples of twin practical numbers

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.3, p. 723–734.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_3_723_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 723-734

On 5-Tuples of Twin Practical Numbers.

GIUSEPPE MELFI

Sunto. – *Un intero positivo m si dice pratico se ogni intero n con $1 < n < m$ può essere espresso come una somma di divisori distinti positivi di m. In questo articolo è affrontato il problema dell'esistenza di infinite cinquine di numeri pratici della forma $(m-6, m-2, m, m+2, m+6)$.*

1. – Introduction.

In this paper we deal with a recent topic in elementary number theory, namely the theory of practical numbers. As extensively pointed out in [6], some properties of practical numbers appear to be close to those of primes, although practical numbers are defined in a completely different way. In particular, practical numbers apparently show some irregularities of distribution which resemble those of primes.

DEFINITION 1. – *A positive integer m is said to be practical if every n with $1 < n < m$ is a sum of distinct positive divisors of m.*

This definition is due to Srinivasan [11], who also pointed out the first properties of practical numbers in his short note. After him, several authors dealt with various aspects of the theory of practical numbers. Stewart [12] proved the following structure theorem: an integer $m \geqslant 2$, $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, with primes $q_1 < q_2 < \cdots < q_k$ and integers $\alpha_i \geqslant 1$, is practical if and only if $q_1 = 2$ and, for $i = 2, 3, \ldots, k$,

$$q_i \leqslant \sigma(q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_{i-1}^{\alpha_{i-1}}) + 1 ,$$

where $\sigma(n)$ denotes the sum of the positive divisors of n.

Let $P(x)$ be the counting function of practical numbers:

$$P(x) = \sum_{\substack{m \leqslant x \\ m \text{ practical}}} 1 .$$

Erdös announced in [1] that practical numbers have zero asymptotic density, i.e., $P(x) = o(x)$. Hausman and Shapiro [4] showed that

$$P(x) \ll \frac{x}{(\log x)^{\beta}}$$

for any $\beta < (1/2)(1 - 1/\log 2)^2 \simeq 0.0979$. On the other hand, Margenstern ([5], [6]) proved that

$$P(x) \gg \frac{x}{\exp\{1/(2\log 2)(\log\log x)^2 + 3\log\log x\}}\,.$$

Tenenbaum ([13], [14]) improved the above upper and lower bounds as follows:

$$\frac{x}{\log x}(\log\log x)^{-5/3-\varepsilon} \ll_{\varepsilon} P(x) \ll \frac{x}{\log x}\log\log x \log\log\log\log x\,.$$

Recently Saias [10] improved the above estimates by providing upper and lower bounds of Chebishev's type:

$$c_1 \frac{x}{\log x} < P(x) < c_2 \frac{x}{\log x}$$

for suitable effectively computable constants c_1 and c_2. This is in accordance with the asymptotic behaviour conjectured by Margenstern in [5]:

CONJECTURE 1. – *There exists a constant λ such that*

$$P(x) \sim \lambda \frac{x}{\log x}\,.$$

Margenstern's computations suggest $\lambda \simeq 1.341$ for the above conjecture.

Among other things, a Goldbach-type result holds for practical numbers: every even positive integer is a sum of two practical numbers [7, Theorem 1].

Here we are interested in finite sequences of consecutive practical numbers. There exist infinitely many pairs $(m, m+2)$ of twin practical numbers (see also [6, Théorème 6] for a more general result), although it looks difficult to estimate the asymptotic behaviour of their counting function. In [8, Theorem 6] the author constructed a sequence $\{m_n\}_{n\geqslant 1}$ of practical numbers such that m_n+2 is also practical for every n, and such that $m_{n+1}/m_n<2$. In [7, Theorem 1] we get a slightly better estimate: $m_{n+1}/m_n<3/2$. Both estimates give

$$\sum_{\substack{m\leqslant x\\ m,\, m+2\ \text{practical}}} 1 \gg \log x\,,$$

but this estimate is very far from Margenstern's conjecture:

CONJECTURE 2. – *Let* $P_2(x)=\sum_{\substack{m\leqslant x\\ m,\, m+2\ \text{practical}}} 1$. *For a suitable constant* λ_2

$$P_2(x)\sim\lambda_2\,\frac{x}{(\log x)^2}\,.$$

As is well-known, there is an analogous celebrated conjecture of Hardy and Littlewood [3, Section 22.20, p. 371–373] for $\pi_2(x)$, the counting function of the pairs of twin primes.

The author proved in [7] that there exist infinitely many triplets of practical numbers of the form $(m-2,\, m,\, m+2)$. As a consequence of that proof one gets

$$\sum_{\substack{m\leqslant x\\ m-2,\, m,\, m+2\ \text{practical}}} 1 \gg \log\log x\,,$$

very far from the following conjecture of Erdös [2]:

CONJECTURE 3. – *There exists a positive constant c such that*

$$\sum_{\substack{m\leqslant x\\ m-2,\, m,\, m+2\ \text{practical}}} 1 \gg \frac{x}{(\log x)^c}\,.$$

It is shown in [6] that for any even $m>2$, one at least of $m,\, m+2,\, m+4,\, m+6$ is not practical. In fact, at least one of them is $\not\equiv 0 \bmod 3$ and $\not\equiv 0 \bmod 4$, hence of the form $2q_1^{\alpha_1}\cdots q_k^{\alpha_k}$ with odd primes $q_1<q_2<\cdots<q_k$ and $q_1\geqslant 5$, in contradiction with Stewart's structure theorem.

On the other hand, explicit computations suggest the following conjecture, first stated in [8]:

CONJECTURE 4. – *There exist infinitely many* 5*-tuples of practical numbers of the form* $(m-6,\, m-2,\, m,\, m+2,\, m+6)$.

In Table 1 a short table of the first m's such that $m-6$, $m-2$, m, $m+2$, $m+6$ are practical numbers is shown.

TABLE 1. – *The first m's such that $m-6, m-2, m, m+2, m+6$ are practical numbers.*

No.	m	No.	m	No.	m
1	18	13	52578	25	359658
2	30	14	67938	26	432822
3	198	15	88506	27	526878
4	306	16	92202	28	533370
5	462	17	96222	29	584166
6	1482	18	123006	30	659934
7	2550	19	131070	31	1032858
8	4422	20	219102	32	1051650
9	17298	21	226182	33	1140414
10	23322	22	237190	34	1142658
11	23550	23	277506	35	1243170
12	40350	24	312702	36	1255422

In this paper we discuss this conjecture and reduce it to a very reasonable, although unproved, Diophantine property of a certain pair of integer sequences.

2. – Arithmetical tools.

A reasonable attempt to attack Conjecture 4 might be to ask whether there exist infinitely many n such that $2\cdot 3\cdot(3^{n-1}-1)$, $2\cdot(3^n-1)$, $2\cdot 3^n$, $2\cdot(3^n+1)$, $2\cdot 3\cdot(3^{n-1}+1)$ are practical numbers: in fact these 5-tuples are of the form of our conjecture, and this approach is similar to the problem of the triplets that the author solved in [7].

We begin with the study of some arithmetical questions related to our approach for Conjecture 4.

LEMMA 1. – *If m is a practical number, and n is a positive integer not exceeding $\sigma(m)+1$, then mn is a practical number. In particular, for $1\leqslant n\leqslant 2m$, mn is practical.*

PROOF. – This lemma is a corollary of Stewart's structure theorem. See also [6, Corollaire 1]. QED

Let φ be the Euler totient function, and let ϕ_n be the cyclotomic polynomial for $\exp\{2\pi i/n\}$.

LEMMA 2. – *For every positive integer* $n > 1$, *we have*

$$\varphi(n)\log\frac{4}{\sqrt{3}} < \log\phi_n(3) < \varphi(n)\log\frac{9}{2}\,.$$

PROOF. – By [9, Satz], for every integer $n > 1$ one has

$$\left(\frac{16}{27}\right)^{2^{\nu(n)-2}} 3^{\varphi(n)} < \phi_n(3) < \left(\frac{3}{2}\right)^{2^{\nu(n)-1}} 3^{\varphi(n)}, \tag{1}$$

where $\nu(n)$ is the number of distinct prime factors of n. Note that, for $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$ with primes $q_1 < \cdots < q_k$ and positive integers $\alpha_1, \ldots, \alpha_k$, one has

$$2^{\nu(n)-1} = \overbrace{2\cdot 2\cdots 2}^{k-1 \text{ times}} \leqslant \varphi(q_2^{\alpha_2})\,\varphi(q_3^{\alpha_3})\cdots\varphi(q_k^{\alpha_k}) \leqslant \varphi(n)\,,$$

hence, by (1), the statement easily follows. QED

DEFINITION 2. – *Let* $(\mathcal{D}, \preccurlyeq)$ *be an ordered finite set of positive integers. We say that* $d \in \mathcal{D}$ *is admissible if*

$$\sum_{\delta \prec d} \varphi(\delta)\log\frac{4}{\sqrt{3}} > \varphi(d)\log\frac{9}{2}\,,$$

where, as usual, by $\delta \prec d$ *we mean* $\delta \preccurlyeq d$ *and* $\delta \neq d$.

Note that this definition depends on the arrangement of the elements of $\mathcal{D}$, and, when it will be opportune, we shall indicate the set $\mathcal{D}$ and the arrangement «$\preccurlyeq$» for which d is admissible.

LEMMA 3. – *Let* $(\mathcal{D}, \leqslant)$ *be a finite set of positive integers, ordered with the usual increasing order of positive integers. Suppose that* $d \in \mathcal{D}$ *is admissible for* $(\mathcal{D}, \leqslant)$. *Let* q *be a positive integer. Let* $\mathcal{D}(q)$ *be the set of its divisors and* $\mathcal{D}' = \mathcal{D}(q)\cdot\mathcal{D}$. *Then* qd *is admissible for* $(\mathcal{D}', \leqslant)$.

PROOF. – We can assume that q is a prime. Since d is admissible for $(\mathcal{D}, \leqslant)$, there exist $d_1, \ldots, d_l$ with $\max_{1 \leqslant i \leqslant l}\{d_i\} < d$ such that

$$\sum_{i=1}^{l} \varphi(d_i)\log\frac{4}{\sqrt{3}} > \varphi(d)\log\frac{9}{2}\,.$$

We can assume that $(d_i, q) = 1$ for $i \leqslant h$, and that $q \mid d_i$ for $i > h$. Now we take $l + h$ terms of $\mathcal{D}'$ smaller than dq as follows: for $1 \leqslant i \leqslant h$ we take d_i and qd_i.

Notice that

$$\varphi(d_i) + \varphi(qd_i) = q\varphi(d_i).$$

For $i > h$ we take qd_i. In this case

$$\varphi(qd_i) = q\varphi(d_i).$$

Since q is a prime, $d_1, d_2, \ldots, d_h, qd_1, qd_2, \ldots, qd_l$ are distinct and smaller than qd. Further

$$\left(\sum_{i=1}^{l} \varphi(qd_i) + \sum_{i=1}^{h} \varphi(d_i) \right) \log \frac{4}{\sqrt{3}} = q \sum_{i=1}^{l} \varphi(d_i) \log \frac{4}{\sqrt{3}} >$$

$$q\varphi(d) \log \frac{9}{2} \geqslant \varphi(dq) \log \frac{9}{2},$$

and this proves the admissibility of dq for $(\mathcal{D}', \leqslant)$. QED

LEMMA 4. – *Let M be a positive integer and let $(\mathcal{D}, \leqslant)$ be an ordered finite set of positive integers. Suppose that $M \cdot \prod_{\delta < d} \phi_\delta(3)$ is practical and that for $\delta \geqslant d$, δ is admissible. Then $M \cdot \prod_{\delta \in \mathcal{D}} \phi_\delta(3)$ is practical.*

PROOF. – We prove this lemma by finite induction. Let $b \geqslant d$ and suppose that $M \cdot \prod_{\delta < b} \phi_\delta(3)$ is practical. Our aim is to show that $M \cdot \prod_{\delta \leqslant b} \phi_\delta(3)$ is practical. We have

$$M \cdot \prod_{\delta \leqslant b} \phi_\delta(3) = M \cdot \prod_{\delta < b} \phi_\delta(3) \cdot \phi_b(3).$$

Since b is admissible, one has

$$\log \phi_b(3) < \varphi(b) \log \frac{9}{2} < \sum_{\delta < b} \varphi(\delta) \log \frac{4}{\sqrt{3}} <$$

$$\sum_{\delta < b} \log \phi_\delta(3) = \log \prod_{\delta < b} \phi_\delta(3) \leqslant \log \left(M \prod_{\delta < b} \phi_\delta(3) \right),$$

i.e., $\phi_b(3) \leqslant 2M \prod_{\delta < b} \phi_\delta(3)$, and, by Lemma 1, this completes the proof. QED

3. – Main result.

We now define two auxiliary sequences $\{m_n^{(e)}\}_{n \geqslant 1}$ and $\{m_n^{(o)}\}_{n \geqslant 1}$ of increasing positive integers. Let $\{p_n\}_{n \geqslant 1}$ be the increasing sequence of primes, and let

$$\begin{cases} m_1^{(e)} = 2 \\ m_2^{(e)} = 10 \qquad (= 2\cdot 5) \\ m_3^{(e)} = 110 \qquad (= 2\cdot 5\cdot 11) \\ m_n^{(e)} = \begin{cases} m_{n-1}^{(e)}\cdot p_{2n} & \text{if } m_{n-1}^{(e)} < m_{n-1}^{(o)} \text{ and } n \geqslant 4 \\ m_{n-1}^{(e)}\cdot p_{2n-1} & \text{if } m_{n-1}^{(e)} > m_{n-1}^{(o)} \text{ and } n \geqslant 4 \end{cases} \end{cases}$$

and

$$\begin{cases} m_1^{(o)} = 3 \\ m_2^{(o)} = 21 \qquad (= 3\cdot 7) \\ m_3^{(o)} = 273 \qquad (= 3\cdot 7\cdot 13) \\ m_n^{(o)} = \begin{cases} m_{n-1}^{(o)}\cdot p_{2n} & \text{if } m_{n-1}^{(o)} < m_{n-1}^{(e)} \text{ and } n \geqslant 4 \\ m_{n-1}^{(o)}\cdot p_{2n-1} & \text{if } m_{n-1}^{(o)} > m_{n-1}^{(e)} \text{ and } n \geqslant 4\,. \end{cases} \end{cases}$$

Remark that $\lim_{n\to\infty} m_n^{(e)}/m_n^{(o)} = 1$ and that $(m_n^{(e)}, m_n^{(o)}) = 1$ for every n. We can now prove the following

PROPOSITION 1. – *There exists an effectively computable constant c with $0 < c < 1$ such that for sufficiently large n and for every odd positive integer $r < c\min\{m_n^{(e)}, m_n^{(o)}\}$, the numbers*

$$\text{(i)}\quad 6\cdot(3^{rm_n^{(o)}} - 1) \qquad\qquad \text{(iii)}\quad 2\cdot(3^{rm_n^{(e)}} + 1)$$

$$\text{(ii)}\quad 2\cdot(3^{rm_n^{(e)}} - 1) \qquad\qquad \text{(iv)}\quad 6\cdot(3^{rm_n^{(o)}} + 1)$$

are all practical numbers.

PROOF. – We begin by proving the above proposition for $r = 1$. The proof is similar for each of the above four cases. We shall prove that, for each number (i), (ii), (iii), (iv) and for sufficiently large n, there exists an arrangement «$\preccurlyeq$» of divisors $\mathcal{D}_n$ (divisors of $m_n^{(o)}$, divisors of $m_n^{(e)}$, divisors of $2m_n^{(e)}$ which are not divisors of $m_n^{(e)}$, and divisors of $2m_n^{(o)}$ which are not divisors of $m_n^{(o)}$ respectively) and a finite set $\mathcal{A} \subseteq \mathcal{D}_n$, independent of n, and composed by suitable terms at the beginning of the arrangement of $\mathcal{D}_n$, such that every term of $\mathcal{D}_n - \mathcal{A}$ is admissible, and $M\cdot\prod_{d\in\mathcal{A}}\phi_d(3)$ is practical (with $M = 6$ in case (i) and (iv), and with $M = 2$ in case (ii) and (iii)). Since each number (i), (ii), (iii), (iv) is of the form $M\cdot\prod_{d\in\mathcal{D}_n}\phi_d(3)$ and $M\cdot\prod_{d\in\mathcal{A}}\phi_d(3)$ is practical, by Lemma 4 we achieve the proof.

(i) We have

$$6\cdot(3^{m_n^{(o)}}-1)=6\cdot\prod_{d\mid m_n^{(o)}}\phi_d(3).$$

Let $n>2$ and

$$A_1(n)=\prod_{\substack{d\mid m_n^{(o)}\\ d\leqslant 23}}\phi_d(3);\quad B_1(n)=\prod_{\substack{d\mid m_n^{(o)}\\ 23<d<m_n^{(o)}/3}}\phi_d(3);\quad C_1(n)=\phi_{m_n^{(o)}/3}(3)\cdot\phi_{m_n^{(o)}}(3).$$

We have $6(3^{m_n^{(o)}}-1)=6A_1(n)\,B_1(n)\,C_1(n)$. For sufficiently large n, $A_1(n)$ does not depend on n, since

$$A_1(n)=\phi_1(3)\,\phi_3(3)\,\phi_7(3)\,\phi_{13}(3)\,\phi_{17}(3)\,\phi_{21}(3)\,\phi_{23}(3).$$

Hence, for sufficiently large n

$$6A_1(n)=2^2\cdot 3\cdot 13\cdot 47\cdot 1093\cdot 1871\cdot 34511\cdot 368089\cdot 797161\cdot 1001523179,$$

which is a practical number by the structure theorem. The next step is to prove that $6A_1(n)\,B_1(n)$ is practical.

For $n=5,6,7,8$ one can directly check that every divisor d of $m_n^{(o)}$ with $17<d<m_n^{(o)}/3$ is admissible for the increasing arrangement of the divisors of $m_n^{(o)}$, hence, by Lemma 4, $6A_1(n)\,B_1(n)$ is practical. Let $n\geqslant 8$, and assume that there exists an arrangement «$\preccurlyeq$» of the divisors $\mathcal{D}_n$ of $m_n^{(o)}$ such that every divisor d with $17<d<m_n^{(o)}/3$ is admissible for $(\mathcal{D}_n,\preccurlyeq)$. Let $p=m_{n+1}^{(o)}/m_n^{(o)}$, and define the following arrangement, again denoted by «$\preccurlyeq$», of the divisors $\mathcal{D}_{n+1}$ of $m_{n+1}^{(o)}$. Note that $\mathcal{D}_{n+1}\supset\mathcal{D}_n$. First, we arrange the ordered finite sequence $\mathcal{D}_n$ excluding $m_n^{(o)}/3$ and $m_n^{(o)}$; then we arrange $p\mathcal{D}_n$ again excluding $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$; finally, we arrange the ordered set of the four numbers $m_n^{(o)}/3$, $m_n^{(o)}$, $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$.

For the first set of divisors d of $m_{n+1}^{(o)}$ it is obvious that every $d>17$ is admissible for $(\mathcal{D}_{n+1},\preccurlyeq)$ since d is admissible for $(\mathcal{D}_n,\preccurlyeq)$. By Lemma 3, this implies that, for the second set of divisors, every divisor of $m_{n+1}^{(o)}$ of the form dp with $d\mid m_n^{(o)}$ and $d>17$ (in this set $d<m_n^{(o)}/3<m_{n+1}^{(o)}/3$) is admissible. If a divisor of this set is of the form dp with $d\mid m_n^{(o)}$ and $d=1,3,7,13$ or 17 and $n\geqslant 8$, we have

$$\varphi(dp)\log\frac{9}{2}\leqslant 16\cdot(p-1)\log\frac{9}{2}<\left(m_n^{(o)}-\varphi(m_n^{(o)})-\varphi\left(\frac{m_n^{(o)}}{3}\right)\right)\log\frac{4}{\sqrt{3}}=$$

$$=\sum_{\substack{d'\mid m_n^{(o)}\\ d'<m_n^{(o)}/3}}\varphi(d')\log\frac{4}{\sqrt{3}},$$

and in our arrangement every d' such that $d' \mid m_n^{(o)}$, $d' < m_n^{(o)}/3$ precedes dp, hence dp is admissible.

In order to prove the admissibility of every divisor d of $m_{n+1}^{(o)}$ with $17 < d < m_{n+1}^{(o)}/3$ we need to prove that $m_n^{(o)}/3$ and $m_n^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preccurlyeq)$. Since $n \geqslant 8$ we have $p \geqslant 61$, hence $p - 1 > 6 \log \frac{9}{2} / \log \frac{4}{\sqrt{3}}$. This implies that

$$\varphi\left(\frac{m_n^{(o)}}{3}\right) \log \frac{9}{2} < \varphi(m_n^{(o)}) \log \frac{9}{2} < \varphi\left(\frac{m_n^{(o)}}{7} p\right) \log \frac{4}{\sqrt{3}},$$

i.e., both $m_n^{(o)}/3$ and $m_n^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preccurlyeq)$.

To complete the proof of (i) we now prove that for sufficiently large n, $m_{n+1}^{(o)}/3$ and $m_{n+1}^{(o)}$ are admissible for $(\mathcal{D}_{n+1}, \preccurlyeq)$, so by Lemma 4, $6 \cdot (3^{m_{n+1}^{(o)}} - 1)$ is practical. In fact, since $\varphi(m_n^{(o)} p) = o(m_n^{(o)} p)$, for sufficiently large n we have

$$\sum_{\substack{d \mid m_{n+1}^{(o)} \\ d < m_{n+1}^{(o)}/3}} \varphi(d) \log \frac{4}{\sqrt{3}} = \left(m_n^{(o)} p - \varphi(m_n^{(o)} p) - \varphi\left(\frac{m_n^{(o)} p}{3}\right)\right) \log \frac{4}{\sqrt{3}} >$$

$$\varphi(m_n^{(o)} p) \log \frac{9}{2} = \max\left\{\varphi\left(\frac{m_n^{(o)} p}{3}\right), \varphi(m_n^{(o)} p)\right\} \log \frac{9}{2},$$

as required.

(ii) We have

$$2 \cdot (3^{m_n^{(e)}} - 1) = 2 \cdot \prod_{d \mid m_n^{(e)}} \phi_d(3).$$

Let $n > 2$ and

$$A_2(n) = \prod_{\substack{d \mid m_n^{(e)} \\ d \leqslant 29}} \phi_d(3); \quad B_2(n) = \prod_{\substack{d \mid m_n^{(e)} \\ 29 < d < m_n^{(e)}/2}} \phi_d(3); \quad C_2(n) = \phi_{m_n^{(e)}/2}(3) \cdot \phi_{m_n^{(e)}}(3),$$

hence $2(3^{m_n^{(e)}} - 1) = 2A_2(n)\, B_2(n)\, C_2(n)$. For sufficiently large n, $A_2(n)$ does not depend on n, since

$$A_2(n) = \phi_1(3)\, \phi_2(3)\, \phi_5(3)\, \phi_{10}(3)\, \phi_{11}(3)\, \phi_{19}(3)\, \phi_{22}(3)\, \phi_{29}(3).$$

Hence, for sufficiently large n

$$2A_2(n) = 2^4 \cdot 11^2 \cdot 23 \cdot 59 \cdot 61 \cdot 67 \cdot 661 \cdot 1597 \cdot 3851 \cdot 28537 \cdot 363889 \cdot 20381027,$$

which is a practical number by the structure theorem. The remaining part of (ii) is similar to (i).

(iii) We have

$$2\cdot(3^{m_n^{(e)}}+1)=2\cdot\prod_{\substack{d\mid 2m_n^{(e)}\\ d\nmid m_n^{(e)}}}\phi_d(3).$$

Let $n>3$ and

$$A_3(n)=\prod_{\substack{d\mid 2m_n^{(e)}\\ d\nmid m_n^{(e)}\\ d\leqslant 148}}\phi_d(3);\quad B_3(n)=\prod_{\substack{d\mid 2m_n^{(e)}\\ d\nmid m_n^{(e)}\\ 148<d<2m_n^{(e)}/5}}\phi_d(3);\quad C_3(n)=\phi_{2m_n^{(e)}/5}(3)\cdot\phi_{2m_n^{(e)}}(3),$$

hence $2(3^{m_n^{(e)}}+1)=2A_3(n)\,B_3(n)\,C_3(n)$. For sufficiently large n, $A_3(n)$ does not depend on n, since $A_3(n)=\phi_4(3)\,\phi_{20}(3)\,\phi_{44}(3)\,\phi_{76}(3)\,\phi_{116}(3)\,\phi_{148}(3)$. Hence, for sufficiently large n

$$2A_3(n)=2^2\cdot5^2\cdot149\cdot1181\cdot5501\cdot12413\cdot570461\cdot953861\cdot5301533\cdot$$
$$25480398173\cdot3794512766652900052301\!3\cdot142659759801404920771391593\,,$$

which is a practical number by the structure theorem. The remaining part of (iii) is similar to the preceding cases.

(iv) We have

$$6\cdot(3^{m_n^{(o)}}+1)=6\cdot\prod_{\substack{d\mid 2m_n^{(o)}\\ d\nmid m_n^{(o)}}}\phi_d(3).$$

Let $n>2$ and

$$A_4(n)=\prod_{\substack{d\mid 2m_n^{(o)}\\ d\nmid m_n^{(o)}\\ d\leqslant 34}}\phi_d(3);\quad B_4(n)=\prod_{\substack{d\mid 2m_n^{(o)}\\ d\nmid m_n^{(o)}\\ 34<d<2m_n^{(o)}/3}}\phi_d(3);\quad C_4(n)=\phi_{2m_n^{(o)}/3}(3)\cdot\phi_{2m_n^{(o)}}(3),$$

hence $6(3^{m_n^{(o)}}+1)=6A_4(n)\,B_4(n)\,C_4(n)$. For sufficiently large n, $A_4(n)$ does not depend on n, since $A_4(n)=\phi_2(3)\,\phi_6(3)\,\phi_{14}(3)\,\phi_{26}(3)\,\phi_{34}(3)$. Hence, for sufficiently large n

$$6A_4(n)=2^3\cdot3\cdot7\cdot103\cdot307\cdot547\cdot1021\cdot398581\,,$$

which is a practical number by the structure theorem. The remaining part of (iv) is similar to the preceding cases.

We incidentally provided a second proof of the existence of infinitely many triplets of practical numbers of the form $(m-2,m,m+2)$ with $m=2\cdot3^{m_n^{(e)}}$.

The above arguments are suitable to complete the proof. Whenever $r>1$ is odd, the divisors of $2rm_n^{(e)}$ [$2rm_n^{(o)}$] which are not divisors of $rm_n^{(e)}$ [$rm_n^{(o)}$] contain the divisors of $2m_n^{(e)}$ [$2m_n^{(o)}$] which are not divisors of $m_n^{(e)}$ [$m_n^{(o)}$]. Further, if $\max\{p\mid r\}/m_n^{(e)}$ is sufficiently small, we can prove that (i), (ii), (iii),

(iv) are practical numbers. The computation of the constant c which suffices for our aims is not much important in our opinion, and we omit it. QED

We are ready to prove the following

THEOREM 1. – *At least one of the two following statements holds*:

(a) *There exist only finitely many pairs* $(m_n^{(e)}, m_n^{(o)})$ *such that the Diophantine equation*

$$xm_n^{(e)} - ym_n^{(o)} = 1$$

has a solution in odd integers x, y *and* $0 < x, y < c \min\{m_n^{(e)}, m_n^{(o)}\}$, *where* c *is defined as above.*

(b) *There exist infinitely many* 5-*tuples of practical numbers of the form* $(m-6, m-2, m, m+2, m+6)$.

PROOF. – Suppose that for infinitely many n there exist odd integers x_n, y_n such that $0 < x_n, y_n < c \min\{m_n^{(e)}, m_n^{(o)}\}$ and $x_n m_n^{(e)} - y_n m_n^{(o)} = 1$. Then, for sufficiently large n, the numbers $6(3^{y_n m_n^{(o)}} - 1)$, $2(3^{x_n m_n^{(e)}} - 1)$, $2(3^{x_n m_n^{(e)}} + 1)$, $6(3^{y_n m_n^{(o)}} + 1)$ are practical numbers by Proposition 1. Hence, for $m = 2 \cdot 3^{x_n m_n^{(e)}}$, the numbers $m - 6 = 6(3^{y_n m_n^{(o)}} - 1)$, $m - 2 = 2(3^{x_n m_n^{(e)}} - 1)$, m, $m + 2 = 2(3^{x_n m_n^{(e)}} + 1)$ and $m + 6 = 6(3^{y_n m_n^{(o)}} + 1)$ are practical numbers. QED

We remark that statistical arguments suggest that (a) should be false, although a proof appears to be difficult at first sight.

Acknowledgements. I wish to thank the referee for his/her helpful comments and suggestions.

REFERENCES

[1] P. ERDÖS, *On a Diophantine equation*, Mat. Lapok, **1** (1950), 192-210.
[2] P. ERDÖS, personal communication, 1996 III 19.
[3] G. H. HARDY - E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford (1980).
[4] M. HAUSMAN - H. N. SHAPIRO, *On practical numbers*, Comm. Pure Appl. Math., **37** (1984), 705-713.
[5] M. MARGENSTERN, *Results and conjectures about practical numbers*, C. R. Acad. Sc. Paris, **299** (1984), 895-898.
[6] M. MARGENSTERN, *Les nombres pratiques: théorie, observations et conjectures*, J. Number Theory, **37** (1991), 1-36.

[7] G. MELFI, *On two conjectures about practical numbers*, J. Number Theory, **56** (1996), 205-210.
[8] G. MELFI, *A survey on practical numbers*, Rend. Sem. Mat. Uni. Pol. Torino, **53** (1995), 347-359.
[9] B. RICHTER, *Eine Abschätzung der Werte der Kreisteilungspolynome für reelles Argument*, J. Reine. Angew. Math., **267** (1974), 74-76.
[10] E. SAIAS, *Entiers à diviseurs denses 1*, J. Number Theory, **62** (1997), 163-191.
[11] A. K. SRINIVASAN, *Practical numbers*, Curr. Sci., **6** (1948), 179-180.
[12] B. M. STEWART, *Sums of distinct divisors*, Amer. J. Math., **76** (1954), 779-785.
[13] G. TENENBAUM, *Sur un problème de crible et ses applications*, Ann. Sci. Ec. Norm. Sup. (4), **19** (1986), 1-30.
[14] G. TENENBAUM, *Sur un problème de crible et ses applications*, 2. *Corrigendum et étude du graphe divisoriel*, Ann. Sci. Ec. Norm. Sup. (4), **28** (1995), 115-127.

Université de Lausanne, Institut de Mathématiques
CH-1015 Lausanne, Switzerland
E-mail: Giuseppe.Melfi@ima.unil.ch

Pervenuta in Redazione
il 2 novembre 1998