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On the Asymptotic Expansion of the Airy Function.

FAUSTO SEGALA

Sunto. – *Si prova una nuova formula di rappresentazione per la famosa funzione di Airy. Ne viene data applicazione per la determinazione di certi bounds significativi per la funzione stessa.*

1. – Introduction.

The famous Airy function is defined by

$$\mathrm{Ai}(r) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{(z^3/3 - rz)}\, dz$$

and solves the Airy equation $u'' = ru$.

The Airy function (or the Airy integral) has many and very deep applications to mathematics, physics and engineering.

In 1963 Copson [1] gave the following representation of the Airy function for $r > 0$:

$$\mathrm{Ai}(r) = \frac{1}{2\sqrt{\pi}} r^{-1/4} e^{-2/3\, r^{3/2}} q(r) \tag{1.1}$$

with

$$q(r) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \cos\left(\frac{r^{-3/4} t^3}{3}\right) dt\,. \tag{1.2}$$

Since $q(r) \to 1$ per $r \to \infty$, by (1.1) it follows

$$\mathrm{Ai}(r) \sim \frac{1}{2\sqrt{\pi}} r^{-1/4} e^{-2/3 r^{3/2}}, \qquad r \to \infty\,. \tag{1.3}$$

In this note we give a new representation of $q(r)$. Precisely we prove that

$q(r)$ can be written in the form

$$q(r) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} k\left(\frac{t}{r^{3/4}}\right) dt \tag{1.4}$$

where $k(t)$ has the graph in the picture below:

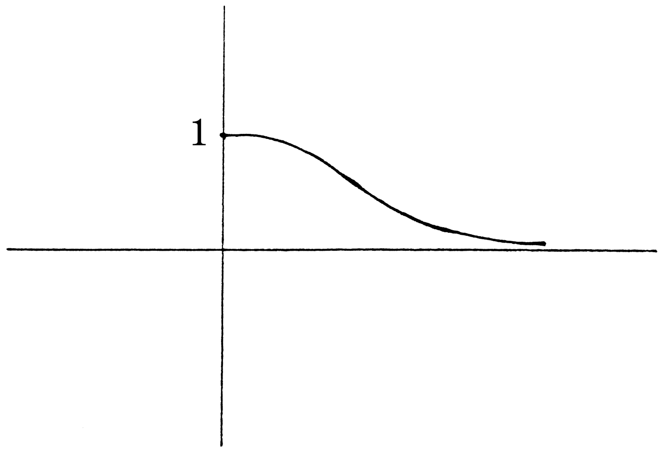

Fig. 1.

As a consequence, $q(r)$ has the graph in the next picture:

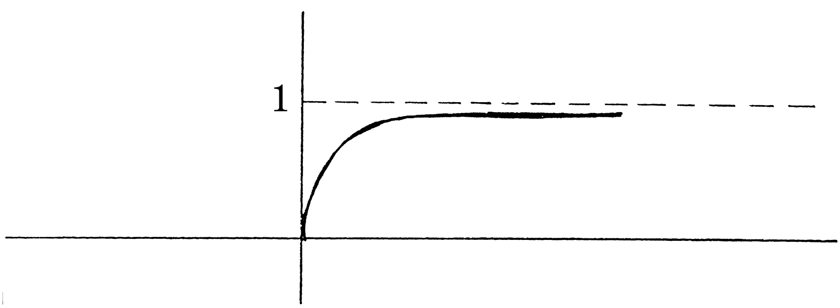

Fig. 2.

From (1.4) it follows in an absolutely elementary way, some fundamental well known properties of the Airy function for positive argument. In fact, by (1.4) we have $\mathrm{Ai}(r) > 0$ on $[0, \infty[$ and (since $\mathrm{Ai}''(r) = r\,\mathrm{Ai}(r)$), $\mathrm{Ai}''(r) > 0$ on $]0, \infty[$. Therefore $\mathrm{Ai}'(r)$ is increasing and since $\mathrm{Ai}'(r) \to 0$ when $r \to \infty$, we obtain $\mathrm{Ai}'(r) < 0$ on $[0, \infty[$. In conclusion, $\mathrm{Ai}(r)$ is positive and decreasing on $[0, \infty[$.

In the section 2 we will prove the formula (1.4) and in the section 3 we will make use of (1.4) in order to obtain bounds for the Airy function and its derivative on the real positive axis.

For small values of the argument, our bounds are better than well known classical asymptotic bounds of the Airy function (see for example Olver [2]).

The proof of (1.4) is an application of the Riemann method of steepest descent.

2. – The proof.

By the change of variable $z = \sqrt{r}\,w$, we can write

$$\mathrm{Ai}(r) = \frac{\sqrt{r}}{2\pi i}\int_{-i\infty}^{+i\infty} e^{r^{3/2}(w^3/3 - w)}\,dw\,. \tag{2.1}$$

We observe that 1 is a critical point of $\phi(w) = w^3/3 - w$ and $\phi(1) = -2/3$. We prove there exists a curve $w = w(t)$ such that $\phi(w(t)) = -2/3 - t^2$, that is

$$w^3 - 3w = 2 - 3t^2. \tag{2.2}$$

By setting $w = x + iy$, we obtain

$$x^3 - 3xy^2 - 3x = -2 - 3t^2, \tag{2.3}$$

$$y(3x^2 - y^2 - 3) = 0 \tag{2.4}$$

and therefore

$$t^2 = \frac{2}{3}(4x^3 - 3x - 1)\,. \tag{2.5}$$

For $t > 0$, the equation (2.5) has one root $x(t) > 1$.

We take $y(t) = \sqrt{3x(t)^2 - 3}$, $w(t) = x(t) \pm iy(t)$.

Therefore if we integrate along $w = w(t)$, from (2.2) we have

$$\mathrm{Ai}(r) = \frac{r^{1/2}}{\pi} e^{-2/3 r^{3/2}} \int_0^\infty e^{-r^{3/2}t^2} y'(t)\,dt =$$

$$\frac{r^{-1/4}}{\pi} e^{-2/3 r^{3/2}} \int_0^\infty e^{-t^2} y'\left(\frac{t}{r^{3/4}}\right) dt = \frac{r^{-1/4}}{2\sqrt{\pi}}\,\frac{2}{\sqrt{\pi}}\, e^{-2/3 r^{3/2}} \int_0^\infty e^{-t^2} k\left(\frac{t}{r^{3/4}}\right) dt$$

where $k(t) = y'(t)$.

From (2.3),(2.4) it follows

$$x' = \frac{t}{4x^2-1}\,, \qquad y' = \frac{3xx'}{y}\,, \tag{2.6}$$

$$y'^2 = \frac{9x^2x'^2}{y^2} = \frac{3x^2x'^2}{x^2-1}\,. \tag{2.7}$$

By inserting (2.5) in the first equation (2.6), we get

$$x'^2 = \frac{8x^3-6x-2}{3(4x^2-1)^2}\,. \tag{2.8}$$

Now, by (2.7) and (2.8), one obtains

$$y'^2 = \frac{x^2}{x^2-1}\,\frac{8x^3-6x-2}{(4x^2-1)^2} = \frac{2x^2(4x^2+4x+1)}{(4x^2-1)^2(x+1)} = \frac{2x^2}{(2x-1)^2(x+1)}\,.$$

Put

$$F(x) = \frac{\sqrt{2}\,x}{(2x-1)\sqrt{x+1}} = \frac{\sqrt{2}x}{(4x^3-3x+1)^{1/2}}\,.$$

By a simple calculation we have

$$F'(x) = -\frac{1}{\sqrt{2}}\,\frac{2x^2+x+2}{(2x-1)^2(x+1)^{3/2}} < 0\,, \qquad x>1\,.$$

Finally, by taking into account that

$$y'(t) = F(x(t))\,, \qquad y''(t) = F'(x(t))\,x'(t)\,,$$

we conclude that y'' is negative and therefore $k = y'$ is decreasing, $k(0) = F(x(0)) = F(1) = 1$.

3. – Bounds of the Airy function.

From (1.4) it follows

$$q(r) = \frac{2}{\sqrt{\pi}}\,3^{-1/6}r^{1/4}\int_0^\infty e^{-t^2}\psi\left(\frac{t}{r^{3/4}}\right)t^{-1/3}\,dt \tag{3.1}$$

with

$$\psi(t) = 3^{1/6}k(t)t^{1/3}. \tag{3.2}$$

In section 2 we proved that $k(t) = F(x(t))$ with

$$F(x) = \frac{\sqrt{2}\,x}{(4x^3 - 3x + 1)^{1/2}}\,. \tag{3.3}$$

By (2.5), we have

$$t = \frac{\sqrt{2}}{\sqrt{3}}(4x^3 - 3x - 1)^{1/2} \tag{3.4}$$

and therefore, after some easy calculations, by using (3.2), (3.3) and (3.4) we get

$$\psi(t) = G(x(t)) \tag{3.5}$$

with

$$G(x) = \frac{x(x^3 - 3/4x - 1/4)^{1/6}}{(x^3 - 3/4x + 1/4)^{1/2}}\,. \tag{3.6}$$

If we put $L = \max\limits_{[1,\,\infty[} G$, we can write

$$r^{-1/4} q(r) \leqslant L\frac{2}{\sqrt{\pi}}3^{-1/6}\int_0^\infty e^{-t^2} t^{-1/3}\,dt = L\frac{3^{-1/6}}{\sqrt{\pi}}\Gamma\left(\frac{1}{3}\right) = 2\sqrt{\pi}\,L\,\mathrm{Ai}(0)$$

and since $L = 1.067, \ldots,$ we conclude with the estimate

$$\frac{1}{2\sqrt{\pi}} r^{-1/4} q(r) \leqslant (1.067, \ldots)\,\mathrm{Ai}(0), \qquad r > 0\,.$$

On the other hand, from (3.1) (or from (1.1)) it follows

$$\lim_{r \to 0} \frac{1}{2\sqrt{\pi}} r^{-1/4} q(r) = \mathrm{Ai}(0)\,.$$

Hence we have

$$\mathrm{Ai}(0) \leqslant \sup_{[0,\,\infty[}\left[\frac{1}{2\sqrt{\pi}} r^{-1/4} q(r)\right] \leqslant (1.067, \ldots)\,\mathrm{Ai}(0)\,. \tag{3.7}$$

Now we examine $\mathrm{Ai}'(r)$. From the calculations developed in section 2, we have

$$\mathrm{Ai}'(r) = -\frac{r^{1/4}}{2\sqrt{\pi}} e^{-2/3\,r^{3/2}} p(r) \tag{3.8}$$

with

$$p(r) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} h\left(\frac{t}{r^{3/4}}\right) dt \tag{3.9}$$

and

$$h(t) = \frac{d}{dt}(xy)(t) = H(x(t)), \tag{3.10}$$

$$H(x) = \sqrt{2}\,\frac{2x^2-1}{\sqrt{x+1}(2x-1)}. \tag{3.11}$$

By setting

$$R(x) = \frac{H(x)}{\sqrt{2x}}, \qquad \mu(t) = \frac{\sqrt{2}}{3^{1/6}}\,\frac{\sqrt{x(t)}-1}{t^{1/3}}, \tag{3.12}$$

from (3.9) we obtain

$$p(r) = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) \sqrt{x\left(\frac{t}{r^{3/4}}\right)}\, dt = \tag{3.13}$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) dt + \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right)\left[\sqrt{x\left(\frac{t}{r^{3/4}}\right)}-1\right] dt =$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) dt +$$

$$\frac{2\cdot 3^{1/6}}{\sqrt{\pi}}\, r^{-1/4} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) \frac{\sqrt{x\left(\frac{t}{r^{3/4}}\right)}-1}{(3^{1/6}/\sqrt{2})(t/r^{3/4})^{1/3}}\, t^{1/3}\, dt =$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) dt + \frac{2\cdot 3^{1/6}}{\sqrt{\pi}}\, r^{-1/4} \int_0^\infty e^{-t^2} R\left(x\left(\frac{t}{r^{3/4}}\right)\right) \mu\left(\frac{t}{r^{3/4}}\right) t^{1/3} dt.$$

By (3.12) and (3.4) one has

$$\mu(t) = Q(x(t)) \tag{3.14}$$

with

$$Q(x) = \frac{\sqrt{x} - 1}{(x^3 - 3/4x - 1/4)^{1/6}} .$$

We have the following two estimates about R and Q:

$$\sup_{[1,\ \infty[} R(x) = \lim_{x \to \infty} R(x) = 1 , \tag{3.15}$$

$$\sup_{[1,\ \infty[} Q(x) = \lim_{x \to \infty} Q(x) = 1 . \tag{3.16}$$

We give the proof of (3.16).

$Q(x^2) = (x-1)/(x^6 - 3/4x^2 - 1/4)^{1/6} < 1$ on $[1,\ \infty[$ is equivalent to $f(x) > 0$ on $[0,\ \infty[$, where

$$f(x) = 24x^5 - 60x^4 + 80x^3 - 63x^2 + 24x - 5 .$$

On the other hand, $f(1) = 0$, $f'(1) = 18$, $f''(1) = 114$,

$$f'''(x) = 480(3x^2 - 3x + 1) > 0 , \qquad \forall x \in \boldsymbol{R} .$$

From (3.13), (3.15), (3.16) we obtain

$$p(r) \leqslant \sqrt{2} + \frac{2 \cdot 3^{1/6}}{\sqrt{\pi}} r^{-1/4} \int_0^\infty e^{-t^2} t^{1/3}\, dt =$$

$$\sqrt{2} + \frac{2\sqrt{\pi} 3^{-1/3}}{\Gamma(1/3)} r^{-1/4} = \sqrt{2} + 2\sqrt{\pi}\,|\mathrm{Ai}'(0)|\, r^{-1/4},$$

that is

$$\frac{r^{1/4} p(r)}{\sqrt{2\pi}} \leqslant \frac{1}{\sqrt{2\pi}} r^{1/4} + |\mathrm{Ai}'(0)| . \tag{3.17}$$

Finally (3.17) and (3.8) imply the estimate

$$|\mathrm{Ai}'(r)| \leqslant |\mathrm{Ai}'(0)|(1 + \alpha r^{1/4})\, e^{-2/3 r^{3/2}}, \qquad r \geqslant 0 , \tag{3.18}$$

with $\alpha = 1/|\mathrm{Ai}'(0)|\sqrt{2\pi}$.

The estimate (3.18) is, in some sense, the best possible in virtue of (3.15) and (3.16). We may compare (3.18) with the known formula

$$|\mathrm{Ai}'(r)| \leqslant \frac{1}{2\sqrt{\pi}} \left(r^{1/4} + \frac{7}{48 r^{3/2}} \right) e^{-2/3 r^{3/2}} \tag{3.19}$$

which is the best possible for $r \to \infty$.

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