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Polynomial Identities of Nil Algebras of Bounded Index.

FRANCESCA BENANTI (*) - VESSELIN DRENSKY (**)

Sunto. – *Lo scopo di questo lavoro è di dare una nuova descrizione del* T*-ideale generato dalla nil-identità $x^n = 0$ come immagine omeomorfa della n-esima potenza tensoriale simmetrica dell'algebra associativa libera $K\langle X\rangle$ su un campo K di caratteristica 0. Come applicazione calcoliamo il carattere delle conseguenze multilineari di grado $\leqslant n+2$ dell'identità $x^n = 0$.*

Introduction.

Let K be a field of characteristic 0. The classical Nagata-Higman theorem [10, 14] states that the polynomial identity $x^n = 0$ implies the identity of nilpotency $x_1 \ldots x_{p(n)} = 0$. In the 40's this theorem was also established by Dubnov and Ivanov [7] but completely overlooked by the mathematical community. The proofs of many results on the structure of PI-algebras involve essentially the Dubnov-Ivanov-Nagata-Higman theorem and these results would have quantitative character if one knows the exact value of the class of nilpotency $p(n)$. The best known bounds for $p(n)$

$$\frac{n(n+1)}{2} \leqslant p(n) \leqslant n^2$$

are due respectively to Kuz'min [13] and Razmyslov [18]. The only exact values of $p(n)$ are known for $n \leqslant 3$ (Dubnov [6]) and $n = 4$ (Vaughan-Lee [21]):

$$p(1) = 1\,, \qquad p(2) = 3\,, \qquad p(3) = 6\,, \qquad p(4) = 10\,.$$

Hence for $n \leqslant 4$ the class of nilpotency reaches the bound of Kuz'min and there is a conjecture that the same holds for any n.

There is a nice relation between the Dubnov-Ivanov-Nagata-Higman theorem and classical invariant theory (we refer to the book by Formanek [9] for details). Let R_n be the algebra generated by the generic $n \times n$ matrices $y_r = \sum_{i,j=1}^{n} \xi_{ij}^{(r)} e_{ij}$, $r = 1, 2, \ldots$, where $\xi_{ij}^{(r)}$ are commuting variables and e_{ij} are

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the matrix units. The general linear group $GL_n(K)$ acts by conjugation on the generic matrices which induces an action on the polynomial algebra

$$\Omega_n = K[\xi_{ij}^{(r)} \,|\, i, j = 1, \ldots, n, \quad r = 1, 2, \ldots].$$

By the Razmyslov-Procesi theory (see Razmyslov [18, Final remark], Procesi [16, Theorem 3.3], [17, Theorem 4.3] and Formanek [8, Theorem 6]), the algebra of invariants $\Omega_n^{GL_n(K)}$ is generated by the set of traces $tr(y_{r_1} \cdots y_{r_k})$ of degree k bounded by the class of nilpotency $p(n)$ of the nil algebras of index n. The bound $k \leqslant p(n)$ is exact. Minimal sets of generators for the invariants of $n \times n$ matrices are known for small n only. The case $n = 2$ has been handled by Siberskii [19]. Abeasis and Pittaluga [1] have suggested an algorithm for finding a minimal set of generators for $\Omega_n^{GL_n(K)}$. Combining computers with calculations by hand they have successfully applied this algorithm to the case of 3×3 matrices.

The purpose of our paper is the better understanding of the T-ideal $\langle x^n \rangle^T$ generated by x^n in the free associative nonunitary algebra $A = K\langle X \rangle$ where $X = \{x_1, x_2, \ldots\}$ is a countable set of variables. Our approach is based on representation theory of the general linear group $GL_m(K)$ which, for applications to PI-algebras is equivalent to the representation theory of S_n (see [4] and Berele [3]). For m fixed, the group $GL_m(K)$ acts canonically on the free algebra $A_m = K\langle x_1, \ldots, x_m \rangle$ in the same way as it acts on the tensor algebra of the m-dimensional vector space with basis $x_1, \ldots, x_m$. Since the T-ideals of A_m are $GL_m(K)$-invariant, a lot of information for $\langle x^n \rangle^T$ can be obtained from its $GL_m(K)$-module structure. *Our first result describes the $GL_m(K)$-module $A_m \cap \langle x^n \rangle^T$ as a homomorphic image of the n-th symmetrized tensor power of A_m.*

As an application we have computed the S_k-character of the multilinear consequences of degree $k \leqslant n + 2$ of the identity $x^n = 0$ and found explicit generators of the irreducible components. In particular, *for $n \geqslant 3$*

$$\chi_{S_{n+1}}(V_{n+1} \cap \langle x^n \rangle^T) = \chi(n+1) + 2\chi(n, 1) + \chi(n-1, 2) + \chi(n-1, 1^2)$$

and for $n \geqslant 6$

$$\begin{aligned}\chi_{S_{n+2}}(V_{n+2} \cap \langle x^n \rangle^T) = \chi(n+2) + 4\chi(n+1, 1) + 5\chi(n, 2) + 5\chi(n, 1^2) + \\ 3\chi(n-1, 3) + 6\chi(n-1, 2, 1) + 3\chi(n-1, 1^3) + \chi(n-2, 4) + \\ \chi(n-2, 3, 1) + 2\chi(n-2, 2^2) + \chi(n-2, 2, 1^2) + \chi(n-2, 1^4),\end{aligned}$$

where V_k is the set of all multilinear polynomials of degree k in the free associative algebra.

It turns out that for $k \leqslant n + 2$ the exact values of the multiplicities of the irreducible S_k-characters in the character sequence of $\langle x^n \rangle^T$ coincide with or are very close to the estimates obtained from the description of $\langle x^n \rangle^T$ as a symmetrized tensor power. In order to calculate the S_k-characters we have developed further the approach from our recent paper [2] where we have found the S_{n+2}-character of the multilinear consequences of degree $n+2$ of the standard polynomial of degree n. As in [2], we have used results of Thrall [20] on plethysms and, especially, on symmetrized tensor powers of irreducible $GL_m(K)$-modules. The complete calculations, including the list of the highest weight vectors of the consequences of degree $n+2$ of $x^n = 0$, are available by request as a Plain TEX file and as a preprint. Since our considerations are based on algorithmic ideas, we believe that one can use them for further investigations. For example one can try to find the exact value of the class of nilpotency of algebras satisfying the identity $x^5 = 0$ and to confirm the above mentioned conjecture for $n = 5$.

1. – Preliminaries.

Let K be a fixed field of characteristic 0. We consider associative, nonunitary K-algebras only. All vector spaces and tensor products are also over K. We denote by $A = K\langle X \rangle$ the free associative algebra of countable rank with a set of generators $X = \{x_1, x_2, \ldots\}$. We consider the free algebra $A_m = K\langle x_1, \ldots, x_m \rangle$ of rank m as a subalgebra of A. An ideal U of A is a T-*ideal* if it is invariant under all endomorphisms of A. We denote by $\langle f \rangle^T$ the T-ideal of A generated by $f \in A$.

The algebras A and A_m are graded, $A = \sum A^{(k)}$, where $A^{(k)}$ is the space of all homogeneous polynomials of degree k and $A_m = \sum A_m^{(k)}$ with $A_m^{(k)} = A_m \cap A^{(k)}$. We shall also use the multigrading $A_m = \sum A_m^{(k_1, \ldots, k_m)}$, where $A_m^{(k_1, \ldots, k_m)}$ is the component homogeneous of degree k_i in each variable x_i. In our considerations, all graded vector spaces are subspaces or tensor products of subspaces of A and A_m with these particular gradings. It is well known that every T-ideal U is a graded subspace of A and $A_m \cap U$ is multigraded.

The general linear group $GL_m = GL_m(K)$ acts canonically from the left on the vector space with basis $\{x_1, \ldots, x_m\}$. This action is extended diagonally on A_m by

$$g(x_{i_1} \ldots x_{i_k}) = g(x_{i_1}) \ldots g(x_{i_k}), \qquad g \in GL_m, \qquad x_{i_1} \ldots x_{i_k} \in A_m,$$

and $A_m \cap U$ is a GL_m-submodule of A_m for any T-ideal U. Similarly, if $V_k = A_k^{(1, \ldots, 1)}$, $k = 1, 2, \ldots$, is the vector space of all multilinear polynomials of degree k in A_k, the symmetric group S_k acts from the left on V_k by

$$\sigma(x_{i_1} \ldots x_{i_k}) = x_{\sigma(i_1)} \ldots x_{\sigma(i_k)}, \qquad \sigma \in S_k, \quad x_{i_1} \ldots x_{i_k} \in V_k.$$

Every T-ideal U of A is generated by its multilinear elements and $V_k \cap U$ is an

S_k-submodule of V_k, $k = 1, 2, \ldots$ The S_k-character

$$\chi_k(U) = \chi_{S_k} V_k/(V_k \cap U), \qquad k = 1, 2, \ldots,$$

is the *k-th cocharacter* of U.

For a background on representation theory of the symmetric and the general linear groups we refer to [11], [14] or [23]. Recall that $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a partition of k (notation $\lambda \vdash k$) if $\lambda_1 \geqslant \ldots \geqslant \lambda_p \geqslant 0$ and $\lambda_1 + \ldots + \lambda_p = k$. The irreducible representations of S_k and the irreducible polynomial representations of GL_m are described by partitions. We denote respectively by $M(\lambda)$, $\chi(\lambda)$ and $W(\lambda)$ the irreducible S_k-module, its character and the irreducible GL_m-module related with λ, assuming that $W(\lambda) = 0$ if $\lambda_{m+1} > 0$. The modules $M(\lambda)$ and $W(\lambda)$ are isomorphic to submodules of V_k and $A_m^{(k)}$, respectively.

By a result of Berele [3] and one of the authors [4], for any T-ideal U the actions of S_k on $V_k \cap U$ and of GL_m on $A_m^{(k)} \cap U$, $k = 1, 2, \ldots$, are equivalent and if the S_k-module $V_k \cap U$ is isomorphic to $\sum_\lambda m(\lambda) M(\lambda)$, $m(\lambda) \in \mathbb{N} \cup \{0\}$, $\lambda \vdash k$, then the GL_m-module $A_m^{(k)} \cap U$ is isomorphic to $\sum_\lambda m(\lambda) W(\lambda)$ with the same multiplicities $m(\lambda)$, for each $\lambda = (\lambda_1, \ldots, \lambda_m)$.

Let $\lambda_{m+1} = 0$. Up to a multiplicative constant there exists a unique generator w_λ of $W(\lambda)$ which is multihomogeneous of degree $(\lambda_1, \ldots, \lambda_m)$. This element is called the *highest weight vector* of $W(\lambda)$ and can be described in the following way. The symmetric group S_k acts from the right on the homogeneous component of degree k of A_m by place permutation

$$(x_{i_1} \ldots x_{i_k})\, \sigma^{-1} = (x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}), \qquad \sigma \in S_k,\ x_{i_1} \ldots x_{i_k} \in A_m^{(k)}.$$

Every nonzero element

$$w_\lambda(x_1, \ldots, x_m) = \left(\prod_{j=1}^{q} s_{p_j}(x_1, \ldots, x_{p_j}) \right) \sum_{\sigma \in S_k} b_\sigma \sigma, \qquad b_\sigma \in K,$$

is the highest weight vector of a submodule $W(\lambda)$ of $A_m^{(k)}$. Here $q = \lambda_1$, p_j is the length of the j-th column of the Young diagram related with λ and

$$s_d(x_1, \ldots, x_d) = \sum_{\pi \in S_d} (\operatorname{sign} \pi) x_{\pi(1)} \ldots x_{\pi(d)}$$

is the standard polynomial of degree d.

For a GL_m-module W we consider the *k-th symmetrized tensor power* $W^{\otimes_s k}$ identifying the tensors $w_1 \otimes \ldots \otimes w_k$ and $w_{\pi(1)} \otimes \ldots \otimes w_{\pi(k)}$, $w_1, \ldots, w_k \in W$, $\pi \in S_k$. In particular, $W \otimes_s W$ is the symmetrized tensor square of W and $w_1 \otimes_s w_2$ is the symmetric product of w_1 and w_2. As in [2], we shall use the Young rule for the tensor product of GL_m-modules and the rule for the decomposition of the symmetrized tensor squares of $W(2)$ and $W(1^2)$ which follows from the results of Thrall on plethysms [20] (see also [14]).

PROPOSITION 1.1. – (i) *The tensor product* $W(\lambda_1, \ldots, \lambda_m) \otimes W(k)$ *is isomorphic to the direct sum of* $W(\mu)$, *where* $\mu = (\mu_1, \ldots, \mu_m)$,

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_m \geq \lambda_m, \qquad \mu_1 + \ldots + \mu_m = \lambda_1 + \ldots + \lambda_m + k\,,$$

i.e. the diagram of μ *is obtained by adding k boxes to the diagram of* λ *in such a way that no two new boxes are in the same column of the diagram of* μ.

(ii) *The following* GL_m*-module isomorphisms hold*

$$W(2) \otimes_s W(2) \cong W(4) \oplus W(2^2), \qquad W(1^2) \otimes_s W(1^2) \cong W(2^2) \oplus W(1^4).$$

PROPOSITION 1.2. – *Let* $W = \sum_{i \geq 1} W_i$ *be a direct sum of irreducible polynomial* GL_m*-modules such that for each* λ *only a finite number of* W_i *are isomorphic to* $W(\lambda)$. *Then, as a* GL_m*-module, the n-th symmetrized tensor power* $W^{\otimes_s n}$ *of W is isomorphic to the direct sum*

$$\sum W_{i_1}^{\otimes_s n_1} \otimes \ldots \otimes W_{i_p}^{\otimes_s n_p},$$

where the summation runs over all $i_1 < \ldots < i_p$, $p \leq n$, *and* $n_1 + \ldots + n_p = n$.

PROOF. – Every irreducible polynomial GL_m-module is a multigraded vector space. We choose an ordered multigraded basis $\{w_1, w_2, \ldots\}$ of W assuming that the inclusions $w_j \in W_p$ and $w_{j+1} \in W_q$ imply that $p \leq q$. Then $W^{\otimes_s n}$ is a graded vector space and its homogeneous components are finite dimensional. There is an obvious isomorphism as graded vector spaces between $W^{\otimes_s n}$ and $\sum W_{i_1}^{\otimes_s n_1} \otimes \ldots \otimes W_{i_p}^{\otimes_s n_p}$, defined by

$$(w_{j_1} \otimes_s \ldots \otimes_s w_{j_{n_1}}) \otimes_s \ldots \otimes_s (w_{j_{n-n_p+1}} \otimes_s \ldots \otimes_s w_{j_n}) \rightarrow$$
$$(w_{j_1} \otimes_s \ldots \otimes_s w_{j_{n_1}}) \otimes \ldots \otimes (w_{j_{n-n_p+1}} \otimes_s \ldots \otimes_s w_{j_n}),$$

where $w_{j_1}, \ldots, w_{j_{n_1}} \in W_{i_1}, j_1 \leq \ldots \leq j_{n_1}, \ldots, w_{j_{n-n_p+1}}, \ldots, w_{j_n} \in W_{i_p}, j_{n-n_p+1} \leq \ldots \leq j_n$, $i_1 < \ldots < i_p$, $n_1 + \ldots + n_p = n$. It is well known that if two GL_m-modules U_1 and U_2 are direct sums of polynomial submodules and the homogeneous components $U_1^{(k)}$ and $U_2^{(k)}$ are finite dimensional for any k, then the isomorphism of U_1 and U_2 as graded vector spaces implies the isomorphism as GL_m-modules. Hence, the modules $W^{\otimes_s n}$ and $\sum W_{i_1}^{\otimes_s n_1} \otimes \ldots \otimes W_{i_p}^{\otimes_s n_p}$ are isomorphic.

COROLLARY 1.3. – *Let* $*$ *be a linear operator acting on the* GL_m*-module*

$$W = W^+(1) \oplus \big(W^+(2) \oplus W^-(1^2)\big) \oplus \big(W^+(3) \oplus W^+(2,1) \oplus W^-(2,1) \oplus W^-(1^3)\big),$$

where as a GL_m*-module* $W^{\pm}(\lambda)$ *is isomorphic to* $W(\lambda)$ *and the operator* $*$ *acts* *on* $W^{\pm}(\lambda)$ *by* $w^* = \pm w$, $w \in W^{\pm}(\lambda)$. *If we extend the action of* $*$ *on* $W^{\otimes_s n}$ *by*

$$(w_1 \otimes_s \ldots \otimes_s w_n)^* = w_1^* \otimes_s \ldots \otimes_s w_n^*,$$

then for $n \geqslant 6$

$$\begin{aligned}
W^{\otimes_s n} \cong\ & W^+(n) \oplus \big(W^+(n+1) \oplus W^+(n,1) \oplus W^+(n-1,2)\big) \oplus \\
& \big(W^-(n,1) \oplus W^-(n-1,1^2)\big) \oplus \big(2W^+(n+2) \oplus 3W^+(n+1,1) \oplus \\
& 5W^+(n,2) \oplus W^+(n,1^2) \oplus 2W^+(n-1,3) \oplus 3W^+(n-1,2,1) \oplus W^+(n-1,1^3) \oplus \\
& W^+(n-2,4) \oplus 2W^+(n-2,2^2) \oplus W^+(n-2,1^4)\big) \oplus \\
& \big(2W^-(n+1,1) \oplus 2W^-(n,2) \oplus 4W^-(n,1^2) \oplus W^-(n-1,3) \oplus 3W^-(n-1,2,1) \oplus \\
& 2W^-(n-1,1^3) \oplus W^-(n-2,3,1) \oplus W^-(n-2,2,1^2)\big) \oplus W',
\end{aligned}$$

where we have denoted by W' *the sum of irreducible* GL_m*-modules which are homogeneous of degree* $> n+2$.

PROOF. – Clearly the GL_m-module $W(1)^{\otimes_s k}$ is isomorphic to the homogeneous component of degree k of the polynomial algebra $K[x_1, \ldots, x_m]$ and $W(1)^{\otimes_s k} \cong W(k)$. Since for every $\lambda = (\lambda_1, \ldots, \lambda_m)$ the irreducible module $W(\lambda)$ is homogeneous of degree $\lambda_1 + \ldots + \lambda_m$, by Proposition 1.2 the homogeneous components of degree $\leqslant n+2$ of $W^{\otimes_s n}$ are decomposed respectively as

$$(W^{\otimes_s n})^{(n)} \cong W^+(1)^{\otimes_s n} \cong W^+(n),$$

$$(W^{\otimes_s n})^{(n+1)} \cong W^+(1)^{\otimes_s (n-1)} \otimes \big(W^+(2) \oplus W^-(1^2)\big) \cong W^+(n-1) \otimes \big(W^+(2) \oplus W^-(1^2)\big),$$

$$\begin{aligned}
(W^{\otimes_s n})^{(n+2)} \cong\ & W^+(1)^{\otimes_s (n-1)} \otimes \big(W^+(3) \oplus W^+(2,1) \oplus W^-(2,1) \oplus W^-(1^3)\big) \oplus \\
& W^+(1)^{\otimes_s (n-2)} \otimes \big(W^+(2)^{\otimes_s 2} \oplus W^+(2) \otimes W^-(1^2) \oplus W^-(1^2)^{\otimes_s 2}\big) \cong \\
& W^+(n-1) \otimes \big(W^+(3) \oplus W^+(2,1) \oplus W^-(2,1) \oplus W^-(1^3)\big) \oplus \\
& W^+(n-2) \otimes \big(W^+(2)^{\otimes_s 2} \oplus W^+(2) \otimes W^-(1^2) \oplus W^-(1^2)^{\otimes_s 2}\big).
\end{aligned}$$

Using the Young rule from Proposition 1.1 (i) and taking into account the action of $*$ we calculate

$$W^+(n-1) \otimes W^+(2) \cong W^+(n+1) \oplus W^+(n,1) \oplus W^+(n-1,2),$$

$$W^+(n-1) \otimes W^-(1^2) \cong W^-(n,1) \oplus W^-(n-1,1^2).$$

Combining the Young rule with the decompositions of the symmetrized tensor squares of $W(2)$ and $W(1^2)$ from Proposition 1.1 (ii) we obtain

$$W^+(n-1)\otimes W^+(3)\cong W^+(n+2)\oplus W^+(n+1,1)\oplus$$
$$W^+(n,2)\oplus W^+(n-1,3),$$

$$W^+(n-1)\otimes W^\pm(2,1)\cong W^\pm(n+1,1)\oplus W^\pm(n,2)\oplus$$
$$W^\pm(n,1^2)\oplus W^\pm(n-1,2,1),$$

$$W^+(n-1)\otimes W^-(1^3)\cong W^-(n,1^2)\oplus W^-(n-1,1^3),$$

$$W^+(n-2)\otimes W^+(2)^{\otimes_s 2}\cong W^+(n-2)\otimes\big(W^+(4)\oplus W^+(2^2)\big)\cong$$
$$W^+(n+2)\oplus W^+(n+1,1)\oplus 2W^+(n,2)\oplus W^+(n-1,3)\oplus$$
$$W^+(n-1,2,1)\oplus W^+(n-2,4)\oplus W^+(n-2,2^2),$$

$$W^+(n-2)\otimes W^+(2)\otimes W^-(1^2)\cong W^+(n-2)\otimes\big(W^-(3,1)\oplus W^-(2,1^2)\big)\cong$$
$$W^-(n+1,1)\oplus W^-(n,2)\oplus 2W^-(n,1^2)\oplus W^-(n-1,3)\oplus$$
$$2W^-(n-1,2,1)\oplus W^-(n-1,1^3)\oplus W^-(n-2,3,1)\oplus W^-(n-2,2,1^2),$$

$$W^+(n-2)\otimes W^-(1^2)^{\otimes_s 2}\cong W^+(n-2)\otimes\big(W^+(2^2)\oplus W^+(1^4)\big)\cong$$
$$W^+(n,2)\oplus W^+(n-1,2,1)\oplus W^+(n-1,1^3)\oplus$$
$$W^+(n-2,2^2)\oplus W^+(n-2,1^4)$$

and the proof is completed by counting the irreducible components in the above tensor products.

2. – Description of the T-ideal of the nil identity.

In this section we shall describe the $GL_m(K)$-module structure of $A_m\cap\langle x^n\rangle^T$ as a homomorphic image of the n-th symmetrized tensor power of A_m. The following result can be obtained from the proof of Nagata of the Dubnov-Ivanov-Nagata-Higman theorem.

LEMMA 2.1. – *Let*

$$h(x_1,\dots,x_n)=\sum_{\pi\in S_n}x_{\pi(1)}\dots x_{\pi(n)}$$

be the complete linearization of the polynomial x^n. Every element of the T-

ideal $\langle x^n \rangle^T$ *is a linear combination of evaluations of* $h(x_1, \ldots, x_n)$ *in* $K\langle X\rangle$.

PROOF. – The multilinear polynomial $h(x_1, \ldots, x_n)$ is a linear combination of elements $(\alpha_1 x_1 + \ldots + \alpha_n x_n)^n$, $\alpha_i \in K$, and $x^n = (1/n!)h(x, \ldots, x)$.

Every element from $\langle x^n \rangle^T$ has the form

$$u = \sum u_i v_i^n w_i,$$

where u_i, v_i, $w_i \in K\langle X\rangle$ and u_i, w_i are allowed to be empty symbols. Hence, it is sufficient to show that the polynomials yx^n, $x^n y$ and $yx^n z$ are linear combinations of u^n, $u \in K\langle x, y, z\rangle$. Recall that the partial linearizations of a polynomial identity $f(x_1, \ldots, x_m)$ are obtained by Vandermonde arguments and are linear combinations of $f(u_1, \ldots, u_m)$, $u_i \in K\langle X\rangle$. Since

$$x^n y = yx^n + [x^n, y] = yx^n + \sum_{i=1}^{n} x^{i-1}[x, y]\, x^{n-i},$$

where $[x, y] = xy - yx$ and $\sum_{i=1}^{n} x^{i-1}[x, y]\, x^{n-i}$ is an evaluation of the partial linearization

$$h_1(x, z) = zx^{n-1} + xzx^{n-2} + \ldots + x^{n-2}zx + x^{n-1}z,$$

it is enough to handle the case yx^n only. Clearly $h_1(x, x^2) = nx^{n+1}$ and x^{n+1} is a linear combination of evaluations of x^n. Linearizing partially x^{n+1} (again linear combinations) we obtain

$$f(x, y) = yx^n + xyx^{n-1} + \ldots + x^{n-1}yx + x^n y$$

and a direct computation shows that

$$f(x, y) - h_1(x, xy) = yx^n.$$

Since

$$yx^n z = (yx^n)\, z = \Big(\sum_i \alpha_i u_i^n\Big)\, z = \sum_i \alpha_i (u_i^n z) = \sum_i \alpha_i \big(\sum_j \beta_j v_{ij}^n\big) = \sum_{ij} \alpha_i \beta_j v_{ij}^n$$

the proof of the lemma is completed.

Now we state the main result of the section.

THEOREM 2.2. – *As a* GL_m*-module the* T*-ideal* $A_m \cap \langle x^n \rangle^T$ *is a homomorphic image of the n-th symmetrized tensor power of* A_m.

PROOF. – By Lemma 2.1 every element of $A_m \cap \langle x^n \rangle^T$ is a linear combination of $h(u_1, \ldots, u_n)$, $u_1, \ldots, u_n \in A_m$, where $h(x_1, \ldots, x_n)$ is the complete linearization of x^n. Since $h(x_{\varrho(1)}, \ldots, x_{\varrho(n)}) = h(x_1, \ldots, x_n)$ for every $\varrho \in S_n$, we can define the following mapping

$$\phi: u_1 \otimes_s \ldots \otimes_s u_n \to h(u_1, \ldots, u_n), \qquad u_1, \ldots, u_n \in A_m.$$

This mapping can be extended to a vector space homomorphism $\tilde{\phi}$ of $A_m^{\otimes_s n}$ onto $A_m \cap \langle x^n \rangle^T$. Obviously $\tilde{\phi}$ is also a GL_m-module homomorphism because for every $g \in GL_m$

$$g\big(\phi(u_1 \otimes_s \ldots \otimes_s u_n)\big) = g\big(h(u_1, \ldots, u_n)\big) =$$

$$h\big(g(u_1), \ldots, g(u_n)\big) = \phi\big(g(u_1) \otimes_s \ldots \otimes_s g(u_n)\big)$$

and this completes the proof of the theorem.

COROLLARY 2.3. – *For $n \geq 6$ the multiplicities of the irreducible S_k-characters of the S_k-modules $V_k \cap \langle x^n \rangle^T$, $k = n+1, n+2$, are bounded from above respectively by the corresponding multiplicities of the S_k-characters*

$$\chi_{S_{n+1}} = \chi(n+1) + 2\chi(n, 1) + \chi(n-1, 2) + \chi(n-1, 1^2),$$

$$\chi_{S_{n+2}} = \chi(n+2) + 5\chi(n+1, 1) + 7\chi(n, 2) + 5\chi(n, 1^2) +$$

$$3\chi(n-1, 3) + 6\chi(n-1, 2, 1) + 3\chi(n-1, 1^3) + \chi(n-2, 4) +$$

$$\chi(n-2, 3, 1) + 2\chi(n-2, 2^2) + \chi(n-2, 2, 1^2) + \chi(n-2, 1^4).$$

PROOF. – By the equivalence between the S_k-module $V_k \cap U$ and the GL_m-module $A_m^{(k)} \cap U$ for any T-ideal U, it is sufficient to find upper bounds for the multiplicities of the irreducible components of the GL_m-modules $A_m^{(k)} \cap \langle x^n \rangle^T$, $k = n+1, n+2$. It is well known that the homogeneous components $A_m^{(k)}$ have the following GL_m-module decomposition:

$$A_m^{(k)} \cong \sum \deg \chi(\lambda)\, W(\lambda),$$

where the summmation runs on all partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ of k and $deg\chi(\lambda)$ is the degree of the corresponding S_k-character. In particular,

$$A_m^{(1)} \cong W(1), \quad A_m^{(2)} \cong W(2) \oplus W(1^2), \quad A_m^{(3)} \cong W(3) \oplus 2W(2, 1) \oplus W(1^3).$$

We apply Theorem 2.2 and Proposition 1.2. Clearly, in the notation of Proposition 1.2 the homogeneous components of degree $k = n+1, n+2$ of $A_m^{\otimes_s n}$ are obtained for $k = n+1$ from

$$p = 2, \quad \deg W_{i_1} = 1, \quad n_1 = n-1, \quad \deg W_{i_2} = 2, \quad n_2 = 1,$$

and, for $k = n+2$, from

$$p = 2, \quad \deg W_{i_1} = 1, \quad n_1 = n-1, \quad \deg W_{i_2} = 3, \quad n_2 = 1,$$

$$p=2, \quad \deg W_{i_1}=1, \quad n_1=n-2, \quad \deg W_{i_2}=2, \quad n_2=2,$$

$$p=3, \quad \deg W_{i_1}=1, \quad n_1=n-2, \quad \deg W_{i_2}=\deg W_{i_3}=2, \quad n_2=n_3=1.$$

Now we apply a weaker version of Corollary 1.3 without taking into account the action of the operator $*$ in its statement and obtain the bounds for all $\lambda \neq (n+2)$. The case $\lambda=(n+2)$ is trivial, because the multiplicity of $W(k)$ in the GL_m-module $A_m^{(k)}$ is equal to 1.

Recall that the linear operator $*$ acting on an algebra R is called an *involution* if it is an algebra antiautomorphism of order 2, i.e. $(r^*)^*=r$ and $(r_1 r_2)^*=r_2^* r_1^*$ for every $r, r_1, r_2 \in R$. Then $R=R^+ \oplus R^-$ as a vector space, where $R^{\pm}=\{r \in R \mid r^*=\pm r\}$ and any $r \in R$ has a decomposition $r=r^+ + r^-$, $r^{\pm}=(1/2)(r \pm r^*) \in R^{\pm}$.

The free associative algebras A and A_m have an involution defined by

$$(x_{i_1} \ldots x_{i_k})^* = x_{i_k} \ldots x_{i_1}.$$

Since the action of $*$ on A_m commutes with the action of GL_m, if $w \in W(\lambda) \subset A_m$ is a highest weight vector and $w=w^+ + w^-$, $w^{\pm} \in A_m^{\pm}$, then w^+ and w^- are also highest weight vectors.

COROLLARY 2.4. – *The* T-*ideal* $\langle x^n \rangle^T$ *is invariant under the involution* $*$ *of the free algebra. For* $n \geq 6$ *the multiplicities of the irreducible* S_k-*characters of the* S_k-*modules* $(V_k \cap \langle x^n \rangle^T)^{\pm}$, $k=n+1, n+2$, *are bounded from above respectively by the corresponding multiplicities of the* S_k-*characters*

$$\chi_{S_{n+1}}^+ = \chi(n+1)+\chi(n,1)+\chi(n-1,2) \quad \textit{for } (V_{n+1} \cap \langle x^n \rangle^T)^+,$$

$$\chi_{S_{n+1}}^- = \chi(n,1)+\chi(n-1,1^2) \quad \textit{for } (V_{n+1} \cap \langle x^n \rangle^T)^-,$$

$$\begin{aligned}\chi_{S_{n+2}}^+ = {} & \chi(n+2)+3\chi(n+1,1)+5\chi(n,2)+\chi(n,1^2)+\\ & 2\chi(n-1,3)+3\chi(n-1,2,1)+\chi(n-1,1^3)+\chi(n-2,4)+\\ & 2\chi(n-2,2^2)+\chi(n-2,1^4) \quad \textit{for } (V_{n+2} \cap \langle x^n \rangle^T)^+,\end{aligned}$$

$$\begin{aligned}\chi_{S_{n+2}}^- = {} & 2\chi(n+1,1)+2\chi(n,2)+4\chi(n,1^2)+\chi(n-1,3)+3\chi(n-1,2,1)+\\ & 2\chi(n-1,1^3)+\chi(n-2,3,1)+\chi(n-2,2,1^2) \quad \textit{for } (V_{n+2} \cap \langle x^n \rangle^T)^-.\end{aligned}$$

PROOF. – The statement that $\langle x^n \rangle^T$ is $*$-invariant is true for any T-ideal generated by $*$-invariant identities. Any $f \in \langle x^n \rangle^T$ can be written as $f = \sum_{i=1}^{p} u_i' v_i^n u_i''$, $u_i', u_i'', v_i \in A$ (some u_i', u_i'' may be empty symbols) and

$$f^* = \sum (u_i'')^* (v_i^*)^n (u_i')^* \in \langle x^n \rangle^T.$$

Hence $\langle x^n \rangle^T$ is $*$-invariant and $(\langle x^n \rangle^T)^{\pm} \subset \langle x^n \rangle^T$. For the second part of the lemma we use the following highest weight vectors of $W(\lambda) \subset A_m^{(k)}$, $k = 1, 2, 3$:

$$w_{(k)}^+ = x_1^k,\ w_{(2,\,1)}^+ = [x_2, x_1, x_1] = x_1 s_2(x_1, x_2) - s_2(x_1, x_2)\, x_1,$$

$$w_{(1^2)}^- = [x_1, x_2],\ w_{(2,\,1)}^- = [x_1, x_1 x_2 + x_2 x_1] = x_1[x_1, x_2] + [x_1, x_2]x_1,$$

$$w_{(1^3)}^- = s_3(x_1, x_2, x_3).$$

Repeating the arguments from the proof of Corollary 2.3 we see that

$$A_m^{(1)} \cong W^+(1), \quad A_m^{(2)} \cong W^+(2) \oplus W^-(1^2),$$

$$A_m^{(3)} \cong W^+(3) \oplus W^+(2, 1) \oplus W^-(2, 1) \oplus W^-(1^3).$$

By Lemma 2.1, every element $f \in \langle x^n \rangle^T$ has the form $f = \sum \alpha_i h(u_{i1}, \ldots, u_{in})$, $\alpha_i \in K$, $u_{ij} \in A$. Since $(h(u_1, \ldots, u_n))^* = h(u_1^*, \ldots, u_n^*)$ we obtain that the action of $*$ on $\langle x^n \rangle^T$ is exactly the same as the action of the linear operator $*$ from Corollary 1.3. Now the proof is completed by Theorem 2.2 and Corollary 1.3.

In the next section we shall show that the bounds given in Corollaries 2.3 and 2.4 are very close to exact.

3. – Cocharacters of small degree.

In this section we shall calculate the $n+1$-st and the $n+2$-nd character of the multilinear polynomials in $\langle x^n \rangle^T$ of degree $n+1$ and $n+2$, respectively. Our approach is similar to the approach from [2], where we have solved an analogous problem for the multilinear consequences of degree $n+2$ of the standard polynomial s_n of degree n. In the case of x^n the calculations are easier than in the case s_n because by Lemma 2.1 all consequences are obtained by substitutions only. In the case of s_n we had to take into account also consequences of the forms $x_{n+1} x_{n+2} s_n(x_1, \ldots, x_n)$ and $x_{n+2} s_n(x_1, \ldots, x_n x_{n+1})$. Involving the involution $*$ of the free algebra we additionally simplify the concrete calculations.

The following lemma is a restatement of [2, Lemma 1.7].

LEMMA 3.1. – *Let Λ be a finite set of partitions and let the sum*

$$W = \sum_{\lambda \in \Lambda} m(\lambda)\, W(\lambda)$$

be a submodule of the GL_m-module A_m. Let $w_i^{(\lambda)} \in W(\lambda) \subset W$, $i = 1, \ldots, m(\lambda)$, $\lambda \in \Lambda$, be a collection of heighest weight vectors which generates the summand $m(\lambda) W(\lambda)$. If for each $\lambda \in \Lambda$ the polynomials $w_i^{(\lambda)}$, $i = 1, \ldots, r(\lambda)$, are linearly *independent and the other polynomials* $w_j^{(\lambda)}$, $j = r(\lambda) + 1, \ldots, m(\lambda)$, *are their linear combinations, then the set*

$$\{w_i^{(\lambda)} \mid i = 1, \ldots, r(\lambda), \lambda \in \Lambda\}$$

generates W *and the multiplicity of* $W(\lambda)$ *in* W *is equal to* $r(\lambda)$.

The following assertion gives a criterion when a polynomial is a highest weight vector. If $f(x_1, \ldots, x_m)$ is a multihomogeneous polynomial, then we denote by $f^{(i)}(x_1, \ldots, x_m, y)$ the linear in y component of $f(x_1, \ldots, x_i + y, \ldots, x_m)$.

PROPOSITION 3.2 (Koshlukov [12]). – *Let* W *be a polynomial* GL_m*-module. A non-zero multihomogeneous element* $f(x_1, \ldots, x_m)$ *of* W *of degree* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *is a highest weight vector if and only if*

$$f^{(i)}(x_1, \ldots, x_m, x_j) = 0, \qquad 1 \leqslant j < i \leqslant m.$$

From now on we denote

$$\psi_k(x_{i_1},\ldots,x_{i_k}) = \sum_{\sigma\in S_k(i_1,\ldots,i_k)} \sum_{j_1+\ldots+j_k\leqslant n-k} x_1^{n-(j_1+\ldots+j_k)-k} x_{\sigma(i_1)} x_1^{j_1} x_{\sigma(i_2)} x_1^{j_2} \ldots x_{\sigma(i_k)} x_1^{j_k}$$

where $S_k(i_1, \ldots, i_k)$ stands for the symmetric group of degree k on the integers $i_1, \ldots, i_k$. Clearly, ψ_k is a partial linearization of x^n. As usually, $u \circ v = uv + vu$. We also keep the notation $w^{\pm}$ for the elements from $A_m^{\pm}$ with respect to the involution $*$ of A_m.

PROPOSITION 3.3. – *Let* λ *be a partition of* $n + 1$. *The highest weight vectors of the irreducible components* $W(\lambda)$ *of the* GL_m*-submodule of the homogeneous consequences in* $A_m^{(n+1)}$ *of the identity* $x^n = 0$, $n \geqslant 3$, *are linear combinations of the following highest weight vectors*:

(i) *For* $\lambda = (n + 1)$

$$w_1^+ = x_1^{n+1};$$

(ii) *For* $\lambda = (n, 1)$

$$w_1^+ = 2\psi_2(x_1^2, x_2) - (n-1)\psi_1(x_1 \circ x_2),$$

$$w_1^- = \psi_1([x_1, x_2]);$$

(iii) *For* $\lambda = (n - 1, 2)$

$$w_1^+ = (n-1)(n-2)\,\psi_1(x_2^2) - (n-2)\,\psi_2(x_1 \circ x_2, x_2) + \psi_3(x_1^2, x_2, x_2);$$

(iv) *For* $\lambda = (n-1, 1^2)$

$$w_1^- = \sum_{\sigma \in A_3} \psi_2([x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}).$$

For each λ *the highest weight vector* $w_1^{\pm}$ *is different from* 0.

PROOF. – We shall briefly explain how we have found the above highest weight vectors. By the proofs of Corollaries 2.3 and 2.4, the consequences of degree $n+1$ of $x^n = 0$ form a homomorphic image of the components of the GL_m-submodule $(W^+(2) \oplus W^-(1^2)) \otimes W^+(n-1)$ of $A_m^{\otimes_s n}$. Since

$$W^+(2) \otimes W^+(n-1) \cong W^+(n+1) \oplus W^+(n, 1) \oplus W^+(n-1, 2),$$

$$W^-(1^2) \otimes W^+(n-1) \cong W^-(n, 1) \oplus W^-(n-1, 1^2),$$

first we find the highest weight vectors of these five irreducible GL_m-modules. Let us denote by ψ'_k the partial linearization of x_1^{n-1}. Applying the algorithm for computing the highest weight vectors of the irreducible components of the tensor product of two GL_m-modules [5, Rule 1.3.4] (or some of its shorter versions) we obtain the highest weight vectors

$$w_\lambda^+(x_1) = x_1^2 \otimes x_1^{n-1}, \qquad \lambda = (n+1),$$

$$w_\lambda^+(x_1, x_2) = (n-1)(x_1 \circ x_2) \otimes x_1^{n-1} - 2x_1^2 \otimes \psi'_1(x_2), \quad \lambda = (n, 1),$$

$$w_\lambda^+(x_1, x_2) = (n-2)(n-1)\, x_2^2 \otimes x_1^{n-1} - (n-2)(x_1 \circ x_2) \otimes \psi'_1(x_2) + x_1^2 \otimes \psi'_2(x_2, x_2), \quad \lambda = (n-1, 2)$$

for the components of $W^+(2) \otimes W^+(n-1)$ and

$$w_\lambda^-(x_1, x_2) = [x_1, x_2] \otimes x_1^{n-1}, \quad \lambda = (n, 1),$$

$$w_\lambda^-(x_1, x_2, x_3) = \sum_{\sigma \in A_3} [x_{\sigma(1)}, x_{\sigma(2)}] \otimes \psi'_1(x_{\sigma(3)}), \quad \lambda = (n-1, 1^2)$$

for the components of $W^-(1^2) \otimes W^+(n-1)$.

For example, we shall show that, for $\lambda = (n-1, 2)$, $w_\lambda^+(x_1, x_2)$ is a highest weight vector. Clearly, $\psi'_1(x_2)$ is of total degree $n-1$, has $n-1$ summands $x_1^i x_2 x_1^{n-2-i}$ (one for each $i = 0, 1, \ldots, n-2$) and $\psi'_1(x_1) = (n-1)x_1^{n-1}$. Similarly, $\psi'_2(x_2, x_3)$ is the partial linearization of $\psi'_1(x_2)$ (i.e. in $\psi'_1(x_2)$ we replace x_1 by $x_1 + x_3$ and consider the linear in x_3 component) and $\psi'_2(x_1, x_2) = \psi'_2(x_2, x_1) = (n-2)\psi'_1(x_2)$. Hence for the partial linearization $(w_\lambda^+)^{(2)}(x_1, x_2, y)$

$$(w_\lambda^+)^{(2)}(x_1, x_2, x_1) = (n-2)(n-1)(x_1 \circ x_2) \otimes x_1^{n-1} - 2(n-2)\, x_1^2 \otimes \psi'_1(x_2) - (n-2)(n-1)(x_1 \circ x_2) \otimes x_1^{n-1} + 2(n-2)\, x_1^2 \otimes \psi'_1(x_2) = 0, \quad \lambda = (n-1, 2)$$

and $w_\lambda^+(x_1, x_2)$ is a highest weight vector by Proposition 3.2. Now we use the GL_m-module homomorphism $\widetilde{\phi}: A_m^{\otimes_s n} \to A_m \cap \langle x^n \rangle^T$ and consider the images $\widetilde{\phi}(w_\lambda^\pm)$ of $w_\lambda^\pm$ in $A_m \cap \langle x^n \rangle^T$. Up to multiplicative constants they are equal to the highest weight vectors given in the statement of the proposition.

Finally, in order to see that the polynomials $w_1^\pm$ are different from 0, we choose a monomial $u = x_{i_1} \dots x_{i_{n+1}}$, calculate the coefficient $\alpha^\pm$ of u in $w_1^\pm$ and show that $\alpha^\pm \neq 0$.

For $\lambda = (n+1)$ obviously $w_1^+ = x_1^{n+1} \neq 0$.

For $\lambda = (n, 1)$ we choose $u = x_2 x_1^n$ and by direct calculations see that $\alpha^+ = n - 1 \neq 0$ and $\alpha^- = -1$, i.e. $w_1^\pm \neq 0$.

For $\lambda = (n-1, 2)$ we choose $u = x_2^2 x_1^{n-1}$ and calculate that $\alpha^+ = n(n-2) \neq 0$.

For $\lambda = (n-1, 1^2)$ we choose $u = x_2 x_3 x_1^{n-1}$ and obtain that $\alpha^- = n \neq 0$.

Now we state the main result of the section.

THEOREM 3.4. – *For $n \geq 6$ the S_k-character of the multilinear consequences of degree $k \leq n+2$ of the identity $x^n = 0$ is the following*:

$$\chi_{S_n}(V_n \cap \langle x^n \rangle^T) = \chi(n),$$

$$\chi_{S_{n+1}}(V_{n+1} \cap \langle x^n \rangle^T) = \chi(n+1) + 2\chi(n, 1) + \chi(n-1, 2) + \chi(n-1, 1^2),$$

$$\begin{aligned}\chi_{S_{n+2}}(V_{n+2} \cap \langle x^n \rangle^T) = {} & \chi(n+2) + 4\chi(n+1, 1) + 5\chi(n, 2) + 5\chi(n, 1^2) + \\ & 3\chi(n-1, 3) + 6\chi(n-1, 2, 1) + 3\chi(n-1, 1^3) + \chi(n-2, 4) + \\ & \chi(n-2, 3, 1) + 2\chi(n-2, 2^2) + \chi(n-2, 2, 1^2) + \chi(n-2, 1^4).\end{aligned}$$

PROOF. – The case $k = n$ is obvious and the case $k = n+1$ follows from Proposition 3.3. For $k = n+2$ we repeate the arguments from the proof of the same proposition. For a partition λ of $n+2$ we find the highest weight vectors of the irreducible components $W(\lambda)$ of the GL_m-submodule of the homogeneous consequences in $A_m^{(n+2)}$ of the identity $x^n = 0$, $n \geq 6$. Every highest weight vector $w^\pm$ is a linear combination of highest weight vectors $w_1^\pm, \dots, w_p^\pm$, where $p = p^\pm(\lambda)$ is the multiplicity prescribed in Corollary 2.4. We shall prove the theorem if we establish that the polynomials $w_1^\pm, \dots, w_p^\pm$ are linearly independent in all cases different from $\lambda = (n+1, 1)$ and $\lambda = (n, 2)$. In the two cases left we have to show that there exist unique linear dependences between the w_j^+'s for $\lambda = (n+1, 1)$ and between the $w_j^\pm$'s for $\lambda = (n, 2)$. We use the same approach for all λ. We fix a partition λ and a sign $\pm$. For the polynomials $w_1^\pm, \dots, w_p^\pm$ for which we claim that are linearly independent in $K\langle X\rangle$, we consider a relation

$$\sum_{j=1}^{p} \xi_j^{\pm} w_j^{\pm}(x_1, \dots, x_m) = 0$$

with unknown coefficients $\xi_j^{\pm} \in K$, $j = 1, \dots, p$. We choose p monomials $u_i = x_{i_1} \dots x_{i_{n+2}}$ and calculate the coefficient α_{ij} of u_i in $w_j^{\pm}$. In this way we obtain p equations

$$\sum_{j=1}^{p} \xi_j^{\pm} \alpha_{ij} = 0, \quad i = 1, \dots, p.$$

Considering these equations as a linear homogeneous system with respect to the unknowns $\xi_j^{\pm}$, we show that the only solution of the system is $\xi_j^{\pm} = 0$, $j = 1, \dots, p$, and this gives that the highest weight vectors w_j are linearly independent. Below we give the polynomials w_j^{+} for $\lambda = (n+1, 1)$ and $w_j^{\pm}$ for $\lambda = (n, 2)$.

For $\lambda = (n+1, 1)$:

$$w_1^{+} = 3\psi_2(x_1^3, x_2) - (n-1)\,\psi_1(x_1^2 x_2 + (x_1 \circ x_2)\, x_1),$$

$$w_2^{+} = \psi_1([x_2, x_1, x_1]),$$

$$w_3^{+} = 2\psi_3(x_1^2, x_1^2, x_2) - (n-2)\,\psi_2(x_1^2, x_1 \circ x_2);$$

For $\lambda = (n, 2)$:

$$w_1^{+} = 3\psi_3(x_1^3, x_2, x_2) - 2(n-2)\,\psi_2(x_1^2 x_2 + (x_1 \circ x_2)\, x_1, x_2) + (n-1)(n-2)\,\psi_1(x_2^2 x_1 + (x_1 \circ x_2)\, x_2),$$

$$w_2^{+} = \psi_2([x_2, x_1, x_1], x_2) + (n-1)\,\psi_1([x_1, x_2, x_2]),$$

$$w_3^{+} = 6\psi_4(x_1^2, x_1^2, x_2, x_2) - 6(n-3)\,\psi_3(x_1^2, x_1 \circ x_2, x_2) + (n-2)(n-3)\,\psi_2(x_1 \circ x_2, x_1 \circ x_2) + 2(n-2)(n-3)\,\psi_2(x_1^2, x_2^2),$$

$$w_4^{+} = 4\psi_2(x_1^2, x_2^2) - \psi_2(x_1 \circ x_2, x_1 \circ x_2),$$

$$w_5^{+} = \psi_2([x_1, x_2], [x_1, x_2]),$$

$$w_1^{-} = \psi_2([x_1^2, x_2], x_2) + (n-1)\,\psi_1([x_2^2, x_1]),$$

$$w_2^{-} = 2\psi_3([x_1, x_2], x_1^2, x_2) - (n-2)\,\psi_2([x_1, x_2], x_1 \circ x_2);$$

We want to see that the highest weight vectors w_1^{+}, w_2^{+} for $\lambda = (n+1, 1)$ and $w_1^{+}, w_2^{+}, w_4^{+}, w_5^{+}$ for $\lambda = (n, 2)$ are linear independent, $w_1^{-} \neq 0$ for

$\lambda = (n, 2)$. The polynomials u_i and their coefficients α_{ij} in $w_j^{\pm}$ are the following:

for $\lambda = (n+1, 1)$,

u_i	w_1^+	w_2^+
$x_2 x_1^{n+1}$	$2(n-1)$	1
$x_1 x_2 x_1^n$	$n-4$	-1

and, for $\lambda = (n, 2)$,

u_i	w_1^+	w_2^+	w_4^+	w_5^+	w_1^-
$x_2^2 x_1^n$	n^2+n-6	n	$4(n-1)$	0	$n-2$
$x_1 x_2^2 x_1^{n-1}$	$2(n^2-n-5)$	$2n-1$	$2(2n-5)$	-2	
$x_1^2 x_2^2 x_1^{n-2}$	$2(n^2-2n-6)$	$2n$	$2(2n-5)$	-2	
$x_2 x_1^2 x_2 x_1^{n-2}$	$-2(n+4)$	1	-4	0	

Now we shall establish the linear dependences

$$(n-4)\, w_1^+ + (n+2)(n-1) w_2^+ - 3 w_3^+ = 0$$

for $\lambda = (n+1, 1)$ and

$$8 w_1^+ + 4(n+1)\, w_2^+ + 2 w_3^+ - n(n+1)\, w_4^+ + 3n(n+1)\, w_5^+ = 0,$$

$$2 w_1^- + w_2^- = 0$$

for $\lambda = (n, 2)$.

For $\lambda = (n+1, 1)$ we express each w_j^+ as a linear combination of monomials $x_1^a x_2 x_1^b$, where $a + b = n + 1$:

$$w_j^+ = \sum \alpha_j(a, b)\, x_1^a x_2 x_1^b,\ \alpha_j(a, b) \in K.$$

Since we consider symmetric polynomials, it is sufficient to show the linear dependence

$$(n-4)\, w_1^+ + (n+2)(n-1)\, w_2^+ - 3 w_3^+ = 0$$

for the coefficients $\alpha_j(a, b)$, $j = 1, 2, 3$, for fixed (a, b) and $a \leqslant b$.

The coefficients $\alpha_j(a, b)$ are given below:

	w_1^+	w_2^+	w_3^+
$a=0$	$2(n-1)$	1	$(n-1)(n-2)$
$a=1$	$n-4$	-1	$-3(n-2)$
$2 \leqslant a \leqslant b$	-6	0	$-2(n-4)$

Now, it is easy to check directly the given linear dependence.

For $\lambda = (n, 2)$ we present w_j^+, $j = 1, \ldots, 5$, and w_1^-, w_2^- as linear combinations of $x_1^a x_2 x_1^b x_2 x_1^c$, $a + b + c = n$, assuming that $a \leqslant c$ in the symmetric and $a < c$ in the skew-symmetric case. Then we verify the linear dependences for each monomial. In the symmetric case the coefficients of w_j^+ are given in the following matrix:

a	b	c	w_1^+	w_2^+	w_3^+	w_4^+	w_5^+
$\geqslant 2$	$\geqslant 2$	$\geqslant 2$	$-6(n+2)$	0	$-4(n^2-5n-12)$	-8	0
1	$\geqslant 2$	$\geqslant 2$	$-2(2n+5)$	-1	$-2(2n^2-7n-21)$	-8	0
$\geqslant 2$	1	$\geqslant 2$	$n^2-5n-12$	$-2n$	$-6(n^2-3n-8)$	-6	2
0	$\geqslant 2$	$\geqslant 2$	$-2(n+4)$	1	$-2(n+3)(n-5)$	-4	0
$\geqslant 2$	0	$\geqslant 2$	$2(n^2-2n-6)$	$2n$	$2(n^3-6n^2+5n+24)$	$2(2n-5)$	-2
1	1	$\geqslant 2$	$(n+2)(n-5)$	$-(2n+1)$	$-6(n^2-2n-7)$	-6	2
1	$\geqslant 2$	1	$-2(n+4)$	-2	$-4(n^2-2n-9)$	-8	0
1	0	$\geqslant 2$	$2(n^2-n-5)$	$2n-1$	$2(n-3)(n^2-3n-7)$	$2(2n-5)$	-2
0	1	$\geqslant 2$	n^2-n-8	$-(2n-1)$	$-2(n-3)(2n+5)$	-2	2
0	$\geqslant 2$	1	-6	0	$-2(n-3)(n+4)$	-4	0
0	0	$\geqslant 2$	$(n-2)(n+3)$	n	$2(n+2)(n-2)(n-3)$	$4(n-1)$	0
0	$\geqslant 2$	0	$2(n-2)$	2	$2(n-2)(n-3)$	-2	-2

The calculations in the skew-symmetric case are similar.

REMARK 3.5. – For $n = 3$, the S_5-character $\chi_{S_5}(V_5 \cap \langle x^3 \rangle^T)$ was calculated in [22]:

$$\chi_{S_5}(V_5 \cap \langle x^3 \rangle^T) = \chi(5) + 4\chi(4, 1) + 4\chi(3, 2) + 5\chi(3, 1^2) + 5\chi(2^2, 1) + 3\chi(2, 1^3) + \chi(1^5).$$

Comparing this character with the S_5-character prescribed by Theorem 3.4 (and omitting the partitions $\lambda \vdash 5$ which do not exist for $n = 3$, e.g. $\lambda = (n-1, 3)$), we obtain that the only differences are for $\lambda = (3, 2)$ and $\lambda = (2^2, 1)$.

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