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Localization of Biholomorphisms for Real Hyperquadrics in $\mathbb{C}^3$: a Computational Approach (*).

G. Dini - A. Selvaggi Primicerio

Sunto. – *Si calcola esplicitamente con l'aiuto di un computer l'espressione di ogni germe di biolomorfismo in un punto di una iperquadrica reale Q in $\mathbb{C}^3$, che porti Q in Q. Tale germe risulta ovviamente una trasformazione lineare fratta, che lascia Q invariante.*

1. – Introduction.

We present in this paper a new approach to prove the following well-known result [11] [5]: *any germ of biholomorphic map at a point of the hyperquadric*

$$Q = \left\{ z = (z_1, z_2, z_3) \in \mathbb{C}^3 : \varrho(z, \bar{z}) = \sum_{i=1}^{3} \varepsilon_i z_i \overline{z_i} - 1 = 0 \right\} \tag{1}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, $\varepsilon_3 = 1$, *mapping Q into Q, extends to a linear rational transformation.*

We make use of Segre hypersurfaces, hyperplanes in our case, to calculate directly with the aid of a computer the form of the considered germ.

The problem of extension of local biholomorphisms from a hypersurface Q to Q or, more generally, betweeen hypersurfaces Q and Q' goes back to Poincaré [10] for $\mathbb{C}^2$. Tanaka [11] solved the problem in $\mathbb{C}^n$ for hyperquadrics (see also [5]). Burns and Shnider [3] gave a counterexample in $\mathbb{C}^2$ to the holomorphic continuation along any paths of a germ of biholomorphism on a real analytic compact strictly pseudoconvex spherical (i.e. locally biholomorphic to a sphere) but not simply connected hypersurface.

If the hypersurfaces are boundaries of bounded domains of $\mathbb{C}^n$, the extension of local biholomorphisms of the hypersurface can lead to global biholomorphisms of the whole domains. In this case one speaks of «localization principle of biholomorphisms» for the domains or, of «localization principle of au-

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tomorphisms» if the domains are the same [6]. Pinchuk [8], [9] established a new proof of the result of Alexander [2] for the ball $\mathbb{B}^n$ and proved the localization principle of biholomorphisms for strongly pseudoconvex domains with real analytic simply connected boundaries. If the domains are pseudoconvex but not strictly then the «localization principle» does not hold without further assumptions [6]. In [7] and [6] we prove the localization principle of biholomorphisms for generalized pseudoellipsoids in $\mathbb{C}^n$ with a condition on the size of the set of weak pseudoconvex points of the boundary.

2. – Preliminaries.

Let Q be the hyperquadric of $\mathbb{C}^3$ as in (1). Improperly we will call «automorphism» of Q the restriction to $\mathbb{C}^3$ of any projective automorphism of the Zariski projective closure $\widetilde{Q}$ of Q in $\mathbb{C}P^3$. The linear fractional transformation given by

$$\Psi(z) = \left(\frac{iz_1}{1+z_3}, \frac{iz_2}{1+z_3}, i\frac{1-z_3}{1+z_3} \right)$$

maps $Q \setminus \{z_3 = -1\}$ on the real hyperquadric

$$H = \left\{ z \in \mathbb{C}^3 \colon \varrho(z, \bar{z}) = \sum_{i=1}^{2} \varepsilon_i z_i \overline{z_i} - \operatorname{Im} z_3 = 0 \right\}.$$

The group $\mathcal{T}$ of «translations», h_a, $a = (a_1, a_2, a_3) \in H$, given by

$$h_a(z) = (\varepsilon_1 z_1 + a_1, \varepsilon_2 z_2 + a_2, z_3 + a_3 + 2i(z_1 \overline{a_1} + z_2 \overline{a_2}))$$

is easily checked to be a subgroup of the group of automorphisms of H, $\operatorname{Aut}(H)$, and that it acts transitively on H. h_a maps 0 to a and ∞ to ∞. Moreover $h_a^{-1}(z) = h_{a'}(z)$, where $a' = (-\varepsilon_1 a_1, -\varepsilon_2 a_2, -\overline{a_3})$. The group $\operatorname{Aut}(H)$ is uniquely determined by the group $\mathcal{T}$ and the isotropy group $\operatorname{Aut}_0(H)$ of the origin. Chern and Moser [5] have characterized the elements of $\operatorname{Aut}_0(H)$ as

$$\begin{cases} z_1^* = \left(\sum_{\alpha=1}^{2} t_\alpha^1 z_\alpha + \tau^1 z_3 \right) t^{-1} \delta^{-1} \\ z_2^* = \left(\sum_{\alpha=1}^{2} t_\alpha^2 z_\alpha + \tau^2 z_3 \right) t^{-1} \delta^{-1} \\ z_3^* = |t|^{-2} z_3 \delta^{-1} \end{cases}$$

where

$$\begin{cases} 1 + t^{-1} \sum_{\alpha=1}^{2} t_\alpha z_\alpha + t^{-1} \tau z_3 = \delta \\ -2i \sum_{\sigma=1}^{2} \varepsilon_\sigma t_\alpha^\sigma \overline{\tau}^\sigma = t_\alpha \\ t\bar{t}^{-1}(t_1^1 t_2^2 - t_1^2 t_2^1) = 1 \qquad \alpha = 1, 2\,. \\ \sum_{\sigma=1}^{2} \varepsilon_\sigma |t_\alpha^\sigma|^2 = \varepsilon_\alpha \\ \varepsilon_1 t_1^1 \overline{t_2^1} + \varepsilon_2 t_1^2 \overline{t_2^2} = 0 \\ -\varepsilon_1 |\tau^1|^2 - \varepsilon_2 |\tau^2|^2 = \mathrm{Im}\,(\tau t^{-1}) \end{cases}$$

The automorphism group of Q, $\mathrm{Aut}\,(Q) = \Psi^{-1} \mathrm{Aut}\,(H)\, \Psi$, acts doubly transitively on Q, (same proof of Abate [1] for $\mathrm{Aut}\,(\mathbb{B}^n)$). Its isotropy subgroup of $e_3 = (0, 0, 1)$, $\mathrm{Aut}_{e_3}(Q) = \Psi^{-1}\, \mathrm{Aut}_0(H)\Psi$, is given by elements of the form

$$\Upsilon(z) = g(z)^{-1}\big(2\,\bar{t}(\tau^1 + t_1^1 z_1 + t_2^1 z_2 - \tau^1 z_3),\, 2\,\bar{t}(\tau^2 + t_1^2 z_1 + t_2^2 z_2 - \tau^2 z_3)\,, \\ -1 + i\bar{t}\tau + \bar{t}t + i\bar{t}t_1 z_1 + i\bar{t}t_2 z_2 + z_3 - i\bar{t}\tau z_3 + \bar{t}tz_3\big) \tag{2}$$

where $g(z) = 1 + i\bar{t}\tau + \bar{t}t + i\bar{t}t_1 z_1 + i\bar{t}t_2 z_2 - z_3 - i\bar{t}\tau z_3 + \bar{t}tz_3$.

3. – Computational steps.

Our proof relies heavily on the aid of a computer: we will explain the steps we take and of the intermediate results we presents only the significant ones. We will follow the idea of Webster [12] (see also [13]).

Let $\mathcal{Q} = \left\{(z, w) \in \mathbb{C}^3 \times \mathbb{C}^3 \colon \varrho(z, \overline{w}) = \sum_{i=1}^{3} \varepsilon_i z_i \overline{w_i} - 1 = 0\right\}$ be the complexification of the hyperquadric Q. For a fixed $w \in \mathbb{C}^3$ the Segre varieties

$$Q_w = \left\{z \in \mathbb{C}^3 \colon \varrho(z, \overline{w}) = \sum_{i=1}^{3} \varepsilon_i z_i \overline{w_i} - 1 = 0\right\}$$

are complex hyperplanes, depending anti-holomorphically on w. Since $\varrho(z, \bar{z})$ is real, $z \in Q_w$ iff $w \in Q_z$.

Now assume that U_p, V_q are open connected neighborhoods of p and q respectively, $p, q \in Q$, and $f = (f_1, f_2, f_3)$: $U_p \mapsto V_q$ is a biholomorphism mapping p to q and $U_p \cap Q$ into $V_q \cap Q$. If $z, w \in U_p$, $\varrho(w, \bar{z}) = 0$ iff $\varrho(f(w), \overline{f(z)}) = 0$, i.e. $w \in Q_z$ iff $f(w) \in Q_{f(z)}$.

To determine the properties of f: $U_p \mapsto V_q$, fix $z \in U_p$ such that $Q_z \cap U_p \neq \emptyset$.

f induces a map $\tilde{f}$: $Q_z \cap U_p \mapsto Q_{f(z)} \cap V_q$ that, for any fixed z, can be factored through algebraic maps as

$$w \xrightarrow{\pi_z(\overline{w})} T_z Q_w \xrightarrow{df^*} T_{f(z)} Q_{f(w)} \xrightarrow{\pi_{f(z)}^{-1}(\overline{f(w)})} f(w) . \tag{3}$$

The first map $\pi_z(\overline{w})$, from Q_z in the projective space of 2-dimensional hyperplanes, associates to w the holomorphic tangent space $T_z(Q_w)$ to Q_w in z and it is antiholomorphic in w. As a consequence of the non degenerancy of the Levi form of the considered hypersurface (see [12] pag. 55), this map is locally invertible, so the third map $\pi_{f(z)}^{-1}(\overline{f(w)})$ is well defined.

Fix a basis of $T_z Q_w$: for simplicity assume that $w_3 \neq 0$ and choose as basis

$$T_j = \frac{\partial \varrho(z, \overline{w})}{\partial z_j} \frac{\partial}{\partial z_3} - \frac{\partial \varrho(z, \overline{w})}{\partial z_3} \frac{\partial}{\partial z_j} = \varepsilon_j \overline{w_j} \frac{\partial}{\partial z_3} - \overline{w_3} \frac{\partial}{\partial z_j}$$

for $j = 1, 2$. Their images under the linear operator df^* are

$$df^* T_j = \sum_{k=1}^{3} \varepsilon_j \frac{\partial \varrho(z, \overline{w})}{\partial z_j} \frac{\partial f_k(z)}{\partial z_3} \frac{\partial}{\partial \zeta_k} - \frac{\partial \varrho(z, \overline{w})}{\partial z_3} \frac{\partial f_k(z)}{\partial z_j} \frac{\partial}{\partial \zeta_k} = \sum_{k=1}^{3} (T_j f_k(z)) \frac{\partial}{\partial \zeta_k} .$$

The tangent space $T_{f(z)}(Q_{f(w)})$ can be identified with the set of vectors $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3$ such that $\sum_{i=1}^{3} \varepsilon_i \zeta_i \overline{f(w_i)} = 0$; in particular, if ζ is identified with $df^* T_j$ for $j = 1$ and $j = 2$, $\sum_{i=1}^{3} \varepsilon_i (T_j f_i(z)) \overline{f(w_i)} = 0$. Hence the third map in (3) gives $\overline{f(w)}$ as the solution of the following system:

$$\begin{cases} \sum_{k=1}^{3} \varepsilon_k f_k(z) \overline{f_k(w)} = 1 , \\ \sum_{k=1}^{3} \varepsilon_k (T_1 f_k(z)) \overline{f_k(w)} = 0 , \\ \sum_{k=1}^{3} \varepsilon_k (T_2 f_k(z)) \overline{f_k(w)} = 0 , \end{cases} \tag{4}$$

and is completely determined in terms of w, z, $f(z)$, and the derivatives of f in z or its conjugates (see also [8], [6]). One can reach the same conclusion just considering that $\sum_{k=1}^{3} \varepsilon_k f_k(z) \overline{f_k(w)} = 1$ is a defining function for Q_w and that T_1, T_2 are tangential operators and use the non degeneracy of the Levi form of the hypersurface.

The unique solution $\overline{f(w)}$ of the linear system (4) may be written as:

$$\overline{f_j(w)} = \frac{\det \Lambda_j}{\det \Lambda}, \qquad j = 1, 2, 3 \tag{5}$$

where

$$\Lambda = \begin{bmatrix} \varepsilon_1 f_1(z) & \varepsilon_2 f_2(z) & \varepsilon_3 f_3(z) \\ \varepsilon_1 T_1(f_1(z)) & \varepsilon_2 T_1(f_2(z)) & \varepsilon_3 T_1(f_3(z)) \\ \varepsilon_1 T_2(f_1(z)) & \varepsilon_2 T_2(f_2(z)) & \varepsilon_3 T_2(f_3(z)) \end{bmatrix}$$

and Λ_j are the 2×2 minors obtained with the last rows of Λ.

As in [12] now fix $z_0, w_0 \in U_p \cap Q$ such that $\varrho(z_0, \overline{w_0}) = 0$ and prove that $\overline{f(w)}$ is algebraic in a neighborhood of w_0 in the following way: one can find points $v_1, v_2 \in Q_{z_0}$ such that the algebraic curve γ defined by $\varrho(z, \overline{v_1}) = \varrho(z, \overline{v_2}) = 0$ is transverse to Q_{w_0}, and intersects it locally in $\{z_0\}$. $\bigcup_{z \in \gamma} Q_z$ fills simply a neighborhood N of w_0 and if $z, z' \in \gamma$ are distinct points $Q_z \cap Q_{z'} = \emptyset$. For any $w \in$ N there exists a unique solution z of

$$\varrho(z, \overline{w}) = \varrho(z, \overline{v_1}) = \varrho(z, \overline{v_2}) = 0 \tag{6}$$

and z is a rational function of $\overline{w}$. Let $Q_i = \{z \in \mathbb{C}^3 \colon \varrho(z, \overline{v_i}) = 0\}$. If $z \in Q_i$ then $f(z)$ is uniquely determined as an algebraic function of z by (3) just replacing z with v_i and w with z; moreover if $z \in \gamma$ also the jacobian matrix of $f(z)$, $(\partial f_k(z)/\partial z_j)$, is uniquely determined as function of z. $\partial/\partial z_j$ can be obtained as linear combination of operators tangential to Q_i with coefficients that are rational functions of z. If one chooses v_i such that Q_i are parallel to coordinate hyperplanes then $\partial/\partial z_j$, $j = 1, 2, 3$, are tangential operators to Q_1 or Q_2. In this case $\partial f_k(z)/\partial z_j$ will depend only on the values of f on Q_1 or on Q_2, that are algebraic functions of z and on the parameters v_i, $f(v_i)$, $\partial f_k(v_i)/\partial z_j$ or their conjugates. Note that all these parameters are not independent: in fact the values of $f(z)$ on $Q_1 \cap Q_2$ can be deduced from those of $f(z)$ on Q_1 or Q_2 and such values have to agree on $Q_1 \cap Q_2$. Analougous situation for derivatives of $f(z)$ tangential to $Q_1 \cap Q_2$. Replacing $f(z)$ and $\partial f_k(z)/\partial z_j$ in the expression (5) of $\overline{f(w)}$ and subsequently z as determined by (6) after conjugation one obtains $f(w)$ as an algebraic function for $w \in$ N and hence for $w \in \mathbb{C}^3$.

We consider the case $\varepsilon_1 = -1$, $\varepsilon_2 = \varepsilon_3 = 1$ as the other cases are similar and reduce to the case of the ball. In any case $f(w)$ turns out to be a linear rational map, leaving Q invariant.

We used the software package «Mathematica» to perform the necessary calculations. Indicating the equations in (4) as equa1,equa2,equa3 we determine an expression of $\overline{f(w)}$ as function of w, $T_i f_k(z)$, $i = 1, 2$, $k = 1, 2, 3$, with a Solve command. Replacing the corresponding values of $T_i f_k(z)$ we get for the components of $\overline{f(w)}$ the following expressions:

$$\overline{f_1(w)} = \Lambda^{-1}\left(-\overline{w_3}\,\frac{\partial f_2(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_1} + \overline{w_2}\,\frac{\partial f_2(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_1} + \overline{w_3}\,\frac{\partial f_2(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_2} + \overline{w_1}\,\frac{\partial f_2(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_2} - \overline{w_2}\,\frac{\partial f_2(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_3} - \overline{w_1}\,\frac{\partial f_2(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_3}\right)$$

$$\overline{f_2(w)} = \Lambda^{-1}\left(\overline{w_3}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_1} - \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_1} - \overline{w_3}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_2} - \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_2} + \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_3} + \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_3}\right)$$

$$\overline{f_3(w)} = \Lambda^{-1}\left(\overline{w_3}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_2(z)}{\partial z_1} - \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_2(z)}{\partial z_1} - \overline{w_3}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_2(z)}{\partial z_2} - \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_2(z)}{\partial z_2} + \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_2(z)}{\partial z_3} + \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_2(z)}{\partial z_3}\right)$$

where

$$\begin{aligned}\Lambda = f_3(z)\Bigg(& \overline{w_3}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_2(z)}{\partial z_1} - \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_2(z)}{\partial z_1} - \overline{w_3}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_2(z)}{\partial z_2} - \\ & \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_2(z)}{\partial z_2} + \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_2(z)}{\partial z_3} + \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_2(z)}{\partial z_3}\Bigg) + \\ & f_2(z)\Bigg(-\overline{w_3}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_1} + \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_1} + \overline{w_3}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_2} + \\ & \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_2} - \overline{w_2}\,\frac{\partial f_1(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_3} - \overline{w_1}\,\frac{\partial f_1(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_3}\Bigg) + \\ & f_1(z)\Bigg(\overline{w_3}\,\frac{\partial f_2(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_1} - \overline{w_2}\,\frac{\partial f_2(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_1} - \overline{w_3}\,\frac{\partial f_2(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_2} - \\ & \overline{w_1}\,\frac{\partial f_2(z)}{\partial z_3}\,\frac{\partial f_3(z)}{\partial z_2} + \overline{w_2}\,\frac{\partial f_2(z)}{\partial z_1}\,\frac{\partial f_3(z)}{\partial z_3} + \overline{w_1}\,\frac{\partial f_2(z)}{\partial z_2}\,\frac{\partial f_3(z)}{\partial z_3}\Bigg).\end{aligned}$$

Such expressions give $\overline{f(z)}$ on Q_1 just replacing z with v_1 and w with z and the same on Q_2.

Owing to the double action on Q of $\mathrm{Aut}(Q)$, one can assume that $v_1 = (0, 0, 1)$, $v_2 = (0, 1, 0)$ and $f(v_i) = v_i$. These conditions imply that $Q_1 = \{z \in \mathbb{C}^3 \colon z_3 = 1\}$, $Q_2 = \{z \in \mathbb{C}^3 \colon z_2 = 1\}$, and as $T_{v_i}(Q_i)$ are fixed by df^*

$$\frac{\partial f_3(v_1)}{\partial z_1} = 0\,, \qquad \frac{\partial f_3(v_1)}{\partial z_2} = 0\,, \qquad \frac{\partial f_2(v_2)}{\partial z_1} = 0\,, \qquad \frac{\partial f_2(v_2)}{\partial z_3} = 0\,.$$

Substituting the chosen values for v_1, v_2, $f(v_1)$, $f(v_2)$ in $f(z)_{|Q_i}$ we establish some relations among the derivatives of f in v_1 and v_2 that allow to eliminate some of the parameters in excess. As $(f_{|Q_1})_{|Q_2} = (f_{|Q_2})_{|Q_1}$,

$$\left(\frac{\partial f_1(v_2)}{\partial z_1}, \frac{\partial f_1(v_2)}{\partial z_3}, \frac{\partial f_3(v_2)}{\partial z_3}, \frac{\partial f_3(v_2)}{\partial z_1}\right) = \Theta\left(\frac{\partial f_1(v_1)}{\partial z_1}, \frac{\partial f_1(v_1)}{\partial z_2}, \frac{\partial f_2(v_1)}{\partial z_2}, \frac{\partial f_2(v_1)}{\partial z_1}\right),$$

where Θ is a constant of proportionality.

As $v_1 \in Q_1$ calculating the derivatives of $\overline{f(z)}_{|Q_1}$ respect to $\overline{z_1}$ and $\overline{z_2}$ one obtains that

$$\left(\overline{\frac{\partial f_1(v_1)}{\partial z_1}}, \overline{\frac{\partial f_1(v_1)}{\partial z_2}}, \overline{\frac{\partial f_2(v_1)}{\partial z_1}}, \overline{\frac{\partial f_2(v_1)}{\partial z_2}}\right) = \lambda\left(\frac{\partial f_2(v_1)}{\partial z_2}, \frac{\partial f_2(v_1)}{\partial z_1}, \frac{\partial f_1(v_1)}{\partial z_2}, \frac{\partial f_1(v_1)}{\partial z_1}\right)$$

with

$$\lambda = \frac{\overline{\dfrac{\partial f_3(v_1)}{\partial z_3}}}{-\dfrac{\partial f_2(v_1)}{\partial z_1}\dfrac{\partial f_1(v_1)}{\partial z_2} + \dfrac{\partial f_1(v_1)}{\partial z_1}\dfrac{\partial f_2(v_1)}{\partial z_2}}\,.$$

We found some relations for $f(z)$ and $\partial f_k(z)/\partial z_j$ when z is in γ, that lead us to apply the following:

$$\begin{cases} f_1(z) \mapsto \dfrac{z_1 \dfrac{\partial f_1(v_1)}{\partial z_1} + \dfrac{\partial f_1(v_1)}{\partial z_2}}{z_1 \dfrac{\partial f_2(v_1)}{\partial z_1} + \dfrac{\partial f_2(v_1)}{\partial z_2}}\,, \end{cases}$$

$$f_2(z) \mapsto 1\,,$$

$$f_3(z) \mapsto 1\,,$$

$$\frac{\partial f_1(z)}{\partial z_1} \mapsto \frac{-\dfrac{\partial f_1(v_1)}{\partial z_2}\,\dfrac{\partial f_2(v_1)}{\partial z_1} + \dfrac{\partial f_1(v_1)}{\partial z_1}\,\dfrac{\partial f_2(v_1)}{\partial z_2}}{\left(z_1\,\dfrac{\partial f_2(v_1)}{\partial z_1} + \dfrac{\partial f_2(v_1)}{\partial z_2}\right)^2}\,,$$

$$\frac{\partial f_1(z)}{\partial z_2} \mapsto \frac{z_1\,\dfrac{\partial f_1(v_1)}{\partial z_1} + \dfrac{\partial f_1(v_1)}{\partial z_2} + z_1\,\dfrac{\partial f_1(v_1)}{\partial z_2}\,\dfrac{\partial f_2(v_1)}{\partial z_1} - z_1\,\dfrac{\partial f_1(v_1)}{\partial z_1}\,\dfrac{\partial f_2(v_1)}{\partial z_2}}{\left(z_1\,\dfrac{\partial f_2(v_1)}{\partial z_1} + \dfrac{\partial f_2(v_1)}{\partial z_2}\right)^2}\,,$$

$$\frac{\partial f_1(z)}{\partial z_3} \mapsto \frac{z_1\,\dfrac{\partial f_1(v_1)}{\partial z_1} + \dfrac{\partial f_1(v_1)}{\partial z_2} + z_1\,\Theta\,\dfrac{\partial f_1(v_1)}{\partial z_2}\,\dfrac{\partial f_2(v_1)}{\partial z_1} - z_1\,\Theta\,\dfrac{\partial f_1(v_1)}{\partial z_1}\,\dfrac{\partial f_2(v_1)}{\partial z_2}}{\Theta\left(z_1\,\dfrac{\partial f_2(v_1)}{\partial z_1} + \dfrac{\partial f_2(v_1)}{\partial z_2}\right)^2}\,,$$

$$\frac{\partial f_2(z)}{\partial z_2} \mapsto \left(z_1\,\frac{\partial f_2(v_1)}{\partial z_1} + \frac{\partial f_2(v_1)}{\partial z_2}\right)^{-1}\,,$$

$$\frac{\partial f_3(z)}{\partial z_3} \mapsto \left(\Theta\left(z_1\,\frac{\partial f_2(v_1)}{\partial z_1} + \frac{\partial f_2(v_1)}{\partial z_2}\right)\right)^{-1}\,,$$

$$\left.\frac{\partial f_2(z)}{\partial z_3} \mapsto 0,\ \frac{\partial f_2(z)}{\partial z_1} \mapsto 0,\ \frac{\partial f_3(z)}{\partial z_1} \mapsto 0,\ \frac{\partial f_3(z)}{\partial z_2} \mapsto 0\right\}.$$

These relations allow to obtain $\overline{f(w)}$ as function of $\overline{w}$, z_1 and some constants. As z satisfies (6), $z_1 = (-1 + \overline{w_2} + \overline{w_3})/\overline{w_1}$ and the dependence on z_1 can be eliminated. Imposing that $f(w) \in Q$ iff $w \in Q$ one can establish that $\Theta = \partial f_3(v_1)/\partial z_3$ and replace it. Simplifying common factors between numerators and denominators and conjugating one at last obtains

$$f_1(w) = h(w)\left(w_1\,\frac{\partial f_1(v_1)}{\partial z_1} - \frac{\partial f_1(v_1)}{\partial z_2} + w_2\,\frac{\partial f_1(v_1)}{\partial z_2} + w_3\,\frac{\partial f_1(v_1)}{\partial z_2}\right),$$

$$f_2(w) = h(w)\left(w_1\,\frac{\partial f_2(v_1)}{\partial z_1} - \frac{\partial f_2(v_1)}{\partial z_2} + w_2\,\frac{\partial f_2(v_1)}{\partial z_2} + w_3\,\frac{\partial f_2(v_1)}{\partial z_2} + \frac{\partial f_3(v_1)}{\partial z_3} - w_3\,\frac{\partial f_3(v_1)}{\partial z_3}\right),$$

$$f_3(w) = h(w)\left(1 - w_2 + w_1 \frac{\partial f_2(v_1)}{\partial z_1} - \frac{\partial f_2(v_1)}{\partial z_2} + w_2 \frac{\partial f_2(v_1)}{\partial z_2} + w_3 \frac{\partial f_2(v_1)}{\partial z_2}\right),$$

where

$$h(w) = \left(1 - w_2 + w_1 \frac{\partial f_2(v_1)}{\partial z_1} - \frac{\partial f_2(v_1)}{\partial z_2} + w_2 \frac{\partial f_2(v_1)}{\partial z_2} + w_3 \frac{\partial f_2(v_1)}{\partial z_2} + \frac{\partial f_3(v_1)}{\partial z_3} - w_3 \frac{\partial f_3(v_1)}{\partial z_3}\right)^{-1}.$$

Hence $f(w)$ is a linear rational transformation that leaves Q invariant and induces on $\mathbb{C}P^3$ an automorphism of $\widetilde{Q}$.

One can obtain $f(w)$ as an element in (2). $\Upsilon \in \mathrm{Aut}_{e_3}(Q) \cap \mathrm{Aut}_{e_2}(Q)$ iff, in (2), the following relations hold:

$$\tau^2 = \bar{t}^{-1} - t_2^2, \qquad \tau = -i\bar{t}^{-1} - t_2 + it, \qquad \tau^1 = -t_2^1. \tag{7}$$

Taking into account (7) and the following rules for the coefficients in (2)

$$\frac{t_1^1}{t} \mapsto \frac{\partial f_1(v_1)}{\partial z_1}, \quad \frac{t_2^1}{t} \mapsto \frac{\partial f_1(v_1)}{\partial z_2}, \quad \frac{t_1^2}{t} \mapsto \frac{\partial f_2(v_1)}{\partial z_1},$$

$$\frac{t_2^2}{t} \mapsto \frac{\partial f_2(v_1)}{\partial z_2}, \quad \frac{\tau^1}{t} \mapsto -\frac{\partial f_1(v_1)}{\partial z_2},$$

$$\frac{\tau^2}{t} \mapsto -\frac{\partial f_2(v_1)}{\partial z_2} + \frac{\partial f_3(v_1)}{\partial z_3}, \quad \bar{t} \mapsto \frac{1}{t\dfrac{\partial f_3(v_1)}{\partial z_3}}$$

$\Upsilon(w)$ turns to $f(w)$.

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