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Multibump Solutions for Hamiltonian Systems with Fast and Slow Forcing.

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Sunto. – *Si dimostra l'esistenza di infinite soluzioni «multi-bump» – e conseguentemente il comportamento caotico – per una classe di sistemi Hamiltoniani del secondo ordine della forma* $-\ddot{q}+q=(g_1(\omega t)+g_2(t/\omega))\,V'(q)$ *per* ω *sufficientemente piccolo. Qui* $q\in\mathbb{R}^n$, g_1 *e* g_2 *sono funzioni strettamente positive e periodiche e* V *è un potenziale superquadratico (ad esempio* $V(q)=|q|^4$*).*

1. – Introduction.

In this paper we prove that the Hamiltonian system

$$(\mathrm{HS}_\omega)\qquad -\ddot{q}+q=\left(g_1(\omega t)+g_2\left(\frac{t}{\omega}\right)\right)V'(q)$$

has, for $\omega\neq 0$ small, a «chaotic» behavior under the following assumptions:

(V1) $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$;

(V2) $V(0)=V'(0)=V''(0)=0$;

(V3) Exists $\mu>2$ such that $0<\mu V(x)\leqslant V'(x)\cdot x$ for all $x\neq 0$,

while the assumptions on $a_\omega(t)=g_1(\omega t)+g_2(t/\omega)$ are:

(a1) Exist $0<k_1<k_2$ such that $k_1\leqslant a_\omega(t)\leqslant k_2$ for all $t\in\mathbb{R}$;

(a2) g_1 is 1-periodic in t, and g_2 is a T-periodic function with zero mean;

(a3) Exist $\alpha_0>0$ $\beta>0$ such that $g_1'(t)\geqslant\alpha_0$ for all $t\in[0,\beta]$.

REMARK 1.1. – Let us point out that it not necessary that g_2 has zero mean. Indeed, if the mean of g_2 is $m\neq 0$, then we can consider $\tilde{g}_1=g_1+m$ and $\tilde{g}_2=g_2-m$, and assumptions (a1)-(a3) hold for $a_\omega(t)=g_1(\omega t)+g_2(t/\omega)=\tilde{g}_1(\omega t)+\tilde{g}_2(t/\omega)$.

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We also remark that the interval $[0, \beta]$ in assumption (a3) can be replaced by any other interval in $[0, 1]$, and that the existence of such an interval is trivial if we assume that g_1 is nonconstant.

REMARK 1.2. – We believe that our approach works, with minor changes, also when g_1 is an almost periodic function.

About g_2, what we really need in the proof is that it is sufficiently small (with respect to g_1) in some topology, for example in the weak* topology, as it is the case in the situation we consider (indeed $g_2(\cdot/\omega) \to 0$ in such a topology).

To better illustrate what we mean by «chaotic» behavior, let us remark that equation (HS_ω) depends on time in a quasi-periodic fashion. For this reason the most widely accepted definition of «chaos» (that is, conjugation to a Bernoulli shift) does not make sense (indeed such a conjugation implies, for example, existence of periodic solutions).

What we will show is that our system admits a class of multi-bump solutions (see theorem 1.4 for the precise result). For systems depending periodically on time this fact implies the existence of an approximate Bernoulli shift, and positivity of topological entropy (see [25]). So we think that this fact is a good indication of a chaotic behavior in our setting.

Results on existence of chaotic behavior for systems like (HS_ω) date back to [22] and [18]. Indeed many paper have studied the behavior of (HS_ω) under different set of assumptions on the time dependence. In more recent years the study of such a class of systems has been done also using variational techniques (or a mixture of variational and perturbative techniques).

In the papers [8, 24, 25, 10] the periodic case is considered, while the papers [5, 9, 20, 23, 26] study the almost periodic case. In all this papers there is no small parameter. The «typical» result is: there exist infinitely many solutions. Moreover, if a nondegeneracy condition hold (not easy to check), multi-bump solutions exist. In particular the results of [20] apply in our situation, for every $\omega \neq 0$, and imply existence of infinitely many homoclinic solutions for (HS_ω). Existence of a chaotic behavior follows provided an additional nondegenerate condition hold. Remark that we do not need such a condition (for ω small).

The case $g_2 = 0$, and ω small, has been studied in the paper [14], and later extended by [2, 3, 7, 11, 15, 16, 21]. All these papers deals with existence of one or more solution for ω small. g_1 does not have to be periodic, or quasi-periodic in this situation. The typical result is that such a system has, for ω small, solutions (one or more, possibly multi-bump ones) which concentrates near critical points of g_1. For these results to hold, some information on the «limit» problem (corresponding in such a case to $g_1(t) = a \in \mathbb{R}$) is needed. Neither of these results are applicable in our case, even if $g_2 = 0$. Indeed in the papers

[2, 3, 7, 14, 16, 21] one assumes the limit problem has a nondegenerate manifold of solutions, while the papers [11, 15] require an additional assumption (which implies that the limit problem has the Mountain Pass critical level as the lowest critical level).

The case $g_2 \neq 0$, and ω small, has been studied in the paper [1]. In that paper it is proved existence of chaotic behavior when $a_\omega(t) = g_1(\omega t) + g_2(t)$, g_1 and g_2 almost periodic and ω small. An additional assumption with respect to our setting is required (which implies that the limit problem has the Mountain Pass critical level as the lowest critical level-a crucial point in their proof), on the other hand no smallness condition on the term g_2 is required. It is not clear to us if their method can be adapted to deal with equation (HS_ω).

Finally, let us mention that a very interesting problem, which partly motivated our study, arise when $g_1(t) = a \in \mathbb{R}$ is a constant function, the «true» rapidly oscillating problem (case that we do not cover). The problem has been widely investigate under the assumption that the unperturbed problem, corresponding to $g_2 = 0$, has a homoclinic solution. In this case it is known that the separatrix splitting is exponentially small in ω, and existence of a chaotic behavior does not follow using the usual Melnikov techniques. More refined analysis are required, and some result can be obtained when everything is analytic. For a discussion of the problem, see, for example, [4, 12, 13].

Let us now state more precisely our result. In order to do that, we have to introduce some notation and recall some results on quasi-periodic functions.

NOTATION 1.3. – First of all, for us a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ is quasi-periodic if $g(t) = f(\omega_1 t, \omega_2 t, \ldots, \omega_n t)$ for some function $f\colon \mathbb{R}^n \to \mathbb{R}$, f 1-periodic in each of its variables. Given a quasi-periodic function g, we say that $\tau \in \mathbb{R}$ is an ε-period for g if $|g(\tau + t) - g(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$. We will denote by $P(g, \varepsilon)$ the set of ε-periods of g. It is possible to show that for all $\varepsilon > 0$ there exists a $\lambda_\varepsilon > 0$ such that $[a, a + \lambda_\varepsilon] \cap P(g, \varepsilon) \neq \emptyset$ for all $a \in \mathbb{R}$ (i.e. the ε-periods are λ_ε-dense in $\mathbb{R}$).

Given ω, ε, $N \in \mathbb{R}^+$ and $k \in \mathbb{N}$, we will set

$$\mathcal{P}(\omega, N, \varepsilon, k) = \{\vec{p} \in \mathbb{R}^k \,|\, p_i \in P(a_\omega, \varepsilon),\ \ p_{i+1} - p_i \geq N\}.$$

and, given $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon, k)$, we let $p_0 = -\infty$, $p_{k+1} = +\infty$ and

$$I_i = [p_i, p_{i+1}], \qquad i = 0, \ldots k.$$

We also introduce, for $u \in H$ and $A \subset H$, (following [25])

$$\operatorname{dist}(u, A) = \sup_{0 \leq i \leq k} \inf_{v \in A} \|u - v\|_{H^1(I_i, \mathbb{R}^n)}.$$

If $\tau \in \mathbb{R}$ and $v \in H$, we let $(\tau * v)(\cdot) = v(\cdot - \tau)$; if $\vec{p} \in \mathbb{R}^k$, we let $(\vec{p} * v)(\cdot) = \sum_{i=1}^{k} v(\cdot - p_i)$

while if $A \subset H$, we let

$$\vec{p} * A = \left\{ u = \sum_{i=1}^{k} u_i(\cdot - p_i) \;\middle|\; u_i \in A \right\}. \tag{1.6}$$

We are now in position to state our theorem.

THEOREM 1.4. – *Suppose* (V1), (V2), (V3), (a1), (a2) *and* (a3) *hold. Then there exist a* $\omega_0 > 0$ *such that for all* $0 < |\omega| < \omega_0$ *there exists a nonempty set* $\mathcal{A}$ *of homoclinic solutions of* (HS_ω), *compact in the* $C^2(\mathbb{R}; \mathbb{R}^n)$ *and in the* $H^1(\mathbb{R}; \mathbb{R}^n)$ *topology.*

In correspondence to such a set $\mathcal{A}$ *and to each* $r > 0$ *exist* $\varepsilon > 0$ *and* $N > 0$ *such that for every* $k \in \mathbb{N}$, *for every* $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon, k)$ *there exists a solution* $v_{\vec{p}}$ *of* (HS_ω) such that

$$\mathrm{dist}\,(v_{\vec{p}}, \vec{p} * \mathcal{A}) < r\,.$$

2. – Variational setting.

In this section we will introduce the variational problem corresponding to (HS_ω), we show that it has a Mountain Pass geometry.

REMARK 2.1. – Let us remark that, as a consequence of assumption (V2), we have that exist δ_1 and δ_2, $\delta_2 > \delta_1 > 0$, such that

$$V(x) \leqslant \frac{1}{4k_2}|x|^2 \quad \text{and} \quad |V'(x)| \leqslant \frac{1}{4k_2}|x| \quad \text{for all } |x| \leqslant \delta_1. \tag{2.1}$$

$$V'(x)\cdot x \leqslant \frac{1}{2k_2}|x|^2 \quad \text{for all } |x| \leqslant \delta_2. \tag{2.2}$$

We also remark, that, for any $r > 0$ exist K_r such that

$$V(x) \leqslant \frac{K_r}{2}|x|^2 \quad \text{and} \quad |V'(x)| \leqslant \frac{K_r}{2}|x| \quad \text{for all } |x| \leqslant r\,. \tag{2.3}$$

We introduce the Hilbert space

$$H = H^1(\mathbb{R}, \mathbb{R}^n) = \{u \in L^2 \mid \dot{u} \in L^2\},$$

with scalar product and norm given by

$$\langle u, v\rangle = \int_{-\infty}^{+\infty} [\dot{u}\cdot\dot{v} + u\cdot v]\,dt \qquad \|u\|^2 = \langle u, v\rangle.$$

We recall that $H \subset L^\infty$, and that the following inequality holds

$$\|u\|_\infty \leq C_\infty \|u\| \quad \text{for all } u \in H . \tag{2.4}$$

The homoclinic solutions of (HS_ω) are in one to one correspondence with the critical points of the functional

$$f_\omega(u) = \frac{1}{2}\int_{-\infty}^{+\infty} [\,|\dot{u}|^2 + |u|^2]\,dt - \int_{-\infty}^{+\infty} a_\omega(t)\,V(u)\,dt$$

in H. We will denote with $\mathcal{K}_\omega$ the set of nontrivial critical points of f_ω.

An easy consequence of our assumptions is that the functional has the Mountain Pass geometry.

LEMMA 2.2 (Mountain Pass Geometry. – *There exist* $r > 0$ *and* $\alpha > 0$, *not depending on* ω, *such that*

$$f_\omega(u) \geq \alpha > f_\omega(0) = 0 \quad \textit{for all } \|u\| = r$$

and for all $u \neq 0$

$$f_\omega(\lambda u) \to -\infty \quad \textit{as } \lambda \to +\infty .$$

This allows us to introduce the Mountain Pass level, defined as

$$c_\omega = \inf_{\gamma \in \Gamma_\omega} \max_{t \in [0,1]} f_\omega(\gamma(t)) ,$$

where $\Gamma_\omega = \{\gamma \in C([0,1], H) \mid \gamma(0) = 0 \text{ and } f_\omega(\gamma(1)) < 0\}$.

By the Mountain Pass theorem we get that $c_\omega > 0$ and that there exists a Palais Smale sequence for f_ω at level c_ω, namely,

$$u_n \in H \quad \text{such that} \quad f_\omega(u_n) \to c_\omega > 0 \text{ and } \nabla f_\omega(u_n) \to 0 .$$

We remark that in general the Palais Smale condition does not hold as one can easily verify when $a_\omega(t)$ is a periodic function. In the next section we give a precise description of this lack of compactness.

3. – Palais Smale sequences.

In this section we give some results concerning PS sequences. These results are a consequence of the concentration-compactness method of Lions, see [17], and are contained in most of the above quoted paper. So we will just recall the results, and refer to [9], whose setting is very close to the one here, for the proofs we omit.

First of all, by assumption (V3) we easily get that the Palais Smale sequences are bounded and at non negative level.

LEMMA 3.1. – *Let $u_n \in H$ be a Palais Smale sequence for f_ω. Then u_n is bounded and* $\liminf_n f_\omega(u_n) \geq 0$. *In particular, $f_\omega(u) > 0$ if u is a nontrivial critical point.*

PROOF. – By (V3) we have that

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \leq f_\omega(u_n) + \frac{1}{\mu}\|\nabla f_\omega(u_n)\|\,\|u_n\|\,. \tag{3.1}$$

LEMMA 3.2. – *Let $u_n \in H$ be a* PS *sequence for f_ω at level b, weakly convergent to $u \in H$. Then $\nabla f_\omega(u) = 0$*

LEMMA 3.3. – *Let $u_n \in H$ be a* PS *sequence for f_ω not strongly convergent to* 0, *then* $\limsup \|u_n\|_\infty \geq \delta_1$ *and there exists a sequence $t_n \in \mathbb{R}$ such that, up to a subsequence, $u_n(\cdot - t_n) \to v$ strongly in $H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ with $\|v\|_\infty \geq \delta_1$.*

In particular for every ω, for every u solution of (HS_ω), $u \not\equiv 0$, we have that

$$\|u\|_\infty > \delta_1\,. \tag{3.2}$$

Non we give a local compactness property and a characterization of the lack of compactness.

LEMMA 3.4. – *Let $u_n \in H$ be a* PS *sequence for f_ω such that $u_n \to v$ weakly in H. If there exists $T > 0$ such that*

$$\sup_{|t| > T} |u_n(t)| \leq \delta_1 \qquad \text{for all } n \in \mathbb{N}\,,$$

then $u_n \to v$ strongly in H (up to subsequences).

LEMMA 3.5. – *Let $u_n \in H$ be a* PS *sequence for f_ω. Then there are $v_0 \in \mathcal{K}_\omega \cup \{0\}$, $k \in \mathbb{N} \cup \{0\}$, $v_1, \ldots, v_k \in H$, with $\|v_j\|_\infty \geq \delta_1$ and sequences $t_n^1, \ldots, t_n^k \in \mathbb{R}$ such that, up to a subsequence, as $n \to +\infty$, $|t_n^j| \to +\infty$, $t_n^{j+1} - t_n^j \to +\infty$, for all $j = 1, \ldots, k$, and*

$$\left\| u_n - \left(v_0 + \sum_{i=1}^k v_i(\cdot - t_n^i)\right)\right\| \to 0\,.$$

REMARK 3.6. – The function v_i can be characterized as homoclinic solution of a problem at infinity, that is, as solution of the equation

$$-\ddot{v}_i + v_i = \tilde{a}(t)\, V'(v_i)$$

for some $\tilde{a}(\cdot) = \lim_{n \to \infty} a_\omega(\cdot - t_n^i)$ (such limit exists if a_ω is almost periodic).

Now we introduce two functions $T^{\pm}: H \to [-\infty, +\infty]$ defined as follows. For any $u \in H$

$$T^{+}(u) = \sup \{ t \in \mathbb{R} \mid |u(t)| = \delta_1 \} ,$$

$$T^{-}(u) = \inf \{ t \in \mid |u(t)| = \delta_1 \} .$$

with the agreement that $T^{\pm}(u) = \mp \infty$ if $\|u\|_{\infty} < \delta_1$.

First of all note that arguing as in lemma 3.4 we have the following compactness property.

LEMMA 3.7. – *Let $u_n \in H$ be a* PS *sequence for f_{ω}. If the sequence $T^{+}(u_n)$ is bounded then, up to a subsequence, $u_n \to v$ in $H^1([-R, +\infty), \mathbb{R}^N)$ for any $R \in \mathbb{R}$.*

Similarly, if $T^{-}(u_n)$ is bounded then $u_n \to v$ in $H^1([-\infty, R, \mathbb{R})^N)$ for any $R \in \mathbb{R}$.

Now we prove a continuity property of the functions $T^{\pm}$ on Palais Smale sequences. Precisely, we have:

LEMMA 3.8. – *Let $u_n \in H$ be a* PS *sequence for f_{ω} such that $u_n \to v$ weakly in H. If the sequence $T^{+}(u_n)$ $(T^{-}(u_n))$ is bounded then (up to a subsequence) $T^{+}(u_n) \to T^{+}(v)$ $(T^{-}(u_n) \to T^{-}(v))$.*

PROOF. – Let us assume $T^{+}(u_n)$ is bounded. Then $v \not\equiv 0$ and, from lemma 3.2 and 3.3 we deduce that $v \in \mathcal{K}_{\omega}$ and $\|v\|_{\infty} \geqslant \delta_1$. Therefore $T^{+}(v) \in \mathbb{R}$. If there exists $R > 0$ such that $|T^{+}(u_n)| < R$, we have in particular that there exists a subsequence of $T^{+}(u_n)$, that we denote again by $T^{+}(u_n)$, that converges to some $\bar{t} \in \mathbb{R}$. We claim that $\bar{t} = T^{+}(v)$. Indeed, by lemma 3.7 we have that $u_n \to v$ (up to a subsequence) in $H^1([-R, +\infty), \mathbb{R}^N)$ and by the continuous Sobolev embedding it converges in $L^{\infty}([-R, +\infty), \mathbb{R}^N)$. Therefore, $\bar{t} \leqslant T^{+}(v)$ plainly follows. Now, arguing by contradiction, let us suppose that $\bar{t} < T^{+}(v)$. By continuity there exists $\varrho \in (0, 1/2(T^{+}(v) - \bar{t}))$ such that $|v(t)| \leqslant \delta_2$ for any $t \in [T^{+}(v) - \varrho, T^{+}(v)]$ and, since $\ddot{v} = v - a_{\omega}(t) V'(v)$, by (2.2), there exists $a > 0$ such that $d/(dt)|v(t)|^2 \leqslant -a$ for all $t \in [T^{+}(v) - \varrho, T^{+}(v)]$. Hence we get

$$|v(T^{+}(v) - \varrho)|^2 = \delta_1^2 - \int_{T^{+}(v) - \varrho}^{T^{+}(v)} \frac{d}{dt} |v(t)|^2 \, dt \geqslant \delta_1^2 + a\varrho .$$

Hence there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geqslant \bar{n}$ we have $|u_n(T^{+}(v) - \varrho)| > \delta_1$ and $T^{+}(u_n) > T^{+}(v) - \varrho$, that is a contradiction. Exactly the same argument applies for the sequence $T^{-}(u_n)$. ■

4. – Estimates on solutions.

LEMMA 4.1. – *Let* $b > 0$.

Then there exist $\delta_3 > 0$ *and* $\delta_4 > 0$ *such that for every* ω, *for every* $u \in H$ *solution of* (HS_ω) *such that* $f_\omega(u) \leq b$ *one has*

$$(a) \quad \|u\| \leq \delta_3 = \sqrt{\frac{2\mu b}{\mu - 2}}\,;$$

$$(b) \quad \|\dot{u}\|_\infty \leq \delta_4 \equiv \delta_3(1 + C_\infty) + k_2 \sup\left\{|V'(x)| \,\big|\, |x| \leq C_\infty \delta_3\right\}.$$

PROOF. – Just observe that from

$$b \geq f_\omega(u) - \frac{1}{\mu}\langle \nabla f_\omega(u), u\rangle =$$

$$= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2 + \frac{1}{\mu}\int a_\omega(t)(V'(u)\cdot u - \mu V(u)) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2$$

one deduces point (a) of the lemma.

Since, by (a), $\|u\| \leq \delta_3 = \sqrt{2\mu b/(\mu - 2)}$, we get $\|u\|_\infty \leq C_\infty \delta_3$. Take any $t \in \mathbb{R}$, and let $n \in \mathbb{N}$ be such that $t \in [n, n+1] \subset \mathbb{R}$. From

$$\delta_3^2 \geq \int_{\mathbb{R}} |\dot{u}|^2 + |u|^2 \geq \int_n^{n+1} |\dot{u}|^2 \geq \inf_{s \in [n, n+1]} |\dot{u}(s)|^2 = |\dot{u}(\theta)|^2$$

we find that in each interval of length one there exists a point θ such that $|\dot{u}(\theta)| \leq \delta_3$.

Then, using the equation (HS_ω), we find

$$|\dot{u}(t)| \leq |\dot{u}(\theta)| + |\dot{u}(t) - \dot{u}(\theta)| \leq |\dot{u}(\theta)| + |t - \theta|\,|\ddot{u}(\xi)| \leq$$

$$\leq \delta_3 + |u(\xi|) + k_2|V'(u(\xi))| \leq \delta_3 + C_\infty \delta_3 + k_2 \sup_{|x| \leq C_\infty \delta_3} |V'(x)| = \delta_4.$$

LEMMA 4.2. – *Let* δ_1 *and* δ_2 *be as in* (2.1) *and* (2.2), $\omega \in \mathbb{R}$, $b > 0$ *and* u *a solution of* (HS_ω) *such that* $f_\omega(u) \leq b$. *Assume there exist* $[a, c] \subset \mathbb{R}$ *such that*

1. $|u(t)| \geq \delta_1$ *for all* $t \in [a, c]$;
2. $|u(a)| = |u(c)| = \delta_1$.

Then

$$(a)\quad c-a \geqslant \frac{(\mu-2)(\delta_2-\delta_1)^2}{\mu b};$$

$$(b)\quad \int_a^c [\,|\dot u|^2+|u|^2] \geqslant 2\sqrt{2}(\delta_2-\delta_1)\,\delta_1;$$

$$(c)\quad \int_a^c V(u) \geqslant \frac{(\mu-2)(\delta_2-\delta_1)^2}{\mu b} \inf_{|x|\geqslant\delta_1} V(x).$$

PROOF. – Let $\bar t \in [a, c]$ be a local maximum of $t \mapsto |u(t)|$. Then $|u(\bar t)| > \delta_2$ since

$$\frac{1}{2}\frac{d^2}{dt^2}|u(t)|^2 = |\dot u(t)|^2 + |u(t)|^2 - a_\omega(t)\, V'(u(t))\cdot u(t) >$$

$$> |u(t)|^2 - k_2\frac{1}{2k_2}|u(t)|^2 > 0\,.$$

whenever $|u(t)| \leqslant \delta_2$.

Then

$$\delta_2-\delta_1 \leqslant |u(\bar t)| - |u(a)| \leqslant |u(\bar t)-u(a)| \leqslant \left|\int_a^{\bar t} \dot u(s)\,ds\right| \leqslant$$

$$\leqslant \sqrt{\bar t - a}\left(\int_a^{\bar t} |\dot u(s)|^2 ds\right)^{1/2}$$

and we obtain

$$\int_a^{\bar t} |\dot u(s)^2 ds \geqslant \frac{(\delta_2-\delta_1)^2}{\bar t - a}\,.$$

Reasoning similarly on the interval $[\bar t, c]$, one finds

$$\int_a^c |\dot u(s)|^2 ds \geqslant 2\frac{(\delta_2-\delta_1)^2}{c-a}\,.$$

Since $|u(t)| \geqslant \delta_1$ in $[a, c]$, we deduce that

$$\int_a^c |\dot{u}(s)|^2 + |u(s)|^2 ds \geqslant 2\,\frac{(\delta_2 - \delta_1)^2}{c-a} + (c-a)\,\delta_1^2. \tag{4.1}$$

We know from lemma 4.1 that

$$\frac{2\mu b}{\mu - 2} \geqslant \int_a^c |\dot{u}(s)|^2 + |u(s)|^2 ds \geqslant 2\,\frac{(\delta_2 - \delta_1)^2}{c-a}$$

from which we deduce

$$c - a \geqslant \frac{(\mu-2)(\delta_2 - \delta_1)^2}{\mu b}.$$

Always from (4.1) we find that

$$\int_a^c |\dot{u}(s)|^2 + |u(s)|^2 ds \geqslant 2\sqrt{2}(\delta_2 - \delta_1)\,\delta_1.$$

(remark that the estimate is independent from b).

Finally

$$\int_a^c V(u(s))\,ds \geqslant (c-a) \inf_{|x| \geqslant \delta_1} V(x) \geqslant \frac{(\mu-2)(\delta_2-\delta_1)^2}{\mu b} \inf_{|x| \geqslant \delta_1} V(x),$$

and the lemma is proved. ■

LEMMA 4.3. – *Let u be a solution of* (HS$_\omega$) *such that $|u(t)| \leqslant \delta_1$ for all $t \in [a, c]$. Assume that $\Delta = c - a \geqslant 1$.*

Then, for all $t \in [a, c]$,

$$|u(t)|^2 \leqslant \frac{|u(a)|^2 - |u(c)|^2 e^{-\Delta}}{1 - e^{-2\Delta}}\, e^{-(t-a)} + \frac{|u(c)|^2 - |u(a)|^2 e^{-\Delta}}{1 - e^{-2\Delta}}\, e^{(t-c)}. \tag{4.2}$$

As a consequence, setting $m = (a+c)/2$, we have

$$|u(t)|^2 \leqslant 4\delta_1^2 e^{-\Delta/2} \cosh\,(t-m) \tag{4.3}$$

and

$$|\dot{u}(t)| \leqslant 9\delta_1 e^{-\Delta/4}\left(1 + \sinh\frac{(t-m)}{2}\right). \tag{4.4}$$

PROOF. – Let

$$L = -\frac{d^2}{dt^2} + 1$$

and

$$w(t) = \frac{|u(a)|^2 - |u(c)|^2 e^{-\Delta}}{1 - e^{-2(c-a)}} e^{-(t-a)} + \frac{|u(c)|^2 - |u(a)|^2 e^{-\Delta}}{1 - e^{-2(c-a)}} e^{(t-c)}.$$

It is immediate to check that w solves the boundary value problem

$$\begin{cases} Lw = 0 & \text{in } [a, c], \\ w(a) = |u(a)|^2, \\ w(c) = |u(c)|^2. \end{cases} \tag{4.5}$$

Let u be a solution of (HS_ω) such that $|u(t)| \leq \delta_1$ for all $t \in [a, c]$. Then, using (HS_ω) and (2.1), we deduce that

$$\begin{aligned} L(|u(t)|^2) &= -2|\dot{u}(t)|^2 - 2u(t)\cdot\ddot{u}(t) + |u(t)|^2 \leq \\ &\leq 2u(t)\cdot(-u(t) + a_\omega(t)\,V'(u(t))) + |u(t)|^2 \leq \\ &\leq -2|u(t)|^2 + 2k_2 V'(u(t))\cdot u(t) + |u(t)|^2 \leq \\ &\leq -|u(t)|^2 + |u(t)|^2 \leq 0 \end{aligned}$$

for all $t \in [a, c]$. Then

$$\begin{cases} L(w - |u|^2) \geq 0 & \text{in } (a, c), \\ w(t) - |u(t)|^2 = 0 & \text{for } t = a, c\,. \end{cases}$$

By maximum principle we obtain

$$|u(t)|^2 \leq w(t) \qquad \text{for all } t \in [a, c].$$

Since

$$w(t) \leq \frac{|u(a)|^2}{1 - e^{-2\Delta}} e^{-(t-a)} + \frac{|u(c)|^2}{1 - e^{-2\Delta}} e^{(t-c)}$$

(4.3) follows since $|u(a)| \leq \delta_1$ and $|u(c)| \leq \delta_1$. As a consequence we have that

$$|u(t)| \leq 2\sqrt{2}\delta_1 e^{-\Delta/4} \cosh\frac{t-m}{2}\,. \tag{4.6}$$

To prove (4.4), we remark that

$$|\dot{u}(m)| = \left| u(m+1) - u(m) - \int_m^{m+1} dt \int_m^t \ddot{u}(\tau)\, d\tau \right| \leqslant$$

$$\leqslant |u(m+1)| + |u(m)| + \int_m^{m+1} dt \int_m^t |u(\tau|) - a_\omega(\tau)\, V'(u(\tau))\, d\tau \leqslant$$

$$\leqslant 2\sqrt{2}\delta_1 e^{-\Delta/4}(\cosh(1/2) + \cosh(0)) + \frac{3}{2}\int_m^{m+1} dt \int_m^t |u(\tau)|\, d\tau \leqslant$$

$$\leqslant 2\sqrt{2}\delta_1 e^{-\Delta/4}(3\cosh(1/2) - 1) \leqslant 9\delta_1 e^{-\Delta/4}$$

(we have used the bounds (4.6) and (2.1) together with the equation (HS_ω)). Finally, from

$$\dot{u}(t) = \dot{u}(m) + \int_m^t \ddot{u}(s)\, ds$$

and the above bounds one deduces (4.4). ∎

LEMMA 4.4. – *Let $b \in \mathbb{R}$. Then exists $\omega_0 > 0$ such that for all $0 < |\omega| < \omega_0$ if $u \in H$ is such that $f_\omega(u) \leqslant b$. Assume that*

$$|u(\beta/4\omega)| = \delta_1 \ \textit{ and } \ |u(t)| < \delta_1 \ \textit{ for all } \ t < \beta/4\omega$$

or that

$$|u(3\beta/4\omega)| = \delta_1 \ \textit{ and } \ |u(t)| < \delta_1 \ \textit{ for all } \ t > 3\beta/4\omega.$$

Then u is not a solution of (HS_ω).

PROOF. – Let us assume that u satisfying the above conditions is a solution of (HS_ω). We will find a contradiction.

Let

$$A = \left\{ t \in \left[0, \frac{\beta}{\omega}\right] \,\middle|\, |u(t)| \geqslant \delta_1 \right\}.$$

A is not empty, and from our assumptions it follows that it contains an interval of the form $[\beta/4\omega, \beta']$ since $(\beta/4\omega)$ in A cannot be a local maximum of $|u(t)|^2$. Indeed the equation implies that

$$\frac{d^2}{dt^2}|u(t)|^2 \geqslant |u(t)|^2 - k_2 V'(u(t))\cdot u(t) > 0 \quad \text{for all } t \text{ such that } |u(t)| \leqslant \delta_1.$$

Then, using lemma 4.1, we deduce

$$\delta_3^2 \geqslant \int |u|^2 \geqslant \int_A |u|^2 \geqslant \delta_1^2 \operatorname{meas}(A)$$

and hence

$$\operatorname{meas}(A) \leqslant \frac{\delta_3^2}{\delta_1^2}.$$

Remark that the constant δ_3^2/δ_1^2 does not depend on ω.

We now claim that exists ω_0 such that for all $|\omega| < \omega_0$ we can find a closed interval $[\alpha_\omega, \beta_\omega]$ satisfying

(a) $[\alpha_\omega, \beta_\omega] \subset [\beta/4\omega, \beta/\omega]$;

(b) exists k_0 independent of ω such that $\beta_\omega - \alpha_\omega \geqslant k_0/\omega$;

(c) $|u(t)| \leqslant \delta_1$ for all $t \in [\alpha_\omega, \beta_\omega]$.

Since u is a continuous function, $A = \bigcup_{i=1}^{l} [a_i, b_i]$. Remark that $a_1 = \beta/4\omega$ and we also set $a_{l+1} = \beta/\omega$ if $b_l < \beta/\omega$. Since $\operatorname{meas}(A) \leqslant \delta_3^2/\delta_1^2$, in $[\beta/4\omega, \beta/\omega] \setminus A$ there is at least one interval (b_i, a_{i+1}) such that

$$a_{i+1} - b_i \geqslant \frac{1}{l}\left(\frac{\beta}{\omega} - \frac{\beta}{4\omega} - \frac{\delta_3^2}{\delta_1^2}\right) = \frac{1}{l}\left(\frac{3\beta}{4\omega} - \frac{\delta_3^2}{\delta_1^2}\right)$$

and the claim is proved provided we can estimate l independently of ω by choosing $[\alpha_\omega, \beta_\omega]$ to be the largest interval in $\beta/4\omega, \beta/\omega] \setminus A$.

In order to bound l independently of ω, we simply use, in each interval $[a_i, b_i]$, point (b) of lemma 4.1 together with point (a) of lemma 4.1. We find

$$\delta_3^2 \geqslant \sum_{i=1}^{l} \int_{a_i}^{b_i} |\dot{u}|^2 + |u|^2 \geqslant l 2\sqrt{2}(\delta_2 - \delta_1)\,\delta_1$$

and the claim follows provided $\omega < \omega_0$, ω_0 sufficiently small. Corresponding to such an interval $[\alpha_\omega, \beta_\omega]$ we define $\Delta = \beta_\omega - \alpha_\omega$ and $m = (\alpha_\omega + \beta_\omega)/2$. Observe that we are in position to apply lemma 4.3 to such an interval.

We will now show that such an u cannot be a solution of (HS_ω) by showing that

$$\|\nabla f_\omega(u)\| \geqslant C_1 \omega - C_2 \omega^2.$$

then the lemma will follow provided ω_0 is sufficiently small. In the following, C and $\widetilde{C}$ will denote positive constants, independent of ω, which can vary from line to line.

Let m be as above and define $\phi\colon \mathbb{R} \to \mathbb{R}$ by setting

$$\phi(t) = \begin{cases} 1\,, & t \leqslant m\,, \\ m - t + 1\,, & m \leqslant t \leqslant m + 1\,, \\ 0\,, & t \geqslant m + 1\,. \end{cases}$$

Since u is a solution of (HS_ω), it is regular; in particular $\dot{u} \in H$ (indeed $\dot{u} \in L^2$ and $\ddot{u} = u - a_\omega V'(u) \in L^2$). Then also $v = \phi\dot{u} \in H$. Let us estimate

$$\text{(4.7)} \qquad \langle \nabla f_\omega(u), v\rangle = \int_{-\infty}^{+\infty} \dot{u}\cdot\dot{v} + u\cdot v - \int_{-\infty}^{+\infty} a_\omega(t)\, V'(u)\cdot v =$$

$$= \int_{-\infty}^{+m} \dot{u}\cdot\ddot{u} + u\cdot\dot{u} - \int_{-\infty}^{m} a_\omega(t)\, V'(u)\cdot\dot{u} +$$

$$+ \int_{m}^{m+1} \dot{u}\cdot(\phi\ddot{u} - \dot{u}) + u\cdot\phi\dot{u} - \int_{m}^{m+1} a_\omega(t)\, V'(u)\cdot\phi\dot{u}\,.$$

We begin estimating the term

$$\int_{m}^{m+1} \dot{u}\cdot(\phi\ddot{u} - \dot{u}) + u\cdot\phi\dot{u} - \int_{m}^{m+1} a_\omega(t)\, V'(u)\cdot\phi\dot{u}\,.$$

Using the exponential estimates of lemma 4.3 together with the equation (HS_ω), one easily deduces that this term is smaller then $\widetilde{C}e^{-\Delta/2} \leqslant C_1 e^{-k_0/\omega}$ for some constant C_1 independent of ω.

In order to estimate the other term of (4.7) we perform an integration by part, finding

$$\text{(4.8)} \qquad \int_{-\infty}^{+m} \dot{u}\cdot\ddot{u} + u\cdot\dot{u} - \int_{-\infty}^{m} a_\omega(t) V'(u)\cdot\dot{u} =$$

$$= \frac{1}{2}\,|\,\dot{u}(m)\,|^2 + \frac{1}{2}\,|u(m)\,|^2 - a_\omega(m)\, V(u(m)) + \int_{-\infty}^{m} \dot{a}_\omega(t)\, V(u(t))\,.$$

Always by the estimates in lemma 4.3, one finds that the term $\frac{1}{2}|\,\dot{u}(m)\,|^2 + \frac{1}{2}|u(m)\,|^2 - a_\omega(m)\, V(u(m))$ can be bounded by $C_1 e^{-k_0/\omega}$ (eventually taking C_1 larger). So we are left with the term

$$\int_{-\infty}^{m} \dot{a}_\omega(t)\, V(u(t))\,.$$

Recalling that $a_\omega(t) = g_1(\omega t) + g_2(t/\omega)$, this term breaks down to

$$\omega \int_{-\infty}^{m} \dot{g}_1(\omega t)\, V(u(t))\, dt + \frac{1}{\omega} \int_{-\infty}^{m} \dot{g}_2\left(\frac{t}{\omega}\right) V(u(t))\, dt \,.$$

Let us show that the term $\frac{1}{\omega} \int_{-\infty}^{m} \dot{g}_2\left(\frac{t}{\omega}\right) V(u(t))\, dt$ is smaller then $C\omega^2$ provided ω is small. Given any periodic, zero mean function h, denote by $(\Pi h)(t)$ the zero mean primitive of h. We have that, for some constant k_1, $\|\Pi^j g_2\|_\infty \leq k_1$ for $j = 0, 1, 2$. Integrating by part we obtain

$$\frac{1}{\omega} \int_{-\infty}^{m} \dot{g}_2\left(\frac{t}{\omega}\right) V(u)\, dt = g_2\left(\frac{m}{\omega}\right) V(u(m)) - \int_{-\infty}^{m} g_2\left(\frac{t}{\omega}\right) V'(u)\cdot \dot{u}\, dt =$$

$$= g_2\left(\frac{m}{\omega}\right) V(u(m)) - \omega(\Pi g_2)\left(\frac{m}{\omega}\right) V'(u(m))\cdot \dot{u}(m) +$$

$$+ \omega \int_{-\infty}^{m} (\Pi g_2)\left(\frac{t}{\omega}\right) [V''(u)\, \dot{u}\cdot\dot{u} + V'(u)\cdot(u - a_\omega(t)\, V'(u))]\, dt \,.$$

The terms $g_2(m/\omega)\, V(u(m))$ and $\omega(\Pi g_2)(m/\omega)\, V'(u(m))\cdot\dot{u}(m)$ are exponentially small in $1/\omega$. In order to evaluate the integral, we perform an additional integration by part. We obtain the boundary terms (exponentially small as before) and the integral term

$$\omega^2 \int_{-\infty}^{m} (\Pi^2 g_2)\left(\frac{t}{\omega}\right) \frac{d}{dt}[V''(u)\, \dot{u}\cdot\dot{u} + V'(u)\cdot u - a_\omega V'(u)\cdot V'(u)]\, dt \,.$$

Let us examine the terms arising from this integral. First observe that $\|u\|_\infty$ and $\|\dot{u}\|_\infty$ are bounded independently of ω; this implies that also $V(u(t))$ as well as all $V'(u(t))$, $V''(u(t))$ and $V'''(u(t))$ are bounded independently of ω. Then

$$|V'''(u)[\dot{u}, \dot{u}, \dot{u}]| \leq C|\dot{u}|^2,$$

while

$$|V''(u)\, \ddot{u}\cdot\dot{u}| = |V''(u)\, u\cdot\dot{u} - V''(u)\, a_\omega V'(u)\cdot\dot{u}| \leq C|u||\dot{u}|$$

(we have used the equation and the fact that $|V'(u)| \leq K_{\|u\|_\infty} |u|$, see (2.3)). Similarly we have that

$$\left|\frac{d}{dt} V'(u)\cdot u\right| \leq C|\dot{u}||u|\,.$$

We also have that

$$|a_\omega V''(u)\,\dot u\cdot V'(u)| \leqslant C|\dot u||u|$$

and that

$$|\omega \dot g_1(\omega t)\,V'(u)\cdot V'(u)| \leqslant \omega C|u|^2.$$

Let us examine now

$$\omega^2 \int_{-\infty}^{m} (\Pi^2 g_2)\left(\frac{t}{\omega}\right)\frac{1}{\omega}\,\dot g_2\left(\frac{t}{\omega}\right)|V'(u)|^2\,dt\,.$$

Observe that $\dot g_2(t)(\Pi^2 g_2)(t)$ is a zero mean, T-periodic function, hence the function $G \equiv \Pi(\dot g_2(\Pi^2 g_2))$ is a periodic (and hence bounded) zero mean function. Then, performing another integration by part, we obtain

$$\omega \int_{-\infty}^{m} (\Pi^2 g_2)\left(\frac{t}{\omega}\right)\dot g_2\left(\frac{t}{\omega}\right)|V'(u)|^2\,dt =$$

$$= \omega^2 G\left(\frac{m}{t}\right)|V'(u(m))|^2 - 2\omega^2 \int_{-\infty}^{m} G\left(\frac{t}{\omega}\right)V'(u)\cdot V''(u)\,\dot u\,dt$$

and since

$$\left| G\left(\frac{t}{\omega}\right)V'(u)\cdot V''(u)\,\dot u\right| \leqslant C|u||\dot u|$$

we finally find, collecting all the pieces, that

$$\left|\frac{1}{\omega}\int_{-\infty}^{m} \dot g_2\left(\frac{t}{\omega}\right)V(u)\,dt\right| \leqslant \tilde C\omega^2 \int_{-\infty}^{m} (|u|^2 + |u||\dot u| + |\dot u|^2) \leqslant C\omega^2$$

where C does not depend on ω.

Let us now examine the last term in $\langle \nabla f_\omega(u), v\rangle$, the term

$$\omega \int_{-\infty}^{m} \dot g_{1\omega}(\omega t)\,V(u(t))\,dt\,.$$

We have that $|u(t)| \leqslant \delta_1$ for all $t \in [0, \beta/4\omega]$ and $u(t) \to 0$ as $t \to -\infty$, hence, by the lemma 4.3,

$$|u(t)|^2 \leqslant \delta_1 e^{-(\beta/4\omega - t)}$$

for all $t \leqslant \beta/4\omega$. We deduce that

$$\left| \int_{-\infty}^{0} \dot{g}_1(\omega t)\, V(u(t))\, dt \right| \leqslant C \int_{-\infty}^{0} |u|^2 \leqslant Ce^{-\beta/4\omega}.$$

From assumption (a3), we deduce, using the estimate (c) of lemma 4.2, which we can apply in the interval $[\beta/4\omega, b_1] \subset [0, m]$

$$\int_0^m \dot{g}_1(\omega t)\, V(u(t))\, dt \geqslant \alpha_0 \int_0^m V(u(t)) \geqslant \alpha_0 \int_{\beta/4\omega}^{b_1} V(u(t)) \geqslant C\,.$$

All the above estimates imply that

$$C_1 \omega - C_2 \omega^2 \leqslant \langle \nabla f_\omega, v \rangle \leqslant \| \nabla f_\omega(v) \| \|v\| \,.$$

To conclude, it is enough to show that $\|v\|$ is bounded from above independently of ω. This is a consequence of point (a) of lemma 4.1.

The other part of the lemma follow in a similar fashion. ∎

5. – Multibump solutions.

From now on we fix w, $0 < |\omega| < \omega_0$ and $c^* > c_\omega$. By lemmas 4.4 and 3.8 we deduce the following property:

(*) there exist $t_+, t_- \in \mathbb{R}$, $\eta > 0$ and $\bar{\mu} > 0$ such that $\|\nabla f_\omega(u)\| \geqslant \bar{\mu}$ for any $u \in \{f_\omega \leqslant c^*\}$ for which $T^+(u) \in I^+ \equiv [t^+ - \eta, t^+ + \eta]$ or $T^-(u) \in I^- \equiv [t^- - \eta, t^- + \eta]$.

Property (*) is actually the nondegeneracy condition needed in [20] to prove the existence of multibump solutions for (HS_ω), therefore the technique used there apply. In the following we just sketch how property (*) allow us to prove the existence of multibump solutions, referring to [20] and [19] for all the proofs we omit.

Let us fix $h^* \in (0, 1/4(c^* - c_\omega))$, and define

$$\mathfrak{I}^\pm = \{u \in H \,|\, T^\pm(u) \in I^\pm\} \cap \{u \in H \,|\, f_\omega(u) \leqslant c_\omega + h^*\}\,.$$

Let us also define, for $\tau \in \mathbb{R}$,

$$\mathfrak{I}^\pm_\tau = \{u \in H \,|\, T^\pm(u) + \tau \in I^\pm\} \cap \{u \in H \,|\, f_\omega(u) \leqslant c_\omega + h^*\}\,.$$

Since $a_\varepsilon(t)$ depends quasi-periodically in time, we get the existence of a sequence $\tau_n \to \pm\infty$ as $n \to \pm\infty$ for which we have an estimate from below for the norm of the gradient of f_ω in the set $\bigcup_{n \in \mathbb{Z}} \mathfrak{I}^+_{\tau_n} \cup \bigcup_{n \in \mathbb{Z}} \mathfrak{I}^-_{\tau_n}$.

Indeed, the quasi-periodicity of the potential reflects on the functional in the following way:

LEMMA 5.1. – *For any* $\lambda > 0$ *and* $R > 0$ *there exists* $\varepsilon > 0$ *such that for any* $u \in H$ *such that* $\|u\| \leqslant R$, *if* $\tau \in P(a_\omega, \varepsilon)$ *then,*

(*a*) $|\|\nabla f_\omega(u)\| - \|\nabla f_\omega(\tau * u)\|| \leqslant \lambda$;

(*b*) $|f_\omega(u) - f_\omega(\tau * u)| \leqslant \lambda$.

PROOF. – For any $\tau \in P(a_\omega, \varepsilon)$ we have

$$|\langle \nabla f_\omega(u) - \nabla f_\omega(\tau * u), h\rangle| \leqslant \int_{\mathbb{R}} |(a_\omega(t) - a_\omega(t+\tau)||V'(u)||h| \leqslant$$

$$\leqslant \sup_{t \in \mathbb{R}} |(a_\omega(t) - a_\omega(t+\tau)| \int_{\mathbb{R}} |V'(u)||h| \leqslant C\varepsilon \int_{\mathbb{R}} |u||h| \leqslant C\varepsilon\|h\|$$

and (*a*) plainly follows. The proof of (*b*) is analogous. ∎

Therefore, we get

LEMMA 5.2. – *There exist* $\mu_0 > 0$, $\varepsilon_0 > 0$ *and a sequence* $\tau_n \in P(a_\omega, \varepsilon_0)$, $\tau_n \to \pm\infty$, *as* $n \to \pm\infty$, *with* $\tau_i \leqslant \tau_{i+1}$, *for all* $i \in \mathbb{Z}$, *and* $\tau_0 = 0$, *such that*:

$$\|\nabla f_\omega(u)\| \geqslant \mu_0 \quad \textit{for all } u \in \left(\bigcup_{n \in \mathbb{Z}} \mathfrak{J}^+_{\tau_n}\right) \cup \left(\bigcup_{n \in \mathbb{Z}} \mathfrak{J}^-_{\tau_n}\right).$$

PROOF. – First of all note that, by (3.1), there exists $R > 0$ such that $\|\nabla f_\omega(u)\| \geqslant \bar{\mu}$ for all $\|u\| \geqslant R$ such that $f_\omega(u) \leqslant c^*$, where $\bar{\mu}$ is given by (*).
Moreover, by lemma 5.1 for any $\lambda > 0$ there exists $\varepsilon > 0$ such that if $\|u\| \leqslant R$ and $\tau \in P(a_\omega, \varepsilon)$ then $|\|\nabla f_\omega(u)\| - \|\nabla f_\omega(\tau * u)\|| \leqslant \lambda$ and $|f_\omega(u) - f_\omega(\tau * u)| \leqslant \lambda$.
Therefore, choosing $\lambda \leqslant (1/2) \min\{\bar{\mu}, c^* - (c_\omega + h^*)\}$ there exists ε_0 and a sequence $\tau_n \in P(a_\omega, \varepsilon_0)$, $\tau_n \to \pm\infty$ as $n \to \pm\infty$, and $\tau_i \leqslant \tau_{i+1}$, for any $i \in \mathbb{Z}$, such that the lemma follows with $\mu_0 = \bar{\mu}/2$.

Then, given $\nu \in (0, \mu_0)$ we define the set

$$A^\nu = \{u \in H \,|\, \|\nabla f_\omega(u)\| < \nu \text{ and } f_\omega(u) \leqslant c_\omega + h^*\}.$$

Moreover, for $\varrho \in (0, C_\infty \delta_1)$, $i, j \in \mathbb{Z}$ and $\tau_i \in P(a_\omega, \varepsilon)$ given by lemma 5.2, we consider the sets

$$\mathcal{U}_\varrho^\nu = A^\nu \cap \{u \in H \mid \|u\| < \varrho\},$$

$$A_{ij}^\nu = A^\nu \cap \{u \in H \mid T^+(u) \in [t^+ + \eta - \tau_i, t^+ - \eta - \tau_{i-1}] \text{ and } T^-(u) \in [t^- + \eta - \tau_j, t^- - \eta - \tau_{j-1}]\}.$$

By lemma 3.7 we have the following compactness result:

LEMMA 5.3. – *Let $u_n \in X$ be a* PS *sequence for f_ω. If $u_n \in A_{ij}^\nu$, for some fixed $i, j \in \mathbb{Z}$, then u_n is precompact.*

Clearly $A_{ij}^\nu \cap A_{i'j'}^\nu = \emptyset$ for all $(i, j) \neq (i', j')$. Moreover, for ν sufficiently small the sets $A_{ij}^\nu \subset H$ are uniformly disjoint. Precisely, we have:

LEMMA 5.4. – *There exist $\bar{\nu} \in (0, \mu_0)$ and $\tilde{\varrho} > 2\bar{\nu}$ such that:*

$$A^{\bar{\nu}} = \mathcal{U}_{\tilde{\varrho}}^{\bar{\nu}} \cup \bigcup_{i, j \in \mathbb{Z}} A_{ij}^{\bar{\nu}}$$

$$\mathcal{U}_{\tilde{\varrho}}^{\bar{\nu}} \subset \left\{ f_\omega \leqslant \frac{c_\omega}{2} \right\}$$

and

$$\|u - v\| \geqslant r_0 \quad \textit{for all } u \in A_{ij}^{\bar{\nu}},\ \ v \in A_{i'j'}^{\bar{\nu}} \cup \mathcal{U}_{\tilde{\varrho}}^{\nu} \ \textit{ if } (i, j) \neq (i', j').$$

In the following we denote $A_{ij} = A_{ij}^{\bar{\nu}}$ for $i, j \in \mathbb{Z}$ and $\mathcal{U}_0 = \mathcal{U}_{\tilde{\varrho}}^{\bar{\nu}}$, and given a set $A \subset H$, we define $B_r(A) = \{u \in H \mid \inf_{v \in A} \|u - v\| < r\}$.

Then, by lemma 5.4 and a deformation argument, we have

LEMMA 5.5. – *For any $r \in (0, r_0/2)$ there exists $\varDelta_r > 0$ such that for any $h \in (0, \min\{h^*, \varDelta_r\})$ there is a path $\gamma \in \Gamma_\omega$ and a finite number of sets $A_{i_1 j_1}, \ldots, A_{i_k j_k} \subset A^{\bar{\nu}}$ for which*

(*a*) $\max_{s \in [0, 1]} f_\omega(\gamma(s)) < c_\omega + h$;

(*b*) *if* $\gamma(s) \notin \bigcup_{p=1}^{k} B_r(A_{i_p j_p})$ *then* $f_\omega(\gamma(s)) \leqslant c_\omega - \varDelta_r$.

By lemma 5.5 we can define a local minimax in a region where the Palais Smale condition holds.

We now fix $\bar{r} \in (0, r_0/4)$ and $\bar{h} \in (0, \min\{h^*, \varDelta_{\bar{r}}\})$. We take $\gamma \in \Gamma_\omega$ and sets $A_{i_1 j_1}, \ldots, A_{i_k j_k} \subset A^{\bar{\nu}}$, satisfying (*a*) and (*b*) of lemma 5.5 for the chosen values $\bar{r}$ and $\bar{h}$. By the definition of the minimax level c_ω, there exists $p \in \{1, \ldots, k\}$ such that, setting $A_p = A_{i_p j_p}$, there exist $s_1, s_2 \in [0, 1]$ for which

$u_0 = \gamma(s_1)$, $u_1 = \gamma(s_2) \in \partial B_{\bar{r}}(A_p) \cap \{f_\omega \leqslant c_\omega - \Delta_{\bar{r}}\}$, $\gamma(s) \in B_{\bar{r}}(A_p)$ for any $s \in (s_1, s_2)$ and u_0, u_1 are not connectible in $\{f_\omega < c_\omega\}$.

Setting $\mathcal{B} \equiv \overline{B_{\bar{r}}(A_p)} \cup \{f_\omega \leqslant c_\omega - 1/2\Delta_{\bar{r}}\}$, let us consider the class

$$\bar{\Gamma} = \{\gamma \in C([0,1], H) \,|\, \gamma(0) = u_0,\ \gamma(1) = u_1,\ \gamma([0,1]) \subset \mathcal{B}\}.$$

Since $\bar{\Gamma} \neq \emptyset$ we define $\bar{c}_\omega = \inf_{\gamma \in \bar{\Gamma}} \max_{s \in [0,1]} f_\omega(\gamma(s))$ and we have $c_\omega \leqslant \bar{c}_\omega < c_\omega + \bar{h}$.

Let us denote by K_A the compact set of critical points $\mathcal{K}_\omega \cap A_p$. By lemma 5.5 we have that K_A has a variational characterization as follows:

LEMMA 5.6. – *For any $r \in (0, \bar{r}/4)$ there exists $h_r > 0$ such that for any $h \in (0, h_r)$ there is a path $\gamma \in C([0,1], H)$ satisfying the following properties*:

(*a*) $\gamma(0), \gamma(1) \in \partial B_{r/2}(K_A)$, *and they are not path-connectible in* $B_{\bar{r}}(A_p) \cap \{f_\omega < \bar{c}_\omega\}$;

(*b*) $\gamma([0,1]) \subset \bar{B}_{r/2}(K_A) \cap \{f_\omega \leqslant \bar{c}_\omega + h\}$;

(*c*) $\gamma([0,1]) \cap (B_{r/2}(K_A) \setminus B_{r/4}(K_A)) \subset \{f_\omega \leqslant \bar{c}_\omega - 1/2 h_r\}$;

(*b*) $\operatorname{supp} \gamma(\theta) \subset [-R, R]$ *for any* $\theta \in [0,1]$, R *being a positive constant independent on* θ.

Since a_ω depends quasi-periodically in time, by lemma 5.1 we have also:

LEMMA 5.7. – *For any $r \in (0, \bar{r}/4)$ and $h \in (0, h_r)$ there is $\varepsilon_1 > 0$ such that for any $\tau \in P(a_\omega, \varepsilon_1)$ there is a path $\gamma_\tau \in C([0,1], H)$ satisfying the following properties*:

(*a*) $\gamma_\tau(0), \gamma_\tau(1) \in \partial B_{r/2}(\tau * K_A)$, and they are not path-connectible in $B_{\bar{r}}(\tau * A_p) \cap \{f_\omega < \bar{c}_\omega\}$;

(*b*) $\gamma_\tau([0,1]) \subset \bar{B}_{r/2}(\tau * K_A) \cap \{f_\omega \leqslant \bar{c}_\omega + 3/2 h\}$;

(*c*) $\gamma_\tau([0,1]) \cap (B_{r/2}(\tau * K_A) \setminus B_{r/4}(\tau * K_A)) \subset \{f_\omega \leqslant \bar{c}_\omega - 1/4 h_r\}$;

(*d*) $\operatorname{supp} \gamma_\tau(\theta) \subset [-R + \tau, R + \tau]$ *for any* $\theta \in [0,1]$, R *being a positive constant independent on* θ.

Moreover, we have

LEMMA 5.8. – *For any $r \in (0, \bar{r}/4)$ there exists $\mu_r > 0$ and $\varepsilon_2 > 0$ such that for any $\tau \in P(a_\omega, \varepsilon_2)$*:

$$\|\nabla f_\omega(u)\| \geqslant \mu_r \qquad \text{for any } u \in (B_{\bar{r}}(\tau \circ A_p) \cap \{f_\omega \leqslant \bar{c}_\omega + h^*\}) \setminus B_{r/4}(\tau * K_A).$$

Finally, we state a last preliminary property we need in order to apply the Séré technique to prove the existence of multibump solutions.

Let us consider the set $f_\omega(K_A) \subset \mathbb{R}$. Then, thanks to the behavior at the origin and to the regularity of the potential, we have the following property:

LEMMA 5.9. – *$[0, c^*] \setminus f_\omega(K_A)$ is open and dense in $[0, c^*]$.*

The proof of this lemma, that can be found in [19], use a Morse reduction (see e.g. [6]) and the Sard's theorem.

The lemmas 5.7, 5.8 and 5.9 are the key ingredient to prove our main result:

THEOREM 5.10. – *Suppose* (V1), (V2), (V3), (a1), (a2) *and* (a3) *hold. For any $r > 0$ there exist $N > 0$ and $\varepsilon > 0$ such that for every $k \in \mathbb{N}$, for every $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon, k)$ there exist a solution $v_{\vec{p}}$ of* (HS_ω) *such that*

$$\operatorname{dist}(v_{\vec{p}}, \vec{p} * K_A) < r .$$

PROOF. – Given $r > 0$ and $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon, k)$, let us denote $\mathcal{B}_r(K_A; \vec{p}) = \{u \in H \,|\, \operatorname{dist}(u, \vec{p} \circ K_A) < r\}$. Given $N > 0$ we define $\Lambda_N = \dfrac{\sqrt{N}}{2}(\sqrt{N}/2 + 1)$. Then, we introduce the intervals $M_i = (p_i + \Lambda_N, p_{i+1} - \Lambda_N)$, $M = \bigcup_{i=1}^{k} M_i$ and for any $\delta > 0$ the set $\mathcal{M}_\delta = \{u \in H \,|\, \|u\|^2_{M_i} \leq \delta\}$.

Arguing by contradiction, there is $r > 0$ such that for any $N > 0$ and $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon, k)$ for which $\mathcal{B}_r(K_A; \vec{p}) \cap \mathcal{K}_\omega = \emptyset$.

Thanks to lemmas 5.8 and 5.9 we can construct in $\mathcal{B}_r(K_A; \vec{p})$ a common pseudogradient vector field for f_ω and the truncated functionals

$$f_{\omega, i}(u) = \int_{I_i} \frac{1}{2}(|\dot{u}|^2 + |u|^2) - \int_{I_i} a_\omega(t)\, V(u)$$

(see [20] for a proof).

Let us fix r_1, r_2, r_3 for which $(2/3)\, r < r_1 < r_2 < r_3 < (5/6)\, r$. By lemma 5.9, for any $h \in (0, h_r)$ one can choose c_+, c_- arbitrarily close and $\lambda > 0$ such that the intervals $[c_- - \lambda, c_- + 2\lambda] \subset (\bar{c}_\omega - h, \bar{c}_\omega - h/2)$, $[c_+ - \lambda, c_+ + 2\lambda] \subset (\bar{c}_\omega + h/2, \bar{c}_\omega + h)$ verify

$$u \in B_r(K_A) \cap \{c_\pm - \lambda \leq f_\varepsilon \leq c_\pm + 2\lambda\} \Rightarrow \|\nabla f_\omega(u)\| \geq \nu$$

for some $\nu > 0$.

Then, we have

LEMMA 5.11. – *There exist $\mu_r > 0$, $\varepsilon_r > 0$ and $\bar{\delta} > 0$ such that: $\forall \delta \in (0, \bar{\delta})$ there exists $N > 0$ for which for any $k \in \mathbb{N}$ and $\vec{p} \in \mathcal{P}(\omega, N, \varepsilon_r, k)$, there exists a locally Lipschitz continuous function $\mathcal{W}: H \to H$ which verifies*

(W1) $\max_{1 \leq j \leq k} \| \mathcal{W}(u) \|_{I_j} \leq 1$, $\langle \nabla f_\omega(u), \mathcal{W}(u) \rangle \geq 0$, $\forall u \in H$, $\mathcal{W}(u) = 0$, $\forall u \in H \setminus \mathcal{B}_{r_3}(K_A; \vec{p})$;

(W2) $\langle \nabla f_{\omega, i}(u), \mathcal{W}(u) \rangle \geq \mu_r$ *if* $r_1 \leq \inf_{v \in K_A} \| u - v(\cdot - p_i) \|_{I_i} \leq r_2$, $u \in \mathcal{B}_{r_2}(K_A; \vec{p}) \cap \{ f_{\omega, i} \leq c_+ \}$;

(W3) $\langle \nabla f_{\omega, i}(u), \mathcal{W}(u) \rangle \geq 0$, $\forall u \in \{ c_+ \leq f_{\omega, i} \leq c_+ + \lambda \} \cup \{ c_- \leq f_{\omega, i} \leq c_- + \lambda \}$;

(W4) $\langle u, \mathcal{W}(u) \rangle_{M_j} \geq 0$ $\forall j \in \{0, \ldots, k\}$ *if* $u \in H \setminus \mathcal{M}_\delta$.
Moreover if $\mathcal{K}_\omega \cap \mathcal{B}_{r_3}(K_A; \vec{p}) = \emptyset$ *then there exists* $\mu_{\vec{p}} > 0$ *such that*

(W5) $\langle \nabla f_\omega(u), \mathcal{W}(u) \rangle \geq \mu_{\vec{p}}$ $\forall u \in \mathcal{B}_{r_2}(K_A; \vec{p})$.

Then, fixed suitable values of the parameters $c_\pm$, h and choosing consequently the value of $\delta > 0$ and $N > 0$, we consider the flow associated to the pseudogradient vector field given by lemma 5.11:

$$\begin{cases} \dfrac{d\eta}{ds} = -\mathcal{W}(\eta), \\ \eta(0, u) = u . \end{cases}$$

Since $\mathcal{W}$ is a bounded locally Lipschitz vector field, for any $u \in H$ there exists a solutions $\eta(\cdot, u) \in C(\mathbb{R}^+, H)$, depending continuously on $u \in H$.

We now consider the surface $G: Q = [0, 1]^k \to H$ defined as $G(\theta) = \sum_{i=1}^{k} \gamma_{p_i}(\theta_i)$, for $\theta = (\theta_1, \ldots, \theta_k) \in Q$ and γ_{p_i} given by lemma 5.7 for a suitable value of h.

We consider the deformation $\eta(s, G(\theta))$ under the flow. We get that there exists $\bar{s} > 0$ such that, setting $\overline{G}(\theta) = \eta(\bar{s}, G(\theta))$, we have the following properties:

(a) $\eta(s, G(\theta)) = G(\theta)$ for any $\theta \in \partial Q$ and for any $s \in \mathbb{R}^+$;

(b) there exists $i \in \{1, \ldots, k\}$ and $\xi \in C([0, 1], Q)$ such that $\xi(0) \in \{\theta_i = 0\}$, $\xi(1) \in \{\theta_i = 1\}$ and $f_{\omega, i}(\overline{G}(\theta)) < c_- + \delta$, for any $\theta \in \xi([0, 1])$;

(c) $\eta(s, G(Q)) \subseteq \mathcal{M}_\delta$ for any $s \in \mathbb{R}^+$.

Thanks to the above properties, we finally get a contradiction. Indeed, let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\sup_{t \in \mathbb{R}} |\dot{\chi}(t)| \leq 1$ be such that $\chi(t) = 1$ if $t \in I_i \setminus M$ and $\chi(t) = 0$ if $t \in \mathbb{R} \setminus I_i$, where the index $i \in \{1, \ldots, k\}$ is given by (b). Then, we define a path $g: [0, 1] \to H$ by setting $g(s) = \chi \overline{G}(\xi(s))$ for $s \in [0, 1]$.

By (a) we have that $g(0) = \gamma_{p_i}(0)$ and $g(1) = \gamma_{p_i}(1)$.

Moreover, $g([0, 1]) \subset B_{\bar{r}}(p_i * A_p)$ and, since δ can be chosen sufficiently small by (b) and (c), we get $f_\omega(g(s)) < \bar{c}_\omega$, for any $s \in [0, 1]$. A contradiction with property (a) of lemma 5.7. ∎

Finally, as a corollary, we get the existence of an uncountable set of bounded motions of the system (HS_ω).

COROLLARY 5.12. – *For any $r > 0$ there exist $\varepsilon_r > 0$ and $N_r > 0$ such that given a (bi-infinite) sequence $p_j \in P(a_\omega, \varepsilon_r)$ with $p_{j+1} - p_j \geq N_r$ there exists a solution v of* (HS_ω), *which verifies*

$$\inf_{u \in p_j * K_A} \|v - u\|_{C^1(I_j, \mathbb{R}^n)} < r, \qquad \forall j \in \mathbb{Z}.$$

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