Bollettino Unione Matematica Italiana

Luciano Carbone, Antonio Corbo Esposito, Riccardo De Arcangelis

Homogenization of Neumann problems for unbounded integral functionals

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.2, p. 463–491.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_2_463_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 463-491

Homogenization of Neumann Problems for Unbounded Integral Functionals

LUCIANO CARBONE - ANTONIO CORBO ESPOSITO - RICCARDO DE ARCANGELIS

Sunto. – *Si studia l'omogeneizzazione di problemi di tipo Neumann per funzionali integrali del Calcolo delle Variazioni definiti su funzioni soggette a vincoli puntuali di tipo oscillante sul gradiente, in ipotesi minimali sui vincoli. I risultati ottenuti sono dedotti mediante l'introduzione di nuove tecniche di Γ-convergenza, unitamente ad un risultato di ricostruzione per funzioni affini a tratti, che permettono la dimostrazione di un teorema generale di omogeneizzazione per funzionali integrali a valori reali estesi.*

0. – Introduction.

In the book [BLP] the authors proposed a general issue about the homogenization of certain families of Dirichlet, Neumann and mixed minimum problems for integral functionals of the Calculus of Variations defined on functions subject to pointwise oscillating constraints on the gradient (cf. [A2], [DM] and [SP] beside the above quoted book [BLP] for general references on homogenization theory).

A slightly simplified version of the issue relative to Dirichlet and Neumann problems deals with the study, for every regular bounded open set Ω, $\beta \in L^\infty(\Omega)$, $\lambda > \|\beta\|_{L^\infty(\Omega)}$, of the asymptotic behaviour as ε tends to 0 of the families $\{i_\varepsilon^\infty(\Omega, \beta)\}_{\varepsilon>0}$ and $\{j_\varepsilon^\infty(\Omega, \beta, \lambda)\}_{\varepsilon>0}$ of the minimum values

(0.1) $\quad i_\varepsilon^\infty(\Omega, \beta) =$

$$\min\left\{\int_\Omega \phi\left(\frac{x}{\varepsilon}, Du\right) dx + \int_\Omega \beta u dx \colon u \in W_0^{1,\infty}(\Omega), |Du(x)| \leqslant m\left(\frac{x}{\varepsilon}\right) \text{ for a.e. } x \in \Omega\right\},$$

(0.2) $\quad j_\varepsilon^\infty(\Omega, \beta, \lambda) =$

$$\min\left\{\int_\Omega \phi\left(\frac{x}{\varepsilon}, Du\right) dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \colon u \in W^{1,\infty}(\Omega),\right.$$

$$\left. |Du(x)| \leqslant m\left(\frac{x}{\varepsilon}\right) \text{ for a.e. } x \in \Omega\right\},$$

where ϕ and m are functions verifying (here and in the sequel $Y =]0, 1[^n$)

$$(0.3)\qquad \begin{cases} \phi : (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto \phi(x, z) \in [0, +\infty[, \\ \phi(\cdot, z) \text{ measurable and } Y\text{-periodic for every } z \in \boldsymbol{R}^n, \\ \phi(x, \cdot) \text{ convex for a.e. } x \in \boldsymbol{R}^n, \end{cases}$$

$$(0.4)\qquad \begin{cases} m : x \in \boldsymbol{R}^n \mapsto m(x) \in [0, +\infty], \\ m \text{ measurable and } Y\text{-periodic}, \end{cases}$$

$$(0.5)\qquad m \in L^{\infty}(Y).$$

Once introduced the convex, lower semicontinuous function $\phi^{\infty}_{\text{hom}}$ defined by

$$(0.6)\qquad \phi^{\infty}_{\text{hom}} : z \in \boldsymbol{R}^n \mapsto$$

$$\min \left\{ \int_Y \phi(y, z + Dv)\, dy : v \in W^{1,\infty}_{\text{loc}}(\boldsymbol{R}^n),\ v\ Y\text{-periodic}, \right.$$

$$\left. |z + Dv(y)| \leq m(y) \text{ for a.e. } y \in Y \right\},$$

the convergence of the families $\{i^{\infty}_{\varepsilon}(\Omega, \beta)\}_{\varepsilon > 0}$ and $\{j^{\infty}_{\varepsilon}(\Omega, \beta, \lambda)\}_{\varepsilon > 0}$ respectively to

$$(0.7)\quad i^{\infty}_{\text{hom}}(\Omega, \beta) = \min \left\{ \int_{\Omega} \phi^{\infty}_{\text{hom}}(Du)\, dx + \int_{\Omega} \beta u\, dx : u \in W^{1,\infty}_0(\Omega) \right\},$$

$$(0.8)\quad j^{\infty}_{\text{hom}}(\Omega, \beta, \lambda) = \min \left\{ \int_{\Omega} \phi^{\infty}_{\text{hom}}(Du)\, dx + \int_{\Omega} \beta u\, dx + \lambda \int_{\Omega} |u|\, dx : u \in W^{1,\infty}(\Omega) \right\}$$

has been conjectured.

We observe explicitly that, beside (0.1) and (0.2), also (0.7) and (0.8) are gradient constrained problems; in fact, since $\phi^{\infty}_{\text{hom}}$ in (0.6) may take the value $+\infty$, the functions u that make the integrals in (0.7) and (0.8) finite must be such that $Du(x) \in \operatorname{dom} \phi^{\infty}_{\text{hom}}$ for a.e. $x \in \Omega$, $\operatorname{dom} \phi^{\infty}_{\text{hom}}$ being the closed convex, bounded subset of $\boldsymbol{R}^n$ given by

$$\operatorname{dom} \phi^{\infty}_{\text{hom}} = \{ z \in \boldsymbol{R}^n : \phi_{\text{hom}}(z) < +\infty \} =$$

$$\{ z \in \boldsymbol{R}^n : \text{there exists } v \in W^{1,\infty}_{\text{loc}}(\boldsymbol{R}^n),\ v\ Y\text{-periodic such that } |z + Dv(y)| \leq m(y)$$

$$\text{for a.e. } y \in Y \}.$$

The case of Dirichlet problems has been treated in some papers (cf. [C1] ÷ [C4], [CS1], [CS2], [F]) under some additional assumptions on m and finally in [CEDA2] by requiring only (0.3) ÷ (0.5). On the other side in [CS3], [DAV], [DAGP] and [CDA2] some attempts have been made in order to study the above Dirichlet problems when (0.5) is dropped, nevertheless the study of the general case in which m verifies only (0.4) has not been completely achieved.

On the contrary the case of Neumann problems has been less studied in literature and it has been treated only in [CDA2] under suitable summability hypotheses on m and in [DAGP] when $m(\boldsymbol{R}^n) = \{0, +\infty\}$.

In the present paper we want to study just the case of Neumann problems by assuming minimal hypotheses on m.

We prove that if ϕ, m are as in (0.3), (0.4) and one of the following assumptions (0.9) or (0.10) is fulfilled

$$(0.9) \qquad |z|^p \leqslant \phi(x,z) \quad \text{for a.e. } x \in \boldsymbol{R}^n, \text{ every } z \in \boldsymbol{R}^n \text{ and some } p \in]1, +\infty[,$$

$$(0.10) \qquad m \in L^p(Y) \quad \text{for some } p \in [1, +\infty],$$

then for every convex bounded open set Ω, $\beta \in L^\infty(\Omega)$, $\lambda \geqslant \|\beta\|_{L^\infty(\Omega)}$ the values $j_\varepsilon^p(\Omega, \beta, \lambda)$ given for every $\varepsilon > 0$ by

$$(0.11) \qquad j_\varepsilon^p(\Omega, \beta, \lambda) = \min\left\{\int_\Omega \phi\left(\frac{x}{\varepsilon}, Du\right) dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx :\ u \in W^{1,p}(\Omega),\right.$$
$$\left. |Du(x)| \leqslant m\left(\frac{x}{\varepsilon}\right) \text{ for a.e. } x \in \Omega\right\}$$

converge as ε tends to 0 to

$$(0.12) \qquad j_{\mathrm{hom}}^p(\Omega, \beta, \lambda) =$$
$$\min\left\{\int_\Omega \phi_{\mathrm{hom}}^p(Du)\, dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx :\ u \in W^{1,p}(\Omega)\right\},$$

where ϕ is the convex, lower semicontinuous function defined by

$$(0.13) \qquad \phi_{\mathrm{hom}}^p : z \in \boldsymbol{R}^n \mapsto \min\left\{\int_Y \phi(y, z + Dv)\, dy :\ v \in W_{\mathrm{loc}}^{1,p}(\boldsymbol{R}^n),\right.$$
$$\left. v\ Y\text{-periodic},\ |z, Dv(y)| \leqslant m(y) \text{ for a.e. } y \in Y\right\}.$$

Moreover, if (0.10) holds, $\operatorname{dom}\phi^p_{\text{hom}}$ turns out to be bounded and

$$j^p_{\text{hom}}(\Omega, \beta, \lambda) = \min\left\{\int_\Omega \phi^p_{\text{hom}}(Du)\,dx + \int_\Omega \beta u\,dx + \lambda\int_\Omega |u|\,dx\colon u \in W^{1,\infty}(\Omega)\right\}.$$

Finally, if $\{\varepsilon_h\}$ is a sequence of positive numbers converging to 0 and, for every $h \in \boldsymbol{N}$, u_h is a solution of $j^p_{\varepsilon_h}(\Omega, \beta, \lambda)$ then $\{u_h\}$ is compact in $L^p(\Omega)$ and its converging subsequences converge to minimizers of $j^p_{\text{hom}}(\Omega, \beta, \lambda)$ (Corollary 4.2).

In order to obtain the above results we introduce a new technique based on Γ-convergence theory (cf. [DG], [DGF]), together with a reconstruction argument for the limit problem in (0.12) proposed in [A1], [A2], and a representation result for piecewise affine functions (Theorem 2.1). Such technique allows us to prove very general results for Neumann problems, and can also be applied in the case of Dirichlet problems to get some more refined theorems under additional assumptions, weaker than those existing in literature, but not minimal (cf. [DMD]). The problem of the study of the Dirichlet case under assumptions comparable to the ones proposed in the present paper remains open.

The convergence theorem we prove is deduced by a general homogenization result for unbounded integral functionals, i.e. of the type of those in (0.2) but with integrands taking extended real values and possibly being not finite on large classes of regular functions, (Theorem 3.10), that can be applied also to the treatment of further classes of perturbations of the first integral in (0.11).

We also observe that we are able to consider only minimum problems directly on $W^{1,p}$-spaces and not infimum problems on classes of more regular functions, as for example differentiable or Lipschitz continuous functions. This remark is not trivial since it is known that in general, both in the unconstrained and constrained case, a Lavrentieff phenomenon may occur and the homogenization processes may lead to different results (cf. [CEDA1], [CESC]).

Finally we remark that variational problems of the type considered in this paper are interesting also from a physical point of view: for instance the problem of the homogenization of the elastic-plastic torsion of a cylindrical bar (cf. [An], [B], [BS], [CR], [DL], [DLi], [GL], [L], [T]) can be framed in this context.

In conclusion we also want to thank the referee for the useful remarks and comments.

1. – Notations and preliminary results.

We first recall the notion and the main properties of Γ^--convergence, we refer to [DG], [DGF], [Bu] and [DM] for a complete exposition on the subject.

Let (U, τ) be a topological space, for every $u \in U$ let us denote by $\mathcal{J}(u)$ the set of the neighborhoods of u in τ.

Let $E \subseteq \boldsymbol{R}$, θ be a cluster point of E and let, for every $\varepsilon \in E$, F_ε be a functional from U to $[-\infty, +\infty]$.

DEFINITION 1.1. – *We define the functionals*

$$\Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon \colon u \in U \mapsto \sup_{I \in \mathcal{J}(u)} \liminf_{\varepsilon \to \theta} \inf_{v \in I} F_\varepsilon(v),$$

$$\Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon \colon u \in U \mapsto \sup_{I \in \mathcal{J}(u)} \limsup_{\varepsilon \to \theta} \inf_{v \in I} F_\varepsilon(v).$$

If at $u \in U$ it results $\Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon(u) = \Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon(u)$ we say that the family $\{F_\varepsilon\}_{\varepsilon \in E}$ $\Gamma^-(\tau)$-converges at u as ε goes to θ, and we define the $\Gamma^-(\tau)$-limit at u as

$$\Gamma^-(\tau) \lim_{\varepsilon \to \theta} F_\varepsilon(u) = \Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon(u) = \Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon(u).$$

It is clear that

(1.1) $\quad \Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon(u) \leqslant \Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon(u) \quad$ for every $u \in U$,

moreover it is well known (see [Bu]) that

(1.2) $\quad$ the functionals $\Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon$ *and* $\Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon$

are τ-lower semicontinuous on U.

If $E = \boldsymbol{N}$ and (U, τ) satisfies also the first countability axiom, the functionals $\Gamma^-(\tau) \liminf_{\varepsilon \to \theta} F_\varepsilon$ and $\Gamma^-(\tau) \limsup_{\varepsilon \to \theta} F_\varepsilon$ can be characterized in the following way (cf. [DGF])

(1.3) $\quad \Gamma^-(\tau) \liminf_{h \to +\infty} F_h(u) = \min\left\{\liminf_{h \to +\infty} F_h(v_h) \colon v_h \to u\right\} \quad$ for every $u \in U$,

(1.4) $\quad \Gamma^-(\tau) \limsup_{h \to +\infty} F_h(u) = \min\left\{\limsup_{h \to +\infty} F_h(v_h) \colon v_h \to u\right\} \quad$ for every $u \in U$.

We explicitly recall that (1.3) holds if and only if it results

(1.5) $\quad \Gamma^-(\tau) \liminf_{h \to +\infty} F_h(u) \leqslant \liminf_{h \to +\infty} F_h(v_h) \quad$ for every $u \in U$ and every $v_h \to u$

and

(1.6) $\quad \Gamma^-(\tau) \liminf_{h \to +\infty} F_h(u) \geqslant \liminf_{h \to +\infty} F_h(u_h)$

for every $u \in U$ and at least one sequence $u_h \to u$.

Analogously (1.4) holds if and only if (1.5) and (1.6) are fulfilled with $\Gamma^-(\tau)\liminf_{h\to+\infty} F_h(u)$ replaced by $\Gamma^-(\tau)\limsup_{h\to+\infty} F_h(u)$ and the operator "$\liminf_{h\to+\infty}$" replaced by "$\limsup_{h\to+\infty}$".

If $E\subseteq \boldsymbol{R}$ and θ is a cluster point of E it is easy to verify that

$$(1.7)\qquad \Gamma^-(\tau)\limsup_{\varepsilon\to\theta} F_\varepsilon(u) =$$

$$\sup\left\{\Gamma^-(\tau)\limsup_{h\to+\infty} F_{\varepsilon_h}(u) : \{\varepsilon_h\}\subseteq E,\ \varepsilon_h\to\theta\right\}\quad \text{for every } u\in U,$$

moreover, if in addition (U,τ) satisfies also the first countability axiom, it results that (cf. for example [DAG])

$$(1.8)\qquad \Gamma^-(\tau)\liminf_{\varepsilon\to\theta} F_\varepsilon(u) =$$

$$\min\left\{\Gamma^-(\tau)\liminf_{h\to+\infty} F_{\varepsilon_h}(u) : \{\varepsilon_h\}\subseteq E,\ \varepsilon_h\to\theta\right\}\quad \text{for every } u\in U.$$

The following results are proved in [DGF].

PROPOSITION 1.2. – *Let $\{F_h\}$ be a sequence of functionals from U to $[-\infty, +\infty]$, and assume that the limit $\Gamma^-(\tau)\lim_{h\to+\infty} F_h$ exists on U, then for every τ-continuous functional G from U to $\boldsymbol{R}$ it results*

$$G(u)+\Gamma^-(\tau)\lim_{h\to+\infty} F_h(u) = \Gamma^-(\tau)\lim_{h\to+\infty}\{G(u)+F_h(u)\}\qquad \text{for every } u\in U.$$

Let $\{F_h\}$ be a sequence of functionals from U to $[-\infty, +\infty]$. We say that the functionals F_h are τ-equicoercive on U if for every real number λ there exists a compact subset K_λ of U such that $\{u\in U: F_h(u)\leqslant\lambda\}\subseteq K_\lambda$ for every $h\in\boldsymbol{N}$.

THEOREM 1.3. – *Let $\{F_h\}$ be a sequence of equicoercive functionals on U, and assume that the limit $\Gamma^-(\tau)\lim_{h\to+\infty} F_h(u)$ exists for every $u\in U$, then $\Gamma^-(\tau)\lim_{h\to+\infty} F_h$ has a minimum on U and*

$$\min_{v\in U}\Gamma^-(\tau)\lim_{h\to+\infty} F_h(v) = \lim_{h\to+\infty}\inf_{v\in U} F_h(v).$$

Moreover if $\{u_h\}$ is a sequence such that $u_h\to u$ and $\lim_{h\to+\infty}\{F_h(u_h) - \inf_{v\in U} F_h(v)\} = 0$, then u is a minimum point for $\Gamma^-(\tau)\lim_{h\to+\infty} F_h$ on U.

For every subset E of $\boldsymbol{R}^n$ we define χ_E to be the characteristic function of E defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in \boldsymbol{R}^n \setminus E$, moreover, if in addition E is also Lebesgue measurable, we denote by $|E|$ its measure.

Given $x_0 \in \boldsymbol{R}^n$, $r > 0$ we set $B_r(x_0) = \{x \in \boldsymbol{R}^n : |x - x_0| < r\}$.

For every nonempty subset C of $\boldsymbol{R}^n$ we denote by aff (C) the affine hull of C, i.e. the smallest affine set containing C. It is known that if in addition C is also convex and is regarded as a subset of aff (C), then C possesses interior points in the topology induced on aff (C) by the natural one of $\boldsymbol{R}^n$; the set of such interior points is called the relative interior of C and is denoted by ri (C). Obviously when aff $(C) = \boldsymbol{R}^n$ then ri $(C) = C^\circ$.

For every $d \in \{1, \dots, n\}$ we denote by $0^{(d)}$ the null vector in $\boldsymbol{R}^d$ and by Pr_d the projection operator from $\boldsymbol{R}^n$ to $\boldsymbol{R}^d$ defined by $Pr_d : (x_1, \dots, x_n) \in \boldsymbol{R}^n \mapsto (x_1, \dots, x_d) \in \boldsymbol{R}^d$.

For every $z \in \boldsymbol{R}^n$ we denote by u_z the linear function defined by $u_z(x) = z \cdot x$ for every $x \in \boldsymbol{R}^n$, and, if $p \in [1, +\infty]$, we set $W^{1,p}_{\text{per}}(Y) = \{u \in W^{1,p}_{\text{loc}}(\boldsymbol{R}^n) : u$ is Y-periodic$\}$.

Let u be a continuous function on $\boldsymbol{R}^n$, we say that u is piecewise affine if

$$u(x) = \sum_{j=1}^{m} (u_{z_j}(x) + s_j)\,\chi_{P_j}(x) \qquad \text{for every } x \in \boldsymbol{R}^n, \tag{1.9}$$

where $m \in \boldsymbol{N}$, $z_1, \dots, z_m \in \boldsymbol{R}^n$, $s_1, \dots, s_m \in \boldsymbol{R}$ and $P_1, \dots, P_m$ are pairwise disjoint polyhedra, i.e. finite intersections of half spaces, such that $\bigcup_{j=1}^{m} P_j = \boldsymbol{R}^n$. We denote by $PA(\boldsymbol{R}^n)$ the set of the piecewise affine functions on $\boldsymbol{R}^n$.

We denote by $\mathcal{A}$ the set of the bounded open subsets of $\boldsymbol{R}^n$ and, for every A, $B \in \mathcal{A}$, we say that $A \subset\subset B$ if $\overline{A} \subseteq B$.

Given a set function $\alpha : \mathcal{A} \to [-\infty, +\infty]$ we say that α is increasing if $\alpha(\phi) = 0$ and $\alpha(A_1) \leq \alpha(A_2)$ whenever $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq A_2$. For every increasing set function α we define the inner regular envelope α_- of α as

$$\alpha_- : A \in \mathcal{A} \mapsto \sup\{\alpha(B) : B \in \mathcal{A},\ B \subset\subset A\}.$$

Finally, given a real function $f : \boldsymbol{R}^n \to]-\infty, +\infty]$, we set $\operatorname{dom} f = \{z \in \boldsymbol{R}^n : f(z) < +\infty\}$, define the bipolar f^{**} of f as

$$f^{**} : z \in \boldsymbol{R}^n \mapsto \sup\{a \cdot z + b : a \in \boldsymbol{R}^n,\ b \in \boldsymbol{R},\ a \cdot \xi + b \leq f(\xi) \text{ for every } \xi \in \boldsymbol{R}^n\} \tag{1.10}$$

and recall that f^{**} turns out to be the greatest convex lower semicontinuous function on $\boldsymbol{R}^n$ less than or equal to f.

We now define the functionals we are going to consider in this paper.

Let $\mathcal{L}_n$ and $\mathcal{B}_n$ be respectively the σ-algebras of the Lebesgue measurable and of the Borel subsets of $\boldsymbol{R}^n$, let f be a function verifying

$$(1.11)\qquad \begin{cases} f\colon (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto f(x, z) \in [0, +\infty], \\ f\ \mathcal{L}_n \otimes \mathcal{B}_n \text{ measurable}, \\ f(\cdot, z)\ Y\text{-periodic for every } z \in \boldsymbol{R}^n, \\ f(x, \cdot) \text{ convex and lower semicontinuous for a.e. } x \in \boldsymbol{R}^n, \end{cases}$$

and let $p \in [1, +\infty]$, then for every $\Omega \in \mathcal{A}$, $u \in W^{1,p}(\Omega)$, $\varepsilon > 0$ the function $f\left(\frac{\cdot}{\varepsilon} Du(\cdot)\right)$ is nonnegative and measurable on Ω and hence the functional $F_\varepsilon(\Omega, \cdot)$

$$(1.12)\qquad F_\varepsilon(\Omega, \cdot)\colon u \in L^p(\Omega) \mapsto \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, Du\right) dx & \text{if } u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n), \\ +\infty & \text{if } u \in L^p(\Omega) \backslash W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n), \end{cases}$$

turns out to be well defined. We observe explicitly that the functional in (1.12) involves a constraint on the gradients of the admissible functions, in fact the elements u of $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$ that make the integral in (1.12) finite satisfy the constraint $Du(x) \in \operatorname{dom} f\left(\frac{x}{\varepsilon}, \cdot\right)$ for a.e. $x \in \Omega$.

For every $\Omega \in \mathcal{A}$ let us set

$$(1.13)\qquad F'_{\mathrm{hom}}(\Omega, u) = \Gamma^-(L^p(\Omega)) \liminf_{\varepsilon \to 0} F_\varepsilon(\Omega, u),$$

$$F''_{\mathrm{hom}}(\Omega, u) = \Gamma^-(L^p(\Omega)) \limsup_{\varepsilon \to 0} F_\varepsilon(\Omega, u), \qquad \text{for every } u \in L^p(\Omega),$$

and define the function f_{hom} by

$$(1.14)\qquad f_{\mathrm{hom}}\colon z \in \boldsymbol{R}^n \mapsto \inf\left\{\int_Y f(y, z + Dv)\, dy \colon v \in W^{1,p}_{\mathrm{per}}(Y)\right\},$$

then it is clear that f_{hom} may take the value $+\infty$ and that, by (1.11), f_{hom} turns out to be convex.

Our goal is to prove that, if φ_f is the Y-periodic function given by

$$(1.15)\qquad \varphi_f\colon x \in \boldsymbol{R}^n \mapsto \sup\{|z| : f(x, z) < +\infty\}$$

and if one of the following assumptions

$$(1.16)\qquad |z|^p \leq f(x, z) \quad \text{for a.e. } x \in \boldsymbol{R}^n, \text{ every } z \in \boldsymbol{R}^n \text{ and some } p \in]1, +\infty[$$

or

$$(1.17)\qquad \varphi_f \in L^p(Y) \quad \text{for some } p \in [1, +\infty]$$

is fulfilled, then for every convex bounded open set Ω, $u \in W^{1,p}(\Omega)$ the functionals in (1.13) are equal and their common value agrees with the integral $\int_\Omega f_{\text{hom}}(Du)\,dx$; obviously this last integral includes the gradient constraint $Du(x) \in \operatorname{dom} f_{\text{hom}}$ for a.e. $x \in \Omega$, where

$$\operatorname{dom} f_{\text{hom}} = \left\{ z \in \boldsymbol{R}^n : \text{there exists } v \in W^{1,p}_{\text{per}}(Y) \text{ with } \int_Y f(y, Dv)\,dy < +\infty \right\}. \tag{1.18}$$

We recall that when f in (1.11) is just real valued, the following homogenization result holds (cf. [A2], [BLP], [CS], [CEDA1], [DM], [DGS], [M], [MS], [MT], [SP], [T], [ZKON]).

THEOREM 1.4. – *Let f be as in* (1.11), $p \in [1, +\infty]$, F_ε $(\varepsilon > 0)$ *be given by* (1.12) *and f_{hom} by* (1.14). *In addition assume that*

i) $p < +\infty$

$f(x, z) \leqslant \Lambda(1 + |z|^p)$ *for a.e.* $x \in \boldsymbol{R}^n$, *every* $z \in \boldsymbol{R}^n$ *and some* $\Lambda > 0$,

ii) *if* $p = +\infty$

$$\textit{for every } z \in \boldsymbol{R}^n \; f(\cdot, z) \in L^\infty(Y),$$

then for every $\Omega \in \mathcal{A}$ with Lipschitz boundary the family $\{F_\varepsilon(\Omega, \cdot)\}_{\varepsilon>0}$ $\Gamma^-(L^p(\Omega))$-converges on $W^{1,p}(\Omega)$ as ε goes to 0 *and*

$$\Gamma^-(L^p(\Omega)) \lim_{\varepsilon \to 0} F_\varepsilon(\Omega, u) = \int_\Omega f_{\text{hom}}(Du)\,dx \quad \textit{for every } u \in W^{1,p}(\Omega).$$

Finally we prove the following result that will be used in the sequel.

PROPOSITION 1.5. – *Let f be as in* (1.11), φ_f *be given by* (1.15), *and let f_{hom} be defined by* (1.14). *Assume that $\varphi_f \in L^1(Y)$, then $\operatorname{dom} f_{\text{hom}}$ is bounded and*

$$|z| \leqslant \int_Y \varphi_f\,dx \quad \textit{for every } z \in \operatorname{dom} f_{\text{hom}}.$$

PROOF. – Let $z \in \operatorname{dom} f_{\text{hom}}$ and let $w \in W^{1,p}_{\text{per}}(Y)$ be such that $\int_Y f(y, z + Dw)\,dy < +\infty$, then obviously $f(x, z + Dw(x)) < +\infty$ for a.e. $x \in Y$ and

$$|z + Dw(x)| \leqslant \varphi_f(x) \quad \text{for a.e. } x \in Y, \tag{1.19}$$

therefore, by using the Y-periodicity of w and (1.19), we have

$$|z| = \left| \int_Y z\,dx \right| \leqslant \int_Y |z + Dw(x)|\,dx + \left| \int_Y Dw(x)\,dx \right| =$$

$$\int_Y |z + Dw(x)|\,dx \leqslant \int_Y \varphi_f(x)\,dx\,,$$

from which the thesis follows. ■

2. – A representation result for piecewise affine functions.

Given $x, y \in \boldsymbol{R}^n$ we set $\sigma[x, y] = \{tx + (1-t)\,y : t \in [0, 1]\}$; similar definitions are given for $\sigma]x, y]$, $\sigma[x, y[$, $\sigma]x, y[$.

In the present section we want to prove a representation result concerning piecewise affine functions, more precisely that every piecewise affine function $u = \sum_{j=1}^{m} (u_{z_j} + s_j)\,\chi_{P_j}$ can be represented in a convex open set Ω by means of a finite combination of last upper bounds and greatest lower bounds of those of its "components" $u_{z_j} + s_j$ for which $P_j \cap \Omega$ is nonempty.

Finally we discuss some examples showing that in general the convexity assumption on Ω cannot be dropped.

The result we prove is the following (cf. also [CEDA3] for the analogous in the one dimensional case).

THEOREM 2.1. – *Let* $u = \sum_{j=1}^{m} (u_{z_j} + s_j)\,\chi_{P_j}$ *be in* $PA(\boldsymbol{R}^n)$, *then for every convex open set* Ω *there exist* $k \in \boldsymbol{N}$ *and* $N_1, \ldots, N_k \subseteq \{j \in \{1, \ldots, m\} : P_j \cap \Omega \neq \emptyset\}$ *such that*

$$u(x) = \sup_{i \in \{1, \ldots, k\}} \inf_{j \in N_i} (u_{z_j}(x) + s_j) \quad \text{for every } x \in \Omega\,. \tag{2.1}$$

PROOF. – Let Ω be a convex open set, I be the set of the indexes corresponding to the different components of u, namely $I = \{1\} \cup \{j \in \{2, \ldots, m\} : u_{z_j} + s_j \neq u_{z_i} + s_i \text{ for every } i \in \{1, \ldots, j-1\}\}$ and, for every $\alpha \in I$, set $E_\alpha = \{j \in \{1, \ldots, m\} : u_{z_j} + s_j = u_{z_\alpha} + s_\alpha\}$.

Let $A = \boldsymbol{R}^n \setminus \{x \in \boldsymbol{R}^n :$ there exist $\alpha, \beta \in I$ with $\alpha \neq \beta$ and $u_{z_\alpha}(x) + s_\alpha = u_{z_\beta}(x) + s_\beta\}$, then it is clear that A is open, dense in $\boldsymbol{R}^n$ and that it possesses a finite number of connected components, say $A_1, \ldots, A_h$, that turn out to be pairwise disjoint open polyhedra with $A = \bigcup_{\nu=1}^{h} A_\nu$, moreover let $k \in \boldsymbol{N}$ and $\nu_1, \ldots, \nu_k \in \{1, \ldots, h\}$ be such that $A_\nu \cap \Omega \neq \emptyset$ if and only if $\nu \in \{\nu_1, \ldots, \nu_k\}$.

Then, by using the connectedness of $A_{\nu_1}, \dots, A_{\nu_k}$, it is easy to see that

(2.2) for every $i \in \{1, \dots, k\}$ there exists $\alpha(i) \in I$, $j(i) \in E_{\alpha(i)}$

such that $u = u_{z_{\alpha(i)}} + s_{\alpha(i)}$ in A_{ν_i}, $P_{j(i)} \cap \Omega \neq \emptyset$ and $u = u_{z_{j(i)}} + s_{j(i)}$ in $P_{j(i)}$.

Let us prove that for every $i \in \{1, \dots, k\}$ there exists a nonempty subset M_i of I such that by defining

$$v_i \colon x \in \boldsymbol{R}^n \mapsto \inf_{\alpha \in M_i} (u_{z_\alpha}(x) + s_\alpha) \tag{2.3}$$

then

$$v_i(x) = u(x) \quad \text{for every } x \in A_{v_i} \tag{2.4}$$

and

$$v_i(x) \leqslant u(x) \quad \text{for every } x \in \Omega\,. \tag{2.5}$$

To do this let $i \in \{1, \dots, k\}$, set

$$M_i = \{\alpha \in I \colon u_{z_\alpha}(x) + s_\alpha \geqslant u(x) \text{ for every } x \in A_{\nu_i}$$
$$\text{and there exists } j \in E_\alpha \text{ with } P_j \cap \Omega \neq \emptyset\}\,,$$

and observe that by (2.2) $M_i \neq \emptyset$ and that (2.4) holds.

Let us prove (2.5), by contradiction let us assume that (2.5) does not hold, then the set $B = \{x \in \Omega \colon v_i(x) > u(x)\}$ turns out to be open and nonempty.

Obviously, being A open and dense in $\boldsymbol{R}^n$, it results that $B \cap A \neq \emptyset$. Let $x_1 \in B \cap A$, $\varepsilon > 0$ be such that $B_\varepsilon(x_1) \subseteq B \cap A$ and take $x_0 \in A_{\nu_i} \cap \Omega$.

Since $\boldsymbol{R}^n \setminus A$ is made up by a finite numbers of pieces of flat surfaces and $B_\varepsilon(x_1) \subseteq B \cap A$, we can choose $x_2 \in B_\varepsilon(x_1)$ such that

(2.6) $\sigma[x_0, x_2] \cap (\Omega \setminus A)$ contains only a finite numbers of points

and again $v_i(x_2) > u(x_2)$.

Let $S = \{x \in \sigma[x_0, x_2] : v_i \leqslant u \text{ on } \sigma[x_0, x]\}$, then obviously S turns out to be closed and by (2.4) $x_0 \in S$. Let $x^* \in S$ be such that $|x_0 - x^*| = \max_{x \in S} |x_0 - x|$, then

$$u(x^*) = v_i(x^*)\,. \tag{2.7}$$

Being Ω convex it turns out that $x^* \in \Omega$ but, in general, it is not sure that $x^* \in A$, nevertheless, by using also (2.6), we can find $x_3 \in \sigma[x^*, x_2]$ such that $\sigma]x^*, x_3] \subseteq A \cap \Omega$. Since now $x_3 \in A$, there exists one and only one index $\alpha^* \in I$ for which $u(x_3) = u_{z_{\alpha^*}}(x_3) + s_{\alpha^*}$, moreover, being u affine on $\sigma[x^*, x_3]$, we also have that

$$u = u_{z_{\alpha^*}} + s_{\alpha^*} \qquad \text{on } \sigma[x^*, x_3]. \tag{2.8}$$

On the other side the fact that $x_3 \notin S$ yields the existence of $x_4 \in \sigma]x^*, x_3]$ such that $v_i(x_4) > u(x_4)$ and by this, taking also (2.8) into account, we conclude that

$$u_{z_{\alpha^*}}(x_4) + s_{\alpha^*} = u(x_4) < v_i(x_4). \tag{2.9}$$

We now observe that (2.3) implies that if $\alpha \in M_i$ then $v_i(x) \leq u_{z_\alpha}(x) + s_\alpha$ for every $x \in \boldsymbol{R}^n$, therefore by (2.9) we deduce that $\alpha^* \notin M_i$. By virtue of this, since by (2.8) and the inclusion of $\sigma]x^*, x_3]$ in Ω there exists at least one $j \in E_{\alpha^*}$ such that $P_j \cap \Omega \neq \emptyset$, we conclude that it must result

$$u_{z_{\alpha^*}}(x) + s_{\alpha^*} < u(x) \qquad \text{for some } x \in A_{\nu_i}. \tag{2.10}$$

At this point we recall that u is affine on A_{ν_i} and that by definition of A_{ν_i} either u is identically equal to $u_{z_{\alpha^*}} + s_{\alpha^*}$ on A_{ν_i} or $u(x) \neq u_{z_{\alpha^*}}(x) + s_{\alpha^*}$ for every $x \in A_{\nu_i}$, therefore by virtue of this, of (2.10) and of (2.4) we obtain

$$u_{z_{\alpha^*}}(x_0) + s_{\alpha^*} < u(x_0) = v_i(x_0). \tag{2.11}$$

By (2.3) v_i turns out to be concave, hence by (2.9) and (2.11) we deduce that

$$u_{z_{\alpha^*}}(x) + s_{\alpha^*} < v_i(x) \qquad \text{for every } x \in \sigma[x_0, x_4]$$

and in particular that

$$u_{z_{\alpha^*}}(x^*) + s_{\alpha^*} < v_i(x^*). \tag{2.12}$$

Inequality (2.12) yields a contradiction since by (2.7) and (2.8) we have that

$$u_{z_{\alpha^*}}(x^*) + s_{\alpha^*} = u(x^*) = v_i(X^*),$$

therefore (2.5) holds.

By (2.4) and (2.5) we conclude that

$$u(x) = \sup_{i \in \{1, \ldots, k\}} v_i(x) \qquad \text{for every } x \in \Omega. \tag{2.13}$$

The theorem is now essentially proved, indeed (2.1) follows by (2.13) if we choose, for every $\alpha \in M_i$, $j(\alpha) \in E_\alpha$ such that $P_{j(\alpha)} \cap \Omega \neq \emptyset$, if we define $N_i = \{j(\alpha) \colon \alpha \in M_i\}$ and observe that $v_i = \inf\limits_{j \in N_i} (u_{z_j} + s_j)$ for every $i \in \{1, \ldots, k\}$. ■

In the example below we show that in general the convexity assumption in Theorem 2.1 cannot be dropped.

EXAMPLE 2.2. – Let $n = 1$ and u be the piecewise affine function given by

$$u\colon x \in \mathbf{R} \mapsto x\chi_{]-\infty, 1[}(x) + (2-x)\,\chi_{[1, 2[}(x) + (x-2)\,\chi_{[2, +\infty[}(x),$$

then, in relation to the notations of Theorem 2.1, $m=3$, $z_1=1$, $s_1=0$, $z_2=-1$, $s_2=2$, $z_3=1$, $s_3=-2$, $P_1=]-\infty, 1[$, $P_2=[1, 2[$ and $P_3=[2, +\infty[$.

Let $\Omega =]0, 1[\cup]2, 3[$, then $\{j \in \{1, 2, 3\}\colon P_j \cap \Omega \neq \emptyset\} = \{1, 3\}$ and Theorem 2.1 cannot hold for such choice of Ω since whenever we take $k \in \mathbf{N}$ and k subsets $N_1, \ldots, N_k$ of $\{1, 3\}$ it is easy to verify that $\sup\limits_{i \in \{1, \ldots, k\}} \inf\limits_{j \in N_i} (u_{z_j}(x) + s_j) = x$ or $\sup\limits_{i \in \{1, \ldots, k\}} \inf\limits_{j \in N_i} (u_{z_j}(x) + s_j) = x - 2$ for every $x \in \mathbf{R}$.

The following example shows that Theorem 2.1 can be false even for connected non convex open sets.

EXAMPLE 2.3. – Let $n = 2$ and u be the piecewise affine function given by

$$u\colon (x, y) \in \mathbf{R}^2 \mapsto y\chi_{P_1}(x, y) + x\chi_{P_2}(x, y) - y\chi_{P_3}(x, y),$$

where $P_1 = \{(x, y) \in \mathbf{R}^2\colon x > 0,\ 0 < y < x\}$, $P_2 = \{(x, y) \in \mathbf{R}^2\colon y \geq |x|\}$, $P_3 = \{(x, y) \in \mathbf{R}^2\colon x < 0,\ 0 < y < -x\}$, then, in relation to the notations of Theorem 2.1, $m = 4$, $P_4 = \{(x, y) \in \mathbf{R}^2\colon y \leq 0\}$, $z_1 = (0, 1)$, $z_2 = (1, 0)$, $z_3 = (0, -1)$, $z_4 = (0, 0)$ and $s_1 = s_2 = s_3 = s_4 = 0$.

Let $\Omega = \{(x, y) \in \mathbf{R}^2\colon -1 < x < 1,\ -1 < y < |x|\}$, then $\{j \in \{1, 2, 3, 4\}\colon P_j \cap \Omega \neq \emptyset\} = \{1, 3, 4\}$ and Theorem 2.1 cannot hold for such choice of Ω since whenever we take $k \in \mathbf{N}$ and k subsets $N_1, \ldots, N_k$ of $\{1, 3, 4\}$ the function $\sup\limits_{i \in \{1, \ldots, k\}} \inf\limits_{j \in N_i} (u_{z_j} + s_j)$ does not depend on the x variable whilst so does u.

REMARK 2.4. – We observe that examples 2.2 and 2.3 are even more shrinking, indeed they prove that convexity assumptions on Ω in Theorem 2.1 cannot be dropped even if one tries to represent on an open set Ω a piecewise affine function $u = \sum\limits_{j=1}^{m} (u_{z_j} + s_j)\,\chi_{P_j}$ by means of combinations in arbitrary order, and not as in (2.1), of last upper bounds and greatest lower bounds of those of its "components" $u_{z_j} + s_j$ for which $P_j \cap \Omega$ is nonempty.

3. – The homogenization formula.

Let f be as in (1.11), $p \in [1, +\infty]$ and let $f_{\hom}$, F_ε $(\varepsilon > 0)$ be given respectively by (1.14) and (1.12).

In the present section we prove that for every convex bounded open set Ω, $u \in W^{1,p}(\Omega)$ the limits $\Gamma^-(L^p(\Omega)) \liminf\limits_{\varepsilon \to 0} F_\varepsilon(\Omega, u)$ and $\Gamma^-(L^p(\Omega)) \limsup\limits_{\varepsilon \to 0} F_\varepsilon(\Omega, u)$ agree with the integral $\int\limits_\Omega f_{\hom}(Du)\,dx$. To do part of this we will follow an argument proposed in [A1] and [A2] by first approximating f from below with an increasing sequence of real valued integrands $\{f_k\}$, by applying known results on homogenization of integral functionals for these integrands and then by taking the limit as k diverges.

For every $k \in \boldsymbol{N}$ let us set

$$\overline{f}_k : (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto \min\{f(x, z), k(1 + |z|)\}, \tag{3.1}$$

$$f_k : (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto (\overline{f}_k(x, \cdot))^{**}(z) \tag{3.2}$$

and

$$f_{k,\,\mathrm{hom}} : z \in \boldsymbol{R}^n \mapsto \inf\left\{\int_Y f_k(y, z + Dv)\, dy : v \in W^{1,p}_{\mathrm{per}}(Y)\right\} \tag{3.3}$$

then $f_{k,\,\mathrm{hom}}$ turns out to be convex and finite on $\boldsymbol{R}^n$; moreover for every $\Omega \in \mathcal{A}$ and $\varepsilon > 0$ let $F_{k,\varepsilon}(\Omega, \cdot)$ be the functionals defined by

$$F_{k,\varepsilon}(\Omega, \cdot) : u \in L^p(\Omega) \mapsto \begin{cases} \displaystyle\int_\Omega f_k\left(\frac{x}{\varepsilon}, Du\right) dx & \text{if } u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n), \\ +\infty & \text{if } u \in L^p(\Omega) \setminus W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n). \end{cases}$$

By virtue of (3.1) and (3.2) it turns out that for every $k \in \boldsymbol{N}$ it results that

$$f_k(x, z) \le k(1 + |z|) \quad \text{for a.e. } x \in \boldsymbol{R}^n \text{ and every } z \in \boldsymbol{R}^n, \tag{3.4}$$

therefore by (1.11), (3.4) and the obvious inequality, if $p < +\infty$,

$$k(1 + |z|) \le 2k(1 + |z|^p) \quad \text{for every } z \in \boldsymbol{R}^n,$$

it turns out that the assumptions of Theorem 1.4 are fulfilled by f_k, hence we have

$$\Gamma^-(L^p(\Omega)) \lim_{\varepsilon \to 0} F_{k,\varepsilon}(\Omega, u) = \int_\Omega f_{k,\,\mathrm{hom}}(Du)\, dx \tag{3.5}$$

$$\text{for every } \Omega \in \mathcal{A} \text{ with Lipschitz boundary, } u \in W^{1,p}(\Omega).$$

Moreover, since obviously

$$F_{k,\varepsilon}(\Omega, u) \le F_\varepsilon(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}, \text{ every } k \in \boldsymbol{N} \text{ and } u \in L^p(\Omega), \tag{3.6}$$

if F'_{hom} is defined in (1.13), by (3.5) and (3.6) we deduce that

$$\sup_{k \in \boldsymbol{N}} \int_\Omega f_{k,\,\mathrm{hom}}(Du)\, dx \le F'_{\mathrm{hom}}(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}, u \in W^{1,p}(\Omega). \tag{3.7}$$

In order to evaluate the left-hand side of (3.7) we need to prove some lemmas.

LEMMA 3.1. – *Let f be as in* (1.11), *and let, for every $k \in \boldsymbol{N}$, f_k be given by* (3.2), *then*

$$\sup_{k \in \boldsymbol{N}} f_k(x, z) = \lim_{k \to +\infty} f_k(x, z) = f(x, z) \quad \text{for a.e. } x \in \boldsymbol{R}^n \text{ and every } z \in \boldsymbol{R}^n. \tag{3.8}$$

PROOF. – By (1.11) there exists a subset N of $\boldsymbol{R}^n$ such that $|N| = 0$ and for every $x \in \boldsymbol{R}^n \setminus N$ $f(x, \cdot)$ is convex.

Let $x \in \boldsymbol{R}^n \setminus N$, $z_0 \in \boldsymbol{R}^n$ and let $\alpha < f(x, z_0)$. Since $f(x, \cdot)$ is convex and lower semicontinuous there exist $a \in \boldsymbol{R}^n$ and $b \in \boldsymbol{R}$ such that

$$\begin{cases} \text{(i)} \ \ a \cdot z + b \leqslant f(x, z) \ \text{ for every } z \in \boldsymbol{R}^n, \\ \text{(ii)} \ \ \alpha < a \cdot z_0 + b \leqslant f(x, z_0). \end{cases} \tag{3.9}$$

Let $k \in \boldsymbol{N}$ be such that

$$a \cdot z + b \leqslant k(1 + |z|) \qquad \text{for every } z \in \boldsymbol{R}^n, \tag{3.10}$$

then by (3.1), (3.9) and (3.10) we obtain that

$$\begin{cases} \text{(i)} \ \ a \cdot z + b \leqslant \overline{f}_k(x, z) \ \text{ for every } z \in \boldsymbol{R}^n, \\ \text{(ii)} \ \ \alpha < a \cdot z_0 + b \leqslant \overline{f}_k(x, z_0). \end{cases} \tag{3.11}$$

At this point by (3.2), (3.11) and (1.10) we obtain that

$$\begin{cases} \text{(i)} \ \ a \cdot z + b \leqslant f_k(x, z) \ \text{ for every } z \in \boldsymbol{R}^n, \\ \text{(ii)} \ \ \alpha < a \cdot z_0 + b \leqslant f_k(x, z_0) \leqslant \overline{f}_k(x, z_0) \leqslant f(x, z_0), \end{cases} \tag{3.12}$$

therefore by (ii) of (3.12) we have proved that for every $\alpha < f(x, z_0)$ there exists $k \in \boldsymbol{N}$ such that

$$\alpha < f_k(x, z_0) \leqslant f(x, z_0). \tag{3.13}$$

By (3.13), taking into account that the sequence $\{f_k(x, z_0)\}$ is increasing, we deduce (3.8). ■

We now prove the analogue of (3.8) for $f_{k,\,\mathrm{hom}}$ and f_{hom}.

LEMMA 3.2. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *f_{hom} be given by* (1.14) *and, for every $k \in \boldsymbol{N}$, $f_{k,\,\mathrm{hom}}$ by* (3.2), *then*

$$\sup_{k \in \boldsymbol{N}} f_{k,\,\mathrm{hom}}(z) = \lim_{k \to +\infty} f_{k,\,\mathrm{hom}}(z) = f_{\mathrm{hom}}(z) \qquad \text{for every } z \in \boldsymbol{R}^n. \tag{3.14}$$

PROOF. – It is clear that for every $z \in \boldsymbol{R}^n$ the sequence $\{f_{k,\,\mathrm{hom}}(z)\}$ in increasing and that by (3.1), (3.2)

$$\sup_{k \in \boldsymbol{N}} f_{k,\,\mathrm{hom}}(z) = \lim_{k \to +\infty} f_{k,\,\mathrm{hom}}(z) \leq f_{\mathrm{hom}}(z) \qquad \text{for every } z \in \boldsymbol{R}^n . \tag{3.15}$$

Let $z \in \boldsymbol{R}^n$, in order to prove the reverse inequality in (3.15) we can assume that the left-hand side of (3.15) is finite, moreover let us first consider the case in which (1.16) holds.

For every $\varepsilon > 0$ and $k \in \boldsymbol{N}$ let $v_k \in W^{1,p}_{\mathrm{per}}(Y)$ such that

$$+\infty > \sup_{k \in \boldsymbol{N}} f_{k,\,\mathrm{hom}}(z) + \varepsilon \geq f_{k,\,\mathrm{hom}}(z) + \varepsilon > \int_Y f_k(y, z + Dv_k)\, dy , \tag{3.16}$$

moreover, since we can always assume that $\int_Y v_k \, dy = 0$ for every $k \in \boldsymbol{N}$, we get from (3.16), (1.16) and Poincaré-Wirtinger inequality that $\{v_k\}$ is bounded in $W^{1,p}(Y)$ and that there exists $v \in W^{1,p}_{\mathrm{per}}(Y)$ such that, up to subsequences, $v_k \to v$ weakly in $W^{1,p}(Y)$.

Let us fix now $k_0 \in \boldsymbol{N}$, then by (3.16), the monotonicity of $\{f_k\}$ and the sequential weak-$W^{1,p}(Y)$-lower semicontinuity of the functional $u \in W^{1,p}(Y) \mapsto \int_Y f_{k_0}(y, z + Du)\, dy$ we obtain that

$$\sup_{k \in \boldsymbol{N}} f_{k,\,\mathrm{hom}}(z) + \varepsilon \geq \liminf_{k \to +\infty} \int_Y f_k(y, z + Dv_k)\, dy \geq \tag{3.17}$$

$$\liminf_{k \to +\infty} \int_Y f_{k_0}(y, z + Dv_k)\, dy \geq \int_Y f_{k_0}(y, z + Dv)\, dy .$$

By (3.17), Lemma 3.1 and the monotone convergence theorem we infer first as k_0 increases to $+\infty$ and then as ε decreases to 0 that

$$\sup_{k \in \boldsymbol{N}} f_{k,\,\mathrm{hom}}(z) \geq \int_Y f(y, z + Dv)\, dy \geq f_{\mathrm{hom}}(z) . \tag{3.18}$$

By (3.15) and (3.18) equalities in (3.14) follow if (1.16) holds.

If (1.17) holds in place of (1.16) the proof remains almost the same, the only difference is in the fact that the weak-$W^{1,p}(Y)$ (weak*-$W^{1,p}(Y)$ if $p = +\infty$) - compactness of $\{v_k\}$ is achieved by observing that $|Dv_k(y)| \leq \varphi_f(y)$ for almost every $y \in Y$ and every $k \in \boldsymbol{N}$ and by using the summability properties of φ_f, and that, if $p = +\infty$, the sequential weak*-$W^{1,\infty}(Y)$-lower semicontinuity of the functional $u \in W^{1,\infty}(Y) \mapsto \int_Y f_{k_0}(y, z + Du)\, dy$ must be taken into account. ■

REMARK 3.3. – We observe that if f is as in (1.11), if (1.16) or (1.17) holds and f_{hom} is given by (1.14), then by Lemma 3.2 it turns out that f_{hom} is lower semicontinuous.

If f and $f_{\rm hom}$ are as in Remark 3.3 and (1.16) or (1.17) holds, by Remark 3.3 it soon follows that for every $\Omega \in \mathcal{A}$, $u \in W^{1,p}(\Omega)$ the function $f_{\rm hom}(Du(\cdot))$ is non-negative and measurable on Ω, and hence that the integral $\int_\Omega f_{\rm hom}(Du)\,dx$ is well defined.

We can now prove the estimate from below.

PROPOSITION 3.4. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *and let* $f_{\rm hom}$, $F'_{\rm hom}$ *be given by* (1.14) *and* (1.13), *then for every* $\Omega \in \mathcal{A}$, $u \in L^p(\Omega)$ *it results*

$$\text{(3.19)} \qquad \begin{cases} \displaystyle\int_\Omega f_{\rm hom}(Du)\,dx \ \ \textit{if } u \in W^{1,p}(\Omega) \\ +\infty \ \ \textit{if } u \in L^p(\Omega)\setminus W^{1,p}(\Omega) \end{cases} \leqslant F'_{\rm hom}(\Omega, u)\,.$$

PROOF. – Let $\Omega \in \mathcal{A}$.

For every $k \in \boldsymbol{N}$ let f_k and $f_{k,\,\rm hom}$ be given respectively by (3.2) and (3.3); since the sequence $\{f_{k,\,\rm hom}\}$ is increasing, by (3.7), the monotone convergence theorem and Lemma 3.2 we infer that

$$\text{(3.20)} \qquad \int_\Omega f_{\rm hom}(Du)\,dx = \lim_{k\to+\infty} \int_\Omega f_{k,\,\rm hom}(Du)\,dx \leqslant F'_{\rm hom}(\Omega, u)$$

$$\text{for every } u \in W^{1,p}(\Omega)\,.$$

Let us observe now that by (1.16) or (1.17) it follows that if $u \in L^p(\Omega)$ is such that $F'_{\rm hom}(\Omega, u) < +\infty$ then there exists a sequence in $W^{1,p}_{\rm loc}(\boldsymbol{R}^n)$ bounded in $W^{1,p}(\Omega)$ and converging to u in $L^p(\Omega)$, this implies that

$$\text{(3.21)} \qquad F'_{\rm hom}(\Omega, u) < +\infty \Rightarrow u \in W^{1,p}(\Omega)\,,$$

therefore by (3.20) and (3.21) inequality (3.19) follows. ■

We now study the estimate from above.

LEMMA 3.5. – *Let f be as in* (1.11), $p \in [1, +\infty]$, *and let* $f_{\rm hom}$, $F''_{\rm hom}$ *be given by* (1.14) *and* (1.13), *then*

$$\text{(3.22)} \qquad F''_{\rm hom}(\Omega, u_z) \leqslant |\Omega|\, f_{\rm hom}(z) \qquad \text{for every } \Omega \in \mathcal{A},\ z \in \boldsymbol{R}^n.$$

PROOF. – Let Ω, z be as above and let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0.

Obviously we can assume that $f_{\rm hom}(z) < +\infty$. By virtue of this and of (1.18), given $\eta > 0$, let $v \in W^{1,p}_{\text{per}}(Y)$ be such that

$$\int_Y f(y, z + Dv)\, dy \leqslant f_{\text{hom}}(z) + \eta \tag{3.23}$$

and define, for every $h \in \boldsymbol{N}$, the function v_h by $v_h = \varepsilon_h v\left(\dfrac{\cdot}{\varepsilon_h}\right)$.

Clearly $v_h \in W^{1,p}_{\text{loc}}(\boldsymbol{R}^n)$ and $v_h \to 0$ in $L^p_{\text{loc}}(\boldsymbol{R}^n)$, moreover by the Y-periodicity of $f(\cdot, z + Dv(\cdot))$ and by (3.23) we have

$$\Gamma^-(L^p(\Omega)) \limsup_{h \to +\infty} F_{\varepsilon_h}(\Omega, u_z) \leqslant \limsup_{h \to +\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, z + Dv_h\right) dx = \tag{3.24}$$

$$\limsup_{h \to +\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, z + Dv\left(\frac{x}{\varepsilon_h}\right)\right) dx = |\Omega| \int_Y f(y, z + Dv)\, dy \leqslant |\Omega| (f_{\text{hom}}(z) + \eta)\,.$$

As η decreases to 0, by (3.24), the arbitrariness of $\{\varepsilon_h\}$ and (1.7) inequality (3.22) soon follows. ∎

In order to extend inequality (3.22) to $PA(\boldsymbol{R}^n)$ we need to prove the following result.

LEMMA 3.6. – *Let f be as in* (1.11), $p \in [1, +\infty]$, *and let F'_{hom}, F''_{hom} be given by* (1.13). *Let $g: \boldsymbol{R}^n \to [0, +\infty]$ be a lower semicontinuous function, $\Omega \in \mathcal{A}$ and $\mathcal{U} \subseteq W^{1,p}_{\text{loc}}(\boldsymbol{R}^n)$ be such that*

$$F'_{\text{hom}}(\Omega, u) \geqslant \int_\Omega g(Du)\, dx \qquad \text{for every } u \in W^{1,p}_{\text{loc}}(\boldsymbol{R}^n) \tag{3.25}$$

and

$$F''_{\text{hom}}(\Omega, u) \leqslant \int_\Omega g(Du)\, dx < +\infty \qquad \textit{for every } u \in \mathcal{U}\,, \tag{3.26}$$

then for every $m \in \boldsymbol{N}$ and every $u_1, \ldots, u_m \in \mathcal{U}$ it results that

$$F''_{\text{hom}}(\Omega, u) \leqslant \int_\Omega g(Du)\, dx < +\infty \quad \textit{with } u = \inf\{u_j : j \in \{1, \ldots, m\}\} \tag*{$(3.27)_m$}$$

and

$$F''_{\text{hom}}(\Omega, u) \leqslant \int_\Omega g(Du)\, dx < +\infty \quad \textit{with } u = \sup\{u_j : j \in \{1, \ldots, m\}\}\,. \tag*{$(3.28)_m$}$$

PROOF. – Let us prove the inequalities in (3.27), the proof for those in (3.28) being similar.

We argue by induction on m.

If $m = 1$ clearly (3.26) implies $(3.27)_1$.

Let now $m \in \boldsymbol{N}$ and prove that $(3.27)_m$ implies $(3.27)_{m+1}$.

Let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0, $u_1, \ldots, u_{m+1} \in \mathcal{U}$, $u = \inf\{u_j : j \in \{1, \ldots, m+1\}\}$, and define the function v by $v = \inf\{u_j : j \in \{1, \ldots, m\}\}$.

Since $u_{m+1} \in \mathcal{U}$, by (3.26), $(3.27)_m$ and (1.7) there exist two sequences $\{u_h^{m+1}\}$ and $\{v_h\}$ in $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that $u_h^{m+1} \to u_{m+1}$ in $L^p(\Omega)$, $v_h \to v$ in $L^p(\Omega)$ and

$$\begin{cases} +\infty > \displaystyle\int_\Omega g(Du_{m+1})\,dx \geq \limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, Du_h^{m+1}\right) dx\,, \\ +\infty > \displaystyle\int_\Omega g(Dv)\,dx \geq \limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, Dv_h\right) dx\,. \end{cases} \tag{3.29}$$

Let us observe now that $\inf\{v_h, u_h^{m+1}\} \to u$ and $\sup\{v_h, u_h^{m+1}\} \to \sup\{v, u_{m+1}\}$ in $L^p(\Omega)$ and that

$$f\left(\frac{x}{\varepsilon_h}, D(\inf\{v_h, u_h^{m+1}\})(x)\right) = \tag{3.30}$$

$$f\left(\frac{x}{\varepsilon_h}, Dv_h(x)\right) + f\left(\frac{x}{\varepsilon_h}, Du_h^{m+1}(x)\right) - f\left(\frac{x}{\varepsilon_h}, D(\sup\{v_h, u_h^{m+1}\})(x)\right)$$

$$\text{for a.e. } x \in \Omega\,,$$

therefore by (3.30), (3.29) and (3.25) we have

$$\begin{aligned} &\Gamma^-(L^p(\Omega)) \limsup_{h\to+\infty} F_{\varepsilon_h}(\Omega, u) \leq \limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, D(\inf\{v_h, u_h^{m+1}\})\right) dx \leq \\ &\limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, Dv_h\right) dx + \limsup_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, Du_h^{m+1}\right) dx - \\ &\liminf_{h\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_h}, D(\sup\{v_h, u_h^{m+1}\})\right) dx \leq \\ &\int_\Omega g(Du_{m+1})\,dx + \int_\Omega g(Dv)\,dx - F'_{\mathrm{hom}}(\Omega, \sup\{v, u_{m+1}\}) \leq \\ &\int_\Omega g(Du_{m+1})\,dx + \int_\Omega g(Dv)\,dx - \int_\Omega g(D(\sup\{v, u_{m+1}\}))\,dx = \int_\Omega g(Du)\,dx\,. \end{aligned} \tag{3.31}$$

By (3.31), the arbitrariness of $\{\varepsilon_h\}$ and (1.7), inequality $(3.27)_{m+1}$ and the thesis follow. ∎

LEMMA 3.7. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *and let* f_{hom}, F''_{hom} *be given by* (1.14) *and* (1.13), *then*

$$(3.32)\quad F''_{\mathrm{hom}}(\Omega, u) \leqslant \int_\Omega f_{\mathrm{hom}}(Du)\,dx \quad \textit{for every convex } \Omega \in \mathcal{A},\ u \in PA(\boldsymbol{R}^n).$$

PROOF. – Let Ω be as in (3.32) and $u \in PA(\boldsymbol{R}^n)$ be as in (1.9), then by Theorem 2.1 we obtain the existence of $k \in \boldsymbol{N}$ and of $N_1, \ldots, N_k \subseteq \{j \in \{1, \ldots, m\}: P_j \cap \Omega \neq \emptyset\}$ such that (2.1) holds.

Let us observe that it is not restrictive to assume that $\int_\Omega f_{\mathrm{hom}}(Du)\,dx < +\infty$, that is

$$(3.33)\qquad f_{\mathrm{hom}}(z_j) < +\infty \qquad \text{for every } j \in \{1, \ldots, m\} \text{ such that } P_j \cap \Omega \neq \emptyset\,.$$

Let $i \in \{1, \ldots, k\}$, α_i be the cardinality of N_i and $v_i = \inf\limits_{j \in N_i}(u_{z_j} + s_j)$, then by Remark 3.3, Proposition 3.4, Lemma 3.5, (3.33) and $(3.27)_{\alpha_i}$ of Lemma 3.6 applied with $g = f_{\mathrm{hom}}$ and $\mathcal{U} = \{u_{z_j} + s_j : j \in N_i\}$ we obtain

$$(3.34)\quad F''_{\mathrm{hom}}(\Omega, v_i) \leqslant \int_\Omega f_{\mathrm{hom}}(Dv_i)\,dx < +\infty \qquad \text{for every } i \in \{1, \ldots, k\}\,.$$

At this point by (3.34) and $(3.28)_k$ of Lemma 3.6 applied with $g = f_{\mathrm{hom}}$ and $\mathcal{U} = \{v_i : i \in \{1, \ldots, k\}\}$ we deduce (3.32). ■

If $\Omega \in \mathcal{A}$, u is in $L^p(\Omega)$ and $F''_{\mathrm{hom}}(\Omega, u)$ is given by (1.13) we denote by $(F''_{\mathrm{hom}})_-(\Omega, u)$ the inner regular envelope at Ω of the increasing set function $F''_{\mathrm{hom}}(\cdot, u)$.

LEMMA 3.8. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *and let* f_{hom}, F''_{hom} *be given by* (1.14) *and* (1.13), *then*

$$(3.35)\quad (F''_{\mathrm{hom}})_-(\Omega, u) \leqslant \int_\Omega f_{\mathrm{hom}}(Du)\,dx \quad \textit{for every convex } \Omega \in \mathcal{A},\ u \in C^1(\boldsymbol{R}^n).$$

PROOF. – Let Ω, u be as in (3.35).

Let us observe that the set $\mathrm{dom}\, f_{\mathrm{hom}}$ is convex and that we can obviously assume that it is nonempty and that $Du(x) \in \mathrm{dom}\, f_{\mathrm{hom}}$ for every $x \in \Omega$.

If $\mathrm{dom}\, f_{\mathrm{hom}}$ contains only a single point then the thesis follows by Lemma 3.5, therefore it is not restrictive to assume that the dimension ν of $\mathrm{aff}(\mathrm{dom}\, f_{\mathrm{hom}})$ is bigger than zero.

We first consider the case in which

$$(3.36)\qquad 0 \in \mathrm{ri}(\mathrm{dom}\, f_{\mathrm{hom}})\,.$$

Let R be the identity matrix if $\nu = n$ and an orthogonal matrix such that

$$(3.37) \qquad R(\operatorname{aff}(\operatorname{dom} f_{\hom})) = \boldsymbol{R}^{\nu} \times \{0^{(n-\nu)}\}$$

if $\nu < n$, and let us define the function $\tilde{u}$ by

$$(3.38) \qquad \tilde{u}\colon y \in \boldsymbol{R}^n \mapsto u(R^{-1} y),$$

then

$$(3.39) \qquad D_y \tilde{u}(y) = R D_x u(R^{-1} y) \quad \text{for every } y \in \boldsymbol{R}^n.$$

By (3.39) and (3.37), since $Du(x) \in \operatorname{dom} f_{\hom}$ for every $x \in \Omega$, we infer that $D\tilde{u}(y)$ has the last $n - \nu$ entries equal to zero for every $y \in R\Omega$ and hence, taking into account the convexity of $R\Omega$, that $\tilde{u}$ depends only on $(y_1, \ldots, y_\nu)$ when $(y_1, \ldots, y_n)$ varies in $R\Omega$.

Let us set $\Omega^{(\nu)} = Pr_\nu(R\Omega)$, let $A, B \in \mathcal{A}$ with A convex, $A \subset\subset B \subset\subset \Omega$ and define $A^{(\nu)} = Pr_\nu(RA)$, $B^{(\nu)} = Pr_\nu(RB)$, then obviously $A^{(\nu)} \subset\subset B^{(\nu)} \subset\subset \Omega^{(\nu)}$.

If $\nu < n$ let σ be the multivalued function defined by $\sigma\colon y \in \Omega^{(\nu)} \mapsto \{z \in \boldsymbol{R}^{n-\nu} : (y, z) \in R\Omega\}$ and let β be a continuous selection of σ, i.e. a function in $C^0(\Omega^{(\nu)}; \boldsymbol{R}^{n-\nu})$ such that $\beta(y) \in \sigma(y)$ for every $y \in \Omega^{(\nu)}$ (take for example $\beta(y)$ to be the barycentre of $\sigma(y)$ for every $y \in \Omega^{(\nu)}$).

Since $(y, \beta(y)) \in R\Omega$ for every $y \in \Omega^{(\nu)}$ and $R\Omega$ is open, we can find a function $\beta_1 \in C^1(\boldsymbol{R}^\nu; \boldsymbol{R}^{n-\nu})$ such that $(y, \beta_1(y)) \in R\Omega$ for every $y \in B^{(\nu)}$.

By virtue of this we can define the function $\hat{u}$ by

$$(3.40) \qquad \hat{u}\colon (y_1, \ldots, y_\nu) \in \boldsymbol{R}^\nu \mapsto \begin{cases} \tilde{u}(y_1, \ldots, y_n) & \text{if } \nu = n, \\ \tilde{u}(y_1, \ldots, y_\nu, \beta_1(y_1, \ldots, y_\nu)) & \text{if } \nu < n, \end{cases}$$

then obviously $\hat{u} \in C^1(\boldsymbol{R}^\nu)$ and, being $Du(x) \in \operatorname{dom} f_{\hom}$ for every $x \in \Omega$, by (3.38), (3.39) and (3.40) we obtain that $D\hat{u}(y) \in Pr_\nu(R(\operatorname{dom} f_{\hom}))$ for every $y \in B^{(\nu)}$.

By virtue of this and by (3.36) we infer that there exists a compact subset H_ν of $\operatorname{ri}(Pr_\nu(R(\operatorname{dom} f_{\hom})))$ such that

$$(3.41) \qquad D(s\hat{u})(y) \in H_\nu \quad \text{for every } s \in]0, 1[\text{ and } y \in B^{(\nu)}.$$

Let $s \in [0, 1[$ and let $\{\hat{u}_h\} \subseteq PA(\boldsymbol{R}^\nu)$ be such that $\hat{u}_h \to s\hat{u}$ in $W^{1,\infty}(\Omega^{(\nu)})$, then by (3.41) we obtain that

$$(3.42) \qquad D\hat{u}_h(y) \in K_\nu \quad \text{for every } y \in A^{(\nu)} \text{ and every } h \in \boldsymbol{N} \text{ large enough},$$

K_ν being a suitable compact subset of $\operatorname{ri}(Pr_\nu(R(\operatorname{dom} f_{\hom})))$.

For every $h \in \boldsymbol{N}$ let us now define the functions $\tilde{u}_h$ and u_h by

$$\tilde{u}_h \colon (y_1, \ldots, y_n) \in \boldsymbol{R}^n \mapsto \widehat{u}_h(y_1, \ldots, y_\nu)$$

and

$$u_h \colon x \in \boldsymbol{R}^n \mapsto \tilde{u}_h(Rx),$$

then obviously $u_h \in PA(\boldsymbol{R}^n)$ for every $h \in \boldsymbol{N}$ and

$$u_h \to su \quad \text{in } W^{1,\infty}(\Omega), \tag{3.43}$$

moreover by (3.42) we deduce the existence of a compact subset K of $\mathrm{ri}(\mathrm{dom}\, f_{\mathrm{hom}})$ such that

$$Du_h(x) \in K \quad \text{for every } x \in A \text{ and every } h \in \boldsymbol{N} \text{ large enough}. \tag{3.44}$$

At this point by the convexity of A and Lemma 3.7 we obtain

$$F''_{\mathrm{hom}}(A, u_h) \leq \int_A f_{\mathrm{hom}}(Du_h)\, dx \quad \text{for every } h \in \boldsymbol{N} \text{ large enough}, \tag{3.45}$$

moreover by (3.43), (3.44) and the local Lipschitz continuity of f_{hom} in $\mathrm{ri}(\mathrm{dom}\, f_{\mathrm{hom}})$ we have

$$\lim_{h \to +\infty} \int_A f_{\mathrm{hom}}(Du_h)\, dx = \int_A f_{\mathrm{hom}}(sDu)\, dx, \tag{3.46}$$

therefore by (3.45), (3.46) and the convexity of f_{hom} we obtain

$$\liminf_{h \to +\infty} F''_{\mathrm{hom}}(A, u_h) \leq s \int_\Omega f_{\mathrm{hom}}(Du)\, dx + (1-s)\,|\Omega| f_{\mathrm{hom}}(0) \tag{3.47}$$

and by (3.43), (3.47) and (1.2) that

$$F''_{\mathrm{hom}}(A, su) \leq \int_\Omega f_{\mathrm{hom}}(Du)\, dx + (1-s)\,|\Omega| f_{\mathrm{hom}}(0). \tag{3.48}$$

Taking now the limits in (3.48) first as s tends to 1 and then as A increases to Ω, by (3.36), (3.48) and the convexity of Ω we obtain (3.35) if (3.36) holds.

In conclusion if (3.36) does not hold we only have to take $z_0 \in \mathrm{ri}(\mathrm{dom}\, f_{\mathrm{hom}})$ and consider the function f_0 defined by $f_0 \colon (x, z) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \mapsto f(x, z_0 + z)$. We have, with the obvious meaning for the symbols adopted,

$$f_{0,\mathrm{hom}}(z) = f_{\mathrm{hom}}(z_0 + z) \quad \text{for every } z \in \boldsymbol{R}^n, \tag{3.49}$$

$$F''_{0,\mathrm{hom}}(\Omega, u) = F''_{\mathrm{hom}}(\Omega, u_{z_0} + u) \quad \text{for every } \Omega \in \mathcal{A} \text{ and } u \in L^p(\Omega) \tag{3.50}$$

and

$$0 \in \mathrm{ri}\,(\mathrm{dom}\, f_{0,\,\mathrm{hom}})\,, \tag{3.51}$$

therefore by (3.51) and (3.35) applied to f_0 we infer that

$$(F''_{0,\,\mathrm{hom}})_-(\Omega,\, u) \leq \int_\Omega f_{0,\,\mathrm{hom}}(Du)\, dx \quad \text{for every } \Omega \in \mathcal{A},\ u \in C^1(\boldsymbol{R}^n)\,. \tag{3.52}$$

In conclusion by (3.52), (3.50) and (3.49), inequality (3.35) follows also in the general case. ■

LEMMA 3.9. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *and let f_{hom}, F''_{hom} be given by* (1.14) *and* (1.13), *then*

$$(F''_{\mathrm{hom}})_-(\Omega,\, u) \leq \int_\Omega f_{\mathrm{hom}}(Du)\, dx \quad \text{for every } \Omega \in \mathcal{A},\ u \in W^{1,\,p}_{\mathrm{loc}}(\boldsymbol{R}^n)\,. \tag{3.53}$$

PROOF. – Let Ω, u be as in (3.53).

Let α be a mollifier, i.e. $\alpha \in C^1(\boldsymbol{R}^n)$, $\alpha \geq 0$, $\mathrm{spt}\,(\alpha) \subseteq B_1(0)$ and $\int_{\boldsymbol{R}^n} \alpha(x)\, dx = 1$. For every $\eta > 0$ and $x \in \boldsymbol{R}^n$ let us set $\alpha_\eta(x) = \dfrac{1}{\eta^n}\alpha\left(\dfrac{x}{\eta}\right)$, denote by $u_\eta(x)$ the regularization of u at x defined by $u_\eta(x) = (\alpha_\eta * u)(x) = \int_{\boldsymbol{R}^n} \alpha_\eta(x-y)\, u(y)\, dy$ and set $\Omega_\eta = \{y \in \Omega : \mathrm{dist}\,(y,\, \partial\Omega) > \eta\}$, then it is well known that for every $\eta > 0$ $u_\eta \in C^1(\boldsymbol{R}^n)$ and that $u_\eta \to u$ in $W^{1,\,p}(\Omega)$ as $\eta \to 0$, moreover, by the convexity of f_{hom} and Jensen inequality, we have

$$\int_{\Omega_\eta} f_{\mathrm{hom}}(Du_\eta)\, dx \leq \int_\Omega f_{\mathrm{hom}}(Du)\, dx \quad \text{for every } \eta > 0\,. \tag{3.54}$$

Let now $A \in \mathcal{A}$ with $A \subset\subset \Omega$, A being also convex, and take $\eta > 0$ so small that $A \subset\subset \Omega_\eta$, then by the convexity of A, Lemma 3.8 and (3.54) we infer

$$(F''_{\mathrm{hom}})_-(A,\, u_\eta) \leq \int_A f_{\mathrm{hom}}(Du_\eta)\, dx \leq \int_{\Omega_\eta} f_{\mathrm{hom}}(Du_\eta)\, dx \leq \tag{3.55}$$

$$\int_\Omega f_{\mathrm{hom}}(Du)\, dx \quad \text{for every } \eta > 0 \text{ small enough}\,.$$

Taking the limits in (3.55) first as η tends to 0 and then as A increases to Ω, by (1.2) and the fact that the set function $(F''_{\mathrm{hom}})_-(\cdot,\, u)$ is increasing, we deduce (3.53). ■

We can now prove the representation result for the $\Gamma^-(L^p(\Omega))$-limit of the functionals in (1.12).

THEOREM 3.10. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *let* F_ε ($\varepsilon > 0$) *be the functionals defined in* (1.12) *and* f_{hom} *be given by* (1.14), *then* f_{hom} *turns out to be convex, lower semicontinuous and for every convex bounded open set* Ω, $u \in L^p(\Omega)$ *the limit* $\Gamma^-(L^p(\Omega)) \lim\limits_{\varepsilon \to 0} F_\varepsilon(\Omega, u)$ *exists and*

$$\Gamma^-(L^p(\Omega)) \lim_{\varepsilon \to 0} F_\varepsilon(\Omega, u) = \begin{cases} \displaystyle\int_\Omega f_{\mathrm{hom}}(Du)\, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{if } u \in L^p(\Omega) \setminus W^{1,p}(\Omega). \end{cases} \tag{3.56}$$

Moreover, if (1.17) *holds,* $\mathrm{dom} f_{\mathrm{hom}}$ *turns out to be bounded and the right-hand side of* (3.56) *is equal to* $\begin{cases} \displaystyle\int_\Omega f_{\mathrm{hom}}(Du)\, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{if } u \in L^p(\Omega) \setminus W^{1,p}(\Omega). \end{cases}$

PROOF. – Let Ω be as above and let $F'_{\mathrm{hom}}(\Omega, \cdot)$, $F''_{\mathrm{hom}}(\Omega, \cdot)$ be given by (1.13).

By (1.11), Remark 3.3 and Proposition 1.5 the properties of f_{hom} and of $\mathrm{dom} f_{\mathrm{hom}}$ follow.

Let us first prove that

$$F''_{\mathrm{hom}}(\Omega, u) \leqslant \int_\Omega f_{\mathrm{hom}}(Du)\, dx \qquad \text{for every } u \in W^{1,p}(\Omega). \tag{3.57}$$

To do this we can assume that $\mathrm{dom} f_{\mathrm{hom}} \neq \emptyset$, moreover, as in Lemma 3.8, it is not restrictive to assume that

$$0 \in \mathrm{dom}\, f_{\mathrm{hom}}. \tag{3.58}$$

Let $u \in W^{1,p}(\Omega)$, then, since Ω has Lipschitz boundary, we can extend u to a function in $W^{1,p}(\boldsymbol{R}^n)$ and call again u such extension.

Let $x_0 \in \Omega$, $t > 1$ and let u_t be the function defined by $u_t \colon x \in \boldsymbol{R}^n \mapsto u(x_0 + (x - x_0)/t)$, then obviously $u_t \in W^{1,p}(\boldsymbol{R}^n)$ and $u_t \to u$ in $L^p(\Omega)$ as $t \to 1$; moreover by the convexity of Ω and Lemma 3.9 we have

$$F''_{\mathrm{hom}}(\Omega, u_t) \leqslant (F''_{\mathrm{hom}})_-(x_0 + t(\Omega - x_0), u_t) \leqslant \int_{x_0 + t(\Omega - x_0)} f_{\mathrm{hom}}(Du_t)\, dx\,. \tag{3.59}$$

By performing the change of variable $y = x_0 + (x - x_0)/t$ in the right-hand side of (3.59) and by exploiting the convexity of $f_{\rm hom}$ we obtain

$$
(3.60) \qquad F''_{\rm hom}(\Omega, u_t) \leqslant
$$
$$
t^n \int_\Omega f_{\rm hom}\left(\frac{1}{t} Du\right) dy \leqslant t^{n-1} \int_\Omega f_{\rm hom}(Du)\, dy + t^n \left(1 - \frac{1}{t}\right) |\Omega| f_{\rm hom}(0),
$$

therefore by (1.2) and (3.58) we infer (3.57) by taking the limit as $t \to 1$ in (3.60).

Let now $u \in L^p(\Omega)$, then by (3.57) we have immediately

$$
(3.61) \qquad F''_{\rm hom}(\Omega, u) \leqslant \begin{cases} \displaystyle\int_\Omega f_{\rm hom}(Du)\, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{if } u \in L^p(\Omega) \setminus W^{1,p}(\Omega), \end{cases}
$$

therefore by (3.61), Proposition 3.4 and (1.1) equality (3.56) follows.

Finally, if (1.17) holds, Proposition 1.5 yields the boundedness of $\operatorname{dom} f_{\rm hom}$ from which, together with (3.56), the last statement of the thesis follows. ■

4. – Convergence of minima.

In the present section we deduce by Theorem 3.10 a result on the convergence of minima for Neumann problems for the functionals in (1.12) and, as a corollary, the convergence result exposed in the introduction.

THEOREM 4.1. – *Let f be as in* (1.11) *verifying* (1.16) *or* (1.17), *and let $f_{\rm hom}$ be given by* (1.14), *then $f_{\rm hom}$ turns out to be convex, lower semicontinuous and for every convex bounded open set Ω, $\beta \in L^\infty(\Omega)$, $\lambda > \|\beta\|_{L^\infty(\Omega)}$ the values*

$$
(4.1) \qquad m_\varepsilon(\Omega, \beta, \lambda) =
$$
$$
\min\left\{\int_\Omega f\left(\frac{x}{\varepsilon}, Du\right) dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \colon u \in W^{1,p}(\Omega)\right\} \qquad \varepsilon > 0
$$

converge as ε tends to 0 to

$$
m_{\rm hom}(\Omega, \beta, \lambda) = \min\left\{\int_\Omega f_{\rm hom}(Du)\, dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \colon u \in W^{1,p}(\Omega)\right\}.
$$

Moreover, if (1.17) *holds,* $\operatorname{dom} f_{\rm hom}$ *turns out to be bounded and*

$$
(4.2) \qquad m_{\rm hom}(\Omega, \beta, \lambda) = \min\left\{\int_\Omega f_{\rm hom}(Du)\, dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \colon u \in W^{1,\infty}(\Omega)\right\}.
$$

Finally, if $\{\varepsilon_h\}$ is a sequence of positive numbers converging to 0 *and, for every $h \in \boldsymbol{N}$, u_h is a solution of $m_{\varepsilon_h}(\Omega, \beta, \lambda)$ then $\{u_h\}$ is compact in $L^p(\Omega)$ and its converging subsequences converge to minimizers of $m_{\rm hom}(\Omega, \beta, \lambda)$.*

PROOF. – Let Ω, β, λ be as above and, for every $\varepsilon > 0$, let F_ε be given by (1.12).

The properties of f_{hom} and of $\operatorname{dom} f_{\text{hom}}$ follow by Theorem 3.10.

Let Ω be a convex bounded open set and $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0, then by Theorem 3.10, (1.7), (1.8) and (1.1) we deduce that

$$(4.3) \qquad \Gamma^-(L^p(\Omega)) \lim_{h \to +\infty} F_{\varepsilon_h}(\Omega, u) = \int_\Omega f_{\text{hom}}(Du)\, dx \quad \text{for every } u \in W^{1,p}(\Omega),$$

therefore by (4.3), the $L^p(\Omega)$-continuity of the functional $u \in L^p(\Omega) \mapsto \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx$ and Proposition 1.2 we infer that

$$(4.4) \qquad \Gamma^-(L^p(\Omega)) \lim_{h \to +\infty} \left\{ F_{\varepsilon_h}(\Omega, u) + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \right\} =$$

$$\int_\Omega f_{\text{hom}}(Du)\, dx + \int_\Omega \beta u\, dx + \lambda \int_\Omega |u|\, dx \quad \text{for every } u \in W^{1,p}(\Omega).$$

At this point we observe that by (1.16) or (1.17) and the choices of β and λ the functionals in brackets in (4.4) are $L^p(\Omega)$-lower semicontinuous and $L^p(\Omega)$-equicoercive on $W^{1,p}(\Omega)$, hence by (4.4) and Theorem 1.3 we deduce that minima in (4.1) do exist, that

$$(4.5) \qquad \lim_{h \to +\infty} m_{\varepsilon_h}(\Omega, \beta, \lambda) = m_{\text{hom}}(\Omega, \beta, \lambda)$$

and that if for every $h \in \boldsymbol{N}$ u_h is a solution of $m_{\varepsilon_h}(\Omega, \beta, \lambda)$ then $\{u_h\}$ is compact in $L^p(\Omega)$ and its converging subsequences converge to minimizers of $m_{\text{hom}}(\Omega, \beta, \lambda)$.

In conclusion, being $\{\varepsilon_h\}$ arbitrarily chosen, by (4.5) we deduce that

$$\lim_{\varepsilon \to 0} m_\varepsilon(\Omega, \beta, \lambda) = m_{\text{hom}}(\Omega, \beta, \lambda).$$

Finally, if (1.17) holds, by the boundedness of $\operatorname{dom} f_{\text{hom}}$ (4.2) soon follows. ∎

We now prove the convergence result stated in the introduction. To do this we denote by $I_{[0,+\infty[}$ the function defined by $I_{[0,+\infty[}(t) = 0$ if $t \in [0, +\infty[$, $I_{[0,+\infty[}(t) = +\infty$ if $t \in]-\infty, 0[$.

COROLLARY 4.2. – *Let* ϕ, m *be as in* (0.3), (0.4) *verifying* (0.9) *or* (0.10), *and let* ϕ^p_{hom} *be given by* (0.13), *then* ϕ^p_{hom} *turns out to be convex, lower semicontinuous and for every convex bounded open set* Ω, $\beta \in L^\infty(\Omega)$, $\lambda > \|\beta\|_{L^\infty(\Omega)}$ *the values* $\{j^p_\varepsilon(\Omega, \beta, \lambda)\}_{\varepsilon > 0}$ *in* (0.11) *converge as* ε *tends to* 0 *to* $j^p_{\text{hom}}(\Omega, \beta, \lambda)$ *in* (0.12).

Moreover, if (0.10) *holds,* $\mathrm{dom}\,\phi^p_{\mathrm{hom}}$ *turns out to be bounded and*

$$j^p_{\mathrm{hom}}(\Omega, \beta, \lambda) = \min\left\{\int_\Omega \phi^p_{\mathrm{hom}}(Du)\,dx + \int_\Omega \beta u\,dx + \lambda\int_\Omega |u|\,dx \colon u \in W^{1,\infty}(\Omega)\right\}.$$

Finally, if $\{\varepsilon_h\}$ *is a sequence of positive numbers converging to* 0 *and, for every* $h \in \boldsymbol{N}$, u_h *is a solution of* $j^p_{\varepsilon_h}(\Omega, \beta, \lambda)$ *then* $\{u_h\}$ *is compact in* $L^p(\Omega)$ *and its converging subsequences converge to minimizers of* $j^p_{\mathrm{hom}}(\Omega, \beta, \lambda)$.

PROOF. – If ϕ and m are given as in (0.3) and (0.4), by setting $f(x, z) = \phi(x, z) + I_{[0, +\infty[}(m(x) - |z|)$ it turns out that f verifies (1.11), that φ_f in (1.15) agrees with m and that f_{hom} in (1.14) is equal to ϕ^p_{hom}, moreover by (0.9) or (0.10) it soon follows that (1.16) or (1.17) are fulfilled.

By virtue of this we obtain that the assumptions of Theorem 4.1 are fulfilled, and the thesis follows by Theorem 4.1. ■

Acknowledgments. Work performed as part of the research project "Relaxation and Homogenization Methods in the Study of Composite Materials" of the Progetto Strategico C.N.R. - 1996 "Applicazioni della Matematica per la Tecnologia e la Società".

REFERENCES

[An] B. D. ANNIN, *Existence and uniqueness of the solution of the elastic-plastic torsion problem for a cylindrical bar of oval cross-section*, J. Appl. Math. Mech., **29** (1965), 1038-1047.

[A1] H. ATTOUCH, *Introduction a l'homogenisation d'inequations variationnelles*, Rend. Sem. Mat. Univ. Politecn. Torino, **40** (1982), 1-23.

[A2] H. ATTOUCH, *Variational Convergence for Functions and Operators*, Pitman (1984).

[BLP] A. BENSOUSSAN - J. L. LIONS - G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl., **5**, North Holland (1978).

[B] H. BREZIS, *Multiplicateur de Lagrange en torsion elasto-plastique*, Arch. Rational Mech. Anal., **49** (1973), 32-40.

[BS] H. BREZIS - M. SIBONY, *Équivalence de deux inéquations variationnelles et applications*, Arch. Rational Mech. Anal., **41** (1971), 254-265.

[Bu] G. BUTTAZZO, *Su una definizione generale dei Γ-limiti*, Boll. Un. Mat. Ital. (5), **14**-B (1977), 722-744.

[CR] A. CAFFARELLI - N. M. RIVIERE, *On the Lipschitz character of the stress tensor when twisting an elastic-plastic bar*, Arch. Rational Mech. Anal., **69** (1979), 31-36.

[C1] L. CARBONE, *Sur la convergence des intégrales du type de l'énergie sur des fonctions a gradient borné*, J. Math. Pures Appl., **56** (1977), 79-84.

[C2] L. CARBONE, *Γ-convergence d'intégrales sur des functions avec des contraintes sur le gradient*, Comm. Partial Differential Equations, **2** (1977), 627-651.

[C3] L. Carbone, *Sull'omogeneizzazione di un problema variazionale con vincoli sul gradiente*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (8), Mat. Appl., **63** (1977), 10-14.

[C4] L. Carbone, *Sur un problème d'homogénéisation avec des contraintes sur le gradient*, J. Math. Pures Appl., **58** (1979), 275-297.

[CDA1] L. Carbone - R. de Arcangelis, *On integral representation, relaxation and homogenization for unbounded functionals*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9), Mat. Appl., 8 (1997), 129-135.

[CDA2] L. Carbone - R. de Arcangelis, *Unbounded functionals: applications to the homogenization of gradient constrained problems*, to appear on Ricerche Mat..

[CS1] L. Carbone - S. Salerno, *On a problem of homogenization with quickly oscillating constraints on the gradient*, J. Math. Anal. Appl., **90** (1982), 219-250.

[CS2] L. Carbone - S. Salerno, *Further results on a problem of homogenization with constraints on the gradient*, J. Analyse Math., **44** (1984/85), 1-20.

[CS3] L. Carbone - S. Salerno, *Homogenization with unbounded constraints on the gradient*, Nonlinear Anal., **9** (1985), 431-444.

[CS] L. Carbone - C. Sbordone, *Some properties of Γ-limits of integral functionals*, Ann. Mat. Pura Appl. (4), **122** (1979), 1-60.

[CSJP] D. Cioranescu - J. Saint Jean Paulin, *Homogenization in open sets with holes*, J. Math. Anal. Appl., **71** (1979), 590-607.

[CEDA1] A. Corbo Esposito - R. de Arcangelis, *The Lavrentieff phenomenon and different processes of homogenization*, Comm. Partial Differential Equations, **17** (1992), 1503-1538.

[CEDA2] A. Corbo Esposito - R. de Arcangelis, *Homogenization of Dirichlet problems with nonnegative bounded constraints on the gradient*, J. Analyse Math., **64** (1994), 53-96.

[CEDA3] A. Corbo Esposito - R. de Arcangelis, *Some notes on a characterization of function sets described by constraints on the gradient*, in *Nonlinear Variational Problems and Partial Differential Equations*, A. Marino and M.K.V. Murthy editors, Pitman Res. Notes Math. Series, **320**, Longman Scientific & Technical (1995), 182-196.

[CESC] A. Corbo Esposito - F. Serra Cassano, *A Lavrentieff phenomenon for problems of homogenization with constraints on the gradient*, Ricerche Mat., **46** (1997), 127-159.

[DM] G. Dal Maso, *An Introduction to Γ-Convergence*, Progr. Nonlinear Differential Equations Appl., 8, Birkhäuser (1993).

[DAG] R. de Arcangelis - G. Gargiulo, *Homogenization of integral functionals with linear growth defined on vector-valued functions*, NoDEA Nonlinear Differential Equations Appl., **2** (1995), 371-416.

[DAGP] R. de Arcangelis - A. Gaudiello - G. Paderni, *Some cases of homogenization of linearly coercive gradient constrained variational problems*, Math. Models Methods Appl. Sci., **6** (1996), 901-940.

[DAV] R. de Arcangelis - A. Vitolo, *Some cases of homogenization with unbounded oscillating constraints on the gradient*, Asymptotic Anal., **5** (1992), 397-428.

[DG] E. de Giorgi, *G-operators and Γ-convergence;* in *Proceedings of the International Congress of Mathematicians*, Warszawa, August 16-24, 1983, PWN, Warszawa (1984), 1175-1191.

[DGF] E. de Giorgi - T. Franzoni, *Su un tipo di convergenza variazionale*, Rend. Sem. Mat. Brescia, **3** (1979), 63-101.

[DGS] E. de Giorgi - S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital. (4), 8 (1973), 391-411.

[DMD] U. de Maio - T. Durante, *Homogenization of Dirichlet problems for some types of integral functionals*, to appear on Ricerche Mat..

[DL] G. Duvaut - H. Lanchon, *Sur la solution du problème de torsion élasto-plastique d'une barre cylindrique de section quelconque*, C. R. Acad. Sci. Paris Sér. I Math., **264** (1967), 520-523.

[DLi] G. Duvaut - J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag (1976).

[F] N. Fusco, *Γ-convergenza unidimensionale*, Boll. Un. Mat. Ital. (5), **16**-B (1979), 74-86.

[GL] R. Glowinski - H. Lanchon, *Torsion élastoplastique d'une barre cylindrique de section multiconnexe*, J. Mécanique, **12** (1973), 151-171.

[L] H. Lanchon, *Torsion élastoplastique d'une barre cylindrique de section simplement ou multiplement connexe*, J. Mécanique, **13** (1974), 267-320.

[M] P. Marcellini, *Periodic solutions and homogenization of non linear variational problems*, Ann. Mat. Pura Appl. (4), **117** (1978), 139-152.

[MS] P. Marcellini - C. Sbordone, *Homogenization of non-uniformly elliptic operators*, Appl. Analysis, 8 (1978), 101-113.

[MT] F. Murat - L. Tartar, *H-convergence*, in *Topics in the Mathematical Modelling of Composite Materials*, A. Cherkaev and R. Kohn editors, Progr. Nonlinear Differential Equations Appl., **31**, Birkäuser (1997), 21-44.

[SP] E. Sanchez-Palencia, *Nonhomogeneous Media and Vibration Theory*, Lecture Notes in Phys., **127**, Springer-Verlag (1980).

[T] T. W. Ting, *Elastic-plastic torsion of simply connected cylindrical bars*, Indiana Univ. Math. J., **20** (1971), 1047-1076.

[ZKON] V. V. Zhikov - S. M. Kozlov - O. A. Oleinik - Kha T'en Ngoan, *Averaging and G-convergence of differential operators*, Russian Math. Surveys, **34** (1979), 69-147.

L. Carbone: Università di Napoli "Federico II"
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
via Cintia, Complesso Monte S. Angelo, 80126 Napoli, Italy
E-mail: carbone@biol.dgbm.unina.it

A. Corbo Esposito: Università di Cassino, Dipartimento di Ingegneria Industriale
via Di Biasio 43, 03043 Cassino (FR), Italy
E-mail: corbo@ing.unicas.it

R. De Arcangelis: Università di Napoli "Federico II"
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
via Cintia, Complesso Monte S. Angelo, 80126 Napoli, Italy
E-mail: dearcang@matna2.dma.umina.it

Pervenuta in Redazione
il 23 luglio 1998