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On the Variety of Quadrics of Rank Four Containing a Projective Curve (*)

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Sunto. – *Sia $X \subset \mathbb{P}(H^0(X, L)^*)$ una curva proeittiva e lissa, generali nel senso di Brill-Noether, indichiamo con $\mathcal{R}_4(X)$ l'insieme algebrico di quadrici di rango 4 contenendo a X. In questo lavoro noi descriviamo birazionalmente i componenti irriducibile di $\mathcal{R}_4(X)$.*

1. – Introduction.

Let X be a smooth, complete, and irreducible curve of genus $g \geq 3$ over $\mathbb{C}$, general in the Brill-Noether Theory sense.

Let L be a line bundle on X, $\mathcal{L} = \mathcal{O}_X(D)$, the associated invertible sheaf, $d = \deg L \geq 2(2g-1)$. Under these hypotheses the map $\phi_L : X \to \mathbb{P}(H^0(X, L)^*)$ is an embedding.

We shall denote without distinction by X both the abstract smooth curve and its image under the map ϕ_L.

Define:

$$\mathcal{R}_n(X) := \{Q \in \mathbb{P}S^2(H^0(X, L)^*) \mid \operatorname{rank} Q \leq n \text{ and } X \subset Q\},$$

the set of quadrics of rank less or equal to n containing the image of X in $\mathbb{P}(H^0(X, L)^*)$.

The aim of this paper is to study the variety $\mathcal{R}_4(X)$. First of all, note that $\mathcal{R}_4(X)$ is a closed algebraic set in $\mathbb{P}S^2(H^0(X, L))$. Indeed, it is the intersection of $\mathcal{Q}(X)$, the linear variety of quadrics containing X and $\mathcal{R}_4$, the variety of rank four quadrics in $\mathbb{P}H^0(X, L)^*$. Our main result states that $\mathcal{R}_4(X)$ is a pure dimensional variety of dimension $2d - 3g - 4$, the irreducible components of $\mathcal{R}_4(X)$ are described in terms of fibered products on the Picard variety of X of some varieties parametrizing pencils on X.

More precisely, a pair (g_a^1, g_b^1) satisfying $a + b = d$ and $|g_a^1 + g_b^1| = |D|$ will be denoted by $(g_a^1, g_b^1)_0$. The study of the variety $\mathcal{R}_4(X)$ is equivalent to the study of the pairs $(g_a^1, g_b^1)_0$, as the following lemmas explain:

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LEMMA 1.1. – *Given* $Q \in \mathcal{R}_4(X)$, *denote* $B = X . \operatorname{Sing} Q$, *then*:

1) *Given any pair* $(g_a^1, g_b^1)_0$, *it can be constructed an element* $Q \in \mathcal{R}_4(X)$ *such that* $L_\lambda . X - B = g_a^1 - B_a$ *and* $M_\mu . X - B = g_b^1 - B_b$, *where* L_λ *and* M_μ *are the rulings of* Q, , *and* B_a *(resp.* B_b*) denotes the base locus of* g_a^1 *(resp.* g_b^1*).*

2) *Conversely, given* $Q \in \mathcal{R}_4(X)$, *such that* $X . \operatorname{Sing} Q = \emptyset$, *we obtain a pair* $(g_a^1, g_b^1)_0$ *by intersecting* X *with* L_λ *and* M_μ. *If* $a \neq b$ *this is the inverse of the previous map for the case of base point free pencils.*

3) *If* $a = b$, *the pairs* $(g_a^1, \bar{g}_a^1)_0$ *and* $(\bar{g}_a^1, g_a^1)_0$ *determine the same element of* $\mathcal{R}_4(X)$.

PROOF. – 1) Let s_1, s_2 be a basis of the two dimensional space of sections $V_a \subset H^0(X, M)$ determined by g_a^1, ($M = \mathcal{O}_X(D_a)$, $D_a \in |g_a^1|$), analogously let t_1, t_2 be a basis of $V_b \subset H^0(X, M')$, $M' = L \otimes M^{-1}$.

Then $s_i \otimes t_j \in H^0(X, L)$ and $Q_{ab} = \{\det(s_i \otimes t_j) = 0\}$ is a rank four quadric in the space $\mathbb{P}(H^0(X, L)^*)$. It is trivial to check that $Q_{ab} \in \mathcal{R}_4(X)$. The rulings of Q_{ab} are given by the equations:

$$\begin{cases} s_1(t_1 + \lambda t_2) = 0\,, \\ s_2(\lambda t_2 + t_1) = 0 \end{cases}$$

and

$$\begin{cases} t_1(s_1 + \mu s_2) = 0\,, \\ t_2(\mu s_2 + s_1) = 0\,. \end{cases}$$

So, we see that $L_\lambda . X = g_a^1 + B_b$ and $M_\mu . X = g_b^1 + B_a$.

On the other hand $B_a + B_b = B$. Indeed, $B_a + B_b$ coincides with $(L_\lambda \cap M_\mu). X$, and for any (λ, μ), $(L_\lambda \cap M_\mu) = \operatorname{Sing} Q$. This proves the first part of the lemma.

2) Let $Q \in \mathcal{R}_4(X)$. Let L_λ, M_μ, be the two rulings of Q. The intersections $L_\lambda . Q$, $M_\mu . Q$ determine two pencils on X: g_a^1, g_b^1. If $q \in Q$ the tangent hyperplane $T_q Q = [L_\lambda + M_\mu]$ for some values of λ and μ ([] denotes the linear span). So, we must have $|g_a^1 + g_b^1| = |D|$, where $L = \mathcal{O}_X(D)$. In particular, $a + b = \deg L$.

It is easy to check that on the set of base locus free pencils with $a \neq b$ these two maps are inverse to each other.

3) It is obvious. ■

LEMMA 1.2. – *A pair* $(g_a^1, g_b^1)_0$ *determines a rank* 3 *quadric if and* *only if* $g_b^1 + B_a = g_a^1 + B_b$. *In particular, if* $B_b = \emptyset$, *then* $(g_a^1, g_b^1)_0$ *gives rise to a rank* 3 *quadric if and only if* $a = d/2$ *and* $g_a^1 = g_b^1$.

PROOF. – Assume $(g_a^1, g_b^1)_0$ gives rise to a rank 3 quadric, then we must have a relation of the type

$$(s_1 \otimes t_2) = \lambda (s_2 \otimes t_1), \qquad \lambda \in \mathbb{C}.$$

From this it follows that $g_b^1 + B_a = g_a^1 + B_b$. The other implication is trivial. If $B_b = \emptyset$, $g_b^1 + B_a = g_a^1$ is impossible, unless $a = b$ and $\deg B_a = 0$. ∎

Previous lemmas give the set theoretical description of the variety $\mathcal{R}_4(X)$. After this, we need an algebro-geometric description; this description will be developed in the next section. As a Corollary of Theorem 2.1 we obtain that the linear space $\mathcal{Q}(X)$ of quadrics in $\mathbb{P}H^0(X, L)^*$ containing X intersects the variety $\mathcal{R}_4$ of quadrics of rank four with the expected dimension.

In section 3 we study $\mathcal{R}_4(X)$ in the case X is the bicanonical curve of genus 3. In this case we are not under the hypothesis $\deg L \geqslant 2(2g-1)$ and some strange features occur.

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2. – Construction of $\mathcal{R}_4(X)$.

THEOREM 2.1. – $\mathcal{R}_4(X)$ *is a pure dimensional variety of dimension* $2d - 3g - 4$. *If* g *is odd* $\mathcal{R}_4(X)$ *is the union of irreducible components* V_a,

$$\mathcal{R}_4(X) = \bigcup_a V_a,$$

where $(g+3)/2 \leqslant a \leqslant [d/2]$.

If g *is even, then* $\mathcal{R}_4(X)$ *is the union of irreducible components*:

$$\mathcal{R}_4(X) = \bigcup_a V_a \bigcup_j V_{g+2/2, j},$$

where $(g+2)/2 + 1 \leqslant a \leqslant [d/2]$, $1 \leqslant j \leqslant g!/(g/2)!(g/2+1)!$.

PROOF. – We know that the elements $Q \in \mathcal{R}_4(X)$ are parametrized by pairs (g_a^1, g_b^1) such that $|g_a^1 + g_b^1| = |D|$; we need to give this parametrization algebraically.

We assume in the sequel, without loss of generality, that $a \leqslant b$. Let $G_a^1(X)$ be the variety of linear system on X of degree a and projective dimension 1. As X is general in the Brill Noether sense, $G_a^1(X)$ will be: non-singular and irreducible of dimension $\varrho = 2a - g - 2$, if $\varrho > 0$; $g!/(g/2)!(g/2+1)!$ reduced points if $\varrho = 0$ and empty otherwise. Thus we have $G_a^1(X) \neq \emptyset$ if and only if $[g + 2/2] \leqslant a$ ([1]). Moreover, $a \leqslant b$ implies $a \leqslant d/2$. We denote by S_a the universal bundle on $G_a^1(X)$.

Consider on $\mathrm{Pic}^a(X)$ the locally free sheaf $\mathcal{L}_a' = \mathcal{L}_a^{-1} \otimes \pi_1^* L$, where

$$\pi_1 \colon X \times \mathrm{Pic}^a(X) \to X$$

is the projection onto the first factor.

By the assumption on the degree of L and the convention $a \leqslant b$ we have that $\pi_{2,*} \mathcal{L}_a'$ is a locally free sheaf on $\mathrm{Pic}^a(X)$ ([4], page 53). Let $G(2, \pi_{2,*} \mathcal{L}_a')$ be the corresponding Grassmanian. If S_a' denotes the universal bundle on $G(2, \pi_{2,*} \mathcal{L}_a')$ we have an inclusion:

$$0 \to S_a' \to p'^* \pi_{2,*} \mathcal{L}_a', \tag{2.1}$$

where $p' \colon G(2, \pi_{2,*} \mathcal{L}_a') \to \mathrm{Pic}^a(X)$ is the natural projection. In order to simplify the notation let us call $G_a'(X)$ to $G(2, \pi_{2,*} \mathcal{L}_a')$. The projections

$$\begin{tikzcd} G_a^1(X) \arrow[d, "p"'] & \\ \mathrm{Pic}^a(X) & G_a'(X) \arrow[l, "p'"] \end{tikzcd},$$

allow us to define the fibered product

$$G_a(X) := G_a^1(X) \times_{\mathrm{Pic}^a(X)} G_a'(X).$$

We obtain a commutative diagram:

$$\begin{tikzcd} G_a \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "\pi"] & G_a'(X) \arrow[d, "p'"] \\ G_a^1(X) \arrow[r, "p"'] & \mathrm{Pic}^a(X). \end{tikzcd}$$

Consider the natural multiplication map:

$$0 \to p_1^* S_a \otimes p_2^* S_a' \to H^0(X, L) \otimes \mathcal{O}_{G_a(X)}.$$

Let s_1, s_2, (resp. t_1, t_2) be local generators on an open set $U \subset G_a(X)$ of the locally free sheaf of rank two S_a, (resp. S_a'). Define

$$\phi_U\colon U \to (S^2 H^0(X, L)),$$

$$u \to \det(s_i \otimes t_j)(u).$$

It is easy to see that if s_1', s_2', t_1', t_2' are other sets of local generators then

$$\det(s_i' \otimes t_j') = \det A \cdot \det B \cdot \det(s_i \otimes t_j),$$

where A, $B \in GL(2, \mathcal{O}_U^*)$ satisfies

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = A \begin{pmatrix} s_1' \\ s_2' \end{pmatrix}, \qquad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = B \begin{pmatrix} t_1' \\ t_2' \end{pmatrix}.$$

So, we have a projective morphism:

$$\Phi_a\colon G_a(X) \to \mathbb{P}(S^2 H^0(X, L)).$$

The image of a point of $G_a(X)$ is, just by construction, an element of $\mathcal{R}_4(X)$.

Call $V_a = \Phi_a(G_a(X))$, our next claim is:

LEMMA 2.6. – *a) For all a, $G_a(X)$ is an irreducible variety of dimension* $2d - 3g - 4$,

b) Φ_a is a birational morphism for all $a \neq d/2$; if $a = d/2$, Φ_a is generically a two-sheeted covering.

c) If $a \neq a'$, then $V_a \neq V_a'$.

PROOF OF LEMMA 2.6. – We start by remarking that, if $Z_0 \subset G_a(X)$ is the set corresponding to (g_a^1, g_b^1) with either $B_a \neq \emptyset$ or $B_b \neq \emptyset$ (we use the same notation as in Lemma 1.1), and $Z_1 \subset G_a(X)$ is the set of elements in $G_a(X)$ such that the associated quadric is of rank 3, then $Z_0 \cup Z_1$ is a proper closed set.

Indeed, consider on $G_a(X) \times X$ the natural evaluation map:

$$\sigma_1^*(p'^* S_a' \otimes p^* S_a) \to (p' \times id)^*(\pi^* L),$$

(σ_1 being the projection of $G_a(X) \times X$ onto the first factor). The support of the cokernel of this map corresponds to (g_a^1, g_b^1, p) such that p is a base point of either g_a^1 or g_b^1. This is a closed proper set of $G_a(X) \times X$ and its projection on $G_a(X)$ is Z_0, which is, thus, a proper closed set.

By lemma 1.2 $Z_1 \subset Z_0$, unless $a = d/2$ and $g_a^1 = g_b^1$. In this case we must consider, moreover, $Z_1' \subset Z_1$, the diagonal of the projection $G_a(X) \to \mathrm{Pic}^a(X)$, which is, again, a closed set, since the projection is a proper morphism. ∎

A point of $G^1_a(X)$ will be denoted by (M, W), with $M \in \mathrm{Pic}^a(X)$ and $W \subset H^0(X, M)$ a subspace of dimension two.

a) Consider the projection

$$p_1 \colon G_a(X) \to G^1_a(X),$$

since X is general in $\mathcal{M}_g$, $G^1_a(X)$ is irreducible and non-singular; moreover, the fiber over a point $(M, W) \in G^1_a(X)$ is the Grassmanian $G(2, H^0(X, M'))$, where $M' = M^{-1} \otimes L$.

Once again, as $\deg M' = b > 2g-1$ the dimension of the fibers of p_1 is constant; furthermore, each fiber is non-singular and irreducible. We conclude that $G_a(X)$ is irreducible and

$$\dim G_a(X) = \dim G(2, H^0(X, M')) + \dim G^1_a(X) = 2(b-g-1) + 2a - g - 2\,.$$

So, we obtain part *a*).

b) By Lemma 1.1 it will be sufficient to prove that on the open set of pairs of free base locus linear series determining a rank four quadric in $\mathbb{P}H^0(X, L)^*$ the map Φ_a separates tangent vectors.

In order to study the injectivity of the tangent maps we must analize the behavior of the composite map $\Phi_a \circ \alpha$, with

$$\alpha \in \mathrm{Hom}\,(\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon)^2, G_a(X), w)\,, \qquad w \in G_a(X)\,.$$

Following our notation for elements in $G^1_a(X)$ and $G'_a(X)$, w will be denoted by $w = (M, W, M', W')$.

The morphism α gives rise to an infinitesimal first order deformation of w. If U_α is an open covering of X and $g_{\alpha\beta}$ the transition functions of M, a first order deformation is given by a family M_ε of line bundles with transition functions

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta}(1 + \varepsilon\phi_{\alpha\beta})\,, \qquad \{\phi_{\alpha\beta}\} \in H^1(X, O_X)\,. \tag{2.2}$$

Moreover, if s_1, s_2 are a basis of W we must have relations

$$\begin{cases} \tilde{s}_{1\alpha} = s_{1\alpha} + \varepsilon s'_{1\alpha}\,, \\ \tilde{s}_{2\alpha} = s_{2\alpha} + \varepsilon s'_{2\alpha}\,, \end{cases} \tag{2.3}$$

satisfying $\tilde{s}_{i\alpha} = \tilde{g}_{\alpha\beta}\tilde{s}_{i\beta}$.

Analogously, we must have a family M'_ε of degree b line bundles with transition functions

$$\tilde{h}_{\alpha\beta} = g^{-1}_{\alpha\beta} f_{\alpha\beta}(1 + \varepsilon\varphi_{\alpha\beta})\,,$$

where $f_{\alpha\beta}$ are the transition functions of L. Moreover, $M_\varepsilon \otimes M'_\varepsilon \cong L$ impose the condition $\varphi_{\alpha\beta} = -\phi_\alpha$.

Finally, if t_1, t_2 is a basis of W' we must have the relations:

$$\tilde{t}_{i\alpha} = t_{i\alpha} + \varepsilon t'_{i\alpha}\,, \qquad \tilde{t}_{i\alpha} = \tilde{h}_{\alpha\beta}\tilde{t}_{i\beta}\,.$$

The transition conditions imply:

$$s'_{i\alpha} = g_{\alpha\beta}s'_{i\beta} + \phi_{\alpha\beta}g_{\alpha\beta}s_{i\beta}\,, \qquad t'_{i\alpha} = h_{\alpha\beta}t'_{i\beta} - \phi_{\alpha\beta}g_{\alpha\beta}t_{i\beta}\,.$$

The composition of Φ_a and α will be given by $\det(\tilde{s}_i \otimes \tilde{t}_j)$. Locally the expression is:

$$\det(\tilde{s}_{i\alpha} \otimes \tilde{t}_{j\alpha}) = \det(s_{i\alpha} \otimes t_{j\alpha}) + \varepsilon E_\alpha(s_i, t_j, s'_i, t'_j)\,,$$

where

$$E_\alpha(s_i, t_j, s'_i, t'_j) = w_{11}^\alpha w_{22}^{\alpha'} + w_{11}^{\alpha'} w_{22}^\alpha - w_{12}^\alpha w_{21}^{\alpha'} - w_{12}^{\alpha'} w_{21}^\alpha\,,$$

$w_{ij}^\alpha = s_{i\alpha} \otimes t_{j\alpha}$, $w_{ij}^{\alpha'} = s'_{i\alpha} \otimes t_{j\alpha} + s_{i\alpha} \otimes t'_{j\alpha}$.

On $U_\alpha \cap U_\beta$ the following identities hold:

$$s'_{i\alpha} \otimes t_{j\alpha} + s_{i\alpha} \otimes t'_{j\alpha} = f_{\alpha\beta}(s'_{i\beta} \otimes t_\beta + s_{i\beta} \otimes t'_\beta)\,,$$

thus, the data $\{w_{ij}^\alpha\}$ gives rise to an element of $H^0(X, L)$. Then, we conclude that the tangent space to Φ_a at $\det(w_{ij})$ will be the vector space:

$$\{E = w_{11}w'_{22} + w_{22}w'_{11} - w_{12}w'_{21} - w_{21}w'_{12}\}/\det(w_{ij})\,.$$

We claim that the relation

$$w_{11}w'_{22} + w_{22}w'_{11} - w_{12}w'_{21} - w_{21}w'_{12} = k\det(w_{ij})\,, \tag{2.4}$$

implies that the associated infinitesimal deformation is trivial.

Indeed, if $\det(w_{ij}) = 0$ is a rank four quadric then (2.4) implies $w'_{ij} = w_{ij}$. Thus, we have locally:

$$s'_{i\alpha} \otimes t_{j\alpha} + s_{i\alpha} \otimes t'_{j\alpha} = s_{i\alpha} \otimes t_{j\alpha}$$

but then $s_{i\alpha}(p) = 0$ implies $s'_{i\alpha}(p) \otimes t_{1\alpha}(p) = s'_{i\alpha}(p) \otimes t_{2\alpha}(p) = 0$ and on the set of free base locus pencils this implies $s'_{i\alpha}(p) = 0$. On the other hand $\tilde{s}_{i\alpha} = s_{i\alpha} + \varepsilon s'_{i\alpha}$ (2.3), so $(\tilde{s}_i)_0 \geqslant (s_i)_0$, since $\deg M = \deg M_\varepsilon$; this implies $M = M_\varepsilon$ and $\tilde{s}_i = k \cdot s_i$, $k \in \mathbb{C}$. The same argument is valid for t_j.

c) Let us calculate the intersection of V_a and $V_{a'}$. Assume $a' > a$. Let $Q = \omega_{11}\omega_{22} - \omega_{12}\omega_{21}$ be a quadric in the intersection $V_a \cap V_{a'}$. The two rulings of Q are given by the equations:

$$\begin{cases}\omega_{11} = \lambda\omega_{12}\,,\\ \lambda\omega_{22} = \omega_{21}\,,\end{cases} \qquad \begin{cases}\omega_{11} = \mu\omega_{21}\,,\\ \mu\omega_{22} = \omega_{12}\,,\end{cases} \tag{2.5}$$

since Q is in the image of Φ_a, $\omega_{ij} = s_i \otimes t_j$, where $s_i \in H^0(X, M)$, $\deg M = a$ and $t_j \in H^0(X, L \otimes M^{-1})$. The intersection of (2.5) with X determines a pair $(g_a^1 + B_b, g_b^1 + B_a)$, where g_a^1 is given by the zeroes of $\lambda s_1 + \mu s_2$, g_b^1 is given by the zeroes of $\lambda t_1 + \mu t_2$, B is the base locus of g_b^1 and A the base locus of g_a^1. On the other hand, thinking of Q as a point in the image of $\Phi_{a'}$, we obtain $\bar{s}_i \in H^0(X, \overline{M})$, $\deg \overline{M} = a'$ $\bar{t}_j \in H^0(X, L \otimes \overline{M}^{-1})$, $\deg \overline{M} = a'$ determining a pair $(g_{a'}^1 + B'_b, g_{b'}^1 + B'_a)$. Intersecting the rulings of Q with X we obtain:

$$g_a^1 + B_b = g_{a'}^1 + B'_b .$$

From $a' > a$ it follows that either B_b or B'_b is non-empty and $\deg B_b > \deg B'_b$. We conclude that $\deg B_b > 0$. Using an analogous reasoning for the relation $g_b^1 + B_a = g_{b'}^1 + B'_b$ it follows that $\deg B'_a > 0$. Thus, we conclude that if $Q \in V_a \cap V_{a'}$ $(a' > a)$, then both g_b^1 and $g_{a'}^1$ have non-empty base locus.

Now, if $V_a = V_{a'}$, then for every $(g_{a'}^1, g_{b'}^1) \in G_{a'}$, $g_{a'}^1$ has non-empty base locus, but it is impossible because for X general the set

$$\{g_{a'}^1 \in G_{a'}^1(X) \mid g_{a'}^1 \text{ has non-empty base locus}\}$$

is a proper closed set. ■

In the case that g is even and $a = g + 2/2$, the same construction can be applied to it. In this case the varieties $G_{g+2/2,\, j}$ correspond to the product $p_j \times G(2, H^0(X, M'))$ where the p_j are the points representing the $g_{g+2/2}^1$ on X.

Finally, we observe that the Lemma implies Theorem 2.1. In fact, *a*) and *b*) imply that V_a are irreducible families in $\mathcal{R}_4(X)$, *c*) implies that these varieties, being all of the same dimension, are irreducible components of $\mathcal{R}_4(X)$. By Lemma 1.1 every element of $\mathcal{R}_4(X)$ occurs in one of these components. ■

As an easy consequence of Theorem 2.1 we obtain that $\mathcal{Q}(X)$ and $\mathcal{R}_4$ intersects in the expected dimension:

COROLLARY 2.13. – $\dim \mathcal{R}_4(X) = \dim \mathcal{Q}(X) + \dim \mathcal{R}_4 - \dim S^2 H^0(X, L) - 1.$

PROOF. – We know that $\mathcal{R}_4(X)$ is pure dimensional of dimension $2d - 3g - 4$. On the other hand the dimension of $\mathcal{Q}(X) = \dim S^2 H^0(X, L) - h^0(X, L^{\otimes 2})$. This follows from the fact that under our choice for d, the projective curve X is projectively normal ([3], page 520). The dimension of $\mathcal{R}_4$ is well known to be $(N(N+1))/2 - 1 - ((N-3)(N-4))/2$, where $N = d - g + 1 = h^0(X, L)$ ([2], page 299). The Corollary follows after a simple computation. ■

3. – The bicanonical curve of genus 3.

In this section we study the structure of $\mathcal{R}_4(X)$ in the case where X is the image under the bicanonical embedding of a non-hyperelliptic curve of genus 3. This example was the original motivation for studying the generic case treated above. Moreover, this discussion illustrates how the general theory of the previous section can be used, with some ad hoc modification, to study particular cases which do not satisfy the hypothesis of Theorem 2.1.

Thus, our L will be the bicanonical sheaf $L = \omega_X^2$. This is a degree 8 sheaf, then, we are not in the hypothesis of Theorem 2.1, since $8 < 2(2g-1)$.

As before, $\mathcal{R}_4(X)$ is parametrized by two varieties: $G_3(X)$ and $G_4(X)$. The first strange feature in this case is that $G_4(X)$ is not irreducible:

CLAIM 3.1. – *$G_4(X)$ is the union of two irreducible components $\mathcal{G}_1$ and $\mathcal{G}_2$, where the image of $\mathcal{G}_1$ under Φ_3 is birationally equivalent to the Kummer variety of X and $\mathcal{G}_2$ is isomorphic to $S^2(\mathbb{P}H^0(X, \omega_X)^*)$.*

In fact, in this case $G_4^1(X) \simeq G_4'(X) \simeq \widetilde{\mathrm{Pic}}^4(X)$, the blowing-up of $\mathrm{Pic}^4(X)$ with center at $[\omega_X]$, since for every degree 4 line M on X, $h^0(X, M) = 2$ if and only if $M \neq \omega_X$. The fibered product $G_4(X) \xrightarrow{p} \mathrm{Pic}^4(X)$ is described away from the fiber on $[\omega_X]$ as the pairs $\{(M, M') \mid M, M' \in \mathrm{Pic}^4(X), M \otimes M' = \omega_X^2\}$, so we conclude that the open set $U = G_4(X) - p^{-1}([\omega_X])$ is 3-dimensional.

On the other hand the fiber on $[\omega_X]$ coincides with the symmetric product $S^2(G(2, H^0(X, \omega_X)))$. But $G(2, H^0(X, \omega_X)) = \mathbb{P}H^0(X, \omega_X)^*$, since $g(X) = 3$. This fiber is, thus, a four dimensional variety and we conclude that $G_4(X)$ split out in to two irreducible components: the closure of the open set U, that will be denoted by $\mathcal{G}_1$ and the fiber on $[\omega_X]$, that will be denoted by $\mathcal{G}_2$. That the image of $\mathcal{G}_1$ is birationally equivalent to the Kummer variety of X is almost immediate from the previous description via the identification

$$\mathrm{Pic}^4(X) \xrightarrow{\sim} \mathrm{Pic}^0(X), \qquad M \to M \otimes \omega_X^{-1}. \qquad \blacksquare$$

Call V_1 (resp. V_2) to the image of $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) under the morphism Φ_4. Another exceptional property is:

CLAIM 3.2. – *The image V_3 of $G_3(X)$ under the map Φ_3 is contained in V_2.*

First of all, we will show that for every element $(g_3^1, g_5^1) \in G_3(X)$, g_5^1 has a nonempty base locus. In fact, every g_3^1 on X is given by a pencil of lines through $p \in X'$, where X' is the canonical curve. The condition $|g_3^1 + g_5^1| = \omega_X^2$ implies that the eight points in any divisor in this linear system must be on a conic, so $|g_5^1| = |\omega_X| + p$. This proves that every element of V_3 is included in V_2. ■

The conclusions can be summarized as follows:

THEOREM 3.3. – *Let X be a bicanonical non-hyperelliptic curve of genus* 3. *Then* $\mathcal{R}_4(X) = V_1 \cup V_2$, *where* $\dim V_1 = 3$ *and* $\dim V_4 = 4$. V_1 *is birationally equivalent to the Kummer variety of X, and* V_2 *is birationally equivalent to the symmetric product* $S^2(\mathbb{P}H(0, \omega_X)^*)$.

A more detailed description of the projective properties of the varieties V_1 and V_2 can be found in [6].

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