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# A Canonical Map between Hecke Algebras.

ANDREA MORI (*) - LEA TERRACINI (*)

**Sunto.** – *Sia $D$ un corpo di quaternioni indefinito su $\boldsymbol{Q}$ di discriminante $\Delta$ e sia $\Gamma$ il gruppo moltiplicativo degli elementi di norma* 1 *in un ordine di Eichler di $D$ di livello primo con $\Delta$. Consideriamo lo spazio $S_k(\Gamma)$ delle forme cuspidali di peso $k$ rispetto a $\Gamma$ e la corrispondente algebra di Hecke $\boldsymbol{H}^D$. Utilizzando una versione della corrispondenza di Jacquet-Langlands tra rappresentazioni automorfe di $D^\times$ e di* $\mathrm{GL}_2$, *realizziamo $\boldsymbol{H}^D$ come quoziente dell'algebra di Hecke classica di livello $N\Delta$. Questo risultato permette di ottenere informazioni sulla struttura dell'algebra $\boldsymbol{H}^D$ e di definire una struttura intera per lo spazio $S_k(\Gamma)$.*

## Introduction.

In their study of congruences between modular forms, Ribet [16], and Diamond and Taylor [3], [4], use the fact that a certain quotient of the classical weight 2 Hecke algebra can be identified with an algebra constructed analogously from a quaternion division algebra. Experts regard this identification as true in any weight, because it depends solely on the Jacquet-Langlands correspondence. Nonetheless, to the best of our knowledge, no proof has ever appeared in the literature.

Let $D$ be an indefinite quaternion algebra over $\boldsymbol{Q}$, of discriminant $\Delta \neq 1$. Let $R$ be an Eichler order of level $N$ in $D$, and let $\Gamma$ be the multiplicative group of the elements in $R$ of norm 1. Also, let $\chi$ be a Dirichlet character modulo $N$. As for the classical (i.e. $\mathrm{GL}_2$) case, it is defined an Hecke algebra $\boldsymbol{H}^D(N, \chi)$ acting on the space $S_k(\Gamma, \chi)$ of $\Gamma$-modular forms of weight $k$ and character $\chi$. The main result of this paper is a canonical identification between $\boldsymbol{H}^D(N, \chi)$ and the quotient of the classical Hecke algebra of level $N\Delta$ and character $\chi$, obtained by restricting the Hecke operators to the $\Delta$-new cuspforms.

Our methods are purely representation theoretic. The identification is deduced from a non-canonical translation of the Jacquet-Langlands correspondence in terms of cuspforms (not just representations). We take the opportunity to write down in detail, in the preliminary sections, a few well-known facts relevant to the Jacquet-Langlands correspondence, whose proofs are usually omitted.

A byproduct of our construction is that the space $S_k(\Gamma, \chi)$ inherits from

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$S_k(\Gamma_0(N), \chi)$ a family of Hecke invariant $\boldsymbol{Z}[\chi]$-structures, all Hecke isomorphic to each other. Also, the usual duality properties hold for these structures.

The study of the relations between these structures and the integral structures that can be defined more directly using either the geometry of the Shimura curve, [15], or group cohomology, as in [18], Chapter 8, is the subject of ongoing research. It involves a deeper analysis of their structure as Hecke modules, in particular with regard to the Gorenstein property for the relevant Hecke algebras. A starting point in this direction is Ribet's work [17].

NOTATIONS AND CONVENTIONS. – The symbols $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ denote, as always, respectively the natural numbers, the integers and the fields of rational, real and complex numbers. The symbol $\boldsymbol{A}$ denotes the ring of rational adeles.

Let $p$ be a prime number. The ring of $p$-adic integers and the field of $p$-adic numbers are denoted respectively $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$. Given a $\boldsymbol{Q}$-vector space $V$ and a lattice $\Lambda \subset V$ let $\Lambda_p = \Lambda \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_p$ and $V_p = V \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$.

For an algebraic group $G$ and a ring $R$, $G(R)$, or $G_R$, denotes the group of $R$-rational points of $G$. We shall use a special notation for $G = \mathrm{GL}_2$, namely we let $K_p = \mathrm{GL}_2(\boldsymbol{Z}_p)$. Moreover, for an $m \in \boldsymbol{N}$ let

$$K_p(m) = \left\{ \begin{pmatrix} a & b \\ mc & d \end{pmatrix} \text{ such that } a,\ b,\ c,\ d \in \boldsymbol{Z}_p \text{ and } ad - bmc \not\equiv 0 \mod p \right\}.$$

The Borel subgroup $B$ of $\mathrm{GL}_2$ is the subgroup of upper-triangular matrices, i. e.

$$B(R) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(R) \right\}.$$

We shall use the special notation $B_p$ for $B(\boldsymbol{Q}_p)$.

Finally, let $\mathcal{H} = \{x + iy \in \boldsymbol{C} \text{ such that } y > 0\}$ and $\mathrm{GL}_2^+(\boldsymbol{R}) = \{g \in \mathrm{GL}_2(\boldsymbol{R})$ such that $\det g > 0\}$. The group $\mathrm{GL}_2^+(\boldsymbol{R})$ acts on $\mathcal{H}$ via linear fractional transformations, namely

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}\ .$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\boldsymbol{R})$ and $z \in \mathcal{H}$ let $j(g, z) = cz + d$.

## 1. – Modular forms.

Let $D$ be a $\boldsymbol{Q}$-central quaternion algebra with reduced norm $\nu$. The algebra $D$ supports a canonical anti-involution, the quaternionic conjugation, which we shall denote $\alpha \mapsto \overline{\alpha}$. The set $\mathcal{S}$ consisting of the primes $p$ (including $p = \infty$ and taking $\boldsymbol{Q}_\infty = \boldsymbol{R}$) such that $D_p$ is the unique quaternion division algebra over $\boldsymbol{Q}_p$ is finite and even. The discriminant $\Delta = \Delta(D)$ is defined as the product of the finite primes in $\mathcal{S}$. The isomorphism class of $D$ is completely determined by its discriminant. We shall assume that $D$ is *indefinite*, i. e. that $\infty \notin \mathcal{S}$.

Fix an Eichler order $R$ in $D$ of level $N$, with $(N, \Delta) = 1$, and set

$$\Gamma = \Gamma_R := \{\gamma \in R \text{ such that } \nu(\gamma) = 1\}.$$

Since $D$ is indefinite any two Eichler orders of level $N$ are conjugate. In particular, if $\Delta = 1$ (i. e. $D = M_2(\boldsymbol{Q})$) we may assume that

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \text{ such that } c \equiv 0 \bmod N \right\}.$$

For each prime $p$ not dividing $\Delta$ (including $p = \infty$) fix an isomorphism

$$i_p\colon D_p \to M_2(\boldsymbol{Q}_p)$$

such that for finite $p$,

$$i_p(R_p) = \begin{cases} M_2(\boldsymbol{Z}_p), & \text{if } p \nmid N\Delta, \\ \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in M_2(\boldsymbol{Z}_p) \text{ such that } a, b, c, d \in \boldsymbol{Z}_p \right\}, & \text{if } p \mid N, \end{cases} \tag{1.2}$$

(when $\Delta = 1$, $i_p$ is the identity). From now on, by an abuse of notation, we shall write $\alpha$ for $i_p(\alpha)$ whenever no confusion may arise. Recall that $D_{\boldsymbol{A}}^\times = D^\times \left( \mathrm{GL}_2^+(\boldsymbol{R}) \times \prod_q R_q^\times \right)$, e.g. [14], Theorem 5.2.11. For each prime $p$, including the primes dividing $\Delta$, set

$$E_p = \begin{cases} \{r \in R_p \text{ such that } \nu(r) \neq 0\}, & \text{if } p \nmid N, \\ \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in R_p \text{ such that } a \in \boldsymbol{Z}_p^\times \text{ and } ad - Nbc \neq 0 \right\}, & \text{if } p \mid N, \end{cases}$$

and let

$$E_R = R \cap E_R(\boldsymbol{A})$$

where $E_R(\boldsymbol{A}) = \left( \mathrm{GL}_2^+(\boldsymbol{R}) \times \prod_p E_p \right) \cap D_{\boldsymbol{A}}^\times$. Note that

$$\Gamma \subset E_R \subset E_R(\boldsymbol{A}). \tag{1.3}$$

Let $\chi$ be a Dirichlet character modulo $N$. It gives rise to a Größencharachter whose local component at $p$ is denoted $\chi_p$. For $\gamma = (\gamma_\infty, \gamma_p) \in E_R(\boldsymbol{A})$, write

$\gamma_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}$ for $p$ not dividing $\Delta$ and define

$$\chi(\gamma) = \prod_{p|N} \overline{\chi}_p(a_p). \tag{1.4}$$

Because of (1.3), $\chi$ restricts to a character of $\Gamma$. Observe that if $\Delta = 1$ the extension of $\chi$ to $\Gamma_0(N)$ can be defined directly as $\chi(\gamma) = \chi([d])$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $[d]$ is the class of $d$ modulo $N$. This definition is consistent with (1.4) because $[a][d] = 1$.

The map $i_\infty$ in (1.1) identifies $\Gamma$ with a discrete subgroup of $\mathrm{SL}_2(\boldsymbol{R})$, and we can make $\Gamma$ act on the upper halfplane $\mathcal{H}$ via the usual fractional transformations. The quotient space $\Gamma \backslash \mathcal{H}$ is a Riemann surface which is compact when $\Delta > 1$ and not compact when $\Delta = 1$. In the latter case, it can be compactified by adding in a suitable way a finite set of points called *cusps*. For a detailed proof of these facts see [14], chapters 1 and 5, or [18], chapters 1 and 9.

DEFINITION 1. – *A modular form $f$ of weight $k \in \boldsymbol{Z}$ and character $\chi$ for $\Gamma$, is a holomorphic function $f\colon \mathcal{H} \to \boldsymbol{C}$ such that:*

(1) *the identity $f_{|\gamma}(z) := f(\gamma(z)) j(i_\infty(\gamma), z)^{-k} \overline{\chi}(\gamma) = f(z)$ holds for all $\gamma \in \Gamma$.*

(2) *$f$ extends holomorphically in a neighborhood of each cusp.*

Note that the second condition is non-empty only when $\Delta = 1$ and in this case $f$ is called a *cuspform* if it vanishes at the cusps. When $\Delta \neq 1$ we shall use the terms modular forms and cuspforms interchangeably. Since the space of cuspforms (any $\Delta$) does not depend, up to isomorphism, on the choice of the particular order $R$ of level $N$, we generally speak of cuspforms of *level* $N$ and denote this space $S_k^D = S_k^D(N, \chi)$ (we shall drop the superscript $D$ if $\Delta = 1$). It is a well-known fact that the spaces $S_k^D$ are finite dimensional and actually trivial for $k < 0$.

Let us now assume $\Delta = 1$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, each $f \in S_k$ satisfies the identity $f(z+1) = f(z)$ and thus admits an expansion of the form

$$f(z) = \sum_{n \geq 1} a_n(f)\, q^n, \qquad q = e^{2\pi i z}. \tag{1.5}$$

The expression (1.5) is called the $q$-expansion of $f$ at $\infty$. Analogous $q$-expansions can be defined for each cusp.

More general spaces of modular forms and cuspforms can be obtained by replacing $\Gamma$ in Definition 1 with arbitrary Fuchsian groups of the first kind (see [18], Chapter 1). Of particular interest is the group

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\boldsymbol{Z}) \text{ such that } c \equiv 0 \bmod N,\ a \equiv d \equiv 1 \bmod N \right\},$$

which is a normal subgroup of $\Gamma_0(N)$. Note that $\Gamma_0(N)/\Gamma_1(N) \simeq (\boldsymbol{Z}/N\boldsymbol{Z})^\times$, and

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(N, \chi), \tag{1.6}$$

where the sum ranges through the group of characters of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$, i.e. the Dirichlet characters modulo $N$.

## 2. – The Jacquet-Langlands correspondence.

For more facts and details about the material in this section, see [6] and the references quoted therein.

To each cuspform $f \in S_k(N, \chi)$ is associated an automorphic representation $\pi_f$ of the adelization $\mathrm{GL}_2(\boldsymbol{A})$. Consider the function

$$\varphi_f(g_{\boldsymbol{Q}}\, g_\infty\, k) := f(g_\infty(i))\, j(g_\infty, i)^{-k} (\det g_\infty)^{k/2} \overline{\chi}(k)$$

on $\mathrm{GL}_2(\boldsymbol{A})$, where $g_{\boldsymbol{Q}} \in \mathrm{GL}_2(\boldsymbol{Q})$, $g_\infty \in \mathrm{GL}_2^+(\boldsymbol{R})$ and $k \in K_N = \prod_{p \text{ finite}} K_p(N)$. It defines a function, denoted $\varphi_f$ again, in $L^2(\mathrm{GL}_2(\boldsymbol{Q}) \backslash \mathrm{GL}_2(\boldsymbol{A}))$. The adele group $\mathrm{GL}_2(\boldsymbol{A})$ acts on $L^2(\mathrm{GL}_2(\boldsymbol{Q}) \backslash \mathrm{GL}_2(\boldsymbol{A}))$ by right translation: $\varrho(g)\varphi(x) = \varphi(xg)$. Then, the representation $\pi_f$ is the subrepresentation of the regular right representation $\varrho$ of $\mathrm{GL}_2(\boldsymbol{A})$ generated by the right translates of $\varphi_f$.

Similarly, to each $f^D \in S_k^D(N, \chi)$ is associated an automorphic representation $\pi_{f^D}$ of $D_{\boldsymbol{A}}^\times$. Start with the function

$$\varphi_{f^D}(d g_\infty k) := f^D(g_\infty(i))\, j(g_\infty, i)^{-k} (\det g_\infty)^{k/2} \overline{\chi}(k)$$

on $D_{\boldsymbol{A}}^\times$, where $d \in D^\times$, $g_\infty \in \mathrm{GL}_2^+(\boldsymbol{R})$ and $k \in K_N = \prod_{p \text{ finite}} R_p^\times$. As for the split case, $\varphi_{f^D}$ defines a function in $L^2(D^\times \backslash D_{\boldsymbol{A}}^\times)$, and $\pi_{f^D}$ is the subrepresentation of the right regular representation $\varrho$ of $D_{\boldsymbol{A}}^\times$ on $L^2(D^\times \backslash D_{\boldsymbol{A}}^\times)$ generated by the translates of $\varphi_{f^D}$.

The automorphic representations of $GL_2$ and $D^\times$ have the property that they can be decomposed as infinite tensor products of their local components (this is a general fact, see [5]). This means that we can write

$$\pi_f = \bigotimes_p \pi_{f,p}, \qquad \pi_{f^D} = \bigotimes_p \pi_{f^D, p}, \tag{2.1}$$

where, including $p = \infty$, $\pi_{f,p}$ (resp. $\pi_{f^D,p}$) is an admissible representation of $\mathrm{GL}_2(\boldsymbol{Q}_p)$ (resp. of $D_p^\times$). Observe that if $p \mid \Delta$ the representation $\pi_{f^D,p}$ is finite dimensional because $D_p^\times$ modulo its center is compact.

The admissible representations of $\mathrm{GL}_2(\boldsymbol{Q}_p)$ have been completely classified [6], [10]. There are three families of such representations.

(i) The *principal series representations* $\pi(\mu_1, \mu_2)$, where $\mu_1$ and $\mu_2$ are characters of $\boldsymbol{Q}_p^\times$ such that $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$. The representation $\pi(\mu_1, \mu_2)$ is realized as right translation on the space of the locally constant functions $\psi$ on $\mathrm{GL}_2(\boldsymbol{Q}_p)$ such that

$$\psi\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} k\right) = \mu_1(t_1)\,\mu_2(t_2)\left|\frac{t_1}{t_2}\right|^{1/2}\psi(k), \qquad \text{for all } k \in K. \tag{2.2}$$

(ii) The *special series representations* $\sigma(\mu_1, \mu_2)$, where $\mu_1$ and $\mu_2$ are characters of $\boldsymbol{Q}_p^\times$ such that $\mu_1\mu_2^{-1} = |\cdot|^{\pm 1}$. This time the space of locally constant functions on $\mathrm{GL}_2(\boldsymbol{Q}_p)$ satisfying (2.2) is not irreducible and $\sigma(\mu_1, \mu_2)$ is realized as its unique irreducible subquotient.

(iii) The *supercuspidal representations*, characterized by the fact that their matrix coefficients are compactly supported modulo the center.

Special and supercuspidal representations are square-integrable.

Let $p$ be a prime dividing $\Delta$. By making use of the local Weil representation associated with $D_p$ and its local norm, it is possible to associate to each irreducible representation $\pi'_p$ of $D_p^\times$ an irreducible representation $\pi(\pi'_p)$ of $\mathrm{GL}_2(\boldsymbol{Q}_p)$ [10], § 1. On the other hand, if $p$ does not divide $\Delta$ the isomorphism $i_p$ allows to identify the representations of $D_p^\times$ with those of $\mathrm{GL}_2(\boldsymbol{Q}_p)$, i.e. put $\pi(\pi'_p) = \pi'_p$ in this case.

The Jacquet-Langlands correspondence between automorphic representations of $D_{\boldsymbol{A}}^\times$ and automorphic representations of $\mathrm{GL}_2(\boldsymbol{A})$ is defined by

$$\pi' = \otimes\pi'_p \mapsto JL(\pi') := \otimes\pi(\pi'_p).$$

This correspondence preserves central characters and $L$-functions, i.e. $L(\pi', s) = L(JL(\pi'), s)$.

## 3. – Oldforms and newforms.

Let $M$ be a divisor of $N$, $\chi$ a Dirichlet character modulo $M$, and let $S$ be the Eichler order of level $M$ obtained replacing $N$ with $M$ in (1.2). For each divisor $d$ of $N/M$ the order $R' = D \cap \left(\mathrm{GL}_2^+(\boldsymbol{R}) \times \prod_p R'_p\right)$, where

$$R'_p = \begin{cases} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} R_p \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1}, & \text{if } p \nmid \Delta, \\ R_p, & \text{if } p | \Delta, \end{cases}$$

is an other Eichler order of level $N$ contained in $S$. Let $\eta \in D_A^\times$ be the adele such that $\eta_p = i_p^{-1}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ if $p|N$, and $\eta_p = 1$ otherwise. By [14], Theorem 5.2.10, there exists an element $\alpha_d \in D$ (in fact $\alpha_d$ belongs to a maximal order containing $S$) such that $\nu(\alpha_d) = d$ and $i_p(\alpha_d) \equiv \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ modulo $N^2 M_2(\boldsymbol{Z}_p)$ for all $p|N$. Then $\alpha_d^{-1}\eta \in \mathrm{GL}_2^+(\boldsymbol{R}) \times \prod_{p \nmid N} R_p^\times \times \prod_{p|N} (1 + NM_2(\boldsymbol{Z}_p))$ and $R' = \alpha_d R \alpha_d^{-1}$. The association $f(z) \mapsto f(\alpha_d z)\, j(\alpha_d, z)^{-k}$ defines an embedding

$$j_d^D : S_k^D(M, \chi) \to S_k^D(N, \chi). \tag{3.1}$$

Note that if $\Delta = 1$ then $\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ and $j(\alpha_d, z) = 1$, so that the embedding (3.1) is the more familiar $f(z) \mapsto f(dz)$. The subspace of $S_k^D(N, \chi)$ spanned by the images of the maps (3.1) is denoted $S_k^D(N, \chi)^{\mathrm{old}}$. Its elements are called *oldforms*.

The space $S_k^D(N, \chi)$ is endowed with a canonical inner product, the *Petersson product*, defined as

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z)\, \overline{g(z)}\, y^{k-2}\, dx\, dy$$

where $z = x + iy$ and $\mathcal{F}$ is a fundamental domain for the action of $\Gamma$ on $\mathcal{H}$. So, we can consider the orthogonal decomposition

$$S_k^D(N, \chi) = S_k^D(N, \chi)^{\mathrm{old}} \oplus S_k^D(N, \chi)^{\mathrm{new}}, \tag{3.2}$$

where $S_k^D(N, \chi)^{\mathrm{new}} = (S_k^D(N, \chi)^{\mathrm{old}})^\perp$. We shall use later the finer decomposition

$$S_k^D(N, \chi) = \bigoplus_{M|N} \Big[ \bigoplus_{d|N/M} j_d^D (S_k^D(M, \chi)^{\mathrm{new}}) \Big], \tag{3.3}$$

where the blocks in square brackets are orthogonal to each other. Mind that when the conductor of $\chi$ does not divide $M$, the corresponding spaces in (3.3) are trivial. We can obtain more decompositions as in (3.2) by picking a suitable subset of the maps (3.1). Assume that $N = AB$ with $A$ and $B$ positive integers with $(A, B) = 1$. Then considering oldforms constructed only from the divisors $M$ of $N$ with $B \nmid M$, we define a subspace $S_k^D(N, \chi)^{B\text{-old}}$. It gives rise to a decomposition analogous to (3.2) together with its orthogonal $S_k^D(N, \chi)^{B\text{-new}}$. In terms of the decomposition (3.3),

$$S_k^D(N,\chi)^{B\text{-new}} = \bigoplus_{B|M|N}\Big[\bigoplus_{d|N/M} j_d^D(S_k^D(M,\chi)^{\text{new}})\Big]. \tag{3.4}$$

The maps $j_d^D$ can be described in a simple way from the adelic point of view. Namely, If $\eta$ is the idele considered above, then

$$\varphi_{j_d^D(f)} = \pi_f(\eta^{-1})\,\varphi_f\,. \tag{3.5}$$

Indeed, $g_{\mathbf{Q}}g_\infty k\eta^{-1} = (g_{\mathbf{Q}}\alpha_d^{-1})(\alpha_d g_\infty)(\alpha_d k\eta^{-1})$ and we can check that the left and right hand sides of (3.5) coincide. Note that $k' = \alpha_d k\eta^{-1} \in \prod K_p(M)$ and $\overline{\chi}(k) = \overline{\chi}(k')$. The left hand side is $\varphi_{j_d^D(f)}(g_{\mathbf{Q}}g_\infty k) = f(\alpha_d g_\infty(i))j(\alpha_d g_\infty, i)^{-k}\overline{\chi}(k)$. The right hand side is $f(\alpha_d g_\infty(i))j(\alpha_d g_\infty, i)^{-k}\overline{\chi}(k')$.

Let $f\in S_k^D(N,\chi)$ and assume that $\pi_f$ is irreducible. Then there exists a divisor $M$ of $N$ and a $g\in S_k^D(M,\chi)^{\text{new}}$ such that $f=j_d^D(g)$ for some divisor $d$ of $N/M$. The well-defined number $\operatorname{Cond}(f) = M$ is called the *conductor* of $f$. In terms of the representation $\pi_f$, the conductor can be computed locally. Namely, if $\pi_{f,p}$ is the local component of $\pi_f$ for $p\nmid \varDelta$, as in (2.1), with representation space $V_p(f)$, the theory of Atkin-Lehner [1] asserts that there is an $n=n(p)$ such that the space

$$W_p(f) = \left\{ v\in V_p(f) \text{ such that } \pi\begin{pmatrix} a & b\\ c & d\end{pmatrix}v = \overline{\chi}(a)\,v \text{ for all } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in K_p(p^n)\right\} \tag{3.6}$$

is 1-dimensional and that the same space for $K_p(p^m)$ is trivial for all $0\leqslant m<n$. Moreover, if we let $\operatorname{Cond}(\pi_{f,p}) = p^{n(p)}$ then, thanks to our assumption that $(N,\varDelta)=1$,

$$\operatorname{Cond}(f) = \operatorname{Cond}(\pi_f) = \prod_p \operatorname{Cond}(\pi_{f,p})\,. \tag{3.7}$$

Mind that, according to the general theory, $n(p)=0$ for all but a finite number of $p$, so that the product in (3.7) is actually a finite product.

## 4. – Hecke operators.

We shall now recall the definition of the Hecke operators acting on the space $S_k^D(N,\chi)$. Let $p$ be a prime number, choose arbitrarily an element $\alpha\in E_R$ with $\nu(\alpha)=p$ and consider the double coset $\varGamma\alpha\varGamma$. This double coset decomposes as a finite disjoint union of left cosets

$$\varGamma\alpha\varGamma = \bigcup_j \varGamma\alpha_j \tag{4.1}$$

for some set of representatives $\alpha_j \in R$. Let $f \in S_k^D(N, \chi)$. A straightforward computation shows that the formula

$$T_p^D f(z) = p^{k-1} \sum_j \overline{\chi(\alpha_j)}\, j(\alpha_j, z)^{-k} f(\alpha_j(z)) \tag{4.2}$$

defines an element in $S_k^D(N, \chi)$ which does not depend on the choice of the representatives $\alpha_j$. Operators $T_n^D$ for any integer $n \geqslant 1$ can be defined in a similar way [14]. To simplify the notation, we shall drop the superscript $D$ in $T_n^D$ unless $\Delta \neq 1$ and some ambiguity may arise. It turns out that

$$T_{mn} = T_m T_n = T_n T_m \ \text{ if } \ (m, n) = 1 \ \text{ and } \ T_{p^{k+1}} = T_p T_{p^k} - p^{k-1} \chi(p)\, T_{p^{k-1}} \quad \text{if } p \text{ is prime and } k \geqslant 1\,. \tag{4.3}$$

When $\Delta = 1$ (i.e. $\Gamma = \Gamma_0(N)$) it is possible to produce explicit elements $\alpha$ and $\alpha_j$ as above. Namely (4.1) and (4.2) become respectively

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N) = \begin{cases} \displaystyle\bigcup_{b=0}^{p-1} \Gamma_0(N) \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}, & \text{if } p \mid N\,, \\ \displaystyle\bigcup_{b=0}^{p-1} \Gamma_0(N) \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \cup \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } p \nmid N\,, \end{cases} \tag{4.4}$$

and

$$T_p f(z) = \begin{cases} \displaystyle p^{k-1} \sum_{b=0}^{p-1} f\left( \frac{z+b}{p} \right), & \text{if } p \mid N\,, \\ \displaystyle p^{-1} \sum_{b=0}^{p-1} f\left( \frac{z+b}{p} \right) + p^{k-1} \chi(p) f(pz)\,, & \text{if } p \nmid N\,. \end{cases} \tag{4.5}$$

In particular, this permits to compute the action of each $T_p$ (and consequently of $T_n$, for all $n \in \boldsymbol{Z}$, $n \geqslant 1$) on $q$-expansions at $\infty$. Namely,

$$a_m(T_n f) = \sum_{d \mid (m, n)} \chi(d)\, d^{k-1} a_{mn/d^2}(f)\,, \tag{4.6}$$

with the convention that $a_t = 0$ if $t \notin \boldsymbol{Z}$. It is important to remark that, in particular,

$$a_1(T_n(f)) = a_n(f)\,. \tag{4.7}$$

DEFINITION 2. – *The Hecke algebra $\boldsymbol{H}^D = \boldsymbol{H}^D(N, \chi)$ is the $\boldsymbol{Z}[\chi]$-subalgebra of* $\mathrm{End}\,(S_k^D(N, \chi))$ *generated by the operators $T_p^D$, for all primes $p$.*

Equivalently, by formulae (4.3), the Hecke algebra $\boldsymbol{H}^D(N,\chi)$ may be defined as the $\boldsymbol{Z}[\chi]$-subalgebra of $\mathrm{End}(S_k^D(N,\chi))$ generated by the operators $T_n$, for all $n\in\boldsymbol{Z}$, $n\geqslant 1$. Again, we shall drop the superscript $D$ when $\Delta=1$.

Let $p$ be a prime. For $f\in S_k^D(N,\chi)$ and $p\nmid\Delta$ let

$$\tilde{T}_p^D\varphi_f(g)=d_p\int\limits_{K_p(N)}\chi(k_p)\,\varphi_f\left(gk_p\begin{pmatrix}p&0\\0&1\end{pmatrix}\right)dk_p=\sum_j\varphi_f(g\alpha_j)\tag{4.8}$$

where the $\alpha_j$ are representatives of the right cosets in the decomposition $K_p(N)\begin{pmatrix}p&0\\0&1\end{pmatrix}K_p(N)=\bigcup_j\alpha_jK_p(N)$, $d_p=\deg\left(K_p(N)\begin{pmatrix}p&0\\0&1\end{pmatrix}K_p(N)\right)$ is the number of such representatives and the Haar measure $dk_p$ is normalized so that the volume of $K_p(N)$ is 1. If $p\,|\,\Delta$, let

$$\tilde{T}_p^D\varphi_f(g)=\int\limits_{R_p^\times}\varphi_f(gk_p\pi)\,dk_p=\varphi_f(g\pi)\tag{4.9}$$

where $\pi$ is any uniformizer of $D_p^\times$, that is $\nu(\pi)=p$. As for the Hecke operators, we shall write $\tilde{T}_p$ for $\tilde{T}_p^D$ when there is no risk of confusion.

REMARK 3. – The expressions (4.8) and (4.9) show that under the decomposition (2.1), the operator $\tilde{T}_p$ acts only on the $p$-th component of $\pi_f$.

The following result links $T_p$ to $\tilde{T}_p$.

THEOREM 4. – *Let* $f\in S_k^D(N,\chi)$. *Then, for any prime* $p$, $p^{k/2-1}\tilde{T}_p\varphi_f=\chi_p(p)\,\varphi_{T_pf}$.

This result follows from the fact that the action $f\mapsto f_{|\gamma}$ of $\mathrm{GL}_2^+(\boldsymbol{Q})$ on the full space of modular forms and on $L^2(\mathrm{GL}_2(\boldsymbol{Q})\backslash\mathrm{GL}_2(\boldsymbol{A}))$ coincide up to a character. Instead of following this way, we shall give a less illuminating but perhaps more direct proof in the following section.

## 5. – Proof of Theorem 4.

To prove the formula of Theorem 4, we need first the following two preliminary results.

LEMMA 5. – *Let* $k=\begin{pmatrix}x&y\\z&w\end{pmatrix}\in K_p$, and $m=0,1,\dots,p-1$.

*a) If* $p \nmid mz+w$ *let* $b_m \equiv (mx+y)/(mz+w) \bmod p$*. Then*

$$k\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & b_m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x - b_m z & \dfrac{mx+y-b_m(mz+w)}{p} \\ pz & mz+w \end{pmatrix}.$$

*b) If* $p \mid mz+w$*, then*

$$k\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} px & mx+y \\ z & \dfrac{mz+w}{p} \end{pmatrix}.$$

PROOF. – The result is obtained by a straightforward computation. ■

LEMMA 6. – *Consider the decomposition* $K_p(N)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}K_p(N) = \bigcup_{i \in I} \alpha_i K_p(N)$*, and let* $k \in K_p(N)$*. Then there exists a permutation* $\sigma = \sigma(k)$ *of* $I$ *such that* $k\alpha_i = \alpha_{\sigma(i)} k'$ *with* $\chi_p(k) = \chi_p(k')$*.*

PROOF. – Suppose first that $p \nmid N$. The representatives $\alpha_i$ can be chosen either of the form $\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}$ for some $m = 0, 1, \dots, p-1$, or $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Write $k = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in K_p$ and apply the previous lemma. If $p \nmid mz+w$ then $k\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & b_m \\ 0 & 1 \end{pmatrix} k'$ for some $k' \in K_p$, and if $p \mid mz+w$ then $k\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} k'$ for some $k' \in K_p$. Moreover

$$k\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} x & py \\ z/p & w \end{pmatrix}, & \text{if } p \mid z\,, \\ \begin{pmatrix} p & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (x-bz)/p & y-pw \\ z & pw \end{pmatrix}, & \text{if } p \nmid z \text{ and } x \equiv bz \bmod p\,. \end{cases}$$

Now observe that if $p \mid z$ then $p \nmid mz+w$ so that the permutation we look for fixes $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and on the other representatives is the one induced by the permutation $m \mapsto b_m$ of the set $\{0, 1, \dots, p-1\}$. If $p \nmid z$, there exists exactly one $\overline{m}$

such that $p\,|\,\overline{m}z + w$. In this case the permutation is

$$\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} p & b_m \\ 0 & 1 \end{pmatrix} \text{ if } m \neq \overline{m}, \quad \begin{pmatrix} p & \overline{m} \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mapsto \begin{pmatrix} p & x/z \\ 0 & 1 \end{pmatrix}.$$

If $p\,|\,N$ the only $\alpha_i$ that appear are the matrices $\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}$ for $m = 0, 1, \ldots, p-1$. If $k \in K_p(N)$ is as above, we always have $p\,|\,z$ and we can apply the previous argument again. Finally, the equality $\chi_p(k) = \chi_p(k')$ follows by inspection. ■

We now proceed with the proof of Theorem 4. Suppose first that $p \nmid \Delta$ and consider the decomposition $R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times = \bigcup_{i \in I} \alpha_i R_p^\times$. Note that we could be more explicit about the representatives $\alpha_i$, but choose not to be, since we want to treat the cases $p\,|\,N$ and $p \nmid N$ simultaneously.

We shall now use again the decomposition $D_A^\times = D^\times \left( \mathrm{GL}_2^+(\boldsymbol{R}) \times \prod_q R_q^\times \right)$. For each $i \in I$ consider the adele $\tilde{\alpha}_i$ such that $(\tilde{\alpha}_i)_\infty = 1$, $(\tilde{\alpha}_i)_q = 1$ for all $q \neq p$ and $(\tilde{\alpha}_i)_p = \alpha_i$ and write

$$\tilde{\alpha}_i = d^i g_\infty^i k^i, \tag{5.1}$$

with $d^i \in D^\times$, $g_\infty^i \in \mathrm{GL}_2^+(\boldsymbol{R})$ and $k^i = \prod_q k_q^i \in \prod_q R_q^\times$. Put $\delta^i = p(d^i)^{-1} \in D^\times$.

Pick an element $\alpha$ with the property that the double coset $\Gamma\alpha\Gamma$ is the one that defines the Hecke operator $T_p$ as in section 4. Then $\Gamma\alpha\Gamma = \bigcup_{i \in I} \Gamma\delta^i$. This follows from (5.1) because $d^i g_\infty^i = 1$, $d^i k_p^i = \alpha_i$ and $d^i k_q^i = 1$ for all $q \neq p$. Note that $\Gamma\alpha\Gamma = D^\times \cap \left( \mathrm{GL}_2^+(\boldsymbol{R}) \times R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times \times \prod_{q \neq p} R_q^\times \right)$. Thus, if the $\alpha_i$ are representants of the right cosets of $R_p^\times \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} R_p^\times$, the elements $p\alpha_i^{-1}$ are representatives of the left cosets of $R_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} R_p^\times$. Therefore the $\delta^i$ are representatives for the left cosets of $\Gamma\alpha\Gamma$.

Let $g \in D_A^\times$ and write its usual decomposition $g = dg_\infty k$. Using Lemma 6, we write:

$$g\alpha_i = dg_\infty \left( \prod_{q \neq p} k_q \right) k_p \alpha_i = dg_\infty \left( \prod_{q \neq p} k_q \right) \alpha_j k_p' = d(d^j g_\infty^j k^j)\, g_\infty \left( \prod_{q \neq p} k_q \right) k_p' =$$

$$= dd^j g_\infty^j g_\infty \left( \prod_{q \neq p} k_q^j k_q \right) k_p^j k_p' = dd^j g_\infty^j g_\infty k^j k'$$

where $\chi(k') = \chi(k)$ and we set $j = \sigma(i)$. Therefore

$$\varphi_f(g\alpha_i) = f(g_\infty^j g_\infty(i))j(g_\infty^j g_\infty, i)^{-k} \det(g_\infty^j g_\infty)^{k/2}\overline{\chi}(k^j)\,\overline{\chi}(k) =$$

$$f(g_\infty^j g_\infty(i))j(g_\infty^j, g_\infty(i))^{-k} j(g_\infty, i)^{-k} \det(g_\infty^j)^{k/2} \det(g_\infty)^{k/2}\overline{\chi}(k^j)\,\overline{\chi}(k) =$$

$$f(p^{-1}\delta^j g_\infty(i))j(p^{-1}\delta^j, g_\infty(i))^{-k} j(g_\infty, i)^{-k} p^{-k} \det(\delta^j)^{k/2} \det(g_\infty)^{k/2}\overline{\chi}(k^j)\,\overline{\chi}(k) =$$

$$p^{k/2} f(\delta^j g_\infty(i))j(\delta^j, g_\infty(i))^{-k} j(g_\infty, i)^{-k} \det(g_\infty)^{k/2}\overline{\chi}(k^j)\,\overline{\chi}(k).$$

To evaluate the last expression assume that $q|N$ and $p \neq q$. Then $\overline{\chi}_q(k_q^j) = \overline{\chi}_q((d^j)^{-1}) = \chi_q(p^{-1})\,\chi_q(\delta^j)^{-1}$. Note that, according to the chosen conventions, $\chi_q(r) = \overline{\chi}_q\left(\begin{pmatrix} r & 0 \\ 0 & r\end{pmatrix}\right)$, for all $r \in \boldsymbol{Z}_q$. If $p|N$, $\overline{\chi}_p(k_p^j) = \overline{\chi}_p((d^j)^{-1}\alpha_j) = \chi_p(\alpha_j^{-1} d^j) = \chi_p(\delta^j)^{-1}$, since $\chi_p(p\alpha_j^{-1}) = 1$. We conclude that

$$\overline{\chi}(k^j) = \prod_{q|N,\, q \neq p} \chi_q(p^{-1})\,\chi_q(\delta^j)^{-1} = \chi_p(p)\,\chi(\delta^j)^{-1}.$$

Plugging this value into the previous computation yields

$$\varphi_f(g\alpha_i) = p^{k/2} f(\delta^j g_\infty(i))j(\delta^j, g_\infty(i))^{-k} j(g_\infty, i)^{-k} \det(g_\infty)^{k/2}\chi_p(p)\overline{\chi}(\delta^j)\,\overline{\chi}(k).$$

Putting everything together,

$$\widetilde{T}_p\varphi_f(g) = \sum_{i\in I}\varphi_f(g\alpha_i) =$$

$$p^{k/2}\det(g_\infty)^{k/2}\chi_p(p)\,\overline{\chi}(k)\,j(g_\infty, i)^{-k}\sum_{i\in I} f(\delta^j g_\infty(i))j(\delta^j, g_\infty(i))^{-k}\overline{\chi}(\delta^j) =$$

$$p^{1-(k/2)}\det(g_\infty)^{k/2}\chi_p(p)\,\overline{\chi}(k)\,j(g_\infty, i)^{-k}(T_p f)(g_\infty(i)) = p^{1-(k/2)}\chi_p(p)\,\varphi_{T_p f}(g).$$

We now deal with the case $p|\varDelta$. Pick $\alpha \in E_R$ as in section 4 and let $\tilde{\alpha}$ be the adele which is 1 at all places except $\tilde{\alpha}_p = \alpha$. Then

$$\widetilde{T}_p\varphi_f(g) = \varphi_f(g\tilde{\alpha}) = (\text{using the decomposition } g = dg_\infty k \text{ again}) =$$

$$\varphi_f\left((d\alpha)(\alpha^{-1}g_\infty)\left(\prod_{q\neq p}\alpha^{-1}k_q\right)(\alpha^{-1}k_p\alpha)\right) =$$

$$f(\alpha^{-1}g_\infty(i))j(\alpha^{-1}g_\infty, i)^{-k}\det(\alpha^{-1}g_\infty)^{k/2}\overline{\chi}(k)\,\overline{\chi}(\alpha^{-1}\tilde{\alpha}).$$

Since $\varGamma\alpha\varGamma = \varGamma\overline{\alpha}\varGamma$, the last expression is equal to

$$= f(\overline{\alpha}^{-1}g_\infty(i))j(\overline{\alpha}^{-1}g_\infty, i)^{-k}\det((\overline{\alpha})^{-1}g_\infty)^{k/2}\overline{\chi}(k)\,\overline{\chi}(\overline{\alpha}^{-1}\tilde{\overline{\alpha}}) =$$

$$(\text{note that } \overline{\alpha}^{-1} = p^{-1}\alpha) = p^{k/2} f(\alpha g_\infty(i))j(\alpha g_\infty, i)^{-k}\det(g_\infty)^{k/2}\overline{\chi}(k)\,\chi_p(p)\,\overline{\chi}(\alpha).$$

On the other hand, the expression $T_p f(z) = p^{k-1} f(\alpha z) j(\alpha, z)^{-k} \overline{\chi}(\alpha)$ for the Hecke operator shows that $\varphi_{T_p f}(g) = p^{k-1} f(\alpha g_\infty(i)) j(g_\infty, i)^{-k} \det(g_\infty)^{k/2} \chi(\alpha)\, \overline{\chi}(k)$ and the result follows. ■

## 6. – The Jacquet-Langlands correspondence revisited.

Let $f \in S_k^D(N, \chi)$. We have already remarked (section 2) that the local component $\pi_{f,p}$ for $p \mid \Delta$ is finite dimensional. We now give a better statement.

LEMMA 7. – *Suppose that $\pi_f$ is irreducible and $p \mid \Delta$. Then $\pi_{f,p}$ is one dimensional.*

PROOF. – Write $\varphi_f = \sum_{i=1}^{t} \left( \bigotimes_q v_q^i \right) \in \bigotimes_q V_q(f)$. This vector is invariant under the action of the group $K_p^D = \{ x \in D_p^\times \text{ such that } |\nu(x)|_p = 1 \}$ embedded in the $p$-th component of $D_A^\times$. Let $w^i = \bigotimes_{q \neq p} v_q^i$. Up to rewriting $\varphi_f$ with a smaller number of terms, we may assume that the $w^i$ are linearly independent. Then each vector $v_p^i$ is $K_p^D$-invariant, as follows from the identity $\sum_{i=1}^{t} (\pi_f(k) v_p^i - v_p^i) \otimes w^i = 0$. Hence, $W_p(f) = V_p(f)^{K_p^D} \neq (0)$.

Observe that $K_p^D$ is normal in $D_p^\times$. Therefore, the subspace $W_p(f)$ is stable under the action of $D_p^\times$. It follows from the irreducibility that $W_p(f) = V_p(f)$.

Finally, note that $D_p^\times / K_p^D$ is abelian. Thus $\pi_{f,p}$ is an irreducible representation of an abelian group in a finite dimensional vector space. Therefore $\pi_{f,p}$ is one dimensional. ■

REMARK 8. – If $\pi_f$ is irreducible then $f$ is an eigenform for almost all Hecke operators $T_p$. Indeed, the starting argument in the proof of Lemma 7 shows that if $q \nmid N\Delta$ then the vector $v_q^i$ belongs to the Atkin-Lehner space $W_q(f)$. It follows from Lemma 5 that the $q$-th Hecke operator acts as a scalar on this space.

Note that in the course of the proof of Lemma 7 we have defined 1-dimensional local spaces $W_p(f)$ of invariants also for $p \mid \Delta$.

COROLLARY 9. – *If $f$ is an eigenform for $\boldsymbol{H}^D$ then $W_p(f)$ is $\widetilde{T}_p$-stable for all $p$.*

PROOF. – If $f$ is an eigenform for $\boldsymbol{H}^D$, Theorem 4 shows that the function $\varphi_f$ is $\widetilde{T}_p$-eigen for all $p$. The assertion follows at once. ■

The Jacquet-Langlands correspondence is defined in terms of representations. Since the notion of «normalized forms» is meaningless in $S_k^D(N, \chi)$, (when $\Delta \neq 1$) due to the absence of $q$-expansions, it is impossible to define a direct canonical correspondence between forms. Nonetheless, it is possible to define in a non-canonical way a Jacquet-Langlands correspondence which is well-behaved with respect to the action of the Hecke operators.

PROPOSITION 10. – *There is an isomorphism* $S_k^D(N, \chi)^{\text{new}} \simeq S_k(N\Delta, \chi)^{\text{new}}$ *as Hecke modules.*

REMARK 11. – The statement is actually ambigous because the Hecke algebras acting on the two spaces which are identified are different. We simply mean that if $f^D$ corresponds to $f$, then, for all $p$, $T_p^D f^D$ corresponds to $T_p f$.

PROOF. – If $f^D \in S_k^D(N, \chi)$ is a newform which is an eigenvector of the Hecke algebra, $\pi_{f^D}$ is an irreducible automorphic representation of $D_A^\times$ and $JL(\pi_{f^D})$ is an irreducible automorphic representation of $\mathrm{GL}_2(\boldsymbol{A})$. Let $M$ be the conductor of $\mathrm{JL}(\pi_{f^D})$. By [6], Chapter 5, there exists a unique normalized newform $\tilde{f} \in S_k(M, \chi)$, eigenvalue of the Hecke algebra, such that $\pi_{\tilde{f}} = \mathrm{JL}(\pi_{f^D})$.

When $p \mid \Delta$ Lemma 7 applies and so $\pi_{f^D, p} = \psi \circ \nu$, where $\psi$ is a character of $\boldsymbol{Q}_p^\times$, [10] §4. Since $\nu \colon K_p^D \to \boldsymbol{Z}_p^\times$ is surjective, $\psi$ is unramified. It follows from [1], p. 125, that $\pi(\pi_{f^D, p})$ is the special representation $\sigma(\psi \|^{1/2}, \psi \|^{-1/2})$ of $\mathrm{GL}_2(\boldsymbol{Q}_p)$, which has conductor $p$. Since $\mathrm{Cond}\,(\mathrm{JL}\,(\pi_{f^D})) = \prod_{p < \infty} \mathrm{Cond}\,(\pi(\pi_{f^D, p}))$, we conclude that $\mathrm{Cond}\,(\mathrm{JL}\,(\pi_{f^D})) = \Delta\, \mathrm{Cond}\,(\pi_{f^D})$.

To define the identification which is to be proved, start with a basis of $S_k^D(N, \chi)^{\text{new}}$ consisting of eigenforms for $\boldsymbol{H}^D$ and to each $f^D$ in this basis associate the form $\tilde{f} \in S_k(N\Delta, \chi)^{\text{new}}$ as above.

CLAIM. – The form $\tilde{f}$ is an eigenfunction for the Hecke algebra $\boldsymbol{H}$. Moreover the eigenvalues of $T_p^D$ for $f^D$ and of $T_p$ for $\tilde{f}$ coincide for all primes $p$.

PROOF OF CLAIM. – The assertion is clear for $p \nmid \Delta$ because then $W_p(f^D) = W_p(\tilde{f}\,)$ and the operators $\tilde{T}_p$ are the same.

Let now $p$ be a prime dividing $\Delta$. We have already seen that, in consequence of Lemma 7, $\pi_{f^D, p} = \psi \circ \nu$ where $\psi$ is an unramified character of $\boldsymbol{Q}_p^\times$. The formula (4.9) and the Corollary 9 show that on $W_p(f^D)$ the operator $\tilde{T}_p^D$ acts as multiplication by $\psi \circ \nu(\pi) = \psi(p)$.

To compute the eigenvalue of $\tilde{T}_p$ for $\varphi_{\tilde{f}}$ we shall use the explicit model of $\pi_{\tilde{f}, p} = \sigma(\psi \|^{1/2}, \psi \|^{-1/2})$ of [10], §3. Let $\mathcal{B}$ be the space of locally constant $\boldsymbol{C}$-valued functions $\varphi$ on $\mathrm{GL}_2(\boldsymbol{Q}_p)$ such that

$$\varphi\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g\right) = \psi(t_1 t_2) \left| \frac{t_1}{t_2} \right| \varphi(g) \tag{6.1}$$

for all $t_1, t_2 \in \boldsymbol{Q}_p^\times$. The group $\mathrm{GL}_2(\boldsymbol{Q}_p)$ acts on $\mathcal{B}$ by right translation. Then, a model for $\sigma(\psi \|^{1/2}, \psi \|^{-1/2})$ is the subspace of $\mathcal{B}$ consisting of the functions $\varphi$ such that $\int_{K_p} \varphi(k)\, \psi^{-1}(\det k)\, dk = 0$. Since the character $\psi$ is unramified, the condition just stated is actually simpler: it reads

$$\int_{K_p} \varphi(k)\, dk = 0\,. \tag{6.2}$$

Let $\varphi_0$ be the function on $K_p$ defined as

$$\varphi_0(k) = \begin{cases} 1 & \text{if } k \in K_p(p)\,, \\ -1/p & \text{if } k \notin K_p(p)\,. \end{cases}$$

It is a standard fact that the group $\mathrm{GL}_2(\boldsymbol{Q}_p)$ decomposes as $\mathrm{GL}_2(\boldsymbol{Q}_p) = B_p K_p$. Then, formula (6.1) used as definition permits to extend $\varphi_0$ to a function on $\mathcal{B}$ denoted $\varphi_0$ again (it is easy to check that the extension to the whole of $\mathrm{GL}_2$ is well-defined). Now

$$\int_{K_p} \varphi_0(k)\, dk = \int_{K_p(p)} \varphi_0(k)\, dk - \int_{K_p - K_p(p)} \varphi_0(k)\, dk = \mu(K_p(p)) - \frac{1}{p}\, \mu(K_p - K_p(p))\,. \tag{6.3}$$

The formulae $|\mathrm{GL}_2(\boldsymbol{F}_p)| = (p^2 - p)(p^2 - 1)$ and

$$[K\colon K_p(p)] = [\mathrm{GL}_2(\boldsymbol{F}_p)\colon B(\boldsymbol{F}_p)] = \frac{(p^2-p)(p^2-1)}{p(p-1)^2} = p+1$$

yield $\mu(K_p) = (p+1)\,\mu(K_p(p))$ and so (6.3) vanishes. The condition (6.2) is thus met: $\varphi_0$ does belong to $\sigma(\psi\|^{1/2}, \psi\|^{-1/2})$. Observe that $\varphi_0$ is right $K_p(p)$-invariant (but obviously not $K_p$-invariant). According to the general theory of Atkin-Lehner, the space of right $K_p(p)$-invariant functions in $\mathcal{B}$ is 1 dimensional. Therefore, $\varphi_0$ generates $W_p(\tilde{f}\,)$ because $\tilde{f}$ is a newform. In particular $\varphi_0$ is eigen for the $p$-th Hecke operator. To compute the eigenvalue, use (4.8) (and Remark 3) to write

$$\tilde{T}_p \varphi_0(g) = \sum_{m \bmod p} \varphi_0\left(bk \begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}\right), \tag{6.4}$$

where $g = bk$ is again the decomposition $\mathrm{GL}_2(\boldsymbol{Q}_p) = B_p K_p$. If $k \in K_p(p)$, then $p \nmid mz + w$ for all $m = 0, 1, \ldots, p-1$. Apply part $a$) of Lemma 5 to write

$$\sum_{m \bmod p} \varphi_0\left(bk \begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}\right) = \sum_{m \bmod p} \varphi_0\left(b \begin{pmatrix} p & b_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & * \\ pz & * \end{pmatrix}\right) =$$

$$\sum_{m \bmod p} \varphi_0\left(b \begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}\right) = \sum_{m \bmod p} \psi(p)\, |p|\, \varphi_0(b) = \psi(p)\, \varphi_0(g)\,.$$

If, on the other hand, $k \notin K_p(p)$ there exists a unique $\overline{m}$ mod $p$ such that

$\overline{m}z + w \equiv 0 \bmod p$. Now apply part $b$) of Lemma 5 to write

$$\sum_{m \bmod p} \varphi_0\left(bk\begin{pmatrix} p & m \\ 0 & 1 \end{pmatrix}\right) = \varphi_0\left(b\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} * & * \\ z & * \end{pmatrix}\right) +$$

$$\sum_{\substack{m \bmod p \\ m \neq \overline{m}}} \varphi_0\left(b\begin{pmatrix} p & b_m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} * & * \\ pz & * \end{pmatrix}\right) = -p^{-1}\,|p^{-1}|\,\psi(p)\,\varphi_0(b) +$$

$$+(p-1)\,\psi(p)\,|p|\,\varphi_0(b) = -p^{-1}\psi(p)\,\varphi_0(b) = \psi(p)\,\varphi_0(bk) = \psi(p)\,\varphi_0(g).$$

In any event the eigenvalue is $\psi(p)$ and the claim is proved. ■

The association $f^D \mapsto \tilde{f}$ between Hecke eigenforms extends by linearity to a map JL: $S_k^D(N, \chi)^{\mathrm{new}} \to S_k(N\Delta, \chi)^{\mathrm{new}}$.

*Injectivity of* JL. Since the Jacquet-Langlands correspondence between representations is injective, the various representations $\pi_{\tilde{f}}$ constructed above are all distinct. As such, they have distinct systems of Hecke eigenvalues. Therefore the forms $\tilde{f}$ are linearly independent.

*Surjectivity of* JL. Let us start by observing that if $f \in S_k(N\Delta, \chi)^{\mathrm{new}}$ and $p \mid \Delta$, the $p$-th component $\pi_{f,p}$ is special. Indeed, it cannot be supercuspidal because the conductor of a supercuspidal representation is at least $p^2$, but $p^2$ does not divide $\Delta$. It cannot be principal either, because if $\pi_{f,p} = \pi(\mu_1, \mu_2)$, then its central character would be $\mu_1\mu_2$ and its conductor would be $\mathrm{Cond}(\mu_1)\,\mathrm{Cond}(\mu_2)$, see [2]. But, since $\chi$ is a Dirichlet character modulo $N$, which is prime to $p$, the central character of $\pi_{f,p}$ is trivial, so $\mu_2 = \mu_1^{-1}$ and $p = \mathrm{Cond}(\pi_{f,p}) = \mathrm{Cond}(\mu_1)^2$, a contradiction.

Hence $\pi_{f,p}$ is square-integrable, and one knows [6], Theorem 10.5 that the Jacquet-Langlands local correspondence is surjective onto square-integrable representations. Thus, there exists a reprentation $\pi_p$ of $D_p^\times$ such that $\pi(\pi_p) = \pi_{f,p}$. The representation $\pi_p$ must be one dimensional *a fortiori*, or, else, $\pi_{f,p}$ would be supercuspidal [10], Lemma 4.2. As already recalled, a one dimensional representation of $D_p^\times$ is of the form $\psi \circ \nu$, where $\psi$ is a character of $\boldsymbol{Q}_p^\times$. Since $\mathrm{Cond}(\pi_{f,p}) = p$, $\psi$ is unramified, [2]. Thus $\pi_p$ is $K_p^D$-invariant.

Consider the representation $\pi^D = \left(\bigotimes_{p \nmid \Delta} \pi_{f,p}\right) \otimes \left(\bigotimes_{p \mid \Delta} \pi_p\right)$ of $D_{\boldsymbol{A}}^\times$. It follows from the above discussion that $\pi^D = \pi_{f^D}$ for $f^D \in S_k^D(N, \chi)^{\mathrm{new}}$ and that, up to a constant, $\mathrm{JL}(f^D) = f$. ■

Our next goal is to extend the isomorphism of Proposition 10 to a map from the whole space $S_k^D(N, \chi)$. The result is:

THEOREM 12. – *There is an isomorphism* $S_k^D(N, \chi) \simeq S_k(N\Delta, \chi)^{\Delta\text{-new}}$ *as Hecke modules.*

PROOF. – Consider the decomposition (3.3) of $S_k^D(N, \chi)$. Define $\mathrm{JL}(f)$ for $f$ in the old subspace by requiring that $\mathrm{JL} \circ j_d^D = j_d \circ \mathrm{JL}$. The Proposition 10, together with the characterization (3.4) of the $\Delta$-new space, shows that $\mathrm{JL}\colon S_k^D(N, \chi) \xrightarrow{\sim} S_k(N\Delta, \chi)^{\Delta\text{-new}}$ as vector spaces. It is left to prove that $JL$ is an isomorphism of Hecke modules.

It is enough to prove that if $f \in S_k^D(M, \chi)^{\mathrm{new}}$ for some $M | N$, and if $d | (N/M)$ then

$$T_p(\mathrm{JL}(j_d^D f)) = \mathrm{JL}(T_p^D(j_d^D f)) \tag{6.5}$$

for all $p$. Suppose first that $p \nmid N$. In this situation $T_p^D$ and $j_d^D$ commute, so $T_p(\mathrm{JL}(j_d^D f)) = T_p(j_d(\mathrm{JL}(f))) = j_d(T_p(\mathrm{JL}(f))) =$ (since $f$ is a newform) $= j_d(\mathrm{JL}(T_p^D(f))) = \mathrm{JL}(j_d^D(T_p^D(f))) = \mathrm{JL}(T_p^D(j_d^D(f)))$, which proves (6.5).

For $p | N$, let $Y_p^D(f) = \bigoplus_{p^e | (N/M)} \pi_{f,p}(\eta^D_{p^e,p})\, W_p^D(f)$, where $\eta_d^D$ is the idele defined in section 3 (note that $\pi_p(\eta_{b,p})\, W_p = \pi_p(\eta_{p^e,p})\, W_p$ if $p^e \| b$). The space $Y_p^D(f)$ is $\widetilde{T}_p^D$-stable. Under the JL map the function

$$\varphi_{(j_d^D f)} \in \Big(\bigotimes_{p \nmid N} W_p^D(f)\Big) \otimes \Big(\bigotimes_{p|N} Y_p^D(f)\Big)$$

corresponds to

$$\varphi_{j_d(\mathrm{JL}(f))} \in \Big(\bigotimes_{p \nmid N} W_p(\mathrm{JL}(f))\Big) \otimes \Big(\bigotimes_{p|N} Y_p(\mathrm{JL}(f))\Big).$$

In fact, the spaces $Y_p^D(f)$ and $Y_p(\mathrm{JL}(f))$ are actually the same, because, under the Jacquet-Langlands correspondence the local representations coincide at these $p$ (since $d$ and $\Delta$ are coprime) and $\eta^D_{d,p} = \eta_{d,p}$. Moreover, under the identification $Y_p^D(f) = Y_p(\mathrm{JL}(f))$ the $p$-th Hecke operators coincide. The identity (6.5) follows immediately. ∎

## 7. – Consequences for the Hecke algebras.

For a subring $A \subseteq \boldsymbol{C}$ and $M \in \boldsymbol{N}$ let

$$S_k(M, \chi; A) := \{ f \in S_k(M, \chi) \text{ such that } a_n(f) \in A \text{ for all } n \in \boldsymbol{N} \}.$$

We shall use the notation $S_k(A)$ as a shorthand for $S_k(M, \chi; A)$ if there is no ambiguity about the level and the character under consideration.

PROPOSITION 13. – *There exists a basis of* $S_k(M, \chi)$ *in* $S_k(M, \chi; \boldsymbol{Z}[\chi])$.

PROOF. – It is well-know [18], Theorem 3.52, that the space $S_k(\Gamma_1(M))$ has a basis with Fourier coefficients in $\boldsymbol{Z}$. By projecting this basis in $S_k(M, \chi)$ by means of the projector $e_\chi = \phi(M)^{-1} \sum_{g \in (\boldsymbol{Z}/M\boldsymbol{Z})^\times} \overline{\chi}(g)\langle g \rangle$ we obtain a set of generators of $S_k(M, \chi)$ with coefficients in $\boldsymbol{Q}(\chi)$. The operator $\langle g \rangle$ in the expression of the projector is the diamond operator $f \mapsto f_{|\sigma_g}$ where $\sigma_g \in SL_2(\boldsymbol{Z})$ is congruent to $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ modulo $M$. To conclude it is enough to observe that $\phi(M)e_\chi$ is $\boldsymbol{Z}[\chi]$-integral on $q$-expansions. ■

DEFINITION 14. – *Let $A$ be a subalgebra of $\boldsymbol{C}$. Let $W$ be an $A$-submodule of $S_k(A)$ which is Hecke stable. The Hecke algebra of $W$, denoted $\boldsymbol{H}(W)_A$, is the $A$-subalgebra of* $\mathrm{End}_A(W)$ *generated by the operators $T_{p|W}$.*

PROPOSITION 15. – *Let $A$ be a $\boldsymbol{Z}[\chi]$-subalgebra of $\boldsymbol{C}$. Then*

$$S_k(M, \chi; A) = S_k(M, \chi; \boldsymbol{Z}[\chi]) \otimes A \qquad \textit{and} \qquad \boldsymbol{H}_k(M, \chi)_A = \boldsymbol{H}_k(M, \chi) \otimes A .$$

PROOF. – See for instance [9], Theorem 6.3.2, where the assertion on the space of modular forms is proved using cohomology, and the final argument of the proof of [9], Corollary 5.4.1, which shows that the assertion for the Hecke algebra is then automatic. ■

From now on, $A$ will always be a $\boldsymbol{Z}[\chi]$-subalgebra of $\boldsymbol{C}$. Proposition 15 asserts that the spaces of classical cuspforms are naturally endowed with an integral structure. We now list a few useful properties of this integral structure.

Recall that the existence of the $q$-expansion (1.5) of modular forms in $S_k = S_k(M, \chi)$ allows to define a pairing

$$\boldsymbol{H}(M, \chi) \times S_k(\boldsymbol{Z}[\chi]) \to \boldsymbol{Z}[\chi], \quad (h, f) \mapsto a_1(h(f)) . \tag{7.1}$$

This pairing is non-degenerate [9], page 142, and defines maps

$$\varrho_H \colon \boldsymbol{H}_A \to S_k(A)^\vee = \mathrm{Hom}_A(S_k(A), A) \tag{7.2}$$

and

$$\varrho_S \colon S_k(A) \to \boldsymbol{H}_A^\vee = \mathrm{Hom}_A(\boldsymbol{H}_A, A) . \tag{7.3}$$

LEMMA 16.

1) *The map $\varrho_H$ is injective.*

2) *The map $\varrho_S$ is an isomorphism.*

PROOF. – The injectivity of $\varrho_H$ and $\varrho_S$ is the non-degeneracy of the pairing (7.1). The surjectivity of $\varrho_S$ is clear with $A$ replaced by $\boldsymbol{C}$ (or simply by its field of quotients). Let $\lambda \in \boldsymbol{H}_A^\vee \subset \boldsymbol{H}_C^\vee$ and let $f \in S_k$ be such that $\varrho_S(f) = \lambda$. Then $a_n(f) = a_1(T_n(f)) = \lambda(T_n) \in A$ and $f \in S_k(A)$. ■

DEFINITION 17. – *We say that the algebra $A$ is a dualizing algebra* (*for the pair* $(M, \chi)$) *if the map $\varrho_H$ is an isomorphism.*

Note that if $\boldsymbol{H}_A$ is free over $A$, then $\boldsymbol{H}_A \simeq (\boldsymbol{H}_A^\vee)^\vee \simeq S_k(A)^\vee$. Thus $\varrho_H$ is an isomorphism.

REMARK 18. – Any principal ideal domain $A \supseteq \boldsymbol{Z}[\chi]$ (in particular any field) is a dualizing algebra. This follows from the general fact that any finitely generated torsion-free module over a PID is free.

PROPOSITION 19. – *If $\phi(M)^{-1} \in A$ then $A$ is a dualizing algebra for the pair $(M, \chi)$.*

PROOF. – Since $\boldsymbol{Z}$ is a PID, the pairing (7.1) for $\Gamma_1(M)$ defines an isomorphism $\boldsymbol{H}(\Gamma_1(M))_{\boldsymbol{Z}} \simeq S_k(\Gamma_1(M); \boldsymbol{Z})^\vee$. The decomposition (1.6) is defined over $A$, so that there is a surjection $S_k(\Gamma_1(M); A)^\vee \to S_k(M, \chi; A)^\vee$. Thus, a linear form $\lambda$ on $S_k(M, \chi; A)$ can be lifted to an element $h \in \boldsymbol{H}(\Gamma_1(M))_A$. The restriction $h' = h_{|S_k(M, \chi; A)}$ is a well-defined element of $\boldsymbol{H}(M, \chi)_A$ and $\varrho_H(h') = \lambda$. ■

We shall now derive from Theorem 12 some consequences for the Hecke algebra $\boldsymbol{H}^D$. We shall denote $\boldsymbol{H}(N\Delta, \chi)^{\Delta\text{-new}} = \boldsymbol{H}(S_k(N\Delta, \chi)^{\Delta\text{-new}})$.

THEOREM 20. – *There is a canonical isomorphism $\Psi: \boldsymbol{H}(N\Delta, \chi)^{\Delta\text{-new}} \xrightarrow{\sim} \boldsymbol{H}^D(N, \chi)$ of $\boldsymbol{Z}[\chi]$-algebras.*

PROOF. – Observe that the isomorphism proved in Theorem 12 implies the existence of a *canonical* map of $\boldsymbol{Z}[\chi]$-algebras $\boldsymbol{H}(N\Delta, \chi) \to \boldsymbol{H}^D(N, \chi)$, which is simply $T_p \mapsto T_p^D$ and thus obviously surjective. The kernel of this map is the ideal of the elements that restrict to 0 on the space $S_k(N\Delta, \chi)^{\Delta\text{-new}}$. ■

We shall use the isomorphism of Theorem 20 to deduce some properties of the quaternionic Hecke algebra from analogous properties of the full Hecke algebra $\boldsymbol{H}(N, \chi)$.

PROPOSITION 21. – *There exists a basis of $S_k(N\Delta, \chi)^{\Delta\text{-new}}$ in $S_k(N\Delta, \chi; \boldsymbol{Z}[\chi])$.*

PROOF. – By the decomposition (3.4) in our situation, and the fact that the maps $j_d$ preserve the ring of Fourier coefficients, it is enough to prove the assertion for spaces of newforms. To conclude apply Proposition 13, observing that the assertion holds for spaces of oldforms, again by using the maps $j_d$. ■

We shall now consider the following somewhat general situation. Let $S_k = X \oplus Y$ be a decomposition of Hecke modules. Let $X_A := X \cap S_k(A)$ and $Y_A := Y \cap S_k(A)$. We shall assume that

$$X_A(\text{resp. } Y_A) \text{ contains a basis of } X(\text{resp. } Y).$$

Note that under this assumption we have that $X_A \oplus Y_A$ is a $A$-cotorsion submodule of $S_k(A)$, but not necessarily equal to $S_k(A)$. On the other hand, if $A$ is a field the equality $S_k(A) = X_A \oplus Y_A$ is trivial (by a dimension argument) and the projectors $e_X$, $e_Y$ are elements of $\boldsymbol{H}_A$. Indeed, by duality (Lemma 16), $e_X$ corresponds to the linear form that kills $Y_A$ and is $f \mapsto a_1(f)$ on $X_A$ (same for $e_Y$).

Associated to a decomposition $S_k = X \oplus Y$ and an algebra $A$, there is an injective map of Hecke algebras

$$\pi_{X,Y}\colon \boldsymbol{H}_A \to \boldsymbol{H}(X)_A \times \boldsymbol{H}(Y)_A\,,$$

given by restriction of endomorphisms, which is not an isomorphism in general (it is an isomorphism if and only if the projectors $e_X$ and $e_Y$ are defined over $A$).

The pairing (7.1) induces a pairing

$$X_A \times \boldsymbol{H}(X)_A \to A\,, \tag{7.4}$$

and thus maps $\varrho_{H(X)}$ and $\varrho_X$ as above.

PROPOSITION 22.

1) *The map $\varrho_X$ is an isomorphism.*

2) *The map $\varrho_{H(X)}$ is injective.*

PROOF. – Suppose $\varrho_{H(X)}(h) = 0$. Then for every $n \geqslant 1$ and for every $f \in X_A$, $a_n(h(f)) = a_1(h(T_n f)) = 0$, so $h(f) = 0$. If $\varrho_X(f) = 0$ then $a_n(f) = a_1(T_n(f)) = 0$, so $f = 0$. To show the surjectivity of $\varrho_X$, extend a linear form $\phi$ on $\boldsymbol{H}(X)_A$ to $\boldsymbol{H}(X)_{\boldsymbol{C}}$ by linearity and then to $\boldsymbol{H}$ by composing with the canonical quotient map $\pi_X\colon \boldsymbol{H} \to \boldsymbol{H}(X)$. Then by Proposition 16, $\phi(h) = a_1(h(f))$ for some $f \in S_k$ and all $h \in \boldsymbol{H}$. Note that $\phi(e_Y) = 0$ because $e_Y \mapsto 0$ under the canonical quotient map $\pi_X$. Thus $f \in X$. Finally, $a_n(f) = a_1(T_n(f)) = \phi(T_n) \in A$. ■

PROPOSITION 23.

1) *If $\pi_{X,Y}$ is an isomorphism, then $S_k(A) = X_A \oplus Y_A$.*

2) *If $A$ is a dualizing algebra, then $\pi_{X,Y}$ is an isomorphism if and only $S_k(A) = X_A \oplus Y_A$.*

PROOF. – If $\pi_{X,Y}$ is an isomorphism, then the projectors $e_X$, $e_Y$ are in $\boldsymbol{H}_A$, so that $X_A = e_X S_k(A)$ and $Y_A = e_Y S_k(A)$ and the first assertion follows.

Now suppose that $S_k(A) = X_A \oplus Y_A$ and $A$ dualizing. Let $\phi$ be the linear form on $S_k(A)$ which kills $Y$ and that is $f \mapsto a_1(f)$ on $X$. By surjectivity of the map $\varrho_H$, there exists $h \in \boldsymbol{H}_A$ mapping to $\phi$. This element is the projector on $X$ (i.e. $h = e_X$). The same argument yields the projector $e_Y$. Then $\pi_{X,Y}(h) = (e_X h, e_Y h)$ is an isomorphism. $\blacksquare$.

It is well-known that the fact that $X_A \oplus Y_A \neq S_k(A)$ can be rephrased in terms of congruences between $X_A$ and $Y_A$ as follows. Let $a \in A$. There exists a non-zero element in $S_k(A)/(X_A \oplus Y_A)$ killed by $a$ if and only if there are elements $f \in X_A - aS_k(A)$ and $g \in Y_A - aS_k(A)$ such that $f - g \in aS_k(A)$.

The problem of finding independent criteria for the existence of congruences between modular forms has been discussed by several authors, with a particular attention for the case where $A$ is a ring of integers, possibly localized at some prime.

When $X$ is the subspace generated by a newform $f$ together with its Galois conjugates $f^\sigma$, the support of $S_k(A)/(X_A \oplus Y_A)$ has been related by Hida [7], [8], to the special value at $s = k$ of the symmetric square $L$-function $Z(f, s) = \prod_\sigma L_2(f^\sigma, s)$ associated to $f$.

More explicit results have been obtained by Ribet, [16], for weight 2 and trivial character, and by Diamond and Taylor [3], [4]. Let $f$ be a normalized newform in $S_2(Np)$ with $p \nmid N$. Ribet gives a criterion for the existence of a newform in $S_2(N)$ congruent to $f$ modulo a prime of $\overline{\boldsymbol{Q}}$ over $p$ in terms of the local properties of the Galois representation attached to $f$. The work of Diamond and Taylor points to the opposite direction: they start with a normalized newform in $S_k(\Gamma_1(N))$ and find conditions for the existence of a newform $g$ of higher level congruent to $f$ modulo a prime of $\overline{\boldsymbol{Q}}$ over a prime $p \nmid N$. Again, their result is in terms of the Galois representation attached to $f$.

The problem of the determination of congruences between forms of non-prime to $p$ different levels is the subject of recent work of Khare [11], [12], [13].

Finally, it is shown in [17] that an exceptional behaviour of the Galois representation associated to the Shimura curve of discriminant $\Delta = pq$ is responsible for congruences between newforms and oldforms of weight 2 and level $pq$.

We shall now apply the previous results to the quaternionic modular forms and Hecke algebras by taking $X = S_k(N\Delta, \chi)^{\Delta\text{-new}}$ and for $Y$ its Petersson orthogonal subspace $S_k(N\Delta, \chi)^{\Delta\text{-old}}$. Choose one map JL as in Theorem 12 and let

$$S_k^D(A) = \mathrm{JL}^{-1}(S_k(N\Delta, \chi; A)) = \mathrm{JL}^{-1}(S_k(N\Delta, \chi; A)^{\Delta\text{-new}})$$

and $\boldsymbol{H}_A^D$ the $A$-subalgebra of $\mathrm{End}_A(S_k^D(A))$ generated by the operators $T_p^D$. Mind that the space $S_k^D(A)$ does depend on the actual choice of JL.

We can define a pairing $S_k^D(A) \times \boldsymbol{H}_A^D \to A$ by $(f, h) \mapsto a_1(\Psi^{-1}(h)(\mathrm{JL}(f)))$, and the associated maps $\varrho_S^D\colon S_k^D(A) \to (\boldsymbol{H}_A^D)^\vee$ and $\varrho_H^D\colon \boldsymbol{H}_A^D \to S_k^D(A)^\vee$ as usual.

THEOREM 24.

1) *There exists a basis of* $S_k^D$ *in* $S_k^D(A)$.

2) *The map* $\varrho_S^D$ *is an isomorphism. The map* $\varrho_H^D$ *is injective.*

3) *If* $\boldsymbol{H}_A^D$ *is a factor of* $\boldsymbol{H}_A$, *then* $S_k(N\Delta, \chi; A) = S_k(N\Delta, \chi)_A^{\Delta\text{-new}} \oplus S_k(N\Delta, \chi)_A^{\Delta\text{-old}}$.

4) *If* $A$ *is dualizing for* $(N\Delta, \chi)$ *then the converse of* 3) *holds.*

PROOF. – Points 1), 2) and 4) follow at once from Propositions 21, 22 and part 2) of 23 respectively. To prove 3), write $\boldsymbol{H}_A = \boldsymbol{H}^{\Delta\text{-new}} \times H$, where $H$ is an $\boldsymbol{H}_A$ algebra. Dualizing, $S_k(A) = S_k(A)^{\Delta\text{-new}} \oplus M$ for some Hecke module $M$. Then $M \otimes \boldsymbol{C}$ is an Hecke complement of $S_k^{\Delta\text{-new}}$. The subalgebra of $\boldsymbol{H}_A$ generated by the operators corresponding to primes not diving $N\Delta$ acts semisimply on $S_k$, and so $M \otimes \boldsymbol{C} = S_k^{\Delta\text{-old}}$. Thus $M = S_k(A)^{\Delta\text{-old}}$. ■

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