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Minimal Sections of Conic Bundles (*).

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Sunto. – *Sia $p\colon X \to \boldsymbol{P}^2$ un fibrato in coniche standard con curva discriminante Δ di grado d. La varietà delle sezioni minime delle superfici $p^{-1}(C)$, dove C è una curva di grado $d-3$, si spezza in due componenti $\mathcal{C}_+$ e $\mathcal{C}_-$. Si prova che, mediante la mappa di Abel-Jacobi Φ, una di queste componenti domina la Jacobiana intermedia JX, mentre l'altra domina il divisore theta $\Theta \subset JX$. Questi risultati vengono applicati ad alcuni threefold di Fano birazionalmente equivalenti a un fibrato in coniche. In particolare si prova che il generico threefold di Fano di grado dieci è birazionale a una ipersuperficie di tipo* $(2, 2)$ *nel prodotto di Segre di due piani proiettivi.*

0. – Introduction.

Conic bundles - definitions and general results.

(0.1) Let $p\colon X \to S$ be a surjective morphism from the smooth projective threefold X to the smooth surface S. The morphism p is called a *standard conic bundle* if:

(i) for any $s \in S$, the scheme-theoretic fiber $f_s = p^{-1}(s)$ is isomorphic over the residue field $k(s)$ to a conic in $\boldsymbol{P}^2_{k(s)}$;

(ii) for any irreducible curve $C \subset S$ the surface $S_C = p^{-1}(C)$ is irreducible.

(0.2) More generally, let $q\colon Y \to T$ be a rational map from the smooth threefold Y to the smooth surface T. Then q is called a *conic bundle* if the general fiber $f_t = q^{-1}(t)$ is a smooth rational curve over $k(t)$.

(0.3) Two conic bundles $q\colon Y \to T$ and $p\colon X \to S$ are called *birationally equivalent* if there exist birational maps $g\colon Y \to X$ and $h\colon T \to S$ such that $h \circ q = p \circ g$. By results of A. A. Zagorskii and V. G. Sarkisov (see e.g. [Z]).

(0.4) *Any conic bundle is birationally equivalent to a standard one.*

Let $p\colon X \to S$ be a standard conic bundle, let

$$\Delta = \{s \in S\colon p^{-1}(s) \ \text{is singular}\}$$

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be the *discriminant* of p, let $\varDelta \neq \emptyset$, and let $\widetilde{\varDelta}$ be the «double discriminant curve» of p, i.e. the curve parametrizing the components of the fibers $f_s = p^{-1}(s)$, $s \in \varDelta$. Let $\pi: \widetilde{\varDelta} \to \varDelta$ be the corresponding double covering. Then:

(0.5) $\widetilde{\varDelta}$ and $\varDelta$ are curves with at most double points, and $\pi: \widetilde{\varDelta} \to \varDelta$ is a Beauville covering (see [B]). In particular, if $\varDelta$ is smooth then $\widetilde{\varDelta}$ is smooth and π is unbranched.

By results of A. S. Merkur′ev and V. G. Sarkisov ([Mer], [S]):

(0.6) For any Beauville covering $\pi: \widetilde{\varDelta} \to \varDelta$, and for any embedding $\varDelta \subset S$, where S is a smooth rational surface, there exists a standard conic bundle $p: X \to S$ with a discriminant pair $(\widetilde{\varDelta}, \varDelta)$. Any two such standard conic bundles are birationally equivalent over S (see [Isk1, Lemma 1 (iv)]).

(0.7) Throughout this paper we assume that $S = \boldsymbol{P}^2$ and $\varDelta$ is *smooth.*

Let $p: X \to \boldsymbol{P}^2$ be such a standard conic bundle. Being a rational fibration over a rational surface, X is a threefold with a non-effective canonical class, i.e. $h^{3,0}(X) = h^0(X, \Omega^3_X) = 0$. Therefore the complex torus (the Griffiths intermediate jacobian) $J(X)$ of X does not contain a $(3, 0)$-part. In particular

$$J(X) = H^{2,1}(X)^* / (H_3(X, \boldsymbol{Z}) \ \ mod \ \text{torsion})$$

is a principally polarized abelian variety (p.p.a.v.) with a principal polarization (p.p.) defined by the intersection of real 3-chains on X (see [CG]). The divisor Θ of this polarization is called the theta divisor of $J(X)$. Since $p: X \to \boldsymbol{P}^2$ is standard and $\varDelta$ is smooth, the splitting $p^{-1}(s) = \boldsymbol{P}^1 \vee \boldsymbol{P}^1$, $s \in \varDelta$ defines a unbranched double covering $\pi: \widetilde{\varDelta} \to \varDelta$ of the smooth discriminant curve $\varDelta$. Therefore the pair $(\widetilde{\varDelta}, \varDelta)$ defines in a natural way the p.p.a.v. $P(\widetilde{\varDelta}, \varDelta)$—the Prym variety of $\pi: \widetilde{\varDelta} \to \varDelta$, and by the well-known result of Beauville ([B]) $(J(X), \Theta)$ and $P(\widetilde{\varDelta}, \varDelta)$ are isomorphic as p.p.a.v.

(0.8) More generally, let X be a smooth threefold with $h^{3,0} = 0$, let $(J(X), \Theta)$ be the p.p. intermediate jacobian of X, and let $A_1(X)$ be the group of rational equivalence classes of algebraic 1-cycles C on X which are homologous to 0. Then the integrating over the real 3-chains γ s.t. $\delta(\gamma) =$ (the boundary of γ) $= C$, $C \in A_1(X)$ defines the natural map $\Phi: A_1(X) \to J(X)$—the Abel-Jacobi map for X (see e.g. [CG]). In addition, if $\mathcal{C}$ is a smooth family of homologous cycles C on X, and C_0 is a fixed element of $\mathcal{C}$, then the composition of Φ and the cycle-class map $\mathcal{C} \to A_1(X)$, $C \mapsto [C - C_0]$, defines a map $\Phi_{\mathcal{C}}: \mathcal{C} \to J(X)$.

Let $\mathrm{Alb}(\mathcal{C})$ be the Albanese variety of F. By the universal property of the Albanese map $a: \mathcal{C} \to \mathrm{Alb}(\mathcal{C})$, $\Phi_{\mathcal{C}}$ can be factorized through a, and defines the map $\Phi'_{\mathcal{C}}: \mathrm{Alb}(\mathcal{C}) \to J(X)$. Both $\Phi_{\mathcal{C}}$ and $\Phi'_{\mathcal{C}}$ are called the *Abel-Jacobi maps* for the family of 1-cycles $\mathcal{C}$.

For a large class of such threefolds X (especially—for conic bundles), the transpose ${}^t\Phi$ of the Abel-Jacobi map for X defines an isomorphism between the Chow group $A_1(X)$ and $J(X)$ (see [BM]), and one may expect that for some «rich» families of curves $\mathcal{C}$ on X the Abel-Jacobi map $\Phi_{\mathcal{C}}$ will be surjective. Moreover, one can set the following problem:

(*) *Find a family $\mathcal{C}_\theta$ of algebraically equivalent 1-cycles on X such that the Abel-Jacobi map $\Phi_{\mathcal{C}_\theta}$ sends C_θ surjectively onto a copy of the theta divisor Θ.*

Assume the existence of such a family $\mathcal{C}_\theta$. One can formulate the following additional question:

(**) *Describe, in terms of $\mathcal{C}_\theta$ and X, the structure of the general fiber of $\Phi_{\mathcal{C}_\theta}$.*

Summary of the results in the paper.

In this paper we give a positive answer of the problems (*) and (**) if $p\colon X \to \boldsymbol{P}^2$ is a standard conic bundle with a smooth discriminant curve Δ of degree $d > 3$. More concretely, we prove the existence of two naturally defined families $\mathcal{C}_+$ and $\mathcal{C}_-$ of connected 1-cycles C on X, such that their Abel-Jacobi maps Φ_+ and Φ_- send one of these two families onto the intermediate jacobian $J(X)$ and the second—onto a copy of the theta divisor Θ of $J(X)$ (see Theorem (4.4)).

The general element of $\mathcal{C}_{+/-}$ is a smooth curve $C \in X$ which is mapped isomorphically onto the plane curve $p(C)$ of degree $d-3$, and C can be treated as a minimal section of a well-defined ruled surface $S(C)$. In § 2,3 we prove that, independently of the choice of X, the invariant e of the general $S(C)$ is always one of the numbers $(e_+, e_-) = (g(C), g(C)-1)$, being the invariants of the general elements of the even and the odd versal families of ruled surfaces over a curve of genus $g(C)$ (see [Se]). The general $C \in \mathcal{C}_+$ can be treated as a minimal non-isolated section of $S(C)$, and the general $C \in \mathcal{C}_-$—as a minimal isolated section of $S(C)$. This interpretation makes it possible to describe the geometric structure of the general fibers of the Abel-Jacobi maps of $\mathcal{C}_+$ and $\mathcal{C}_-$ on the base of the Lange and Narasimhan's description [LN] of maximal subbundles of rank two vector bundles on curves (see Theorem (5.3)).

In the examples (6.1), (6.2) and (6.3) we find the families $\mathcal{C}_+$ and $\mathcal{C}_-$ for the natural conic bundle structures on the bidegree (2,2) threefold $T \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$, on the nodal quartic double solid (q.d.s.) B, and also—on the less-known nodal Fano 3-fold X_{10} of genus 6. It turns out that for T and for X_{10} the family which parametrizes Θ is $\mathcal{C}_+$, while this family for B is $\mathcal{C}_-$, which answers the question (*) in each of these three cases—see (6.1.3), (6.2.4) and (6.3.7)-(6.3.8). By Theorem (4.4) we know that the «residue» family $\mathcal{C}_-$ for T and X_{10}, and $\mathcal{C}_+$ for B, parametrizes the intermediate jacobian of the variety. Now, the answer of (**) for T, for the nodal B and for the nodal X_{10} follows automatically from Theorem (5.3) — see (6.1.4), (6.2.5) and (6.3.9). For the nodal q.d.s., the same «theta»-family has been found by Clemens in [C] via degeneration from the Tikhomirov's family of Reye sextics which parametrizes Θ for the general q.d.s. (see [T]).

In (6.1.5), (6.2.6) and (6.3.10) we describe natural families of degenerate sections which parametrize the components of stable singularities of Θ for T (see also [Ve] and [I1]), for the nodal B (see [Vo], [C], [De]), and for the nodal X_{10}. In addition, we show that the general nodal X_{10} is birational to a bidegree (2,2) threefold T.

1. – Minimal sections of ruled surfaces.

Here we collect some known facts about ruled surfaces and rank 2 vector bundles over curves (see [H], [LN], [Se]).

(1.1) *Minimal sections of ruled surfaces and maximal subbundles of* rank 2 *vector bundles on curves* (see [H], [LN], [Se]).

Any ruled surface S over a smooth curve C can be represented as a projectivization $\boldsymbol{P}_C(E)$ of a rank 2 vector bundle E over C. Clearly, $\boldsymbol{P}_C(E)$ is a ruled surface for any such E, and $\boldsymbol{P}_C(E) \cong \boldsymbol{P}_C(E')$ *iff* $E = E' \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$; here we identify vector bundles and the associated free sheaves.

Call the bundle E *normalized* if $h^0(E) \geqslant 1$, but $h^0(E \otimes \mathcal{L}) = 0$ for any invertible $\mathcal{L}$ such that $\deg(\mathcal{L}) < 0$ (see [H, Ch. 5, § 2]).

The question is:

(*) *How many normalized* rank 2 *bundles represent the same ruled surface*?

The answer depends on the choice of the curve C (especially—on the genus $g = g(C)$ of C), and on the choice of the ruled surface S over C. Let $p \colon S \to C$ be the natural fiber structure on S. We shall reformulate the question (*) in the terms of sections of p.

(1.2) DEFINITION. – Call the section $C \subset S$ *minimal* if C is a section on S for which the number $(C\,.\,C)_S$ is minimal. Let C be a minimal section of S. The number $e = e(S) = (C\,.\,C)_S$ is an integer invariant of the ruled surface S. The number $e(S)$ coincides with $\deg(E) := \deg(\det(E))$, where E is any normalized rank 2 bundle which represents S (i.e.—such that $S \cong \boldsymbol{P}_C(E)$) (see e.g. [H, Ch. 5, § 2]). We call the number $e = e(S)$ the *invariant* of S.

(1.3). – Remark. – Here, in contrast with the definition in use, we let

$$e(S) := -(\text{the invariant of } S).$$

The new question is:

(**) *How many minimal sections lie on the same ruled surface*?

The two questions are equivalent in the following sense: Let E be normalized and such that $\boldsymbol{P}(E) = S$. By assumption $h^0(E) \geqslant 1$. Therefore E has at least one section $s \in H^0(E)$. The bundle section s defines (and is defined by) an embedding $0 \to \mathcal{O}_C \to E$. The sheaf $\mathcal{L}$, defined by the cokernel of this injection, is invertible, and $\mathcal{L}$ defines in a unique way a minimal section $C = C(s)$ of the ruled surface $S = \boldsymbol{P}_C(E)$ (see e.g. [H, Ch. 5, § 2: (2.6), (2.8)]). If $h^0(E) = 1$, the bundle section $s \in H^0(E)$ is unique, and the corresponding minimal section $C(s)$ is unique. In contrary, if $h^0(E) \geqslant 2$, the map

$$\boldsymbol{P}(H^0(E)) \to \{\text{the minimal sections of } S\}, \qquad s \mapsto C(s),$$

defines a linear system of minimal sections of S (e.g., if S is a quadric). Therefore, the set of minimal sections of S is the same as the projectivized set of the bundle sections of normalized bundles which represent S. In fact, if $g(C) \geqslant 1$ and S is general, then $h^0(E) = 1$ for any normalized E which represents S. In this case the questions (*) and (**) are equivalent.

(1.4) DEFINITION. – Call the line subbundle $\mathcal{M} \subset E$ a *maximal subbundle* of E, if $\mathcal{M}$ is a line subbundle of E of a maximal degree.

Let E be a fixed normalized bundle which represents S, and let $\mathcal{M} \subset E$ be a maximal subbundle of E. Clearly $\deg(\mathcal{M}) \geqslant 0$, since $\mathcal{O}_C \subset E$. Assume that $\deg(\mathcal{M}) > 0$. Then, after tensoring by $\mathcal{M}^{-1}$, we obtain the embedding $\mathcal{O}_C \subset E \otimes \mathcal{M}^{-1}$.

In particular, $h^0(E \otimes \mathcal{M}^{-1}) \geqslant 0$, $E \otimes \mathcal{M}^{-1}$ represents S, and $\deg(E \otimes \mathcal{M}^{-1}) < \deg(E)$. However E is normalized, hence $\deg(E \otimes \mathcal{M}^{-1})$ cannot be less than $\deg(E)$—contradiction. Therefore $\deg(\mathcal{M}) = 0$, and the maximal subbundle $\mathcal{M}$ of E defines the normalized bundle $E \otimes \mathcal{M}^{-1}$ which also represents S.

Therefore, we can reduce the question (*) to the following question:

(***) *How many maximal subbundles has a fixed normalized* rank 2 *bundle E which represents a given ruled surface S?*

REMARK. – The answer of (*)-(***) for S-decomposable, is given in [H, Ch. 5, Examples 2.11.1, 2.11.2, 2.11.3]. In particular, this implies the well known description of the set of minimal sections of a rational ruled surface $p\colon S \to \boldsymbol{P}^1$. For S is indecomposable—see (1.7)-(1.8).

(1.5) LEMMA (see [Se, Theorem 5]). – *Let $S \to C$ and $S' \to C'$ be two ruled surfaces. Then S and S' can be deformed into each other iff C and C' have the same genus, and the invariants $e(S)$ and $e(S')$ have the same parity.*

(1.6) LEMMA (see [Se, Theorem 13]). – *The general surface in the versal deformation of a rational ruled surface is a quadric if e is even, and the surface $\boldsymbol{F}_1$ if e is odd.*

The general surface of a versal deformation of a ruled surface over elliptic base is a surface represented by the unique indecomposable rank 2 *vector bundle of degree* 1 *if* e *is odd, and a decomposable ruled surface represented by a sum of two (non-incident) line bundles of degree* 0 *if* e *is even.*

The general surface of a versal deformation of a ruled surface over a curve of genus $g \geqslant 2$ *is indecomposable. The invariant of such* S *is* $g-1$ *if* $e \equiv g \bmod 2$, *or* g *if* $e \equiv g-1 \bmod 2$.

(1.7) LEMMA (see [H, Ch. 5, Example 2.11.2 and Exer. 2.7]). – *Let* C *be an elliptic curve, and let* S *be the unique indecomposable ruled surface over* C *with invariant* $e(S) = 1$. *Then the set* $\mathcal{C}_+(S)$ *of minimal sections of* S *form a* 1*-dimensional family parametrized by the points of the base* C. *In particular, all the minimal sections of* S *are linearly non equivalent.*

Let C *be an elliptic curve, and let the ruled surface* S *be represented by the normalized bundle* $E = \mathcal{O}_C \oplus \mathcal{L}$, *where* $\deg(\mathcal{L}) = 0$ *and* $\mathcal{L} \neq \mathcal{O}_C$. *Then* S *has exactly two minimal sections: the section* $C = C(s_E)$ *defined by the unique bundle section* s_E *of* E, *and the section* $\overline{C}$ *defined by the unique section* $s_{\overline{E}}$ *of the second normalized bundle* $\overline{E} = \mathcal{O}_C \oplus \mathcal{L}^{-1}$ *which represents* S.

DEFINITION (see [LN, § 1]). – *The line bundle* $\mathcal{D}$ *on* C *of degree* e *is called* an e-secant line bundle *of* $\alpha(C) \in \boldsymbol{P}^n$ *which passes through the point* $[E] \in \boldsymbol{P}^n$, *if the linear system* $|\mathcal{D}|$ *contains an effective divisor* D *such that the space* $\mathrm{Span}(\alpha(D))$ *passes through the point* [E].

DEFINITION. – *Call the section* C_0 *of the ruled surface* S isolated *if* S *contains only a finite number of sections* C *such that* $C^2 = C_0^2$. *Otherwise, call* C_0 non-isolated (*or continual*) *section of* S.

(1.8) LEMMA (see [LN, Proposition 2.4]). – *Let* S *be an indecomposable ruled surface over a curve* C *of genus* $g \geqslant 2$. *Let* E *be a fixed normalized* rank 2 *bundle over* C *which represents* S, *and let* $[E] \in \boldsymbol{P}(H^0(K_C \otimes \mathcal{L}))$ *be the point which corresponds to the extension* $0 \to \mathcal{O}_C \to E \to \mathcal{L} \to 0$ *defined by* E. *Let* $\alpha\colon C \to \boldsymbol{P}(H^0(K_C \otimes \mathcal{L}))$ *be the map defined by the linear system* $|K_C \otimes \mathcal{L}|$, *and let* $\alpha(C)$ *be the image of* C. *Then the set of maximal line subbundles* $\mathcal{M}$ *of* E, *which are different from* $\mathcal{O}_C$, *is naturally isomorphic to the set* $\mathrm{Sec}_e(\alpha(C), [E])$ *of* e*-secant line bundles of* $\alpha(C)$ *which pass through the point* $[E]$.

In particular, if $S = \boldsymbol{P}(E) \to C$ *is «versal»* (*see* (1.6)) *then* α *is an embedding, and*:

$(+)$ *either* $e(S) = g$, *and the family* $\mathrm{Sec}_g(\alpha(C), [E])$ *is* 1*-dimensional; in particular, the minimal sections of* S *are non-isolated.*

$(-)$ *or* $e(S) = g-1$, *and* $\mathrm{Sec}_{g-1}(\alpha(C), [E])$ *is finite; in particular, the minimal sections of* S *are isolated.*

2. – The conic bundle surfaces S_C.

(2.1) Let $p: X \to \boldsymbol{P}^2$ be a standard conic bundle with a smooth discriminant Δ. Without any substantial restriction we may assume that $\deg \Delta > 3$.

Let $C \subset \boldsymbol{P}^2$ be a general plane curve of degree $k < d$. Then $S_C := p^{-1}(C)$ is a smooth surface, and $p: S_C \to C$ defines a conic bundle structure on S_C.

Let x_i, $i = 1, 2, \ldots, kd$ be the intersection points of C and Δ. Then $q_i = p^{-1}(x_i)$ are the degenerate fibers of $p: p^{-1}(C) \to C$. Let l_i and $\bar{l}_i$ be the components of q_i, $i = 1, \ldots, kd$; in particular l_i and $\bar{l}_i$ are (-1)-curves on S_C. Let $I = \{i_1, \ldots, i_n\}$, $i_1 < \ldots < i_n$ be any ordered (possibly empty) subset of $\{1, 2, \ldots, kd\}$. Any such a multiindex I defines a morphism $\sigma_I: S_C \to S_C(I)$, where σ_I is the composition of all the blow-downs of l_i, $i \in I$ and $\bar{l}_j$, $j \in \bar{I} = \{1, \ldots, kd\} - I$. The map $p: S_C \to C$ induces a $\boldsymbol{P}^1$-bundle structure $p_I: S_C(I) \to C$.

(2.2) Let $\sigma_I: S_C \to S_C(I)$, etc., be as above, and let $s_1, \ldots, s_{kd} \in S_C(I)$ be the images of the exceptional curves $l_i \in I$ and $\bar{l}_j \in \bar{I}$. Call the section $C' \subset S_C(I)$ *non-singular* if the sets C' and $\{s_1, \ldots, s_{kd}\}$ are disjoint.

If C' is non-singular, then σ^{-1} maps C' isomorphically onto the proper preimage of C' on S_C. With a possible abuse of the notation, we denote this proper preimage also by C'.

(2.3) Definition. – A *nonsingular section* of the conic bundle surface S_C is defined to be any proper preimage C' of a nonsingular section on some of the ruled surfaces $S_C(I)$ defined by S_C.

(2.4) Remark. – Although any ruled surface has minimal sections, it might be possible that some of $S_C(I)$ has no nonsingular minimal sections.

Let $\boldsymbol{F}_3 = p_0: \boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(-3)) \to \boldsymbol{P}^1$, let C_0 be the minimal section of F, and let the conic bundle surface S be defined by the composition $p = p_0 \circ \sigma: S \to \boldsymbol{P}^1$ where $\sigma: S \to \boldsymbol{F}_3$ is a blow-up of a point $s \in \boldsymbol{F}_3 - C_0$. If $q = l + \bar{l}$ is the singular fiber over s, and if l is the exceptional divisor of σ, then the blow-down of $\bar{l}$ defines a morphism $\bar{\sigma}: S \to \boldsymbol{F}_2$. In this case the unique minimal section C' of $\boldsymbol{F}_2$ is *singular*: the preimage $\sigma^{-1}(C') = C_0' + \bar{l}$, where C_0' is the isomorphic proper preimage of C_0 on S. However:

(2.5) Lemma. – *Any non-singular conic bundle surface $S \to C$ which has degenerate fibers has a non-singular isolated minimal section.*

Proof. – See Remark (2.4) which can be generalized straightforwardly to the case of a conic bundle surface over an arbitrary smooth curve with a non-empty set of degenerate fibers. In fact, if $S(I) \to C$ is one of the ruled minimal models of S over C, for which $e(S(I)) = e_-(S)$ is minimal, then any mininmal section of $S(C(I))$ is non-singular (see e.g. (2.4) where $e_-(S) = -3$).

(2.6) Corollary. – *Let C be a general plane curve of degree $k < d = \deg \Delta$, let $e_- = e(S_C)$ be the minimal invariant of the ruled surfaces $S_C(I)$, and let I be the multiindex for which $e(S_C(I)) = e$. Then $S_C(I)$ has only a finite number of minimal sections, i.e. all the minimal sections of $S_C(I)$ are isolated.*

(2.7) Corollary. – *Let C be a general plane curve of degree $k < d = \deg \Delta$, and let $e_- := \min\{e(S_C(I)): I \subset \{1, 2, \ldots, kd\}\}$. Then $e_- = g-1$, where $g = (k-1)\cdot(k-2)/2$ is the genus of C.*

Proof. – Clearly, the integer e_- is an invariant of the threefold X. This makes it possible to define the family of all these minimal sections on X as follows:

Call a quasi-section of $p: X \to \boldsymbol{P}^2$ any connected 1-cycle C'' on X such that $C'' = C' + F$, where C' is a section of X (i.e. $p: C' \to p(C')$ is an isomorphism), and F is a sum of fibers and components of fibers of p.

Let $U[k] \subset |\mathcal{O}_{\boldsymbol{P}^2}(k)|$ be the set:

$$U[k] = \{C: S_C = p^{-1}(C) \text{ is smooth and } e_-(S_C) = e_-\},$$

and let

$$\mathcal{C}_-[k] = (\text{the closure of}) \ \{C': \mathrm{C} = \mathrm{p}(\mathrm{C}') \in \mathrm{U[k]} \ \& \ \mathrm{C}' \text{ is a nonsingular section of } \mathrm{S_C} \text{ s.t. } C'^2|_{S_C} = e_-\},$$

where the closure is defined in the family of all the quasi-sections of X. On the one hand $\dim \mathcal{C}_-[k] \geqslant \dim U[k] = (k+1)(k+2)/2 - 1 = (k^2+3k)/2$. On the other hand, by (2.6), the general element C' is an isolated section of S_C, where $C = p(C')$, and $S_C = p^{-1}(C)$. In particular $e_- \leqslant g-1$, where $g = (k-1)(k-2)/2$ is the genus of C'. We shall prove this.

Suppose that $e_- \geqslant g$; then $e_- = g$ (see [H, Ch. 5, Exercise (2.5.d)]). Let $S(C') := S_C(I) \to C$ be the ruled surface for which C' is a nonsingular minimal section. Since the invariant $e(S_C(I)) = e_- = g$, the surface $S_C(I)$ must have at least a 1-dimensional family of minimal sections. In order to see this, we use:

(1) for $g = 0$ (i.e. $k = 1, 2$)—the known property that any of the ruling of the smooth quadric is a $\boldsymbol{P}^1$-family of minimal sections;

(2) for $g = 1$ (i.e. $k = 3$)—Lemma (1.7);

(3) for $g \geqslant 2$ (i.e. $k \geqslant 4$)–Lemma (1.8).

Let e.g. $k \geqslant 4$. Then according to (1.8), the ruled surface $S(C') = S_C(I)$ must have at least a 1-dimensional family of minimal sections. Indeed, the «versal» ruled surface of invariant g has a 1-dimensional family of minimal sections (since the family of g-secant planes through $[E]$ for the «versal» surface is exactly 1-dimensional (see (1.8) and [LN]). That is, in all the cases C' can't be isolated. Therefore $e_- \leqslant g-1$.

In order to see that $e_- \geqslant g-1$, we consider the normal bundle sequence for $C' \subset S_C \subset X$:

$$0 \to N_{C'/S_C} \to N_{C'/X} \to N_{S_C/X} \,|\, C' \to 0 \,.$$

On the one hand, the map $p\colon C' \mapsto C = p(C')$ sends family $\mathcal{C}_-[k]$ surjectively onto the open subset $U[k] \subset |\mathcal{O}_{\boldsymbol{P}^2}(k)|$; therefore $\dim \mathcal{C}_-[k] \geqslant \dim |\mathcal{O}_{\boldsymbol{P}^2}(k)| = (k+1)(k+2)/2 - 1 = (k^2+3k)/2$. On the other hand,

$\dim \mathcal{C}_-[k] = \chi(N_{C'/X}) = \chi(N_{C'/S_C}) + \chi(N_{S_C/X}\,|_{C'}) =$

$$(e_- - g + 1) + (k^2 - g + 1) = (e_- + k^2) + 2 - 2g = (e_- + k^2) - (k^2 - 3k) = e_- + 3k \,.$$

Therefore $e_- \geqslant (k^2+3k)/2 - 3k = (k^2-3k)/2 = (k-1)(k-2)/2 - 1 = g-1$.

3. – The families $\mathcal{C}_-[k]$ and $\mathcal{C}_+[k]$.

(3.1) The family $\mathcal{C}_-[k]$ was defined in the proof of (2.7). We call $\mathcal{C}_-[k]$ *the family of isolated minimal sections of* X (over the plane curves of degree k). According to the proof of (2.7), the invariant e_- of this family must be $g-1 = (k^2-3k)/2$, where $g = g(k)$ is the genus of the general plane curve of degree k.

Let $\{1, 2, \ldots, kd\}$ be as in (2.1), and let $I \subset \{1, 2, \ldots, kd\}$ be such that $e(S_C(I)) = e_- = g-1$. Without any restriction we may assume that $I = \emptyset$ (i.e. that the map $\sigma_I = \sigma_\emptyset\colon S_C \to S_C(I) = S_C(\emptyset)$ blows down the (-1)-curves $\bar{l}_1, \ldots, \bar{l}_{kd}$).

Let $J \subset \{1, 2, \ldots, kd\}$ be a multiindex which differs from I by only one entry; in our case $J = \{i\}$ for some $i \in \{1, 2, \ldots, kd\}$. Let $z_i \in S_C(I) = S_C(\emptyset)$ be the image of $\bar{l}_i$ on $S_C(\emptyset)$—see (2.2). Then the surface $S_C(J) = S_C(\{i\})$ is obtained from $S_C(\emptyset)$ by an elementary transformation centered at z_i. Since all the minimal sections of $S_C(\emptyset)$ are nonsingular, the point z_i does not lie on any of these sections. Therefore the ruled surface $S_C(J) = S_C(\{i\})$ has invariant $e_- + 1 = g$ (see e.g. [LN, Lemma 4.3]). In particular, the surface $S_C(J) = S_C(\{i\})$ has at least a 1-dimensional family of minimal sections (see (1.8)). Now, the same arguments as in the proof of (2.7), and simple combinatorial considerations imply the following:

(3.2) PROPOSITION. – *Let C be a general plane curve of degree k, let $S_C = p^{-1}(C)$, and let Σ be the set of all the multiindices $I \subset \{1, 2, \ldots, kd\}$. Then $\Sigma = \Sigma_- \cup \Sigma_+$, s.t.:*

(1) *For any $I \in \Sigma_-$, the ruled surface $S_C(I)$ has invariant $e_- = g-1 = (k^2-3k)/2$.*

(2) *For any $I \in \Sigma_+$, the ruled surface $S_C(I)$ has invariant $e_+ = g = (k-1)\cdot(k-2)/2$.*

(3) *Let* $|I|$ *be the cardinality of* I. *Then* I_1, I_2 *belong to the same component of* Σ *iff* $|I_1| \equiv |I_2|$ (mod 2).

(3.3) *The surfaces* $S(C')$ *and the map* $\psi\colon C' \mapsto L(C')$.

Let $S_\Delta = p^{-1}(\Delta)$ be the preimage of the discriminant curve Δ. The surface S_Δ is ruled by the components l_x and $\bar{l}_x$ of the degenerate fibers of $p\colon X \to \boldsymbol{P}^2$ and these components parametrize the points of the double discriminant curve $\widetilde{\Delta}$. The *Steiner map*

$$St\colon \Delta \to St(\Delta), \qquad x \mapsto St(x) = l_x \cap \bar{l}_x$$

embeds Δ as a double curve of $S_\Delta \subset X$.

Let $C' \subset X$ be a connected curve such that $p\colon C' \to C = p(C')$ is an isomorphism. By definition (2.3) C' is a nonsingular section if C' does not intersect the Steiner curve $St(\Delta)$. Indeed if C' does not intersect $St(\Delta)$ then $C' \cap S_\Delta$ defines the kd lines $l_1, \ldots, l_{kd}$ ($k = \deg C$). If $\bar{l}_i = p^{-1}(p(l_i)) - l_i$ are their complimentary lines, then C' can be regarded as a section of the ruled surface $S(C') := S_C(\emptyset)$ (defined by contracting all the lines $\bar{l}_i$—see § 2). Moreover, the lines l_i, as well their complimentary $\bar{l}_i$ can be regarded as points of $\widetilde{\Delta}$. In particular, if C' is a nonsingular section, and if $\deg p(C) = k$, then $L = L(C') = l_1 + \ldots + l_{kd}$ is a well-defined effective divisor on $\widetilde{\Delta}$.

This way, any nonsingular section C' of X defines:

(1) the effective divisor $L = L(C') = \psi(C') \cong C' \cap S_\Delta$;

(2) the ruled surface $S(C')$ (see above).

Now, (3.2) implies the following:

(3.4) PROPOSITION. – *Let* $p\colon X \to \boldsymbol{P}^2$ *be a smooth standard conic bundle such that the discriminant curve* $\Delta \subset \boldsymbol{P}^2$ *is smooth, and let* $d = \deg \Delta$. *Then, for any* $k < d$, *there exist two families of connected* 1*-cycles on* X: $\mathcal{C}_-[k]$ *and* $\mathcal{C}_+[k]$ *such that*:

(1) *The general element* $C' \in \mathcal{C}_-[k]$ *is a nonsingular isolated section of the conic bundle surface* $S_C = p^{-1}(C)$, $C = p(C')$, *and if* $S(C')$ *is the ruled surface defined in* (3.3) *then* $e(S(C')) = g - 1$, *where* $g = g(C') = g(C) = (k-1)(k-2)/2$.

(2) *The general element* $C' \in \mathcal{C}_+[k]$ *is a nonsingular non-isolated section of the conic bundle surface* $S_C = p^{-1}(C)$, $C = p(C')$, *and* $e(S(C')) = g$.

(3.5) REMARK. – It was proved in (2.7) that $\dim \mathcal{C}_-[k] = \dim |\mathcal{O}_{\boldsymbol{P}^2}(k)|$. Since the image of map $C' \mapsto C = p(C')$ covers the open subset $U[k]$ of $|\mathcal{O}_{\boldsymbol{P}^2}(k)|$, the map p sends $\mathcal{C}_-[k]$ surjectively onto $|\mathcal{O}_{\boldsymbol{P}^2}(k)|$. Similar arguments, based on the normal bundle sequence for $C' \subset S_{p(C')} \subset X$, imply that $\dim \mathcal{C}_+[k] =$

$\dim|\mathcal{O}_{\boldsymbol{P}^2}(k)| + 1$, and the general fiber of the surjective map $p\colon \mathcal{C}_+ \to |\mathcal{O}_{\boldsymbol{P}^2}(k)|$ is 1-dimensional.

4. – The intermediate jacobian $(J(X), \Theta) = P(\tilde{\Delta}, \Delta)$ and the families $\mathcal{C}_+$ and $\mathcal{C}_-$.

(4.0) *The jacobian* $(J(X), \Theta) = P(\tilde{\Delta}, \Delta)$ *and the sets* $\mathrm{Supp}(\Theta)$ *and* $\mathrm{Supp}(P^-)$.

Let $(\tilde{\Delta}, \Delta)$ be the discriminant pair of $p\colon X \to \boldsymbol{P}^2$, and let $\pi\colon \tilde{\Delta} \to \Delta$ be the induced double covering. Since Δ is smooth, $\tilde{\Delta}$ is smooth and π is unbranched—see (0.5).

It is well-known that the principally polarized intermediate jacobian $(J(X), \Theta)$ can be identified with the Prym variety $P(\tilde{\Delta}, \Delta)$ defined by the double covering $\pi\colon \tilde{\Delta} \to \Delta$ (see e.g. [B]). Here we recall the Wirtinger description of $P(\tilde{\Delta}, \Delta)$ by sheaves on $\tilde{\Delta}$ (see e.g. [W]).

Let $d = \deg(\Delta)$, and let $g = (d-1)(d-2)/2 = g(\Delta)$ be the genus of Δ. The map π induces the *Norm map* $Nm\colon \mathbf{Pic}(\tilde{\Delta}) \to \mathbf{Pic}(\Delta)$ (see [ACGH, p. 281]).

Let ω_Δ be the canonical sheaf of Δ. Then the fiber $Nm^{-1}(\omega_\Delta)$ splits into two components:

$$P^+ = \{\mathcal{L} \in \mathbf{Pic}^{2g-2}(\tilde{\Delta})\colon Nm(\mathcal{L}) = \omega_\Delta \ \ \&\ \ h^0(\mathcal{L}) \ \text{even}\}, \quad \text{and}$$

$$P^- = \{\mathcal{L} \in \mathbf{Pic}^{2g-2}(\tilde{\Delta})\colon Nm(\mathcal{L}) = \omega_\Delta \ \ \&\ \ h^0(\mathcal{L}) \ \text{odd}\}.$$

Both P^+ and P^- are translates of the Prym variety $P = P(\tilde{\Delta}, \Delta) \subset J(\tilde{\Delta}) = \mathbf{Pic}^0(\tilde{\Delta})$; P is the connected component of $\mathcal{O}$ in the kernel of $Nm^0\colon \mathbf{Pic}^0(\tilde{\Delta}) \to \mathbf{Pic}^0(\Delta)$.

The general sheaf $\mathcal{L} \in P^+$ is non effective, i.e. the linear system $|\mathcal{L}|$ is empty. The set $\Theta = \{\mathcal{L} \in P^+\colon |\mathcal{L}| \neq \emptyset\} = \{\mathcal{L} \in P^+\colon h^0(\mathcal{L}) \geqslant 2\}$ is a copy of the theta divisor of the p.p.a.v. $P_+ \cong P$. Since the general sheaf $\mathcal{L} \in P^-$ is effective, this suggests to introduce the following two subsets of $S^{2g-2}\tilde{\Delta}$:

$$\mathrm{Supp}(\Theta) = \{L \in |\mathcal{L}|\colon \mathcal{L} \in \Theta\}, \qquad \mathrm{Supp}(P^-) = \{L \in |\mathcal{L}|\colon \mathcal{L} \in P^-\}.$$

Clearly, $\dim \mathrm{Supp}(\Theta) = \dim \mathrm{Supp}(P^-) = \dim(P) = g-1$. Indeed, the general fiber $\phi_{\mathcal{L}}^{-1}(\mathcal{L})$ of the natural map $\phi_{\mathcal{L}}\colon \mathrm{Supp}(\Theta) \to \Theta$ coincides with the linear system $|\mathcal{L}| \cong \boldsymbol{P}^1$, and the general fiber of $\phi_{\mathcal{L}}\colon \mathrm{Supp}(P^-) \to P^-$ is $|\mathcal{L}| \cong \boldsymbol{P}^0$.

We shall use the same notations for the effective sheaf $\mathcal{L}$ and the set of effective divisors $\{L\colon L \in |\mathcal{L}|\}$.

Let $S^{2g-2}\pi\colon S^{2g-2}\tilde{\Delta} \to S^{2g-2}\Delta$ be the $(2g-2)^{-th}$ symmetric power of π, and let $|\omega_\Delta| \cong |\mathcal{O}_\Delta(d-3)| \cong |\mathcal{O}_{\boldsymbol{P}^2}(d-3)| \cong \boldsymbol{P}^{g-1}$ be the canonical system of Δ. We shall use equivalently any of the different interpretations of the elements of this system, as it is written just above.

(4.1) *The canonical families* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ *of non-isolated and isolated minimal sections of* $p: X \to \boldsymbol{P}^2$.

We define:

$$\mathcal{C}_- := \mathcal{C}_-[d-3], \qquad \mathcal{C}_+ := \mathcal{C}_+[d-3].$$

Let $S_\Delta = p^{-1}(\Delta)$. Identify, as usual, the component of a degenerate fiber $l \subset S_\Delta$ and the corresponding point $l \in \widetilde{\Delta}$. Let

$$\psi: \mathcal{C}_+ \cup \mathcal{C}_- \to S^{2g-2}\widetilde{\Delta}, \qquad \psi(C) \mapsto L(C) = C \cap S_\Delta,$$

be the map defined in (3.3). More precisely, by (3.3), ψ is defined on the open subsets $U_{+/-} \subset \mathcal{C}_{+/-}$ of non-singular minimal sections. By (3.4), we can assume in addition that the *open* subset U_+ (resp. U_-) is such that if $C \in U_+$ (resp. if $C \in U_-$) then the surface $S(C)$ is of invariant $e_+ = g(C)$ (resp.—of invariant $e_- = g(C) - 1$). Now, ψ can be defined correctly on $\mathcal{C}_+ - U_+$ and on $\mathcal{C}_- - U_-$, since: (1) The families $\mathcal{C}_{+/-}$ are the closures of $U_{+/-}$ by *algebraically equivalent* connected 1-cycles on X. (2) The map ψ is defined on $U_{+/-}$ by intersection of cycles on X, and since the algebraic equivalence implies numerical equivalence.

Denote by $C_+ = \psi(\mathcal{C}_+)$, and $C_- = \psi(\mathcal{C}_-)$ the ψ-images of $\mathcal{C}_+$ and $\mathcal{C}_-$.

(4.2) LEMMA. – *The non-ordered pairs* $\{C_+, C_-\}$ *and* $\{\mathrm{Supp}(\Theta), \mathrm{Supp}(P^-)\}$ *of subsets of* $S^{2g-2}\widetilde{\Delta}$ *coincide.*

PROOF. – It rests to note that $C_+ \cup C_- = \mathrm{Supp}(\Theta) \cup \mathrm{Supp}(P^-) = \{L \in S^{2g-2}\widetilde{\Delta}: \pi(L) \in |\omega_\Delta|\}$ q.e.d.

(4.3) *The Abel-Jacobi images of the families* $\mathcal{C}_+$ *and* $\mathcal{C}_-$.

Let $J(X) = H^{2,1}(X)^*/(H_3(X, \boldsymbol{Z})$ mod torsion) be the intermediate jacobian of X, provided with the principal polarization Θ_X defined by the intersection of 3-chains on X. It is well known (see [B]) that $(J(X), \Theta_X)$ is isomorphic, as a p.p.a.v., to the Prym variety (P, Θ) of the discriminant pair $(\widetilde{\Delta}, \Delta)$. Let

$$\Phi_+: \mathcal{C}_+ \to J(X) \cong P \quad \text{and} \quad \Phi_-: \mathcal{C}_- \to J(X) \cong P$$

be the Abel-Jacobi maps for the families $\mathcal{C}_+$ and $\mathcal{C}_-$ of algebraically equivalent 1-cycles on X. Let $Z_+ = \Phi_+(\mathcal{C}_+)$ and $Z_- = \Phi_-(\mathcal{C}_-)$ be the images of Φ_+ and Φ_-. We shall prove the following

(4.4) THEOREM. – *One of the following two alternatives always takes place*:

(1) $h^0(\psi(C)) = 2$ *for the general* $C \in \mathcal{C}_+ \Leftrightarrow h^0(\psi(C)) = 1$ *for the general* $C \in \mathcal{C}_-$, *and then:*

(i) Z_+ *is a copy of the theta divisor* Θ_X;

(ii) Z_- *coincides with* $J(X)$.

(2) $h^0(\psi(C)) = 1$ *for the general* $C \in \mathcal{C}_+ \Leftrightarrow h^0(\psi(C)) = 2$ *for the general* $C \in \mathcal{C}_-$, *and then*:

(i) Z_+ *coincides with* $J(X)$;

(ii) Z_- *is a copy of the theta divisor* Θ_X.

REMARK. – The map $\phi = \phi_{\mathcal{L}}$: $\mathrm{Supp}(\Theta) \cup \mathrm{Supp}(P^-) \to \Theta \cup P^-$ introduced above, can be regarded as the (Prym)-Abel-Jacobi map from the sets of algebraically equivalent $(2g-2)$-tuples of points $\mathrm{Supp}(\Theta) \subset S^{2g-2}\tilde{\Delta}$ and $\mathrm{Supp}(P^-) \subset S^{2g-2}\tilde{\Delta}$, to the Prym variety $P \cong J(X)$.

PROOF OF (4.4). – According to Lemma (4.2), $C_+ = \psi(\mathcal{C}_+)$ coincides either with $\mathrm{Supp}(\Theta)$, or with $\mathrm{Supp}(P^-)$. Alternatively, $C_- = \psi(\mathcal{C}_-)$ coincides either with $\mathrm{Supp}(P^-)$, or with $\mathrm{Supp}(\Theta)$.

Let e.g. $C_+ = \mathrm{Supp}(\Theta)$ (= case (1)). Then $h^0(\psi(C)) = 2$ for the general $C \in \mathcal{C}_+$, $h^0(\psi(C)) = 1$ for the general $C \in \mathcal{C}_-$; and we have to see that $Z_+ \cong \Theta$, and $Z_- = J(X) \cong P$.

Let $C \in \mathcal{C}_+$ be general, and let $z = \Phi_+(C) \in J(X)$ be the Abel-Jacobi image of C. Since C is general, C is a nonsingular section of the conic bundle surface $S_{p(C)} \subset X$, and the effective divisor $L = L(C) = \psi(C) \in \mathrm{Supp}(\Theta)$ is well defined.

We can also assume that $p(C)$ is nonsingular, and $p(C)$ intersects Δ transversally. In particular, the effective divisor $L = L(C)$ does not contain multiple points. We shall prove the following

(*) LEMMA. – *Let* C' *and* $C'' \in \mathcal{C}_+$ *be such that* $\psi(C') = \psi(C'') = L$, *and let* $z' = \Phi_+(C')$, $z'' = \Phi_+(C'')$. *Then* $z' = z''$.

PROOF OF (*). – Since $\psi(C') = \psi(C'')$, the curves C' and C'' have the same image by p: $C_0 = p(C') = p(C'')$, and C' and C'' are non-isolated sections of the conic bundle surface $S_{C_0} = p^{-1}(C_0)$. Let $L = l_1 + \ldots + l_{2g-2}$, and $x_i = p(l_i)$, $i = 1, \ldots, 2g-2$. The degenerate fibers of p: $S_{C_0} \to C_0$ are the singular conics $q(x_i) = p^{-1}(x_i) = l_i + \overline{l}_i$. By assumption C' and C'' intersect simply any of the components l_i, and does not intersect any of $\overline{l}_i$.

Let C be any nonsingular section of S_{C_0} such that $\psi(C) = C \cap S_\Delta = L$, e.g. $C = C'$. Then $\mathbf{Div}(S_{C_0}) = p^*(\mathbf{Div}(C_0)) + \boldsymbol{Z}.\, l_1 + \ldots + \boldsymbol{Z}.\, l_{2g-2} + \boldsymbol{Z}.\, C$.

Since $(C' - C'').\, q = 1 - 1 = 0$, and $(C' - C'').\, l_i = 0$, $(i = 1, \ldots, 2g-2)$, the divisor $C' - C''$ belongs to $p^*(\mathbf{Div}(C_0))$; i.e. $C' - C'' = p^*\delta$ for some $\delta \in \mathbf{Div}(C_0)$.

Obviously, $\deg(\delta) = 0$. Represent δ as a difference of two effective divisors (of the same degree): $\delta = \delta_1 - \delta_2$. Without loss of the generality we can assume that the sets $\mathrm{Supp}(\delta_1)$ and $\mathrm{Supp}(\delta_2)$ are disjoint. Therefore, $p^*(C' - C'') =$

$p^{-1}(\delta_1) - p^{-1}(\delta_2)$ is a sum of fibers of p, with positive and negative coefficients, and of total degree 0.

Since all the fibers of $p: X \to \boldsymbol{P}^2$ are rationally equivalent, the rational cycle class $[p^{-1}(\delta)]$, of $p^{-1}(\delta)$, is 0, in the Chow ring $A.(X)$. Since the Abel-Jacobi map for any family of algebraically equivalent 1-cycles on X factors through the cycle class map, the curves C' and C'' have the same Abel-Jacobi image, i.e. $z' = z''$. This proves (*).

It follows from (*) that the Abel-Jacobi map Φ_+ factors through ψ, i.e., there exists a well-defined map $\overline{\Phi}_+ : \mathrm{Supp}(\Theta) \to Z_+$, such that $\Phi = \overline{\Phi}_+ \circ \psi$.

Let $C \in \mathcal{C}_+$ be general, and let $L = L(C) = \psi(C)$. Let $\mathcal{L} = \phi(L)$ be the sheaf defined by the 1-dimensional linear system of effective divisors linearly equivalent to L. Let $\mathcal{C}_+(\mathcal{L}) = \psi^{-1}(|\mathcal{L}|)$ be the preimage of $|\mathcal{L}|$ in $\mathcal{C}_+$. Since Φ_+ factors through ψ, and Φ_+ is a map to an abelian variety (the intermediate jacobian $J(X)$ of X), the map $\overline{\Phi}_+$ contracts rational subsets of $\mathrm{Supp}(\Theta)$ to points. However, $\psi(\mathcal{C}_+(\mathcal{L})) \cong |\mathcal{L}| \cong \boldsymbol{P}^1$. Therefore, there exists a point $z = z(\mathcal{L}) \in Z_+$ such that $\Phi_+(\phi^{-1}(\mathcal{L})) = \Phi_+(\mathcal{C}_+(\mathcal{L})) = \overline{\Phi}_+(|\mathcal{L}|) = \{z\} \subset Z_+$.

Clearly $z = \Phi_+(C)$, and the uniqueness of the sheaf $\mathcal{L}$ defined by C, implies that the correspondence $\Sigma = \{(z, \mathcal{L}): z = \Phi_+(C), \mathcal{L} = \phi \circ \psi(C), C \in \mathcal{C}_+\}$ is generically (1:1).

Let $i: \Sigma \to Z_+$ and $j: \Sigma \to \Theta$ be the natural projections. The general choice of $C \in \mathcal{C}_+$, and the identity $\psi(\mathcal{C}_+) = \mathrm{Supp}(\Theta)$, imply that j is surjective. Therefore Z_+ and Θ are birational. In particular, Z_+ is a divisor in $J(X) \cong P$. It is not hard to see that the map $i \circ j^{-1}: \Theta \to Z_+$ is regular. In fact, let $\mathcal{L}$ be any sheaf which belongs to Θ. The definition of ϕ implies that $\phi^{-1}(\mathcal{L})$ coincides with the linear system $|\mathcal{L}|$, which is an (odd dimensional) projective space. Therefore, $\overline{\Phi}_+$ contracts the connected rational set $\psi^{-1}(\mathcal{L})$ to a unique point $z = z(L)$, i.e. $i \circ j^{-1}$ is regular in $\mathcal{L}$. It follows that Z_+ is biregular to the divisor of principal polarization Θ, i.e. Z_+ is a translate of Θ.

The coincidence $Z_- = J(X)$ follows in a similar way.

In case (2), the only difference is that the general fiber of ψ is finite, since the minimal sections $C \in \mathcal{C}_-$ which majorate the general $L \in \mathrm{Supp}(\Theta)$, are isolated. Theorem 4.4 is proved.

5. – The fibers of the Abel-Jacobi maps Φ_+ and Φ_-.

(5.1) *The general position of the ruled surfaces* $S(C')$.

Let $d = \deg \Delta \geq 4$, and let $g = (d-4)(d-5)/2$ be the genus of the smooth plane curve of degree $d-3$.

Let $C' \in \mathcal{C}_+ \cup \mathcal{C}_-$ be general. In particular, C' is smooth and nonsingular (see (2.2), (2.3)), the ruled surface $S(C')$ (see (3.3)) is well defined, and the invariant $e(S(C')) = g$ (if $C' \in \mathcal{C}_+$), or $e(S(C')) = g - 1$ (if $C' \in \mathcal{C}_-$)—see Corollary (2.6) and Proposition (3.2). It follows from Remark (3.5) that the general fiber of the natural surjective map $p\colon \mathcal{C}_+ \to |\mathcal{O}_{\boldsymbol{P}^2}(d-3)|$ is *one dimensional*, and the general fiber of the same map for $\mathcal{C}_-$ is *finite*. This implies that if $C' \in \mathcal{C}_+$ is general then the family of minimal sections of $S(C')$ is *one-dimensional*, and if $C' \in \mathcal{C}_-$ is general then the set of minimal sections of $S(C')$ is *finite*.

Let e.g. $d = \deg \Delta \geqslant 7$. Then $g \geqslant 3(\geqslant 2)$. Let, as in Lemma (1.8), E be a normalized rank 2 bundle such that $\boldsymbol{P}(E) = S(C')$, let $\alpha(C')$ and $[E]$ be as in (1.8), and let $e = e(S(C'))$ be the invariant of $S(C')$. We say that $[E]$ is in a *general position with respect to* $\alpha(C')$ if the family of e-secant line bundles of $\alpha(C)$ which pass through $[E]$ is of the expected minimal dimension ($=1$ if $e = g$, and $=0$ if $e = g-1$).

The last and Lemma (1.8) imply that if $S(C')$ comes from a general minimal section then $[E]$ is in a general position with respect to $\alpha(C')$.

If $d = 6$ ($\leftrightarrow g = 1$) then we say that $S(C')$ is *general* if $S(C')$ is one of the surfaces described in Lemma (1.7). The general ruled surfaces over $\boldsymbol{P}^1$ are, of course, $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$—see (1.6). By the same arguments as above the ruled surface $S(C')$ is general for the general minimal section C'.

Remember also that if $S_\Delta = p^{-1}(\Delta)$, then $L = \psi(C') = C' \cap S_\Delta \in \operatorname{Supp}(\Theta) \cup \operatorname{Supp}(P^-)$; and also that $C_0 = p(C')$ is the unique plane curve such that $C_0 \cap \Delta = \pi(L)$. Since $S(C')$ does not depend on the general minimal section $C' \subset S(C')$ we let $S(L) := S(C')$ if $L = \psi(C')$.

(5.2) It follows from Theorem (4.4) that the fibers of Φ_+ and Φ_- depend closely on the alternative conclusions: $Z_+ = \Theta$, or $Z_- = \Theta$. The examples show that any of the two alternatives (4.4)(1)-(4.4)(2) can be true, depending on the choice of the conic bundle $p\colon X \to \boldsymbol{P}^2$ (see section 6).

In either of the cases (4.4)(1) and (4.4)(2), the considerations in (5.1), connecting the main results in § 2 and § 3, yield the description of the general fibers of Φ_+ and Φ_-. We shall collect collect these descriptions in the following:

(5.3) THEOREM. – *Description of the general fibers of the Abel-Jacobi maps* Φ_+ *and* Φ_-.

Let $p\colon X \to \boldsymbol{P}^2$ *be a standard conic bundle with a smooth discriminant* Δ *of degree* $d > 3$. *Let* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ *be the two canonical families of non-isolated and isolated minimal sections (see* (4.1)*), and let* $\phi\colon \mathcal{C}_+ \to C_+$, $\phi\colon \mathcal{C}_- \to C_-$, $\psi\colon \operatorname{Supp}(\Theta) \to \Theta$, *and* $\psi\colon \operatorname{Supp}(P^-) \to P^-$ *be the families and the natural maps defined in* (4.1). *Let* $\Phi_+\colon \mathcal{C}_+ \to J(X)$ *and* $\Phi_-\colon \mathcal{C}_- \to J(X)$ *be the Abel-Jacobi maps for* $\mathcal{C}_+$ *and* $\mathcal{C}_-$, *and let* Z_+ *and* Z_- *be the images of* Φ_+ *and* Φ_-.

Then one of the following two alternatives is true:

($\boldsymbol{A}$: $+$) $C_+ = \operatorname{Supp}(\Theta)$, Z_+ *is a translate of* Θ ($\Leftrightarrow C_- = \operatorname{Supp}(P^-)$, $Z_- = J(X) \cong P$).

Let $z \in Z_+$ *be general, and let* $\mathcal{L} = j \circ i^{-1}(z) \in \Theta$ *be the sheaf which corresponds to* z. *Then*:

(1) *The fiber* $\mathcal{C}_+(z) := \Phi_+^{-1}(z)$ *is* 2-*dimensional.*

(2) *The map* ψ *defines on* $\mathcal{C}_+(z)$ *the natural fibration* $\psi(z)\colon \mathcal{C}_+(z) \to |\mathcal{L}| \cong \boldsymbol{P}^1$.

(3) *The general fiber* $\mathcal{C}_+(L) := \psi(z)^{-1}(L)$ *of* $\psi(z)$ *can be described as follows* ($d \geqslant 4$):

Let $C_0(L) \subset \boldsymbol{P}^2$ *be the plane curve of degree* $d-3$ *defined by* L. *Then*

(i) *If* $d = \deg(\Delta) = 4$ *or* 5, *then* $S(L) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, *and* $\mathcal{C}_+(L) \cong$ *the fiber* $\boldsymbol{P}^1$ *of the projection* $p(L)\colon S(L) \to C_0(L) \cong \boldsymbol{P}^1$ *induced by* p;

(ii) *If* $d = \deg(\Delta) = 6$, *then* $p(L)\colon S(L) \to C_0(L)$ *is the only indecomposable ruled surface over the elliptic base* $C_0(L)$, *and the fiber* $\mathcal{C}_+(L)$ *of* $\psi(z)\colon \mathcal{C}_+(z) \to |\mathcal{L}| \cong \boldsymbol{P}^1$ *is isomorphic to* $C_0(L)$. *In particular,* $\mathcal{C}_+(z)$ *is an elliptic fibration over the rational base curve* $|\mathcal{L}|$;

(iii) *Let* $d = \deg(\Delta) \geqslant 7$, *let* $g = d(d-3)/2 + 1$ *be the genus of* $C_0(L)$, *let* $C \in \mathcal{C}_+(L)$ *be general, and let* $S(C)$ *be the ruled surface defined in* (3.3). *Let* E *be a normalized* rank 2 *bundle over* $C_0(L)$ *such that* $S(C) = \boldsymbol{P}_{C_0}(E)$, *and let*

$$0 \to \mathcal{O}_{C_0(L)} \to E \to \mathcal{N} \to 0$$

be the extension defined by C. *Let* $\alpha(C_0) \subset \boldsymbol{P}(H^0(K_{C_0} \otimes \mathcal{N}))$ *be the image of* C_0 *defined by the sheaf* $K_{C_0} \otimes \mathcal{N}$ (*see* (1.8)). *Then* $\boldsymbol{P}(H^0(K_{C_0} \otimes \mathcal{N})) \cong \boldsymbol{P}^{2g-2}$, α *is a regular morphism of degree* 1, *and the point* $[E]$ *defined by this extension is in general position with respect to the set of* g-*secant line bundles of* $\alpha(C_0)$. *Moreover,* $\mathcal{C}_+(L)$ *is birational to the* 1-*dimensional set* $\mathrm{Sec}_g(\alpha(C_0), [E])$ *of* g-*secant planes of* $\alpha(C_0)$ *through the point* $[E]$. *In particular, if* C' *and* $[E']$ *is another pair of this type, then the normalizations of the curves* $\mathrm{Sec}_g(\alpha(C_0), [E])$ *and* $\mathrm{Sec}_g(\alpha'(C_0), [E'])$ *are isomorphic to each other.*

(4) *If* $z \in Z_-$ *is general and* $\mathcal{L} = j \circ i^{-1}(z)$, *then* $|\mathcal{L}| \cong \boldsymbol{P}^0$. *If* $L = L(z)$ *is the unique element of* $|\mathcal{L}|$, *then the fiber* $\Phi_-(z) = \Phi_-^{-1}(z)$ *is discrete and*:

(i) *If* $d = \deg(\Delta) = 4$ *or* 5, *then* $\Phi_-(z)$ *has exactly one element—defined by the unique* (-1)-*section of the ruled surface* $S(L) \cong \boldsymbol{F}_1$.

(ii) *If* $d = \deg(\Delta) = 6$, *then* $\Phi_-(z)$ *has exactly two elements—defined by the two (disjoint) sections of the decomposable ruled surface* S_L *over the elliptic base* $C_0(L)$.

(iii) *Let* $d = \deg(\Delta) \geqslant 7$. *Then the fiber* $\Phi_-(z)$ *is isomorphic to the fiber* $\psi^{-1}(L)$. *Let* C *be some element of this fiber, let* $S(C) = \boldsymbol{P}_{C_0}(E)$ *be as in* (3)(iii), *let*

$$0 \to \mathcal{O}_{C_0(L)} \to E \to \mathcal{N} \to 0\,,$$

be the extension defined by C, *and let* $\alpha(C_0)$ *and* $[E]$ *be as in Lemma* (1.8). *Then*

$|K_C \otimes \mathcal{N}| \cong \boldsymbol{P}^{2g-3}$, *$\alpha$ is generically of degree* 1, *and* $[E]$ *does not lie on an infinite set of* $(g-1)$*-secant planes of* $\alpha(C_0)$. *Moreover, the cardinality of* $\mathcal{C}_-(z)$ *is equal to* $\#\{(g-1)\text{-secant planes of } \alpha(C_0)\} + 1$ (*see* (1.8)).

($\boldsymbol{A}$: $-$) $C_- = \mathrm{Supp}(\Theta)$, $Z_- \cong \Theta(\Leftrightarrow C_+ = \mathrm{Supp}(P^-), Z_- = J(X) \cong P)$.

Then the description of the general fibers of Φ_- *and* Φ_+ *is similar to this from* $(A\colon +)(1)$-(4). *We shall mark only the differences*:

(1)-(2)-(3) *The fiber* $\mathcal{C}_-(z)$ *is* 1*-dimensional. The map* $\psi(z)\colon \mathcal{C}_-(z) \to |\mathcal{L}(z)| \cong \boldsymbol{P}^1$ *is finite and surjective, and the fiber of* $\psi(z)$ *has the same description as the fiber* $\mathcal{C}_-(L) = \psi^{-1}(L)$ *described in* $(A\colon +)(4)$.

(4) *The fiber* $\mathcal{C}_+(z)$ *is* 1*-dimensional. Let* $\mathcal{L} = \mathcal{L}(z) = j \circ i^{-1}(z)$, *and let* $L = L(z)$ *be the unique element of the linear system* $|\mathcal{L}|$. *Then the sets* $\mathcal{C}_+(z)$ *and* $\mathcal{C}_+(L)$ *coincide. In particular, the fiber* $\mathcal{C}_+(z)$ *has the same description as the set* $\mathcal{C}_+(L)$ *described in* $(A\colon +)(1)$-(4).

6. – Examples.

(6.1) The bidegree $(2, 2)$ threefold.

(6.1.1) *The two conic bundle structures on the bidegree* $(2, 2)$ *threefold.*

Let $W \subset \boldsymbol{P}^8$ be the Segre fourfold $\boldsymbol{P}^2 \times \boldsymbol{P}^2$, and let X be an intersection of W with a general quadric, i.e. X is a *bidegree* $(2, 2)$ *threefold.*

Let p and q be the two standard projections from W to $\boldsymbol{P}^2$, (resp.—from X to $\boldsymbol{P}^2$). Clearly, p and q define conic bundle structures on X.

Let $l = [p^*(\mathcal{O}(1)]$ and $h = [q^*(\mathcal{O}(1)]$ be the generators of **Pic** W (resp.—of **Pic** X). Call the 1-cycle C on X a *bidegree* (m, n)*-cycle*, if C has degree m with respect to l, and degree n-w.r. to h.

(6.1.2) *The families* $\mathcal{C}_+$ *and* $\mathcal{C}_-$ *for* p.

Fix the projection, say p. Then $p\colon X \to \boldsymbol{P}^2$ is a standard conic bundle, and the discriminant Δ is a smooth general plane sextic. Therefore, the jacobian $J(X)$ is a 9-dimensional Prym variety. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ are the canonical families of isolated and non-isolated minimal sections for the conic bundle projection p. By Theorem (4.4) the Abel-Jacobi image of one of these two families is a copy of Θ. It is proven in [I1] that the family which parametrizes the theta divisor is $\mathcal{C}_+$. More precisely, the following is true:

(6.1.3) Proposition. – *Let* $\mathcal{C}_+$ *be the canonical* 10*-dimensional family of non-isolated minimal sections, and let* $\mathcal{C}_-$ *be the canonical* 9*-dimensional family of isolated minimal sections for* p. *Then*:

(1) $\mathcal{C}_+ = \mathcal{C}^1_{3,7}$ (= *the family of elliptic curves of bidegree* (3,7) *on* X), $\mathcal{C}_- = \mathcal{C}^1_{3,6}$ (= *the family of elliptic curves of bidegree* (3,6) *on* X), *and*

(2) $\Phi_+(\mathcal{C}^1_{3,7})$ *is a copy of* Θ, $\Phi_-(\mathcal{C}^1_{3,6})$ *coincides with* $J(X)$.

This and Theorem (5.3)(A: +) (see also Lemma (1.7)) imply:

(6.1.4) Corollary. – *The general fiber of* $\Phi_+ : \mathcal{C}^1_{3,7} \to \Theta$ *is an elliptic fibration over* $\boldsymbol{P}^1$. *The surjective map* $\Phi_- : \mathcal{C}^1_{3,6} \to J(X)$ *is generically finite of degree* 2.

(6.1.5) *Parametrization of* $\mathrm{Sing}^{\mathrm{st}}(\Theta)$ *via degenerate sections.*

It can be seen that on the bidegree (2,2) divisor X lies a 6-dimensional family $\mathcal{D} := \mathcal{C}^1_{3,3}$ of bidegree (3,3) elliptic curves. Any of these curves C can be completed by many ways to a quasi-section C + two fibers of $p \in \mathcal{C}^1_{3,7}$. Moreover, the general $C \in \mathcal{D}$ lies in a ruling of a rank 6 quadric $Q \supset X$ such that Q does not contain W. The ruling of Q defines a $\boldsymbol{P}^3$-system of $C_\xi \in \mathcal{D}$ rationally equivalent to C, and the intersection map $\psi: C_\xi \mapsto L_\xi = L(C_\xi) \in \mathrm{Symm}^{18}(\widetilde{\Delta})$ defines a linear system $\mathcal{L} \in \mathrm{Sing}^{\mathrm{st}}(\Theta)$. Moreover (see [Ve], [I1]):

The Abel-Jacobi image $Z = \Phi(\mathcal{D})$ *is biregular to a* 3*-dimensional component of* $\mathrm{Sing}^{\mathrm{st}}(\Theta)$. *The bidegree* (2,2) *threefold* X *coincides with the base locus of the set of tangent cones of* Θ *at the points* $z \in Z$.

Since the fibers of p are rationally equivalent to each other, the last implies:

Let $\Sigma = \{C + f_1 + f_2 : C \in \mathcal{D}$, *and* f_1 *and* f_2 *are fibers of* p *intersecting* $C\}$.

Then $\Sigma \subset \mathcal{C}^+ = \mathcal{C}^1_{3,7}$, *and* $\Phi_+(\Sigma) \cong Z$ *is a* 3*-dimensional component of* $\mathrm{Sing}(\Theta)$.

(6.1.6) Remark (see [Ve]). – Let $\mathcal{M}$ be the moduli space of plane sextics. Let $\mathcal{R}$ be the 19-dimensional space of pairs (Δ, η) where $\Delta \in \mathcal{M}$ is smooth and $\eta \neq \mathcal{O}_\Delta$ is a 2-torsion sheaf on Δ defining a unbranched 2-sheeted covering $\widetilde{\Delta} \to \Delta$.

It was proved by A. Verra that the Torelli theorem does not hold for the Prym map $\varrho: \mathcal{R} \to \mathcal{P} = \varrho(\mathcal{R}) \subset \mathcal{A}_9$ (= the space of p.p. avelian 9-folds), $\varrho(\Delta, \eta) := P(\widetilde{\Delta}, \Delta)$. More precisely (see [Ve]): $\deg \varrho = 2$, and: (i). For the general $P \in \mathcal{P}$ the fiber $\varrho^{-1}(P) = (\Delta, \eta) \cup (\Delta', \eta')$, where (Δ, η) and (Δ', η') are obtained from each other by the classical Dixon correspondence. (ii). There exists a unique bidegree $(2,2)$ threefold X for which the induced by η and η' double coverings $\widetilde{\Delta} \to \Delta$ and $\widetilde{\Delta}' \to \Delta'$ are the same as the double coverings defined by the two conic bundle projections on X. (iii). Let $\mathcal{R}_0 \subset \mathcal{R}$ be the subspace of these (Δ, η) which come from nodal quartic double solids, and let $\mathcal{P}_0 = \varrho(\mathcal{R}_0)$. Then $\mathcal{P}_0 \subset \mathcal{P}$ is a component of the 18-dimensional branch locus of ϱ.

(6.2) THE NODAL QUARTIC DOUBLE SOLID.

(6.2.1) By definition, a quartic double solid (q.d.s.) is a double covering $\varrho: X \to \boldsymbol{P}^3$ branched along a quartic surface $B \subset \boldsymbol{P}^3$.

The parametrization of Θ for the general quartic double solid by the 12-dimensional family of Reye sextics, and the parametrization of $\mathrm{Sing}(\Theta)$ are obtained by Tikhomirov [T] and Voisin [Vo]. Moreover, the results in [C], [De] imply actually the descriptions of Θ and $\mathrm{Sing}(\Theta)$ by means of minimal sections, for the quartic double solids with ≤ 6 nodes.

The «minimal section» approach imply also a natural parametrization also of the intermediate jacobian J of the nodal q.d.s. X.

(6.2.2) *The conic bundle structure on the nodal q.d.s.*

Let S has a simple node o. Denote by o also the node of X—«above» o. Let $\widetilde{B} \subset \widetilde{\boldsymbol{P}} \subset \boldsymbol{P}^8$ be the image of $B \subset \boldsymbol{P}^3$ by the system of quadrics through o, and let $\tilde{\varrho}: \widetilde{X} \to \widetilde{\boldsymbol{P}}$ be the induced double covering branched along $\widetilde{B}$. (The threefold $\widetilde{\boldsymbol{P}}$ is a projection of the Veronese image $\boldsymbol{P}^3_8 \subset \boldsymbol{P}^9$ of $\boldsymbol{P}^3$, through the image of o. In particular, $\widetilde{\boldsymbol{P}}$ contains a plane $\boldsymbol{P}^2_0$, and the inverse map $\sigma: \widetilde{\boldsymbol{P}} \to \boldsymbol{P}^3$ is a blow-down of $\boldsymbol{P}^2_0$ to o. The restriction $\sigma: \widetilde{B} \to B$ is a blow-down of a smooth conic $q_o \subset \widetilde{B}$ to the node o.)

The threefold $\widetilde{\boldsymbol{P}} \cong \boldsymbol{P}_{P^2}(\mathcal{O} \oplus \mathcal{O}(1))$ has a natural projection p_o to $\boldsymbol{P}^2 = \{\text{the lines } l \text{ in } \boldsymbol{P}^3 \text{ through } o\}$, and $\boldsymbol{P}^2_0$ is the exceptional section of the projectivized bundle $\widetilde{\boldsymbol{P}}$. The general fiber $p^{-1}(l)$ of the composition $p = p_o \circ \tilde{\varrho}: \widetilde{X} \to \boldsymbol{P}^2$ is a smooth conic $q(l) = p^{-1}(l) \cong$ (the desingularization of $\varrho^{-1}(l)$ in o).

The restriction $p_o|_{\widetilde{B}}: \widetilde{B} \to \boldsymbol{P}^2$ desingularizes the projection from the quartic B through the node $o = \mathrm{Sing}(B)$. Therefore, $p_o|_{\widetilde{B}}$ is a double covering branched along a smooth plane sextic Δ, and the conic q_o is totally tangent to Δ. Clearly, the fiber $p^{-1}(x)$ is singular for any $x \in \Delta$, and the natural Abel-Jacobi map $\widetilde{\Delta} \to J = J(\widetilde{X})$ induces an isomorphism of p.p.a.v. $P(\widetilde{\Delta}, \Delta) \cong J$ (see [B]).

(6.2.3) *The families $\mathcal{C}_+$ and $\mathcal{C}_-$.*

It is not hard to find the families $\mathcal{C}_+$ and $\mathcal{C}_-$ for p. Since this description does not differ substantially from the general one, we shall state it in a brief:

Since $\widetilde{\boldsymbol{P}} \subset \boldsymbol{P}^8$, the degree map $\deg: \{\text{subschemes of } \widetilde{\boldsymbol{P}}\} \to \boldsymbol{Z}$ is well defined. In particular, $\deg(\boldsymbol{P}) = \deg(\boldsymbol{P}^3_8) - 1 = 7$.

Let $Z \subset \widetilde{X}$ be a subscheme of $\widetilde{X}$. Define $\deg(Z) := \deg(\tilde{\varrho}_*(Z))$.

EXAMPLE. – Let $l \subset \boldsymbol{P}^3$ be a line through o, let $x = [l] \in \boldsymbol{P}^2$ be the point representing l, and let $q(x)$ be the «conic» $q(x) = p^{-1}(x)$. Then $\deg(p^{-1}(x)) = 2$. Indeed, $\tilde{\varrho}_*(q(x)) = 2l'$ where $l' = p_o^{-1}(x) \subset \widetilde{\boldsymbol{P}}$ is the line in $\boldsymbol{P}^8$ which represents the «bundle-fiber» in $\widetilde{\boldsymbol{P}}$ over $[l]$. Note also that l' is the proper preimage of the line $l \subset \boldsymbol{P}^3$ under the blow-down $\sigma_o: \widetilde{\boldsymbol{P}} \to \boldsymbol{P}^3$.

(6.2.4) PROPOSITION.

(1) $\mathcal{C}_+$ *is a component* $\mathcal{C}^1_{10}$ *of the family of elliptic curves of degree* 10 *on* $\widetilde{X}$; $\mathcal{C}_-$ *is a component* $\mathcal{C}^1_9$ *of the family of elliptic curves of degree* 9 *on* $\widetilde{X}$.

(2) *a*) $\psi(\mathcal{C}_-) = \mathrm{Supp}(\Theta)$. Therefore $\Phi_-(\mathcal{C}^1_9)$ is a copy of Θ. *b*) $\Phi_+(\mathcal{C}^1_{10}) = J$.

PROOF. – The proof of (1) is standard.

(2) The general element $C \in \mathcal{C}_-$ lies in a unique S of the 9-dimensional system $|\mathcal{O}_{\widetilde{X}}(1)| = |\tilde{\varrho}^* \mathcal{O}_{\tilde{P}}(1)|$. The elliptic curve C moves in a $\boldsymbol{P}^1$-system C_t in the $K3$-surface S, this way $\psi(C_t) = L_t$ defines a pencil in $Nm^{-1}(K_\Delta)$—see also [C, p. 98]. By Theorem (4.4), this implies *a*) and *b*).

(6.2.5) COROLLARY (see Theorem (5.3)(A: $-$) and Lemma (1.7). – *The general fiber of* $\Phi_-: \mathcal{C}^1_9 \to \Theta$ *is a disjoint union of two smooth rational curves* (*see also* [C]).

The general fiber of $\Phi_+: \mathcal{C}^1_{10} \to J$ *is an elliptic curve.*

(6.2.6) REMARK. – *The component* $Z \subset \mathrm{Sing}(\Theta)$.

Since the result is essentially known (see [Vo], [De], [C]), we shall only state it (*see also* (6.1.5)):

There exist elliptic septics on $\widetilde{X}$, *and the Abel-Jacobi map sends a component* $\mathcal{C}^1_7$ *of this family onto a 4-dimensional variety* Z *isomorphic to a component of stable singularities of* Θ. *Equivalently, if*

$$\Sigma = \{C + f\colon C \in \mathcal{C}^1_7 \ \&\, f \text{ is a fiber of } p \text{ intersecting } C\} \subset \mathcal{C}^1_9 = \mathcal{C}_-,$$

then $\Phi(\Sigma) \cong Z$ (*cf.* (6.1.5)).

(6.3) THE NODAL SECTION OF THE GRASSMANNIAN $G(2, 5)$.

(6.3.0) Any smooth 3-dimensional intersection $X = X_{10}$ of the grassmannian $G = G(2, 5) \subset \boldsymbol{P}^9$ by a subspace $\boldsymbol{P}^7 \subset \boldsymbol{P}^9$ and a quadric Q is a Fano threefold of degree 10 and of index 1.

It turns out that the nodal X_{10} acquires a natural conic bundle structure. We shall describe it, and also we shall find the parametrization of the Abelian part of the intermediate jacobian of $X = X_{10}$, as well the parametrization of Θ by means of the curves on X representing the families $\mathcal{C}_-$ and $\mathcal{C}_+$.

(6.3.1) *Shortly about flops and extremal rays* (see e.g. [Mo], [K], [Isk2]).

DEFINITION. – Let $pr'\colon X' \to X''$ be an indecomposable birational morphism from the smooth 3-fold X' to the normal 3-dimensional variety X'', and let $D' \subset X'$ be an effective divisor such that:

(a') the exceptional set $Ex(pr')$ of pr' is a union of 1-dimensional cycles $l_i' \subset X'$ such that $-K_X'. l_i' = 0$, $\forall i$;

(b') $D'. l_i' < 0$, $\forall i$.

Let the threefold X^+ be smooth, and let the birational isomorphism $\varrho: X' \to X^+$ (over X'') be an isomorphism in codimension 2. Then ϱ is called a D'-flop over X'' if the composition $pr^+ = pr' \circ \varrho^{-1}: X^+ \to X''$ is an indecomposable birational morphism, $Ex(pr^+)$ is a union of 1-dimensional cycles l_i^+, and K_{X^+}, l_i^+ and D^+ (= the proper image of D' on X^+) fulfill the properties:

(a^+) $-K_{X^+}. l_i^+ = 0$, $\forall i$;

(b^+) $D^+. l_i^+ > 0$, $\forall i$.

By [K], if such X', pr', D', etc. fulfill (a'), (b') then a D'-flop always exists, and any sequence of such D'-flops is finite.

Let X^+ be a smooth variety, let $N(X) = \{1 - \text{cycles on } X\} / \equiv \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ be the finite-dimensional real space of numerically equivalence classes of 1-cycles on X^+, and let $\overline{NE}(X^+)$ be the closure of the convex cone generated by the effective 1-cycles on X^+. The half-line $R = \boldsymbol{R}_+ . [C^+]$ is called an *extremal ray* on X^+ if R is an extremal ray of the cone $\overline{NE}(X^+)$ and $-K_{X^+}. C^+ > 0$. The *rational* curve $C^+ \subset X^+$ is called *extremal* if $-K_{X^+}. C \leqslant \dim(X^+) + 1$, and $R = \boldsymbol{R}_+ . [C^+]$ is an extremal ray. By *The Cone Theorem* [Mo], any extremal ray on X^+ is generated by some extremal curve.

The numerically effective divisor $D^+ \subset X^+$ is called a *supporting function* of the extremal ray $R = \boldsymbol{R}_+ . [C^+]$ on X^+ if $D^+ . C^+ = 0$, and if for any effective 1-cycle C on X^+ the identity $D^+ . C = 0$ implies $[C] \in R$.

By [Mo], any extremal ray R on X^+ defines a morphism $\phi_R: X^+ \to Y$, where Y is a normal variety, and such that ϕ contracts all the irreducible curves $[C^+] \in R$, and any extremal ray R on X^+ has a supporting function D''. Moreover, by the *Theorem of Stable Freedom* ([KMM]), the morphism ϕ_R can be defined by $|m. D^+|$ for $m \gg 0$.

Especially, if $\dim X^+ = 3$, $\dim Y = 2$, and $-K_{X^+}. C^+ = 1$ then, by [Mo], $\phi_R: X^+ \to Y$ is a standard conic bundle.

(6.3.2) *The double projection from o—a birational conic bundle structure on X.*

Let (X, o) be a general pair of a nodal X_{10} and a node o on it, let $pr: X \to X''$ be the rational projection from o, let $\sigma: X' \to X$ be the blow-up of o, and let $pr' = pr \circ \sigma: X' \to X''$. Let $Q' = \sigma^{-1}(o) \subset X'$ be the exceptional quadric on X', and let Q'' be the image of Q' on X''. Let H'' be the hyperplane section of X'', (as well the proper preimages of H'' on X and on X'). Let H be the hyperplane section of X (as well its proper preimage on X'). Now, the following are standard properties of the projections (see e.g. [Isk2] discussing the double projection from a line).

(1) $X'' \subset \boldsymbol{P}^6$ is a complete intersection of three quadrics, and $Q'' \subset X''$ is a smooth quadric surface on X''—see e.g. (6.3.5).

(2) There are finite number of lines $l_i \subset X$ such that $o \in l_i$; in fact, their number is 6—see the proof of Lemma (6.3.5)(1).

(3) Let $l_i' \subset X'$ be the proper preimages of l_i on X', and let $x_i' = l_i' \cap Q'$. Then $pr': X' \to X''$ is an indecomposable birational morphism (defined by the linear system of $H'' \sim H - Q'$ on X'), and $Ex(pr')$, $K_{X'}$, l_i', etc. fulfill the property (6.3.1)(a').

(4) Let $x_i'' \in Q''$ be the points $x_i'' = pr'(l_i') = pr(x_i')$. Then $\operatorname{Sing} X'' = \{x_1'', \dots, x_6''\}$, and all these points are simple nodes of X'—see e.g. the proof of Lemma (6.3.5)(1).

(5) If D'' is any effective divisor of the $\boldsymbol{P}^2$-system $|H'' - Q''|$ (on X''), and if D' is the proper preimage of D'' on X', then D' and l_i' fulfill the property (6.3.1)(b')—by the standard properties of blow-ups.

By (6.3.1) there exists a D'-flop $\varrho: X' \to X^+$ over X''.

(6.3.3) *The standard conic bundle structure on* X^+.

Let $\boldsymbol{P}_o^1 = \boldsymbol{P}(\boldsymbol{C}_o^2) \subset \boldsymbol{P}^4 = \boldsymbol{P}(\boldsymbol{C}^5)$ be the line representing the node o, i.e. $o = \boldsymbol{P}(\wedge^2 \boldsymbol{C}_o^2)$, let $\boldsymbol{P}_o^2 := \{\boldsymbol{P}^3 \subset \boldsymbol{P}^4 : \boldsymbol{P}_o^1 \subset \boldsymbol{P}^3\} \subset (\boldsymbol{P}^4)^* = \boldsymbol{P}(\boldsymbol{C}^{5*})$, and let $\boldsymbol{P}^3 \in \boldsymbol{P}_o^2$ be general. Then the cycle $C = C(\boldsymbol{P}^3) := \sigma_{1,1}(\boldsymbol{P}^3) \cap X$, being a complete intersection of a codimension 2 space and a quadric in the grassmannian $\sigma_{1,1}(\boldsymbol{P}^3)$, is a space quartic curve with an ordinary double point at o. Let C', C'' and C^+ be the proper images of C on X', X'' and X^+. Then the irreducible curve $C^+ \subset X^+$ is rational; and if $D^+ \subset X^+$ and $H^+ \subset X^+$ are the proper images of $D' \subset X'$ and of $H \subset X$, then:

(6) $-K_{X^+} . C^+ = H^+ . C^+ = 1$;

(7) $D^+ . C^+ = 0$.

The D'-flop $\varrho: X' \to X^+$ is a composition $\varrho_1 \circ \dots \circ \varrho_6$, $\varrho_i = \tau_i \circ \sigma_i$, where σ_i is the blow-up of l_i', and τ_i is the blow-down $Q_i \to l_i^+ \subset X^+$ of the exceptional quadric $Q_i = \sigma_i^{-1}(l_i')$ along the residue ruling. By construction, D^+ is *numerically effective* on X^+ (since, e.g. $D^+ . l_i^+ = 1 > 0$). By (6) and (7), $R = \boldsymbol{R}_+ . [C^+]$ is an extremal ray defining the standard conic bundle

$$p^+ := \phi_R: X^+ \to \boldsymbol{P}^2 \, (\cong \text{the base } \boldsymbol{P}_o^2 \text{ of the family } \{\boldsymbol{P}^3 : \boldsymbol{P}^3 \supset \boldsymbol{P}_o^1\})$$

(see also the end of (6.3.1)).

Now, it is not hard to see that D^+ is a supporting function for $R = \boldsymbol{R}_+ . [C^+]$. In particular, the map p^+ is defined by some multiple $m . D^+$, and we may assume that m is *minimal* with this property. Then, by the choice of $D^+ =$ (the proper image on X^+ of an effective divisor $D'' \subset X''$ of the $\boldsymbol{P}^2$-system $|\mathcal{O}_{X''}(H'' - Q'')|$), we obtain $m = 1$. Since $H - 2Q' \sim H'' - Q' \sim D'$ (on X'), the rational map $\phi_R \circ \varrho : X' \to \boldsymbol{P}^2$ is defined by the linear system $|H - 2Q'|$ on X'. Equivalently:

COROLLARY. – *The map* $p = \phi_R \circ \varrho \circ \sigma^{-1}: X \to \boldsymbol{P}^2$ *is a rational conic bundle structure on X, defined by the non-complete linear system* $|H - 2.\, o|$ *on* X; *i.e. p is the double projection from* o.

(6.3.4) LEMMA. – *If* (X, o) *is general, then the discriminant curve* Δ *of* p^+ *is a smooth plane sextic.*

PROOF. – By [I2, Lemma (3.2.3)], for the general nodal Gushel threefold the plane curve Δ is a smooth sextic. Therefore the same is true also for the general nodal X_{10}, since any (nodal) Gushel threefold $(X(0), o)$ is a smooth deformation of a family $\{(X(t), o)\}$ of (nodal) X_{10}^{-s}.

(6.3.5) *The nodal* X_{10} *and the plane sextics.*

Let G be the grassmannian of lines in $\boldsymbol{P}^4 = \boldsymbol{P}(\boldsymbol{C}^5)$, and let $I_2(G)$ be the family of quadrics containing the Plücker image of G. Any choice of a coordinates (x_i, e_i) in $\boldsymbol{C}^5$ defines a linear isomorphism $Pf: \boldsymbol{P}^4 \to I_2$, where $Pf(x)$ is the Plücker quadric in $\boldsymbol{P}^9$ with vertex $\boldsymbol{P}^3_x = \sigma_{3,0}(x)$. In particular, all the quadrics containing G are of rank 6. The same is true also for the smooth 4-fold $W = G \cap \boldsymbol{P}^7$.

Let, as above, $\boldsymbol{P}^1_o \subset \boldsymbol{P}^4$ be the line representing the node o of $X = X_{10} = W \cap Q$.

Then $Pf = Pf(\boldsymbol{P}^1_o) = \{Pf(x): x \in \boldsymbol{P}^1_o\}$ is a line of rank 6 quadrics containing W (hence—containing $X \in |\mathcal{O}_W(2)|$), and any such quadric is singular at o. Since X is singular at o, we can choose a quadric $Q \subset \boldsymbol{P}^7$ such that Q is singular at o and $X = W \cap Q$. In this notation, we can identify Q and $Pf(x)$; $x \in \boldsymbol{P}^1_o$, and the projections of these quadrics in $\boldsymbol{P}^6$. Therefore $X'' = pr(X) \subset \boldsymbol{P}^6$ coincides with the base locus of the plane of quadrics $\Pi = \langle Pf, Q \rangle$.

The Hessian *Hess* of X'' is a plane septic, and since $\operatorname{rank} Pf(x) = 6$, $\forall x \in \boldsymbol{P}^1_o$, $Hess = Pf + H_6$, where H_6 ia a plane sextic.

(1) LEMMA. – *Let* $X'' \subset \boldsymbol{P}^6$ *be a base locus of a plane* Π *of quadrics in* $\boldsymbol{P}^6$, *such that the Hessian Hess of* X'' *contains a line* L, *and let* X'' *be otherwise general. Then*:

(a) X'' *contains a quadratic surface* Q'', *and* X'' *is singular at* 6 *points which lie on* Q''. *Moreover*

(b) *For any such* X'', *there exists a nodal* $X = X_{10}$ *such that* X'' *is the same as the projection of* X *from its node* o.

PROOF. – (a) Let W be the base locus of L. Since L is assumed to be general, the vertices $v(Q)$, $Q \in L$ sweep-out a twisted cubic C_v, and since W must be singular along C_v, W contains $\boldsymbol{P}^3 = \operatorname{Span} C_v$. If $Q \in \Pi - L$, then $X'' = Q \cap W$ contains the quadric $Q'' = Q \cap \boldsymbol{P}^3$. Since Q can be general, $\operatorname{Sing} X'' = \operatorname{Sing} W \cap Q =$

$C_v \cap Q = \{x_1'', \ldots, x_6''\}$ (here $6 = \deg(Q).\ \deg(C_v)$), and x_i'' are ordinary nodes of $X'' = W \cap Q$.

(b) X is obtained from X'' by blowing-up $x_1'', \ldots, x_6''$, then by contracting any of the obtained 6 exceptional quadrics L_i along this ruling, the general line of which does not intersect the preimage of Q'', and then by blowing-down the proper preimage of Q'' (which describes, in fact, the opposite of the projection *pr*).

(2) COROLLARY.

(a) *The general reducible plane septic* $H_6 + L$, *such that* $\deg L = 1$, *appears as a component of the Hessian of the projection* X'' *of some nodal* $X = X_{10}$. *Moreover*:

(b) (D. Logachev [L]): *The natural double covering* $\widetilde{H}_6 \to H_6$ *is unbranched, and if* J *is the abelian part of the intermediate* $J(X)$ (= *the abelian part of* $J(X'')$) *then* $J = P(\widetilde{H}_6, H_6)$ *as p.p.a.v.*

PROOF. – (a) It is proved by Beauville and Tjurin (see e.g. [FS, Theorem (0.1)]) that any smooth plane septic can be realized as a Hessian *Hess* of a plane Π of quadrics in $\boldsymbol{P}^6$. By degeneration, the same is true also for $Hess = H_6 + L$. where H_6 is e.g. a smooth plane sextic, and L is a general line in Π. Now (2)(a) follows from (1).

(b) If X'' is a projection from a general nodal X_{10} then the count of the parameters yields that any quadric containing X'' is of rank ≥ 6. Therefore the same is true also for the general X'' containing a smooth quadric surface. In particular, the non-trivial component of $\widetilde{Hess}$: $\widetilde{H}_6 = \{\Lambda : \Lambda$ is a ruling of some $Q \in H_6\}$ is well-defined and $\widetilde{H}_6 \to H_6$ is unbranched. The rest of the proof of (b) repeats the original one (see [B], [Tju]) for the general intersection of three quadrics in $\boldsymbol{P}^6$.

(6.3.6) *The families* $\mathcal{C}_+$ *and* $\mathcal{C}_-$.

Let (X, o) be a general nodal X_{10}. Denote by $\mathcal{C}_d^g[m](X)$ the (possibly empty) family of algebraically equivalent connected 1-cycles C on X such that the general element $C \in \mathcal{C}_d^g[k](X)$ is an irreducible curve $C \subset X$, smooth outside o, of geometric genus g and of degree d, which passes through the node o with multiplicity m.

For example {the $\boldsymbol{P}^2$—family of fibers of p} is a component of $\mathcal{C}_4^0[2](X)$; with a possible abuse in the notation we denote this component also by $\mathcal{C}_4^0[2](X)$. In this notation, the discriminant sextic of p (:= the discriminant sextic of p^+) is

$$\Delta = \{x \in \boldsymbol{P}^2 : f_x = q + \bar{q}, \quad \text{s.t.} \quad q, \bar{q} \in \mathcal{C}_2^0[1](X)\}.$$

By (6.3.5), the discriminant Δ is a smooth plane sextic and (J, Θ) is isomorphic, as a principally polarized abelian variety, to the Prym variety $P(\widetilde{\Delta}, \Delta)$.

Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the two canonical families of minimal sections for the standard conic bundle $p^+ : X^+ \to \boldsymbol{P}^2$. We shall find the images of these families on X'' and on X.

Let $\mathcal{C}^g_d[m](X'')$ be the (possibly empty) family of connected 1-cycles on X'', the general element of which is a smooth irreducible curve C of geometric genus g, of degree d, and such that C intersects simply the quadric Q'' in m points.

Denote by pr: $A_1(X) \to A_1(X'')$ also the natural projection-map from the 1-cycles on X to the 1-cycles on Y defined by the rational projection pr: $X \to X''$. In this notation, it is evident that $pr(\mathcal{C}^g_d[m](X)) = \mathcal{C}^g_{d-m}[m](X'')$, and the existence of one of these families yields the existence of the other.

Denote by p'': $X'' \to \boldsymbol{P}^2$ the birational conic bundle structure on X'' induced by p.

First, we shall find one family of elliptic curves on X'' which are sections of p''.

Let $Q \in H_6$ be a rank 6 quadric which does not lie on the intersection $H_6 \cap Pf$. The quadric Q has two rulings $\Lambda \cong \overline{\Lambda} \cong \boldsymbol{P}^3$, and any of these rulings consists of subspaces $\boldsymbol{P}^3 \subset Q$. Let Λ be one of them, and let $\boldsymbol{P}^3 \in \Lambda$ be a general element of Λ. Then $C = C(\boldsymbol{P}^3) = Y \cap \boldsymbol{P}^3$ is a complete intersection of two quadrics, i.e. – an elliptic quartic on X'', and this elliptic quartic intersects Q'' in *one* point. Indeed, if $\boldsymbol{P}^5 \supset C$ is general then $C = X'' \cap \boldsymbol{P}^5 = C + \overline{C}$ on X'' is a reducible canonical curve of degree 8 on X''. By the formula for the canonical class of the singular canonical curve $C + \overline{C}$, $\overline{C}$ will be an elliptic quartic on X'' intersecting C in four points which lie on the plane $\langle C \cap \overline{C}\rangle$. Clearly $\overline{C}$ is defined, in just the same way, by some $\overline{\boldsymbol{P}^3} \in \overline{\Lambda}$ intersecting $\boldsymbol{P}^3$ along the plane $\langle C \cap \overline{C}\rangle$. In particular, C and $\overline{C}$ have the same intersection degree with Q'', and since the canonical curve $C + \overline{C} = X'' \cap \boldsymbol{P}^5 \subset \boldsymbol{P}^6$ intersects the quadric $Q'' \subset X'' \subset \boldsymbol{P}^6$ in *two* ($= \deg Q''$) points, we conclude that $C \in \mathcal{C}^1_4[1](X'')$.

We shall see that the curves $C^1_4 \in \mathcal{C}^1_4[1](X'')$ are sections of p''.

By (6.3.3), the conic bundle structure p: $X \to \boldsymbol{P}^2$ is the same as the double projection $|\mathcal{O}_X(1-2.\ o)|$ from the node o. Let $C^1_4 \in \mathcal{C}^1_4[1](X'')$ be general, and let C^1_5 be the proper preimage of C^1_4 on X. The curve C^1_5 is an elliptic quintic on X which passes through o. Therefore the double projection (hence p) sends C^1_5 onto a plane cubic in $\boldsymbol{P}^2$. It follows that C^1_4 is a section of p'', and p'' maps C^1_4 isomorphically onto a plane cubic.

Let V be a threefold with isolated singularities, and let $C \subset V$ be a smooth curve on V such that $C \cap \operatorname{Sing} V = \emptyset$. Then the normal bundle $N_{C/V}$ is defined, and by the Hirzebruch-Riemann-Roch formula $\chi(N_{C/V}) = c_1(N_{C/V}) - \deg K_C = -K_V.\ C$.

This, in particular, implies that if $\mathcal{C}^g_d[m](X'') \neq \emptyset$, and if $\mathcal{C}^g_d[m](X'')$ contains a smooth curve C disjoint from $\operatorname{Sing} X'' = \{y_1, \ldots, y_6\}$, then $\dim \mathcal{C}^g_d[m](X'') = d$.

The birational conic bundle structure p'' on X'' is induced by the standard conic bundle structure p^+ on X^+. Since the birational isomorphism $X'' \leftrightarrow X^+$ preserves the general fibers of p'' and p^+, the families $\mathcal{C}_{+/-}(X'')$ for p'' are correctly defined as proper images of the families $\mathcal{C}_{+/-}(X^+)$ on the standard conic bundle $p^+ : X^+ \to \boldsymbol{P}^2$.

Fix a general component $f \in \mathcal{C}_1^0[1](X'')$ of a degenerate fiber of p'', and a general $C_4^1 \in \mathcal{C}_4^1[1](X'')$ intersecting f. Since $p''(C_4^1)$ is a plane cubic, and $\deg \varDelta = 6$, the general element of $\mathcal{C}_{+/-}$, being an isomorphic image of a general element of $\mathcal{C}_{+/-}(X^+)$, is a smooth elliptic curve algebraically equivalent to $C_4^1 + k_{+/-} . f$ for some integer $k_{+/-}$. Moreover, the general f, as well the general C_4^1 intersecting f, are disjoint from $\operatorname{Sing} X''$. Therefore the connected 1-cycle $C_4^1 + k . f$ is disjoint from $\operatorname{Sing} X''$ for any integer k. In particular the general element C of $\mathcal{C}_{+/-}(X'')$ is disjoint from $\operatorname{Sing} X''$. Therefore $\dim \mathcal{C}_{+/-} = \deg (C_4^1 + k_{+/-} . f) = 4 + k_{+/-}$.

From (3.5) we know that $\dim \mathcal{C}_+ = \dim \mathcal{C}_- + 1 = 10$. Therefore $k_+ = 6$, $k_- = 5$, i.e. $\mathcal{C}_+ = \mathcal{C}_{10}^1[7][X'']$ and $\mathcal{C}_- = \mathcal{C}_9^1[6](X'')$. The non-evident existence of a smooth curve from any of these two families is assured by the existence of the families $\mathcal{C}_{+/-}(X^+)$ for the standard conic bundle $p^+ : X^+ \to \boldsymbol{P}^2$. This proves the following

(6.3.7) PROPOSITION. – *Let $p : X \to \boldsymbol{P}^2$ be the rational conic bundle structure on the general nodal $X = X_{10}$ defined by the double projection from the node o, and let X'' be the projection of X from o. Then*

(1) $\mathcal{C}_+ \cong \mathcal{C}_{10}^1[7](X'')$ (= *the family of elliptic curves $C \subset X''$, s.t.* $\deg C = 10$ *and* $C . Q'' = 7$) $\cong \mathcal{C}_{17}^1[7](X)$ (= *the family of curves $C \subset X$, s.t.* $\deg C = 17$, $g(C) = 1$, $\operatorname{Sing} C = o$, *and* $\operatorname{mult}_o(C) = 7$).

(2) $\mathcal{C}_- \cong \mathcal{C}_9^1[6](X'')$ (= *the family of elliptic curves $C \subset X''$, s.t.* $\deg C = 9$ *and* $C . Q'' = 6$) $\cong \mathcal{C}_{15}^1[6](X)$ (= *the family of curves $C \subset X$, s.t.* $\deg C = 15$, $g(C) = 1$, $\operatorname{Sing} C = o$, *and* $\operatorname{mult}_o(C) = 6$).

It rests to find which one of these two families parametrizes $\varTheta$.

(6.3.8) PROPOSITION. – $\boldsymbol{\varPhi}_+(\mathcal{C}_+) = \boldsymbol{\varTheta}$; $\boldsymbol{\varPhi}_-(\mathcal{C}_-) = \boldsymbol{J}$.

PROOF. – By (6.3.3)-(6.3.5), $p^+ : X^+ \to \boldsymbol{P}^2$ is a standard conic bundle, and for the general nodal $X = X_{10}$, the discriminant $\varDelta$ of p^+ is a general smooth plane sextic. Let also $\eta \in \mathbf{Pic}^o_{[2]}(\varDelta)$ be the torsion sheaf defining the double covering $\widetilde{\varDelta} \to \varDelta$ induced by p^+. In particular $\varDelta$ has no totally tangent conics (see also (6.2.2)), and (by [Ve]) there exists a bidegree $(2, 2)$ threefold $T = \boldsymbol{P}^2 \times \boldsymbol{P}^2 \cap$ (a quadric), such that $(\varDelta, \eta)$ is induced by some of the two conic bundle projections on T, say $p_1 : T \to \boldsymbol{P}^2$. By (0.6), the two standard conic bundles $p^+ : X^+ \to \boldsymbol{P}^2$ and $p_1 : T \to \boldsymbol{P}^2$ are birational to each other *over* $\boldsymbol{P}^2$. Since such a birational isomorphism $\alpha : X^+ \to T$ preserves the general fibers of p^+ and of p_1, α preserves also the families $\mathcal{C}_+$ and $\mathcal{C}_-$. Since, for T, the parametrizing family for $\varTheta$ is $\mathcal{C}^+$ (see (6.1.2)-(6.1.3)), the same family must parametrize $\varTheta$ also for X^+ (hence—for X, since the birationality $X \leftrightarrow X^+$ is a composition of a blow-up and an isomorphism in codimension 2, both preserving the general fibers of p^+ and p).

(6.3.9) REMARK. – Proposition (6.3.8) and Theorem (5.3) yield the same description of the general fibers of Φ_+ and Φ_- as for the bidegree (2,2) threefold—see (6.1.4).

(6.3.10) COROLLARY. – *If X is a general X_{10} with a node o, and if X'' is the projection of X from o, then:*

(1) *The Abel-Jacobi image of the family $\mathcal{C}^1_6[3](X'')$ of elliptic sextics $C \subset X''$ such that $C \,.\, Q'' = 3$ is biregular to a 3-dimensional component Z of stable singularities of Θ.*

(2) *If $z \in Z$ is general then the tangent cone Q_z of Θ at z is of rank* 6, *and the base locus of all these cones is the* (*unique*) *anticanonically embedded bidegree* $(2, 2)$ *threefold T birational to X.*

REMARK. – Equivalently, if $\Sigma \subset \mathcal{C}^1_{10}[7](X'')$ is the family of degenerate minimal sections of type $C + q_1 + q_2$, where $C \in \mathcal{C}^1_6[3](X'')$ and $q_1, q_2 \in \mathcal{C}^0_2[2](X'')$, then the Abel-Jacobi image of Σ is Z (see (6.1.5)).

PROOF. – By the proof of (6.3.7), it rests only to find the invariants g, d, m of the family $\mathcal{C}^g_d[m](X'')$ of these curves on X'' which are images of the curves on T which belong to the family $\mathcal{D} = \mathcal{C}^1_{3,\,3}(T)$ (see (6.1.5)).

The birational map $X'' \leftrightarrow T$ (preserving the conic bundle fibrations) sends the 4-dimensional family of sections $\mathcal{C}^1_4[1](X'')$ to 4-dimensional family $\mathcal{E}$ of sections of $p_1 \colon T \to \boldsymbol{P}^2$. Since the birational conic bundle map p'' on X'' projects the general $C^1_4 \in \mathcal{C}^1_4[1](X'')$ isomorphically onto a plane cubic, the general $E \in \mathcal{E}$ is an elliptic curve on T of bidegree $(3, d)$ for some $d \geqslant 1$. Therefore $d = 1$—otherwise $4 = \dim \mathcal{C}^1_4[1](X'') = \dim \mathcal{E} = 3 + d \geqslant 5$.

Let $E \in \mathcal{E}$ be general, and let f be a general fiber of p_1 intersecting E. The birational map $T \leftrightarrow X''$ induced by α, sends f isomorphically onto a fiber $q \in \mathcal{C}^0_2[2](X'')$, and E—onto some $C^1_4 \in \mathcal{C}^1_4[X'']$. Let $D \in \mathcal{D}$ be general. Since any element of $\mathcal{D}$ is numerically equivalent to $E + f$, the isomorphic image $C \subset X''$ of $D \subset T$ is numerically equivalent to $C^1_4 + q$. Therefore $g = 1$, $d = 6$, $m = 3$ q.e.d.

REFERENCES

[ACGH] E. ARBARELLO - M. CORNALBA - P. A. GRIFFITHS - J. HARRIS, *Geometry of Algebraic Curves,* Vol. I., Springer-Verlag, New York (1985).

[B] A. BEAUVILLE, *Variétés de Prym et jacobiennes intermédiaires,* Ann. de l'AENS, 4 ser., **10** (1977), 149-196.

[BM] S. BLOCH - J. P. MURRE, *On the Chow group of certain types of Fano threefolds,* Compositio Math., **39** (1979), 47-105.

[C] H. CLEMENS, *The quartic double solid revisited,* Proc. Symp. Pure Math., **53** (1991), 89-101.

[CG] H. Clemens - P. Griffiths, *The intermediate jacobian of the cubic threefold,* Ann. Math., **95** (1972), 281-356.

[De] O. Debarre, *Sur le theoreme de Torelli pour les solides doubles quartiques,* Compositio Math., **73** (1990), 161-187.

[FS] R. Friedman - R. Smith, *Degenerations of Prym varieties and intersections of three quadrics,* Invent. Math., **85** (1986), 615-635.

[H] R. Hartshorne, *Algebraic Geometry,* Springer-Verlag (1977).

[I1] A. Iliev, *The theta divisor of bidegree* $(2, 2)$ *threefold in* $\boldsymbol{P}^2 \times \boldsymbol{P}^2$, to appear in Pacific J. Math., Vol. **180**, No. 1 (1997), 57-88.

[I2] A. Iliev, *The Fano surface of the Gushel threefold,* Compositio Math., **94** (1994), 81-107.

[Isk1] V. A. Iskovskikh, *On the rationality problem for conic bundles,* Duke Math. J., **54**, 2 (1987), 271-294.

[Isk2] V. A. Iskovskih, *The double projection from a line on threedimensional Fano varieties...,* Math. U.S.S.R. Sbornik, **180**, 2 (1989), 260-278 (in Russian).

[K] J. Kollar, *Flops,* Nagoya Math. J., **113** (1989), 14-36.

[KMM] Y. Kawamata - K. Matsuda - K. Matsuki, *Introduction to the minimal model problem,* Algebraic Geometry, Sendai, June 24-25, 1985: Symp. Tokyo, Amsterdam (1987), 283-360.

[L] D. Logachev, *The Abel-Jacobi isogeny for the Fano threefolds of genus 6,* Constructive Algebraic Geometry, Yaroslavl, Vol. **200** (1982), 67-76 (in Russian).

[LN] H. Lange - M. S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves,* Math. Ann., **266** (1983), 55-72.

[Mo] S. Mori, *Threefolds whose canonical bundles are not numerically effective,* Ann. Math., **116** (1982), 133-176.

[Mer] A. S. Merkur′ev, *On the norm residue symbol of degree* 2, Soviet Math. Dokl., **24** (1981), 546-551.

[S] V. G. Sarkisov, *Birational automorphisms of conical fibrations,* Izv. Akad. Nauk SSSR, Ser. Mat., **44** (1980), 918-945 (in Russian).

[Se] W. Seiler, *Deformations of ruled surfaces,* J. Reine Angew. Math., **426** (1992), 203-219.

[T] A. Tikhomirov, *The Abel-Jacobi map of sextics of genus 3 on double spaces of* $\boldsymbol{P}^3$ *of index two,* Soviet Math. Dokl., **33**, 1 (1986), 204-206.

[Tju] A. Tjurin, *The middle Jacobian of three-dimensional varieties,* J. Soviet Math., **13**, 6 (1980), 707-744.

[Ve] A. Verra, *The Prym map has degree two on plane sextics,* preprint (1991).

[Vo] C. Voisin, *Sur la jacobienne intermédiaire du double solide d'indice deux,* Duke Math. J., **57**, 2 (1988), 629-646.

[W] G. E. Welters, *A theorem of Gieseker-Petri type for Prym varieties,* Ann. Sci. E.N.S., 4 ser., **18** (1985), 671-683.

[Z] A. A. Zagorskii, *Three-dimensional conic bundles,* Math. Notes Akad. Sci. USSR, **21** (1977), 420-427.

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