BOLLETTINO
UNIONE MATEMATICA ITALIANA

STEFANO MORTOLA, ROBERTO PEIRONE

The sum of periodic functions

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.2, p. 393–396.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_2_393_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 393-396

The Sum of Periodic Functions

STEFANO MORTOLA - ROBERTO PEIRONE

Sunto. – *Si prova che ogni polinomio in una variabile reale di grado n è somma di $n+1$ funzioni periodiche, ovviamente non tutte continue, e che ci sono funzioni di una variabile reale che non sono somma di un numero finito di funzioni periodiche.*

The periodic functions play a very important role in Mathematical Analysis. A natural problem regarding them is to write, if possible, a given function from $\boldsymbol{R}$ to $\boldsymbol{R}$ as a sum of periodic functions. It is clear that a function that is the sum of finitely many *continuous* periodic functions is bounded, thus a nonconstant polynomial cannot be written in this form, but in this paper we prove that if the functions are not required to be continuous, then the previous statement is no longer valid. Namely, we prove the following result: every polynomial of degree $n>0$ in one real variable is the sum of $n+1$, but not of n, periodic functions from $\boldsymbol{R}$ to $\boldsymbol{R}$; and also, some functions from $\boldsymbol{R}$ to $\boldsymbol{R}$, for example, $f(x)=e^x$, are not the sum of finitely many periodic functions. It is not difficult to see that, if $f_i\colon \boldsymbol{R}\to\boldsymbol{R}$, $i=1,2$, are periodic of period x_i, then the function $f=f_1+f_2$ satisfies: $f(a+x_1+x_2)-f(a+x_1)-f(a+x_2)+f(a)=0$, $\forall a\in\boldsymbol{R}$ (cf. Theorem 1), thus some functions (for example $f(x)=x^2$), are not the sum of two periodic functions. On the other hand, by writing a real number x in the form $x=\sum_{\alpha\in A} x_\alpha T_\alpha$, where $\{T_\alpha\colon \alpha\in A\}$, ($T_\alpha\neq T_\beta$ if $\alpha\neq\beta$), is a Hamel basis of $\boldsymbol{R}$ over $\boldsymbol{Q}$ and $x_\alpha\in\boldsymbol{Q}$, $\forall\alpha\in A$, it is easy to see that the function $f(x)=x$ is the sum of two periodic functions, for, given $\alpha_1,\alpha_2\in A$, with $\alpha_1\neq\alpha_2$, the function $f_1(x)=x_{\alpha_1}T_{\alpha_1}$ is periodic of period T_{α_2} (for example), and the function $f_2(x)=\sum_{\alpha\in A\setminus\{\alpha_1\}} x_\alpha T_\alpha$ is periodic of period T_{α_1}. In order to get the more general above-stated result, we will characterize the functions $f\colon \boldsymbol{R}\to\boldsymbol{R}$ which are the sum of finitely many periodic functions. The idea of the proof is again that of considering $\boldsymbol{R}$ as a vectorial space over $\boldsymbol{Q}$, but in a more sophisticated way.

We start by fixing some notation. If K is a set we denote by $|K|$ the cardinality of K. Given a function $f\colon \boldsymbol{R}\to\boldsymbol{R}$ and $x_i\in\boldsymbol{R}$, $i=1,\dots,n$, $n>0$, we define a function $\Delta(x_1, \ldots, x_n) f$: $\boldsymbol{R} \to \boldsymbol{R}$ by

$$\Delta(x_1, \ldots, x_n) f(a) = \sum_{K \subseteq \{1, \ldots, n\}} (-1)^{n-|K|} f(a + \sum_{i \in K} x_i) .$$

The following properties are easily verified and will be used in the following:

i) $\Delta(x) f(a) = f(a+x) - f(a)$.

ii) f is periodic of period x if and only if $\Delta(x) f = 0$.

iii) $\Delta(x_1, \ldots, x_n) f = \Delta(x_{\sigma_1}, \ldots, x_{\sigma_n}) f$ for every permutation σ of $\{1, \ldots, n\}$.

iv) $\Delta(x_1, \ldots, x_n)$ is linear from $\boldsymbol{R}^{\boldsymbol{R}}$ to $\boldsymbol{R}^{\boldsymbol{R}}$, $\boldsymbol{R}^{\boldsymbol{R}}$ denoting the set of functions from $\boldsymbol{R}$ to $\boldsymbol{R}$.

v) $\Delta(x_1, \ldots, x_n, x_{n+1}) f = \Delta(x_1, \ldots, x_n)(\Delta(x_{n+1}) f)$.

We now generalize ii) in this way:

THEOREM 1. – *If a function f: $\boldsymbol{R} \to \boldsymbol{R}$ is the sum of n functions f_i: $\boldsymbol{R} \to \boldsymbol{R}$, $i = 1, \ldots, n$, $n > 0$, periodic of period x_i, then $\Delta(x_1, \ldots, x_n) f = 0$.*

The converse is true provided $x_1, \ldots, x_n$ satisfy the following additional condition

$$x_i > 0 \qquad \forall i = 1, \ldots, n , \qquad \frac{x_i}{x_j} \notin \boldsymbol{Q} \qquad \text{if } i \neq j . \tag{1}$$

We must require a condition like this because, for example, the function $f(x) = x$ satisfies $\Delta(1, 2) f = 0$, but it is not the sum of two periodic functions, of periods 1 and 2.

PROOF OF THEOREM 1. – For every $i = 1, \ldots, n$ we have $\Delta(x_i) f_i = 0$, and thus $\Delta(x_1, \ldots, x_n) f_i = 0$. Therefore, $\Delta(x_1, \ldots, x_n) f = \sum_{i=1}^{n} \Delta(x_1, \ldots, x_n) f_i = 0$. ■

To reverse Theorem 1, we need a lemma.

LEMMA. – *Suppose $x, y \in \boldsymbol{R} \setminus \{0\}$, $x/y \notin \boldsymbol{Q}$, and g: $\boldsymbol{R} \to \boldsymbol{R}$ is periodic of period x. Then there exists f: $\boldsymbol{R} \to \boldsymbol{R}$, periodic of period x, such that $\Delta(y) f = g$.*

PROOF. – Let B be a basis of $\boldsymbol{R}$ over $\boldsymbol{Q}$, containing $\{x, y\}$. Now, for every $a \in \boldsymbol{R}$ there exist unique $p(a) \in \boldsymbol{R}$, $n(a) \in \boldsymbol{Z}$ such that $a = p(a) + n(a)\, y$, $p(a)$ of the form $p(a) = b + qy$, with $q \in [0, 1[\cap \boldsymbol{Q}$ and b linear combination with coefficients in $\boldsymbol{Q}$ of elements of $B \backslash \{y\}$. It follows:

$$p(a+y)=p(a), \quad n(a+y)=n(a)+1, \quad p(a+x)=p(a)+x, \quad n(a+x)=n(a).$$

It is easy to see that the function f defined by

$$f(a) = \begin{cases} \sum_{k=0}^{n(a)-1} g(p(a)+ky) & \text{if } n(a) > 0, \\ 0 & \text{if } n(a) = 0, \\ -\sum_{k=n(a)}^{-1} g(p(a)+ky) & \text{if } n(a) < 0 \end{cases}$$

satisfies our requirements. ■

THEOREM 2. – *Suppose* $x_1, \ldots, x_n$, $n > 0$, *satisfy* (1), *and* f: $\boldsymbol{R} \to \boldsymbol{R}$ *satisfies* $\Delta(x_1, \ldots, x_n) f = 0$. *Then there exist* f_i: $\boldsymbol{R} \to \boldsymbol{R}$, *periodic of period* x_i, *such that* $f = \sum_{i=1}^{n} f_i$.

PROOF. – The theorem is clearly true if $n = 1$. Suppose it is true for $n \geqslant 1$ and prove that it is true for $n+1$. Let $x_1, \ldots, x_{n+1}$ satisfy (1) and suppose $\Delta(x_1, \ldots, x_{n+1}) f = 0$. Then $\Delta(x_1, \ldots, x_n)(\Delta(x_{n+1}) f) = 0$, thus by the inductive hypothesis, there exist g_i: $\boldsymbol{R} \to \boldsymbol{R}$ periodic of period x_i such that $\Delta(x_{n+1}) f = \sum_{i=1}^{n} g_i$. By the lemma there exist f_i, periodic of period x_i such that $\Delta(x_{n+1}) f_i = g_i$, for $i = 1, \ldots, n$. Let $g = \sum_{i=1}^{n} f_i$. Then, $\Delta(x_{n+1}) g = \sum_{i=1}^{n} \Delta(x_{n+1}) f_i = \Delta(x_{n+1}) f$, and $\Delta(x_{n+1})(f-g) = 0$. Thus, $f = \sum_{i=1}^{n+1} f_i$, where $f_{n+1} = f - g$ is periodic of period x_{n+1}. Hence the theorem is true for $n+1$. ■

Now, if f_i: $\boldsymbol{R} \to \boldsymbol{R}$, $i = 1, \ldots, n$, $n > 0$, are periodic of period $x_i > 0$, and $x_i / x_j \in \boldsymbol{Q}$ for all i, j, then $\sum_{i=1}^{n} f_i$ is periodic, with period any real number $\neq 0$ which is an integer multiple of all x_i. Thus

THEOREM 3. – *A function* f: $\boldsymbol{R} \to \boldsymbol{R}$ *is the sum of finitely many periodic functions if and only if there exist* $x_1, \ldots, x_n$, $n > 0$, *satisfying* (1) *such that* $\Delta(x_1, \ldots, x_n) f = 0$. ■

In order to see whether a given function from $\boldsymbol{R}$ to $\boldsymbol{R}$ is the sum of n periodic functions, we express $\Delta(x_1, \ldots, x_n) f$ in terms of the nth derivative $f^{(n)}$ of f. By repeated applications of the mean value theorem we get

THEOREM 4. – *Suppose* $f \in C^{\infty}(\boldsymbol{R})$. *Then, for every* $a, x_1, \dots, x_n \in \boldsymbol{R}$, *with* $x_i > 0$ *for* $i = 1, \dots, n$, $n > 0$, *there exists* $\xi \in]a, a + \sum_{i=1}^{n} x_i[$, *such that*

$$\Delta(x_1, \dots, x_n) f(a) = f^{(n)}(\xi) \prod_{i=1}^{n} x_i. \qquad \blacksquare$$

COROLLARY. – *Let* P *be a polynomial in one real variable of degree* $n > 0$. *Then* P *can be written as the sum of* $n+1$ *periodic functions, and the periods of these functions can be chosen arbitrary* $n+1$ *real numbers, satisfying* (1). *On the other hand,* P *is not the sum of* n *periodic functions. The function* $f(x) = e^x$, $f: \boldsymbol{R} \to \boldsymbol{R}$, *is not the sum of finitely many periodic functions.* $\blacksquare$

REMARK. – The question whether every function $f: \boldsymbol{R} \to \boldsymbol{R}$ is the sum of two periodic functions was proposed in an Italian mathematical competition, Cortona, Italy, May 1988.

S. Mortola: Politecnico di Milano, Dipartimento di Matematica
P. Leonardo da Vinci 32, 20133 Milano, Italy
e-mail: stemor@mate.polimi.it

R. Peirone: Università di Roma-Tor Vergata, Dipartimento di Matematica
via della Ricerca Scientifica, 00133, Roma, Italy
e-mail: peirone@mat.uniroma2.it

Pervenuta in Redazione
il 21 Luglio 1997