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On the Magnetohydrodynamic Type Equations in a New Class of Non-Cylindrical Domains.

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Sunto. – *Viene provata l'esistenza e l'unicità delle soluzioni deboli per un sistema di equazioni della magnetoidrodinamica in un dominio variabile. Per la dimostrazione si usano il metodo di Galerkin spettrale e la tecnica introdotta da Dal Passo e Ughi per trattare i problemi con dominio dipendente dal tempo.*

1. – Introduction.

The motion of an incompressible conducting fluid can be studied by the magnetoyhdrodynamics equations, which are a coupling of the Navier-Stokes' equations with the Maxwell's equations. In several physical situations the effect of the electric field can be neglected, see Jackson [5] and Eringen-Maugin [2]. If there is free motion of heavy ions, see Schlüter [14], the equations can be reduced to the following system

$$(1)\quad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \dfrac{\eta}{\varrho_m}\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \dfrac{\mu}{\varrho_m}(\boldsymbol{h}\cdot\nabla)\,\boldsymbol{h} = \boldsymbol{f} - \dfrac{1}{\varrho_m}\nabla\left(p^* + \dfrac{\mu}{2}|\boldsymbol{h}|^2\right) & \text{in } D\,, \\ \dfrac{\partial \boldsymbol{h}}{\partial t} - \dfrac{1}{\mu\sigma}\Delta \boldsymbol{h} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{h} - (\boldsymbol{h}\cdot\nabla)\,\boldsymbol{u} = -\nabla\omega & \text{in } D\,, \\ \nabla\cdot\boldsymbol{u} = 0\,, \qquad \nabla\cdot\boldsymbol{h} = 0 & \text{in } D\,, \\ \boldsymbol{u} = 0\,, \qquad \boldsymbol{h} = 0 & \text{on } \partial D\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{h}(0) = \boldsymbol{h}_0 & \text{in } D(0)\,, \end{cases}$$

where

$$(2)\qquad D = \bigcup_{0<t<T} D(t)\times\{t\}\,, \qquad D(t)\subset \boldsymbol{R}^n\,.$$

In the system (1) $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{h} = (h_1, \ldots, h_n)$ are respectively the vel-

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ocity and the magnetic field; p^* is the unknown hydrostatic pressure and ω is a unknown function, related to the motion of heavy ions, such that $\nabla \times \boldsymbol{j} = -\sigma \nabla \omega$, where $\boldsymbol{j} = (j_1, \ldots, j_n)$ is the density of electric current. Furthermore ϱ_m is the mass density of the fluid, μ is the magnetic permeability of the medium, σ is the electric conductivity and η is the kinematic viscosity of the fluid (this four quantities will be assumed to be positive constants) and $\boldsymbol{f}$ is a given external force field.

In this paper we will consider the problem of the existence of weak solutions in a time dependent domain of $\boldsymbol{R}^n \times [0, T]$. The initial value problem (1) has been studied by several authors in a cylindrical domain. Lassner [4], by using the semi group results of Kato and Fujita [15], proved the local existence and uniqueness of strong solutions. Boldrini and Rojas-Medar [6] and [12] improved this results to global solutions by using the spectral Galerkin method.

For time dependent domains there are some results regarding special kinds of open sets. Lions [7], [8] and Fujita and Sauer [3] proved the existence of solutions for various classes of nonlinear problems (Navier-Stokes' and Boussinesq's equations for example), under the assumption that the domain is non-decreasing in t. Rojas-Medar and Beltrán Barrios [11] proved the local existence of weak solution of problem (1) in the more general situation in which the domain is a ball of variable (not zero) radius. They used the techniques introduced by Dal Passo and Ughi [13]: a suitable change of variable allows to use a Galerkin approximation for another boundary value problem in a domain whose section are not time dependent. The proof ends coming back to the non-cylindrical domain by the inverse of the above change of variable.

By using the same tools and by using suitable Sobolev spaces (needed to deal with our domain) we will prove the existence of global weak solution for a more general class of time dependent domains introduced by Limaco [10].

In section 2 we will set the problem in all the details; in section 3 we will prove an existence theorem (Theorem 2.1) and in section 4 we will give a uniqueness result (Theorem 2.2).

2. – Setting of the problem.

In this section we define all the function spaces we'll use to deal with problems in non-cylindrical domains. This spaces are needed to make rigorous definitions of weak solutions. Some results (namely isomorphisms between suitable Banach spaces) are claimed and since they are not completely trivial we try to make a little survey to make the paper self contained. Since the proofs of this results are a bit technical we postpone them to the appendix, because they are not necessary in a first reading.

We consider Ω a smooth, bounded and connected open subset of $\boldsymbol{R}^n$, $n \geq 2$. We define a family of open sets with parameter t with

(3) $\Omega(t) := (x_1, \ldots, x_n) = (a_1 y_1 R(t) + g_1(t), \ldots, a_n y_n R(t) + g_n(t))\,, \qquad y \in \Omega\,,$

with $a_i > 0$, $i = 1, \ldots, n$ and

(4) $$R(t),\, g(t) \in C^1([0, T]) \quad \text{with} \quad \min_{0 \leq t \leq T} R(t) > 0\,.$$

We can define the non-cylindrical domain D and its boundary as

(5) $$D = \bigcup_{0<t<T} (\Omega(t) \times \{t\})\,, \qquad \partial D = \bigcup_{0<t<T} (\partial\Omega(t) \times \{t\})\,.$$

We set as usual

(6) $$\mathcal{V}(\Omega) = \left\{\phi \in (\mathcal{D}_0^\infty(\Omega))^n \colon \nabla \cdot \phi = 0\right\}\,,$$

and $H(\Omega)$ and $V_s(\Omega)$ will be respectively the closure of $\mathcal{V}$ in $(L^2(\Omega))^n = (H^0(\Omega))^n$ and in $(H^s(\Omega))^n$. The spaces $H^s(\Omega)$, $s \geq 0$ are the usual Sobolev's spaces, with inner product and norm

(7) $$(\boldsymbol{u}, \boldsymbol{w})_s = \sum_{i=1}^n (u_i, w_i)_s, \qquad \|\boldsymbol{u}\|_s = (\boldsymbol{u}, \boldsymbol{u})_s^{1/2},$$

where $(u_i, w_i)_s$ is the standard inner product of $H^s(\Omega)$. We also denote with $(V_s(\Omega))'$ the topological dual of $V_s(\Omega)$ and $V(\Omega) = V_1(\Omega)$; for further details we refer to the books by Adams [1] and Ladyzhenskaya [9].

In the sequel we will use some special spaces, strictly related with the open sets $\Omega(t)$. We define, for $a_i > 0$, $i = 1, \ldots, n$

(8) $$\widetilde{\mathcal{V}}(\Omega) := \left\{\phi \in (\mathcal{D}_0^\infty(\Omega))^n \colon \sum_{i=1}^n \frac{1}{a_i} \frac{\partial \phi_i}{\partial x_i} = 0\right\},$$

and we define, in analogy with the usual spaces used in hydrodynamics, $\widetilde{H}(\Omega)$ and $\widetilde{V}_s(\Omega)$ as the closure of $\widetilde{\mathcal{V}}(\Omega)$ respectively in $(L^2(\Omega))^n$ and in $(H^s(\Omega))^n$.

If we set

(9) $$T : V(\Omega) \to \widetilde{V}(\Omega) \;\; \text{defined as} \;\; (\mathrm{u}_1, \ldots, \mathrm{u}_n) \mapsto (\mathrm{a}_1 \mathrm{u}_1, \ldots, \mathrm{a}_n \mathrm{u}_n)$$

we have the following proposition

PROPOSITION 2.1. – *The map T is an isomorphism between $V(\Omega)$ and $\widetilde{V}(\Omega) = \widetilde{V}_1(\Omega)$.*

Let X a Banach space with norm $\|.\|_X$, we denote, as usual, by $L^p(0, T; X)$, $1 \leq p \leq \infty$ the Banach space of vector valued functions $\boldsymbol{u} : (0, T) \to X$ which are strongly measurable and such that the application $t \to \|\boldsymbol{u}(t)\|_X$, defined a.e. in $(0, T)$ belongs to $L^p(0, T)$. The norm in $L^p(0, T; X)$, is defined by

$$\|\boldsymbol{u}\|_{L^p(0, T; X)} = \left[\int_0^T \|u(t)\|_X^p \, dt \right]^{\frac{1}{p}}, \qquad 1 \leq p < +\infty,$$

and

$$\|\boldsymbol{u}\|_{L^\infty(0, T; X)} = \text{ess} \sup_{0<t<T} \|\boldsymbol{u}(t)\|_X.$$

The generalization of this classical definition to the case of vector valued functions, with values on space of functions defined on $\Omega(t)$ is the following:

DEFINITION. – Let B a ball of $\mathcal{R}^n$ such that $\Omega(t) \subset B$, $\forall t \in [0, T]$, we define

$$L^p(0, T; V(\Omega(t))) := \left\{ u \in L^p(0, T; V(B)) : u(x, t) = 0 \text{ a.e. in } B \backslash \Omega(t),\ t \in (0, T) \right\},$$

with norm

$$\|\boldsymbol{u}\|_{L^p(0, T; V(\Omega(t)))} = \left[\int_0^T \|u(t)\|_{V(\Omega(t))}^p \, dt \right]^{\frac{1}{p}}, \qquad 1 \leq p < +\infty,$$

and

$$\|\boldsymbol{u}\|_{L^\infty(0, T; V(\Omega(t)))} = \text{ess} \sup_{0<t<T} \|\boldsymbol{u}(t)\|_{V(\Omega(t))},$$

and in the same way we also define $L^p(0, T; H(\Omega(t)))$.

We now quote some results that will make clear the meaning of the forthcoming definition of weak solutions of (1).

LEMMA 2.1. – *The Banach space* $L^p(0, T; V(\Omega(t)))$ *is a closed subspace of* $L^p(0, T; V(B))$ *for* $1 \leq p \leq \infty$.

This result is straightforward since let $\{\boldsymbol{u}_n\} \subset L^p(0, T; V(\Omega(t)))$ be such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in the topology of $L^p(0, T; V(B))$. We have that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^r(0, T; L^r(B))$ with $r = \min\{p, 2\}$. Therefore $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^r(B \times (0, T))$, and, up to a subsequence, $\boldsymbol{u}_n \to \boldsymbol{u}$ a.e. in $B \times (0, T)$. From the previous definition we have that $\boldsymbol{u}_n(x, t) = 0$ a.e. in $(B \times (0, T)) \backslash D$, consequently $\boldsymbol{u}(x, t) = 0$ a.e. in $(B \times (0, T)) \backslash D$ and $\boldsymbol{u} \in L^p(0, T; V(D(t)))$.

REMARK 2.1. – Since $\boldsymbol{u}(x, t) \in V(\Omega(t))$ a.e. $t \in (0, T)$ from the definition of $L^p(0, T; V(\Omega(t)))$ we can see that $\boldsymbol{u}(\cdot, t) = 0$ a.e. in $B \backslash \Omega(t)$. If we define

$$\tilde{\boldsymbol{u}}(\cdot, t) = \begin{cases} \boldsymbol{u}(\cdot, t) & \text{in } B, \\ 0 & \mathbb{R}^n \backslash B, \end{cases}$$

we obtain that $\tilde{\boldsymbol{u}}(\cdot, t) \in H^1(\mathbb{R}^n)$ and from the regularity of $\Omega(t)$ we get that $\tilde{\boldsymbol{u}}(\cdot, t)_{|\Omega(t)} = \boldsymbol{u}(\cdot, t) \in H_0^1(\Omega(t))$ and $\nabla \cdot \boldsymbol{u}(\cdot, t) = 0$ in $\Omega(t)$. A similar result holds also for $\boldsymbol{u} \in L^p(0, T; H(\Omega(t)))$.

In the following 4 proposition we explain the basic properties of the Banach spaces defined above

PROPOSITION 2.2. – *Let us consider the definition of D and let us set $f_i(t) = a_i R(t)$, $i = 1, \ldots, n$. The application $\mathcal{T}: L^2(0, T; \tilde{V}(\Omega)) \to L^2(0, T; V(\Omega(t)))$ defined by*

$$(11) \qquad (\mathcal{T}\boldsymbol{v})(x, t) = \boldsymbol{u}(x, t) = \boldsymbol{v}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_n - g_n(t)}{f_n(t)}, t\right)$$

is an isomorphism.

PROPOSITION 2.3. – *The application $\mathcal{T}$ defined by (11) is an isomorphism between $L^\infty(0, T; \tilde{H}(\Omega))$ and $L^\infty(0, T; H(\Omega(t)))$.*

PROPOSITION 2.4. – *The application $\mathcal{S}: p \to g$ defined by*

$$(\mathcal{S}p)(x, t) = g(x, t) = p\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_n - g_n(t)}{f_n(t)}, t\right),$$

is an isomorphism between $L^\infty(\Omega \times (0, T))$ and $L^\infty(D)$.

REMARK 2.2. – We note that

$$\sum_{j=1}^n \int_\Omega \frac{1}{a_j} \frac{\partial \phi}{\partial y_j v_j} dx = -\sum_{j=1}^n \int_\Omega \frac{1}{a_j} \phi \frac{\partial v_j}{\partial y_j} dx = 0, \qquad \forall \phi \in H^1(\Omega), \quad \forall \boldsymbol{v} \in \tilde{\mathrm{V}}(\Omega)$$

and that

$$\sum_{i,j=1}^n \int_\Omega \frac{1}{a_j} u_i \frac{\partial v_i}{\partial y_j} w_i \, dx = -\sum_{i,j=1}^n \int_\Omega \frac{1}{a_j} u_i \frac{\partial w_i}{\partial y_j} v_i \, dx, \qquad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \tilde{V}(\Omega).$$

We also point out that, as in the usual space V, we have

$$(12) \qquad \sum_{i,j=1}^n \int_\Omega \frac{1}{a_j} u_i \frac{\partial v_i}{\partial y_j} v_i \, dx = 0, \qquad \forall \boldsymbol{v} \in \tilde{V}(\Omega).$$

With the definitions above mentioned and from the properties claimed in the Propositions 2.1-2.4 we can define the notion of weak solution for system (1)

DEFINITION. – We say that a couple of vector valued function $(\boldsymbol{u}, \boldsymbol{h})$, defined in D is a weak solution of system (1) if

$$(\boldsymbol{u}, \boldsymbol{h}) \in [L^2(0, T; V(\Omega(t))) \cap L^\infty(0, T; H(\Omega(t)))]^2, \tag{13}$$

$$-\int_D \sum_{i,j=1}^n \left[u_i \frac{\partial \phi_i}{\partial t} - \frac{\eta}{\varrho_m} \frac{\partial u_j}{\partial x_i} \frac{\partial \phi_j}{\partial x_i} + u_j \frac{\partial \phi_i}{\partial x_j} u_i - \frac{\mu}{\varrho_m} h_j \frac{\partial \phi_i}{\partial x_j} h_i \right] dx\, dt = \int_D \sum_{i=1}^n f_i \phi_i \, dx\, dt\,, \tag{14}$$

$$-\iint_D \left[\sum_{i,j=1}^n h_i \frac{\partial \psi}{\partial t} - \frac{1}{\mu\sigma} \frac{\partial h_j}{\partial x_i} \frac{\partial \psi_j}{\partial x_i} + u_j \frac{\partial \psi_i}{\partial x_j} h_i - h_j \frac{\partial \psi_i}{\partial x_j} u_i \right] dx\, dt = 0\,, \tag{15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0\,, \qquad \boldsymbol{h}(0) = \boldsymbol{h}_0, \tag{16}$$

$\forall \phi, \psi \in (C_0^1(D))^n$ with $\nabla \cdot \phi = \nabla \cdot \psi = 0$, the initial conditions have the usual meaning, as in Temam [16]. The main results of this paper are the following theorems that are the natural extension of the results holding for cylindrical domains: we have existence of weak solutions in every time interval $[0, T]$, but uniqueness only in the two dimensional case.

We will prove the following results:

THEOREM 2.1. – *Assume that* $n \geq 2$, $\boldsymbol{f} \in L^2(0, T; H(\Omega(t)))$ *and* $(\boldsymbol{u}_0, \boldsymbol{h}_0) \in [H(\Omega(0))]^2$ *then there exists at least one weak solution of problem* (1).

THEOREM 2.2. – *If* $n = 2$ *the solution* $(\boldsymbol{u}, \boldsymbol{h})$ *obtained in Theorem* 2.1 *is unique. Moreover* $\boldsymbol{u}$ *and* $\boldsymbol{h}$ *are almost everywhere equal to continuous functions from* $[0, T]$ *to* $H(\Omega(t))$ *and*

$$\boldsymbol{u}(t) \overset{t \to 0}{\to} \boldsymbol{u}_0 \qquad \boldsymbol{h}(t) \overset{t \to 0}{\to} \boldsymbol{h}_0 \qquad in\ H(B)\,. \tag{17}$$

3. – Proof of Theorem 2.1.

We define the transformation $\theta : D \to \Omega \times (0, T)$ given by

$$(x_1, \ldots, x_n, t) \mapsto \left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_n - g_n(t)}{f_n(t)}, t \right). \tag{18}$$

Since $R(t)$ and $g_i(t)$ satisfy (4), we easily see that θ is a diffeomorphism and its inverse θ^{-1}: $\Omega \times (0, T) \to D$ satisfies

$$\theta^{-1}(y, t) = (f_1(t)y_1 + g_1(t), \dots, a_n f_n(t) + g_n(t), t).$$

We define the following functions

$$(19)\quad \begin{cases} \boldsymbol{v}(y, t) = \boldsymbol{u}(\theta^{-1}(y, t)), & \boldsymbol{b}(\mathrm{y}, \mathrm{t}) = \boldsymbol{h}(\theta^{-1}(y, t)), \quad \pi(y, t) = p(\theta^{-1}(y, t)), \\ \boldsymbol{F}(y, t) = \boldsymbol{f}(\theta^{-1}(y, t)), & \zeta(y, t) = \omega(\theta^{-1}(y, t)), \quad \boldsymbol{v}_0(y) = \boldsymbol{u}_0(\theta^{-1}(y, 0)), \\ \boldsymbol{b}_0(y) = \boldsymbol{h}_0(\theta^{-1}(y, 0)). \end{cases}$$

By using (19) and by setting

$$\alpha = \frac{\varrho_m}{\mu}, \qquad \nu = \frac{\eta}{\mu}, \qquad \gamma = \frac{1}{\mu\sigma},$$

the system (1) is transformed into

$$(20)\quad \alpha \frac{\partial v_i}{\partial t} - \nu \sum_{j=1}^{n} \frac{1}{f_j^2(t)} \frac{\partial^2 v_i}{\partial y_j^2} - \sum_{j=1}^{n} \frac{1}{f_j(t)} b_j \frac{\partial b_i}{\partial y_j} + \alpha \sum_{j=1}^{n} \frac{1}{f_j(t)} v_j \frac{\partial v_i}{\partial y_j} =$$
$$= \alpha \sum_{j=1}^{n} \frac{f_j'(t)\, y_j + g_j'(t)}{f_j(t)} \frac{\partial v_i}{\partial y_j} + \alpha F_i - \frac{1}{\mu f_i(t)} \frac{\partial \pi}{\partial y_i} + \frac{1}{2 f_i(t)} \frac{\partial |\boldsymbol{b}|^2}{\partial y_i},$$

$$(21)\quad \frac{\partial b_i}{\partial t} - \gamma \sum_{j=1}^{n} \frac{1}{f_j^2(t)} \frac{\partial^2 b_i}{\partial y_j^2} + \sum_{j=1}^{n} \frac{1}{f_j(t)} v_j \frac{\partial b_i}{\partial y_j} - \sum_{j=1}^{n} \frac{1}{f_j(t)} b_j \frac{\partial v_i}{\partial y_j} =$$
$$= \alpha \sum_{j=1}^{n} \frac{f_j'(t)\, y_j + g_j'(t)}{f_j(t)} \frac{\partial b_i}{\partial y_j} - \frac{1}{f_i(t)} \frac{\partial \zeta}{\partial y_i},$$

$$(22)\quad \sum_{i=1}^{n} \frac{1}{a_i} \frac{\partial v_i}{\partial y_i} = 0, \qquad \sum_{i=1}^{n} \frac{1}{a_i} \frac{\partial b_i}{\partial y_i} = 0 \quad \text{in } \Omega \times (0, \mathrm{T}),$$

$$(23)\quad \boldsymbol{v}(y, 0) = \boldsymbol{v}_0, \qquad \boldsymbol{b}(y, 0) = \boldsymbol{b}_0 \quad \text{in } \Omega,$$

$$(24)\quad \boldsymbol{v}(y, t) = 0, \qquad \boldsymbol{b}(y, t) = 0 \quad \text{on } \partial\Omega \times (0, T).$$

We now define the notion of weak solution for problem (20)-(24)

DEFINITION. – We say that a a couple of vector valued function $(\boldsymbol{v}, \boldsymbol{b})$ is a weak solution of system (20)-(24) if

$$(25)\quad (\boldsymbol{v}, \boldsymbol{b}) \in \left[L^2(0, T; \widetilde{V}(\Omega)) \cap L^\infty(0, T; \widetilde{H}(\Omega))\right]^2,$$

$$(26)\quad -\int_0^T \int_\Omega \sum_{i,j=1}^{n} \left[\alpha v_i \frac{\partial \widetilde{\phi}_i}{\partial t} - \nu \frac{\alpha}{f_i^2(t)} \frac{\partial v_j}{\partial y_i} \frac{\partial \widetilde{\phi}_j}{\partial y_i} + \frac{\alpha}{f_i(t)} v_j \frac{\partial \widetilde{\phi}_i}{\partial y_j} v_i - \frac{1}{f_i(t)} b_j \frac{\partial \widetilde{\phi}_i}{\partial y_j} b_i \right] dy\, dt =$$
$$\int_0^T \int_\Omega \sum_{i=1}^{n} \alpha F_i \widetilde{\phi}_i \, dy\, dt,$$

$$-\int_0^T \int_\Omega \left[\sum_{i,j=1}^n b_i \frac{\partial \tilde{\psi}}{\partial t} - \frac{\gamma}{f_i^2(t)} \frac{\partial b_j}{\partial y_i} \frac{\partial \tilde{\psi}_j}{\partial y_i} + \frac{1}{f_i(t)} v_j \frac{\partial \tilde{\psi}_i}{\partial y_j} b_i - \frac{1}{f_i(t)} b_j \frac{\partial \tilde{\psi}_i}{\partial y_j} v_i \right] dy\, dt = 0, \tag{27}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \qquad \boldsymbol{b}(0) = \boldsymbol{b}_0 \tag{28}$$

$\forall \tilde{\phi}, \tilde{\psi} \in (C_0^1(\Omega \times (0, T)))^n$ with $\nabla \cdot \tilde{\phi} = \nabla \cdot \tilde{\psi} = 0$.

We start the proof of Theorem 2.1 by proving the following lemma

LEMMA 3.1. – *Let* $(\boldsymbol{v}_0, \boldsymbol{b}_0) \in [\tilde{H}(\Omega)]^2$ *then for every* $\boldsymbol{F} \in L^2(0, T; \tilde{H}(\Omega))$ *there exists a weak solution of problem* (20)-(24).

PROOF OF LEMMA 3.1. – Fix $s = n/2$ and let $\{\tilde{\omega}^i\}$ be a spectral basis of $\tilde{V}_s(\Omega)$, whose elements are solutions of

$$(\tilde{\omega}^i, v)_s = \lambda_i (\tilde{\omega}^i, v), \qquad \forall v \in \tilde{V}_s(\Omega). \tag{29}$$

Let $\tilde{V}_k$ the subspace of $\tilde{V}_s(\Omega)$ spanned by $\{\tilde{\omega}^1, \ldots, \tilde{\omega}^k\}$ and define the approximations of $\boldsymbol{v}$ and $\boldsymbol{b}$ by means of

$$\boldsymbol{v}^k = \sum_{i=1}^k e_i^k(t)\, \tilde{\omega}^i, \qquad \boldsymbol{b}^k = \sum_{i=1}^k l_i^k(t)\, \tilde{\omega}^i, \tag{30}$$

where the coefficients $e_i^k(t)$, $l_i^k(t)$ will be calculated in such a way that $\boldsymbol{v}^k$ and $\boldsymbol{b}^k$ solve the following system of ordinary differential equations

$$-\int_0^T \int_\Omega \sum_{i,j=1}^n \left[\alpha v_i^k \frac{\partial \tilde{\phi}_i}{\partial t} - \nu \frac{1}{f_i^2(t)} \frac{\partial v_j^k}{\partial y_i} \frac{\partial \tilde{\phi}_j}{\partial y_i} + \right. \tag{31}$$

$$\left. \frac{\alpha}{f_i(t)} v_j^k \frac{\partial \tilde{\phi}_i}{\partial y_j} v_i^k - \frac{1}{f_i(t)} b_j^k \frac{\partial \tilde{\phi}_i}{\partial y_j} b_i^k \right] dy\, dt = \int_0^T \int_\Omega \sum_{i=1}^n \alpha F_i \tilde{\phi}_i \, dy\, dt,$$

$$-\int_0^T \int_\Omega \left[\sum_{i,j=1}^n b_i^k \frac{\partial \tilde{\psi}}{\partial t} - \frac{\gamma}{f_i^2(t)} \frac{\partial b_j^k}{\partial y_i} \frac{\partial \tilde{\psi}_j}{\partial y_i} + \right. \tag{32}$$

$$\left. \frac{1}{f_i(t)} v_j^k \frac{\partial \tilde{\psi}_i}{\partial y_j} b_i^k - \frac{1}{f_i(t)} b_j^k \frac{\partial \tilde{\psi}_i}{\partial y_j} v_i^k \right] dy\, dt = 0,$$

$$\boldsymbol{v}^k(0) = \boldsymbol{v}_0^k, \qquad \boldsymbol{b}^k(0) = \boldsymbol{b}_0^k, \tag{33}$$

for each $\tilde{\phi}, \tilde{\psi} \in \tilde{V}_k$ with

$$\boldsymbol{v}^k(0) \to \boldsymbol{v}_0^k \quad \text{and} \quad \boldsymbol{b}^k(0) \to \boldsymbol{b}_0^k \ \text{ in } \tilde{H}(\Omega). \tag{34}$$

We have to study a system of ordinary differential equations for the coefficients $(e_i^k(t),\ l_i^k(t))$. To prove the existence of a local solution we will prove a-priori estimates independent on t and k, that allows us to take $t_k = T$ (where t_k is the interval of existence of $(e_i^k(t),\ l_i^k(t))$ for a fixed k) and to pass to the limit as k goes to infinity.

By setting $\widetilde{\phi} = \boldsymbol{v}^k$ and $\widetilde{\psi} = \boldsymbol{b}^k$ in (31)-(32) and by adding together the resulting equations, we get

$$\frac{1}{2}\frac{d}{dt}(\alpha\|\boldsymbol{v}^k\|^2 + \|\boldsymbol{b}^k\|^2) + \sum_{i=1}^{n} \frac{\nu}{[f_i(t)]^2}\left\|\frac{\partial v_i^k}{\partial y_i}\right\|^2 + \frac{\gamma}{[f_i(t)]^2}\left\|\frac{\partial b_i^k}{\partial y_i}\right\|^2 = \tag{35}$$

$$= \sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{y_i f_i'(t) + g_i'(t)}{f_i(t)}\right)\frac{\partial v_j^k}{\partial y_i} v_j^k + \sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{y_i f_i'(t) + g_i'(t)}{f_i(t)}\right)\frac{\partial b_j^k}{\partial y_i} b_j^k,$$

since by Remark 2.2 all the «bad» nonlinear terms disappear.

We also note that

$$\sum_{i,j=1}^{n}\int_{\Omega}\frac{1}{f_j^2(t)}\frac{\partial v_i^k}{\partial y_j}\frac{\partial v_i^k}{\partial y_j}dx \geqslant A\|\nabla v^k\|_0^2,$$

where $A = \min\limits_{1\leqslant i\leqslant n,\ 0\leqslant t\leqslant T} f_i(t)$.

By Hölder's inequality we can infer that

$$\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{y_i f_i'(t) + g_i'(t)}{f_i(t)}\right)\frac{\partial v_j^k}{\partial y_i} v_j^k \leqslant \tag{36}$$

$$\max_{1\leqslant i\leqslant n}\left(\|\boldsymbol{y}\|_\infty \frac{|f_i'(t)|}{|f_i(t)|} + \frac{|g_i'(t)|}{|f_i(t)|}\right)\|\nabla \boldsymbol{v}^k\|\,\|\boldsymbol{v}^k\|,$$

where $\|\boldsymbol{y}\|_\infty$ denotes the diameter of Ω. By using the Schwartz's inequality we have immediately

$$|(F,\ \boldsymbol{v}^k)| \leqslant \|F\|\,\|\boldsymbol{v}^k\|. \tag{37}$$

At the end we have

$$\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{v}^k\|^2 + \|\boldsymbol{b}^k\|^2) + \|\nabla\boldsymbol{v}^k\|^2 + \|\nabla\boldsymbol{b}^k\|^2 \leqslant K_1\|\boldsymbol{v}^k\|^2 + K_2\|\boldsymbol{b}^k\|^2, \tag{38}$$

in which K_1 depends on $(\alpha,\ \nu,\ \gamma$, $\min f_i(t)$, $\max f_i'(t)$, $g_i'(t)$ and $\|F\|)$.

By integrating (38) between 0 and t we can conclude in a standard way that $(\boldsymbol{v}^k,\ \boldsymbol{b}^k)$ exists for any $t \in [0,\ T]$ and that

$$\{(\boldsymbol{v}^k,\ \boldsymbol{b}^k)\} \text{ is bounded in } \left[L^2(0,\ T; \widetilde{V}(\Omega)) \cap L^\infty(0,\ T; \widetilde{H}(\Omega))\right]^2. \tag{39}$$

We will now prove that $\{(\boldsymbol{v}^k, \boldsymbol{b}^k)\}$ is bounded in $[L^2(0, T; (\widetilde{V}_s(\Omega))')]^2$.
We define some time dependent operators. We set

$$\langle A(t)\, \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{i,j=1}^{n} \frac{1}{f_i^2(t)} \int_{\Omega} \frac{\partial v_j}{\partial y_i} \frac{\partial w_j}{\partial y_i},$$

$$\langle B(t)\boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{i,j=1}^{n} \frac{1}{f_i(t)} \int_{\Omega} v_i \frac{\partial v_j}{\partial y_i} w_i,$$

$$\langle C(t)\, \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{i,j=1}^{n} \int_{\Omega} \left(\frac{y_i f_i'(t) + g_i'(t)}{f_i(t)} \right) \frac{\partial v_j}{\partial y_i} w_j,$$

for functions for which the integrals are well defined.

For all $\boldsymbol{w} \in \widetilde{V}_s(\Omega)$ we have

$$|\langle A(t)\, \boldsymbol{v}^k, \boldsymbol{w}\rangle| \leqslant \sum_{i,j=1}^{n} \frac{1}{f_i^2(t)} \int_{\Omega} \left| \frac{\partial v_j}{\partial y_i} \frac{\partial w_j}{\partial y_i} \right| \leqslant \frac{1}{\min\limits_{0 \leqslant t \leqslant T,\, 0 \leqslant i \leqslant n} f_i^2(t)} \|\nabla \boldsymbol{v}^k(t)\| \, \|\boldsymbol{w}\|_s.$$

The above inequality implies that

$$\int_0^T \|A(t)\boldsymbol{v}^k\|^2_{(\widetilde{V}_s(\Omega))'}\, dt \leqslant C \int_0^T \|\nabla \boldsymbol{v}^k\|^2 \leqslant C_1, \tag{40}$$

and $\boldsymbol{v}^k$ is bounded in $L^2(0, T; (\widetilde{V}_s(\Omega))')$.

In the same way and by using (36) we obtain that $C(t)\, \boldsymbol{v}^k$ is bounded in $L^2(0, T; (\widetilde{V}_s(\Omega))')$. To prove the boundedness of $\{B(t)\, \boldsymbol{v}^k\}$ we will use the following lemma, see Lions [8].

LEMMA 3.3. – *If* $\{\boldsymbol{v}^k\}$ *is bounded in*

$$L^2(0, T; \widetilde{V}(\Omega)) \cap L^\infty(0, T; \widetilde{H}(\Omega))$$

then $\{\boldsymbol{v}^k\}$ *is bounded in* $L^4(0, T; [L^p(\Omega)]^n)$, *with* $1/p = 1/2 - 1/2n$.

In fact, by using Sobolev's embedding theorem, we get

$$|\langle B(t)\, \boldsymbol{v}^k, \boldsymbol{w}\rangle \leqslant \frac{1}{\min\limits_{0 \leqslant t \leqslant T,\, 0 \leqslant i \leqslant n} f_i^2(t)} \|\boldsymbol{v}^k\|^2_{(L^p(\Omega))^n} \|\nabla \boldsymbol{w}\|_{(L^n(\Omega))^{n \times n}} \leqslant \tag{41}$$

$$\leqslant C\|\boldsymbol{v}^k\|^2_{(L^p(\Omega))^n} \|\nabla \boldsymbol{w}\|_{(H^{s-1}(\Omega))^n} \leqslant C_1 \|\boldsymbol{v}^k\|^2_{(L^p(\Omega))^n} \|\boldsymbol{w}\|_{\widetilde{V}_s(\Omega)},$$

and finally

$$(42) \qquad \int_0^T \|B(t)\, \boldsymbol{v}^k\|^2_{(\tilde{V}_s(\Omega))'} \leq C_1 \int_0^T \|\boldsymbol{v}^k\|^4_{(L^p(\Omega))^n},$$

by using (3.41) we obtain that $\{B(t)\, \boldsymbol{v}^k\}$ is bounded in $L^2(0, T; (V_s(\Omega))')$.

Let us consider the projection $P_k \colon H \to V_k$. We can consider it as an application of $\tilde{V}_s(\Omega)$ into it belongs to $\mathcal{L}(\tilde{V}_s(\Omega); \tilde{V}_s(\Omega))$. The adjoint operator P_k^* belongs to $\mathcal{L}((\tilde{V}_s(\Omega))'; (\tilde{V}_s(\Omega)')$ and $\|P_k^*\| \leq \|P_k\| \leq 1$. From (20) and recalling the definition (29) of the eigenfunctions $\tilde{\omega}^i$ we get

$$\alpha \left\langle \frac{\partial \boldsymbol{v}^k}{\partial t}, \tilde{\omega}^i \right\rangle = \langle (- \nu A(t) - \alpha B(t)\, \boldsymbol{v}^k - \alpha C(t))\, \boldsymbol{v}^{\mathrm{k}} + \mathrm{B(t)}\, \boldsymbol{b}^k + \alpha \boldsymbol{F}, \tilde{\omega}^i \rangle =$$

$$\langle (- \nu A(t) - \alpha B(t)\, \boldsymbol{v}^k - \alpha C(t))\, \boldsymbol{v}^k + B(t)\, \boldsymbol{b}^k + \alpha \boldsymbol{F}, P_k \tilde{\omega}^i \rangle =$$

$$\langle P_k^* ((- \nu A(t) - \alpha B(t)\, \boldsymbol{v}^k - \alpha C(t))\, \boldsymbol{v}^k + B(t) \boldsymbol{b}^k + \alpha \boldsymbol{F}), \tilde{\omega}^i \rangle,$$

and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\tilde{V}_s(\Omega)$ and $(\tilde{V}_s(\Omega))')$.

The previous inequality holds for every $\boldsymbol{w} \in \boldsymbol{v}_k$ and by setting $\boldsymbol{w} = P_k \boldsymbol{v}$, $\boldsymbol{v} \in V_s(\Omega)$ we have

$$(43) \qquad \alpha \frac{\partial \boldsymbol{v}^k}{\partial t} = P_k^* ((- \nu A(t) - \alpha B(t)\, \boldsymbol{v}^k - \alpha C(t))\, \boldsymbol{v}^k + B(t)\, \boldsymbol{b}^k + \alpha \boldsymbol{F}),$$

by using (39) and recalling that $\|P_k^*\| \leq 1$ from (43) we get the boundedness of $\{\partial \boldsymbol{v}^k / \partial t\}$ in $L^2(0, T); (\tilde{V}_s(\Omega))')$.

Working as before we have

$$(44) \qquad \frac{\partial \boldsymbol{b}^k}{\partial t} = \langle P_k^* (- \gamma A(t) - C(t)\, \boldsymbol{b}^k + B_1(t)(\boldsymbol{v}^k, \boldsymbol{b}^k) - B_1(t)(\boldsymbol{b}^k, \boldsymbol{v}^k)),$$

Where we define

$$B_1(t)(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^n \frac{1}{f_i(t)} \int_\Omega v_i \frac{\partial w_j}{\partial y_i} v_i.$$

We observe that

$$|B_1(t)(\boldsymbol{v}^k, \boldsymbol{b}^k)| \leq \|\boldsymbol{v}^k\|_{(L^p(\Omega))^n} \|\boldsymbol{b}^k\|_{(L^p(\Omega))^n} \|\nabla \boldsymbol{b}^k\|_{(L^n(\Omega))^{n \times n}} \leq$$

$$C \|\boldsymbol{v}^k\|_{(L^p(\Omega))^n} \|\boldsymbol{b}^k\|_{(L^p(\Omega))^n} \|\boldsymbol{b}^k\|_{\tilde{V}_s(\Omega)},$$

and finally

$$\int_0^T \|B_1(t)(\boldsymbol{v}^k, \boldsymbol{b}^k)\|^2_{(\widetilde{V}_s(\Omega))'} \leq C\left(\int_0^T \|\boldsymbol{v}^k\|^4_{(L^p(\Omega))^n}\right)^{1/2}\left(\int_0^T \|\boldsymbol{b}^k\|^4_{(L^p(\Omega))^n}\right)^{1/2} \leq C_1 .$$

Similarly we prove that $\{B_1(t)(\boldsymbol{b}^k, \boldsymbol{v}^k)\}$ is bounded in $L^2(0, T; (\widetilde{V}_s(\Omega))')$ and then from (44) we obtain that $\{\partial \boldsymbol{b}^k/\partial t\}$ is bounded in $L^2(0, T); (\widetilde{V}_s(\Omega))')$.

By a standard argument, Temam [16] chap III § 3, there exists

$$(\boldsymbol{v}, \boldsymbol{b}) \in \left[L^2\left(0, T; \widetilde{V}(\Omega)\right) \cap L^\infty\left(0, T; \widetilde{H}(\Omega)\right)\right]^2,$$

such that (up to a subsequence)

$$(\boldsymbol{v}^k, \boldsymbol{b}^k) \rightharpoonup (\boldsymbol{v}, \boldsymbol{b}) \quad \text{in } \left[L^2\left(0, T; \widetilde{V}(\Omega)\right)\right]^2,$$

$$(\boldsymbol{v}^k, \boldsymbol{b}^k) \rightharpoonup (\boldsymbol{v}, \boldsymbol{b}) \quad \text{in } \left[L\ \infty\left(0, T; \widetilde{H}(\Omega)\right)\right]^2,$$

$$(\boldsymbol{v}^k, \boldsymbol{b}^k) \rightharpoonup (\boldsymbol{v}, \boldsymbol{b}) \quad \text{in } \left[L^2\left(0, T; \widetilde{H}(\Omega)\right)\right]^2 \text{ and a.e. in } \Omega \times (0,T),$$

$$\left(\frac{\partial \boldsymbol{v}^k}{\partial t}, \frac{\partial \boldsymbol{b}^k}{\partial t}\right) \rightharpoonup \left(\frac{\partial \boldsymbol{v}}{\partial t}, \frac{\partial \boldsymbol{b}}{\partial t}\right) \quad \text{in } \left[L^2\left(0, T; (\widetilde{V}_s(\Omega))'\right)\right]^2,$$

$$(v_i^k v_j^k, b_i^k b_j^k) \to (v_i v_j, b_i b_j) \quad \text{in } L^q(\Omega \times (0, T)) \;\; q = \min\{2, p/2\}.$$

By using the standard procedure, see Lions [8] can pass to the limit and we end the proof of Lemma 3.1.

PROOF OF THEOREM 2.1. – The proof follows defining, for every divergence free $\phi, \psi \in (\mathcal{D}(\Omega \times (0, T)))^n$,

$$\widetilde{\phi}(y, t) = \prod_{i=1}^n f_i(t)\, \phi(f_1(t)\, y_1 + g_1(t), \ldots, f_n(t)\, y_1 + g_n(t), t)\,, \tag{45}$$

and

$$\widetilde{\psi}(y, t) = \prod_{i=1}^n f_i(t)\, \psi(f_1(t)\, y_1 + g_1(t), \ldots, f_n(t)\, y_1 + g_n(t), t)\,, \tag{46}$$

We can verify that

$$\widetilde{\phi}, \widetilde{\psi} \in (\mathcal{D}(\Omega \times (0, T)))^n \quad \text{and} \quad \sum_{i=1}^n \frac{1}{a_i}\frac{\partial \widetilde{\phi}_i}{\partial y_i} = \sum_{i=1}^n \frac{1}{a_i}\frac{\partial \widetilde{\psi}_i}{\partial y_i} = 0\,.$$

The following properties are satisfied

$$-\int_0^T\int_\Omega \boldsymbol{v}\,\frac{\partial\tilde{\phi}}{\partial t}\,dy\,dt-\sum_{i,j=1}^{n}\int_0^T\int_\Omega\left(\frac{y_i f_i(t)+g_i{}'(t)}{f_i(t)}\right)\frac{\partial \boldsymbol{v}_i}{\partial y_i}\,\tilde{\phi}_i\,dy\,dt,$$

$$=-\int_0^T\int_\Omega \prod_{i=1}^{n} f_i(t)\,\boldsymbol{v}\,\frac{\partial\phi}{\partial t}(f_1(t)y_1+g_1(t),\,\dots,f_n(t)y_1+g_n(t),\,t)\,dy\,dt\,,$$

$$\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{1}{f_i(t)}\,v_i v_j\,\frac{\partial\tilde{\phi}_j}{\partial y_i}=$$

$$\sum_{i,j=1}^{n}\int_0^T\int_\Omega \prod_{i=1}^{n} f_i(t)\,v_i v_j\,\frac{\partial\phi}{\partial x_i}(f_1(t)\,y_1+g_1(t),\,\dots,f_n(t)\,y_1+g_n(t),\,t)\,dy\,dt\,,$$

$$\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{1}{f_i^2(t)}\,\frac{\partial v_j}{\partial y_i}\,\frac{\partial\tilde{\phi}_j}{\partial y_i}\,dy\,dt=$$

$$\sum_{i,j=1}^{n}\int_0^T\int_\Omega \frac{1}{f_i(t)}\,\frac{\partial v_j}{\partial y_i}\,\prod_{i=1}^{n} f_i(t)\,\frac{\partial\phi}{\partial x_i}(f_1(t)\,y_1+g_1(t),\,\dots,f_n(t)\,y_1+g_n(t),\,t)\,dy\,dt\,,$$

By using $\tilde{\phi}$ as test functions in (20) we obtain

$$-\int_0^T\int_\Omega\left(\prod_{i=1}^{n} f_i(t)\right)\boldsymbol{v}\,\frac{\partial\phi}{\partial t}(f_1(t)y_1+g_1(t),\,\dots,f_n(t)y_1+g_n(t),\,t)-$$

$$\sum_{i=1}^{n}\left(\prod_{i=1}^{n} f_i(t)\right)\left(\frac{\partial v_j}{\partial y_i}\,\frac{\partial\phi_j}{\partial y_i}\right)(f_1(t)\,y_1+g_1(t),\,\dots,f_n(t)\,y_1+g_n(t),\,t)-$$

$$\left(\prod_{i=1}^{n} f_i(t)\right)v_i\,\frac{\partial\phi_j}{\partial y_i}(f_1(t)\,y_1+g_1(t),\,\dots,f_n(t)\,y_1+g_n(t),\,t)\,dydt=$$

$$\int_0^T\int_\Omega\left(\prod_{i=1}^{n} f_i(t)\right)\boldsymbol{F}\phi(f_1(t)\,y_1+g_1(t),\,\dots,f_n(t)\,y_1+g_n(t),\,t)\,dy\,dt\,.$$

Since the transformation $\theta^{-1}\colon\ \Omega\times(0,\,T)\to D$ satisfies

$$\boldsymbol{v}(y,\,t)=\boldsymbol{u}(\theta^{-1}(y,\,t))\qquad|\mathrm{Jac}\,\theta^{-1}|=\prod_{i=1}^{n} f_i(t),$$

By using the rule for the variables' change in multiple integrals we obtain

$$-\int_D \boldsymbol{u}\,\frac{\partial\phi}{\partial t}+\sum_{i,j=1}^{n}\frac{\eta}{\varrho_m}\,\frac{\partial u_i}{\partial x_j}\,\frac{\partial\phi_i}{\partial x_j}-\sum_{i,j=1}^{n}u_i u_j\,\frac{\partial\phi_j}{\partial x_i}-\frac{\mu}{\varrho_m}h_i h_j\,\frac{\partial\phi_j}{\partial x_i}\,dx\,dt=\int_D \boldsymbol{f}\phi\,dx\,dt\,,$$

and (14) is satisfied. By using Propositions 2.2 and 2.4 we obtain that $\boldsymbol{u}\in L^2(0,T;V(\Omega(t)))\cap L^\infty(0,T;H(\Omega(t)))$.

In a similar way (by using $\tilde{\psi}$ as test function) we will see that (15) is also satisfied. ■

4. – Proof of Theorem 2.2.

We first prove the regularity result. We observe that the proof of Theorem 2.1 shows that $\partial\boldsymbol{v}/\partial t\in L^2(0,T;(\tilde{V}(\Omega))')$ consequently applying Lemma 1.2 in Temam [16] pag. 260 we obtain that $\boldsymbol{v}$ is almost everywhere equal to a function continuous from $[0,T]$ into $\tilde{H}(\Omega)$. The continuity of $\boldsymbol{b}$ is proved in the same way. We also note that

$$\frac{d}{dt}\|\boldsymbol{v}(t)\|^2_{\tilde{H}(\Omega)}=2\left\langle\frac{\partial\boldsymbol{v}(t)}{\partial t}\,,\,\boldsymbol{v}(t)\right\rangle,\qquad \frac{d}{dt}\|\boldsymbol{b}(t)\|^2_{\tilde{H}(\Omega)}=2\left\langle\frac{\partial\boldsymbol{b}(t)}{\partial t}\,,\,\boldsymbol{b}(t)\right\rangle.$$

Consider two solutions $(\boldsymbol{v}_1,\boldsymbol{b}_1)$ and $(\boldsymbol{v}_2,\boldsymbol{b}_2)$ which are solutions of the problem in the domain $\Omega\times(0,T)$ with the same force and initial datum and define the differences

$$\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2,\qquad \boldsymbol{b}=\boldsymbol{b}_1-\boldsymbol{b}_2.$$

We obtain by difference from (26)-(27) and by using as test functions $(\boldsymbol{v},\boldsymbol{b})$

$$\begin{aligned}(47)\qquad &\frac{\alpha}{2}\,\frac{d}{dt}(\|\boldsymbol{v}\|^2+\|\boldsymbol{b}\|^2)+\nu\sum_{i,j=1}^{n}\int_\Omega\frac{1}{f_i^2(t)}\,\frac{\partial v_i}{\partial y_j}\,\frac{\partial v_i}{\partial y_j}+\nu\sum_{i,j=1}^{n}\int_\Omega\frac{1}{f_i^2(t)}\,\frac{\partial b_i}{\partial y_j}\,\frac{\partial b_i}{\partial y_j}=\\ &\alpha\int_\Omega\frac{1}{f_i(t)}v_j\,\frac{\partial v_{1i}}{\partial y_j}v_i+\int_\Omega\frac{1}{f_i(t)}b_j\,\frac{\partial b_{1i}}{\partial y_j}v_i-\int_\Omega\frac{1}{f_i(t)}v_j\,\frac{\partial b_{2i}}{\partial y_j}b_i+\\ &\int_\Omega\frac{1}{f_i(t)}v_j\,\frac{\partial v_{1i}}{\partial y_j}h_i+\sum_{j=1}^{n}\frac{f_j'(t)\,y_j+g_j'(t)}{f_j(t)}\left(\alpha\,\frac{\partial v_i}{\partial y_j}+\frac{\partial b_i}{\partial y_j}\right),\end{aligned}$$

where we used the results of Remark 2.2 to cancel the bad nonlinear terms. Now we observe that by using the classical inequality

$$\int_\Omega u_j\,\frac{\partial u_i}{\partial y_j}\,v_i\leqslant C\|\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{w}\|^{1/2},$$

and Hölder's and Young inequalities we have

$$\int_\Omega \frac{1}{-} f_i(t) v_j \frac{\partial v_{1i}}{\partial y_j} v_i \leq C \|\boldsymbol{v}\|^2_{L^4(\Omega)} \|\nabla \boldsymbol{v}_1\| \leq \frac{\nu}{A} \|\nabla \boldsymbol{v}\| + C(A) \|\boldsymbol{v}\|^2 \|\nabla \boldsymbol{v}_1\|^2.$$

By using similar inequalities for the other terms and recalling inequality (36) we get

$$\frac{d}{dt}(\|\boldsymbol{v}\|^2 + \|\boldsymbol{b}\|^2) + \|\nabla \boldsymbol{v}\|^2 + \|\nabla \boldsymbol{b}\|^2 \leq C(\|\boldsymbol{v}\|^2 + \|\boldsymbol{b}\|^2)(\|\nabla \boldsymbol{v}_1\|^2 + \|\nabla \boldsymbol{b}_1\|^2 + \|\nabla \boldsymbol{b}_2\|^2)$$

with C a constant that depends only on ν, γ, α and on the minimum of $f_i(t)$, and the maximum of $f_i'(t)$, $g_i(t)$ and $g_i'(t)$.

By using Gronwall's lemma and since

$$(\|\nabla \boldsymbol{v}_1\|^2 + \|\nabla \boldsymbol{b}_1\|^2 + \|\nabla \boldsymbol{b}_2\|^2) \in L^1(0, T),$$

we prove that $\boldsymbol{v} \equiv \boldsymbol{b} \equiv 0$. Using the result (see Proposition 2.2) that the map $\mathcal{T}$ is an isomorphism between $L^2(0, T; \tilde{V}(\Omega))$ and $L^2(0, T; V(\Omega(t)))$ (and also between $L^\infty(0, T; \tilde{H}(\Omega))$ and $L^\infty(0, T; H(\Omega(t))))$ we obtain that $\boldsymbol{u} \equiv \boldsymbol{h} \equiv 0$. This ends the proof of Theorem 2.2 ■

5. – Appendix.

In this appendix we make short proofs of the propositions claimed in section 2 and used to prove the main results of this paper.

We start by proving the first proposition

PROOF OF PROPOSITION 2.1. – We consider the application T defined by (2.10) and we observe that $T_{|\mathcal{V}(\Omega)}: \mathcal{V}(\Omega) \to \tilde{\mathcal{V}}(\Omega)$ is a linear injection. Given $\tilde{\boldsymbol{v}} = (\tilde{v}_1, \ldots, \tilde{v}_n) \in \tilde{\mathcal{V}}(\Omega)$, there exists $\boldsymbol{v} = (v_1/a_1, \ldots, v_n/a_n) \in \mathcal{V}(\Omega)$ such that $T\boldsymbol{v} = \tilde{\boldsymbol{v}}$, then T is also bijective.

For $\boldsymbol{v} \in \mathcal{V}(\Omega)$ we have

$$(1) \qquad \|T\boldsymbol{v}\|^2_{\tilde{V}(\Omega)} = \sum_{i=1}^n \sum_{|\alpha| \leq 1} \int_\Omega |D^\alpha (a_i v_i)|^2 dx = \sum_{i=1}^n a_i^2 \sum_{|\alpha| \leq 1} \int_\Omega |D^\alpha v_i|^2 dx .$$

Therefore if we set

$$K = \left[n \max_{1 \leq i \leq n} a_i^2\right]^{1/2}, \qquad L = \left[n \min_{1 \leq i \leq n} a_i^2\right]^{1/2},$$

we obtain that

$$(2) \qquad L\|\boldsymbol{v}\|_{V(\Omega)} \leq \|T\boldsymbol{v}\|_{\tilde{V}(\Omega)} \leq K\|\boldsymbol{v}\|_{V(\Omega)}.$$

and then by a density argument we prove that the transformation is a bijection between $V(\Omega)$ *and* $\tilde{V}(\Omega)$. ■

To prove Proposition 2.2 we need the following lemma

LEMMA. – For every $t \in (0, T)$ the application

$$\tilde{V}(\Omega) \ni \boldsymbol{v}(y_1, \ldots, y_n) \to \boldsymbol{u}(x) = \boldsymbol{v}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_n - g_n(t)}{f_n(t)}\right) \in V(\Omega(t))$$

is an isomorphism.

PROOF. – The application is well defined since $\boldsymbol{u}(x) \in V(\Omega(t))$, $0 < t < T$. From the definition we have that there exists $\{\boldsymbol{v}_n\} \subset \tilde{\mathcal{V}}(\Omega)$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $(H^1(\Omega))^n$ and since t is fixed we have that

$$\boldsymbol{u}_n(x) = \boldsymbol{v}_n\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_n - g_n(t)}{} f_n(t)\right) \in (\mathcal{D}(\Omega(t)))^n,$$

and an explicit calculation gives $\sum_{i=1}^{n} \frac{\partial u_{ni}}{\partial x_i}(x) = 0$. Then we have

$$\sum_{i,j=1}^{n} \int_{\Omega(t)} \left| \frac{\partial u_{nj}}{\partial x_i} - \frac{\partial u_j}{\partial x_i} \right|^2 dx = \sum_{i,j=1}^{n} \int_{\Omega(t)} \left| \frac{\partial v_{nj}}{\partial y_i}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_1 - g_1(t)}{f_1(t)}\right) - \right.$$

$$\left. \frac{\partial v_j}{\partial y_i}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_1 - g_1(t)}{f_1(t)}\right) \right|^2 \frac{1}{f_i^2(t)} dx \leqslant$$

$$C \sum_{i,j=1}^{n} \int_{\Omega(t)} \frac{1}{\prod_{i=1}^{n} f_i^2(t)} \left| \frac{\partial v_{nj}}{\partial y_i}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_1 - g_1(t)}{f_1(t)}\right) - \right.$$

$$\left. \frac{\partial v_j}{\partial y_i}\left(\frac{x_1 - g_1(t)}{f_1(t)}, \ldots, \frac{x_1 - g_1(t)}{f_1(t)}\right) \right|^2 dx\,.$$

Let us consider $K_t \colon \Omega(t) \to \Omega$

$$(x_1, \ldots, x_n) \to \left(\frac{x_1 - g_1(t)}{} f_1(t), \ldots, \frac{x_1 - g_1(t)}{f_1(t)}\right),$$

for every $t \in [0, T]$ K_t is a diffeomorphism of class C^1 and it satisfies $|\mathrm{Jac}\, K_t| = 1/\prod_{i=1}^{n} f_i(t)$.

Making a change of variables we have

$$\sum_{i,j=1}^{n}\int_{\Omega(t)}\left|\frac{\partial u_{nj}}{\partial x_i}-\frac{\partial u_j}{\partial x_i}\right|dx\leqslant\sum_{i,j=1}^{n}\int_{\Omega(t)}\left||\mathrm{Jac}\,K_t|\left[\frac{\partial v_{nj}}{\partial y_i}(K_t(x))-\frac{\partial v_j}{\partial y_i}(K_t(x))\right]\right|^2 dx=$$

$$C\sum_{i,\,j=1}^{n}\int_{\Omega}\left|\frac{\partial v_{nj}(y)}{\partial y_i}-\frac{\partial v_j(y)}{\partial y_i}\right|^2 dy\to 0\,,$$

from which it follows that $\boldsymbol{u}_n\to\boldsymbol{u}$ in $(H^1(\Omega(t)))^n$ and that

$$C_1\|\boldsymbol{v}\|_{\tilde{V}(\Omega)}\leqslant\|\boldsymbol{u}\|_{V(\Omega(t))}\leqslant C_2\|\boldsymbol{v}\|_{\tilde{V}(\Omega)},$$

and the application $\mathcal{T}$ is an isomorphism. ■

We are now in position to prove Proposition 2.2

PROOF OF PROPOSITION 2.2. – From the previous lemma we have that $\boldsymbol{u}:(0,\,T)\to V(\Omega(t))$ defined by (2.11) is defined a.e. in $(0,\,T)$ end extending $\boldsymbol{u}$ by zero outside $B\backslash\Omega(t)$ it can be considered as an application of $(0,\,T)$ in $V(B)$ and

$$\int_0^T\|\boldsymbol{u}(t)\|^2_{V(B)}\,dt=\int_0^T\|\boldsymbol{u}(t)\|^2_{V(\Omega(t))}\,dt=\sum_{i,\,j=1}^{n}\int_0^T\int_{\Omega(t)}\left|\frac{\partial u_j}{\partial x_i}\right|^2 dx\,dt=$$

$$\sum_{i,\,j=1}^{n}\int_0^T\int_{\Omega(t)}\left|\frac{\partial}{\partial x_i}\left(v_j\left(\frac{x_1-g_1(t)}{f_1(t)},\,\ldots,\,\frac{x_n-g_n(t)}{f_n(t)}\right)\right)\right|^2 dx\,dt=$$

$$\sum_{i,\,j=1}^{n}\int_0^T\int_{\Omega}\left|\frac{\partial v_j}{\partial y_i}\left(\frac{x_1-g_1(t)}{f_1(t)},\,\ldots,\,\frac{x_n-g_n(t)}{f_n(t)}\right)\frac{1}{f_i(t)}\right|^2 dy\,dt\leqslant$$

$$C\sum_{i,\,j=1}^{n}\int_0^T\int_{\Omega}\left|\frac{\partial v_j}{\partial y_i}\right|^2 dy\,dt=C\int_0^T\|\boldsymbol{v}(t)\|^2_{\tilde{V}(\Omega)}\,dt\,,$$

therefore $\|\boldsymbol{u}\|_{L^2(0,\,T;\,V(\Omega(t)))}\leqslant C\|\boldsymbol{v}\|_{L^2(0,\,T;\,\tilde{V}(\Omega))}$. The other inequality can be proved in the same manner and, with a density argument, we conclude the proof of Proposition 2.2. and the proofs of Proposition 2.2-2.3 are now straightforward. ■

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