Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.2, p. 341–352.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_2_341_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 341-352

Unique Factorization in Non-Atomic Integral Domains.

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Sunto. – *In un UFD ogni elemento non unitario* $\neq 0$ *può essere espresso in modo unico nella forma* $up_1^{a_1}\dots p_n^{a_n}$ *dove* u *è un elemento unitario, i* p_i *sono primi non associati e ogni* $a_i \geq 1$. *Per studiare questa fattorizzazione in un ambito non atomico, si prende in esame un certo numero di generalizzazioni della potenza di un primo* p^n. *Per numerose di queste generalizzazioni si prova che si ottiene una forma di fattorizzazione unica e la si mette in relazione, nel caso in cui* R *è un dominio di integrità, con rappresentazioni di carattere finito di* R.

1. – Introduction.

Let R be an integral domain with quotient field K. In a UFD R every nonzero nonunit $x \in R$ may be expressed uniquely as $x = up_1^{a_1}\dots p_n^{a_n}$ where u is a unit, the p_i's are nonassociate primes and each $a_i \geq 1$. In Section 2 a number of generalizations of the notion of a prime power p^n are given. A nonzero nonunit $h \in R$ is defined to be *homogeneous* if $x, y \mid h$ and $(x, y)_v = R$ implies x or y is a unit. We show that a completely primal element h is homogeneous if and only if h is *t-pure*, that is, h is contained in a unique maximal t-ideal. In Section 3 we show that if $x \in R$ is a product of t-pure elements, then x is a product of v-coprime t-pure elements and that this factorization is unique. We study pre-Schreier domains in which every nonzero nonunit element is a product of homogeneous elements. We also investigate the relationship between factorization into the various building blocks from Section 2 and finite character representations $\bigcap R_P$ for R.

In general we follow the notation from [11] or [12]. We will freely use well known results on t-ideals; the results needed are surveyed in [6]. For a nonzero ideal I, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup \{J_v \mid 0 \neq J \subseteq I \text{ is a finitely generated ideal}\}$. An ideal I is called a *t-ideal* if $I = I_t$ and a *maximal t-ideal* is a proper ideal maximal in the set of t-ideals. A maximal t-ideal is prime. We let t-Max (R) denote the set of maximal t-ideals and $\mathrm{Cl}_t(R)$ the *t-class group* of R. Of course, $R = \bigcap_{M \in t\text{-Max}(R)} R_M$. Let $x, y \in R$; then x and y are *(non-)v-coprime* if $(a, b)_v = R$ $((a, b)_v \neq R)$. We use $[x, y]$ to denote the GCD of x and y. If $[x, y] = 1$, we say that x and y are *coprime*. We write $[x, y] \neq 1$ if x and y are not coprime, that is, there is a nonunit $d \in R$ with $d \mid x$ and $d \mid y$. Finally, $G(R)$ denotes the group of divisibility of R with its usual partial order.

2. – Building blocks.

Let R be an integral domain with quotient field K. A nonzero nonunit $x \in R$ is *prime* (resp., *irreducible* or an *atom*) if for $a, b \in R$ $x \mid ab \Rightarrow x \mid a$ or $x \mid b$ (resp., $x = ab \Rightarrow a$ or b is a unit). Now R is a UFD if (1) R is *atomic* (i.e., every nonzero nonunit of R is a product of atoms) and (2) this factorization into atoms is unique up to order and associates. Various generalizations of UFD's have been given where R is an atomic domain satisfying certain conditions on factorization into atoms weaker than (2); for example, see [2]. Now UFD's can also be characterized by the property that every nonzero nonunit is a product of prime elements or equivalently that every nonzero nonunit x can be written in the form $x = up_1^{a_1} \dots p_n^{a_n}$ where u is a unit, $p_1, \dots, p_n$ are nonassociate primes and each $a_i \geq 1$. Each of the $p_i^{a_i}$, in addition to being a power of a prime, has a number of other properties, each of which is subject to generalization. Our goal is to study various generalizations of (unique) factorization into prime powers in integral domains which need not be atomic. In this section we consider different building blocks that can replace the notion of prime power.

Now a primary element x (i.e., (x) is a primary ideal) generalizes the notion of a prime power p^n, p a prime element. Integral domains with the property that every nonzero nonunit is a product of primary elements are called *weakly factorial*; see [5]. Another property of p^n is that if $x, y \mid p^n$, then $x \mid y$ or $y \mid x$. With this in mind, following P. M. Cohn, the third author [13] defined a nonzero nonunit $h \in R$ to be *rigid* if $x, y \mid h \Rightarrow x \mid y$ or $y \mid x$. Evidently an integral domain R is *rigid* (i.e., every nonzero nonunit of R is rigid) if and only if R is a valuation domain. And R is said to be *semi-rigid* if every nonzero nonunit of R is a product of rigid elements. While a primary element h need not be rigid, if (h) is P-primary, then P is a maximal t-ideal [3] [Lemma 1] and hence h is contained in a unique maximal t-ideal. Let P be a prime ideal of R. In [6] we defined an ideal A to be *P-pure* if $A_P \cap R = A$. Certainly if (h) is P-primary, then (h) is P-pure. Let us call a nonzero nonunit element $h \in R$ *t-pure* if (h) is P-pure for some maximal t-ideal P. We next observe that h is t-pure if and only if h is contained in a unique maximal t-ideal.

LEMMA 2.1. – *Let h be a nonzero nonunit of R. Then h is t-pure if and only if h is contained in a unique maximal t-ideal.*

PROOF. ($\Rightarrow$) [6, LEMMA 4.2]. – ($\Leftarrow$) Suppose that M is the only maximal t-ideal containing h. Now $R = \bigcap_{P \in t\text{-Max}(R)} R_P$ gives that

$$hR = \bigcap_{P \in t\text{-Max}(R)} hR_P = \bigcap_{P \in t\text{-Max}(R) - \{M\}} R_P \cap hR_M \supseteq R \cap hR_M ,$$

so that $hR = R \cap hR_M$ and hence (h) is M-pure. ∎

Let us call an integral domain R *t-pure* (resp., *semi-t-pure*) if every nonzero nonunit of R is t-pure (resp., a product of t-pure elements). Evidently R is t-pure if and only if R has a unique maximal t-ideal. Although not given a name there, a number of characterizations of semi-t-pure domains were given in [6, Corollary 4.4]. For example, R is semi-t-pure if and only if $R = \bigcap_{P \in t\text{-Max}(R)} R_P$ has finite character, for distinct P, $Q \in t$-Max(R), $P \cap Q$ does not contain a nonzero prime ideal, and $\mathrm{Cl}_t(R) = 0$.

For a nonzero nonunit h, put $P(h) = \{x \in R \mid (x, h)_v \neq R\}$. So $P(h) = \bigcup \{M \in t\text{-Max}(R) \mid h \in M\}$. Thus $P(h)$ is an ideal, necessarily a maximal t-ideal, if and only if h is t-pure. Also note that $P(h)$ is an ideal if and only if $P(h)$ is closed under addition, that is, if $(x, h)_v \neq R$ and $(y, h)_v \neq R$, then $(x + y, h)_v \neq R$. Suppose that h is contained in a unique maximal t-ideal P and suppose that $x, y \mid h$. If x and y are both nonunits, then $x, y \in P$ (for $x, y \in P(h) = P$) and hence $(x, y)_v \neq R$. Thus if $x, y \mid h$ and $(x, y)_v = R$, then x or y is a unit. We define a nonzero nonunit $h \in R$ to be *homogeneous* (resp., *strongly homogeneous*) if $x, y \mid h$ and $(x, y)_v = R$ (resp., $[x, y] = 1$) implies that x or y is a unit. We say that R is (*strongly*) *homogeneous* if every nonzero nonunit of R is (strongly) homogeneous and that R is *semi-*(*strongly-*)*homogeneous* if each nonzero nonunit of R is a product of (strongly) homogeneous elements. Clearly a rigid element is strongly homogeneous and a strongly homogeneous element is homogeneous. Thus a (semi-)rigid domain is a (semi-)strongly-homogeneous domain and a (semi-)strongly-homogeneous domain is a (semi-)homogeneous domain.

To study factorization in a non-atomic setting, P. M. Cohn [9] defined a nonzero nonunit $h \in R$ to be *primal* if $h \mid xy$ implies $h = h_1 h_2$ where $h_1 \mid x$ and $h_2 \mid y$. Thus an atom is primal if and only if it is prime. While a nonunit factor of a prime, primary, irreducible, rigid, t-pure, strongly homogeneous, or homogeneous element has the same property, a factor of a primal element need not be primal (see below). Cohn defined a nonzero nonunit to be *completely primal* if each nonunit factor of h is primal. Thus a factor of a completely primal element is completely primal. Moreover, the product of two completely primal elements is again completely primal [9, Lemma 2.5]. He also defined an integral domain to be a *Schreier domain* if R is integrally closed and every nonzero nonunit of R is (completely) primal. The third author [16, Lemma 2.5] defined an integral domain to be *pre-Schreier* if every nonzero nonunit is (completely) primal. Perhaps the most important example of a Schreier domain is a GCD domain.

In [4] we defined a nonzero nonunit q to be a *prime quantum* if q satisfies Q_1: For every nonunit $r \mid q$, there exists a natural number n with $q \mid r^n$, Q_2: For every natural number n, if $r \mid q^n$ and $s \mid q^n$, then $r \mid s$ or $s \mid r$, and Q_3: For every natural number n, each element t with $t \mid q$ has the property that if $t \mid ab$, then $t = t_1 t_2$ where $t_1 \mid a$ and $t_2 \mid b$. Thus Q_2 says that each power of q is rigid and Q_3 says that q is completely primal (since a product of completely primal elements is completely primal). A prime power p^n is a prime quantum and a prime quantum q is $P(q)$-primary. We defined an integral domain R to be a *generalized unique fac-*

torization domain (*GUFD*) if every nonzero nonunit of R is a product of prime quanta. Clearly a UFD is a GUFD and a GUFD is weakly factorial and a GCD domain [4].

The third author [15] considered yet another generalization of a prime power. He defined a nonzero nonunit $x \in R$ to be a *packet* if $\sqrt{(x)}$ is prime, that is, there is a unique minimal prime over (x). He studied GCD domains, called *unique representations domains*, with the property that every nonzero nonunit is a product of packets.

The various generalizations of a prime element are indicated in the diagram below (Figure 1).

We note that in Figure 1, only the obvious implications hold. Let $k \subsetneq K$ be a field extension with $[K\colon k] < \infty$, so $R = k + XK[\![X]\!]$ is a one-dimensional local domain. Let $h = X^2$, so h is primary (but not prime nor irreducible) and hence is homogeneous, but h is not strongly homogeneous since $lX | X^2$ for each $l \in K^*$ but $[X, lX] = 1$ for $l \in K - k$. Also, note that h is primal but not completely primal. Hence h is not a prime quantum. Next, let $R = \mathbb{Z}_{(p)} + (X, Y)\,\mathbb{Q}[\![X, Y]\!]$ where p is prime. Let $h = XY$. Then h is strongly homogeneous but not rigid. Now every nonzero nonunit of a valuation domain is rigid, t-pure and a packet, but such an element need not be irreducible nor primary. Finally, let R be a

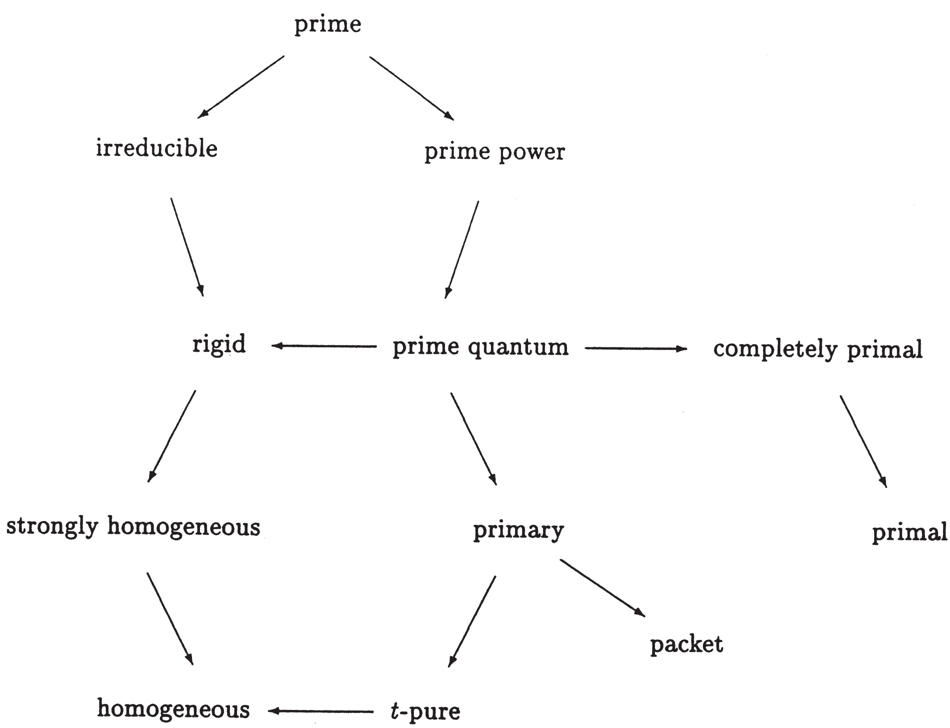

Figure 1.

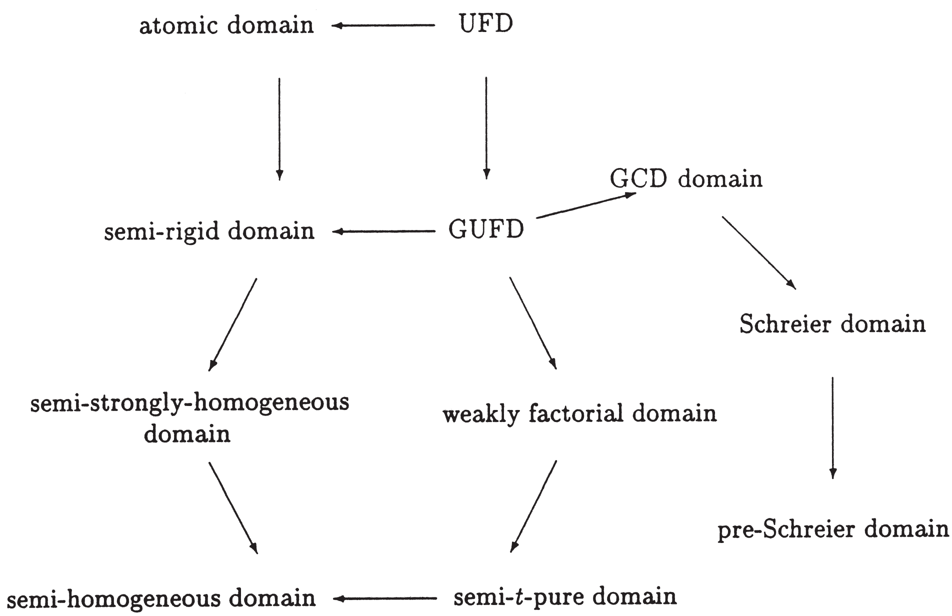

Figure 2.

Dedekind domain with $\mathrm{Cl}(R) = \mathbb{Z}/2\mathbb{Z}$ and let M and N be distinct nonprincipal maximal ideals of R. Then MN is principal, say $MN = (h)$. Now h is irreducible and hence homogeneous but h is not t-pure.

Corresponding to Figure 1, we have Figure 2 above showing the various generalizations of UFD's. Again, the only implications are the obvious ones.

We next show that when dealing with completely primal elements there is no distinction between homogeneous and strongly homogeneous elements.

PROPOSITION 2.2. – *Let $x \in R$ be completely primal and let $0 \neq y \in R$. If $(x, y)_v \neq R$, then there exists a nonunit $h \in R$ with $h | x$ and $h | y$, so $[x, y] \neq 1$. Hence a completely primal element h is homogeneous if and only if it is strongly homogeneous.*

PROOF. – The first statement follows from [8, Corollary 2.2(2)] restated for integral domains. The second statement follows since a factor of a completely primal element is completely primal. ∎

THEOREM 2.3. – *Let $0 \neq h \in R$ be a nonunit completely primal element. Then the following are equivalent*:

(1) *h is homogeneous*,

(2) *h is strongly homogeneous*,

(3) *h is t-pure.*

PROOF. – $(1)\Leftrightarrow(2)$ Proposition 2.2. (3)⇒(1) Lemma 2.1. (1)⇒(3) It suffices to prove that $P(h) = \{x \in R \mid (x, h)_v \neq R\}$ is an ideal for then $P(h)$ is the unique maximal t-ideal containing h. Since $P(h)$ is closed under multiplication, it suffices to show that $(a, h)_v \neq R$ and $(b, h)_v \neq R$ implies that $(a+b, h)_v \neq R$. Since h is completely primal, by Proposition 2.2 there exist nonunits α and β of R with $\alpha \mid a, h$ and $\beta \mid b, h$. Now h is homogeneous and α and β are nonunits, so $(\alpha, \beta)_v \neq R$. Thus α completely primal (for it is a factor of h) gives a nonunit $H \in R$ with $H \mid \alpha, \beta$. Thus $H \mid a+b$ and $H \mid h$ gives $(a+b, h)_v \neq R$.

COROLLARY 2.4. – *Let R be a GCD domain and let $0 \neq h \in R$ be a nonunit. Then the following are equivalent*:

(1) *h is homogeneous,*

(2) *h is strongly homogeneous,*

(3) *h is t-pure,*

(4) *h is rigid.*

PROOF. – Since h is completely primal, (1)-(3) are equivalent. $(4)\Rightarrow(1)$ This holds for any integral domain. $(2)\Rightarrow(4)$ Let h be strongly homogeneous and suppose that $x, y \mid h$. Put $d = [x, y]$. Then $x/d,\ y/d \mid h$ and $[x/d, y/d] = 1$. So x/d or y/d is a unit and so $x \mid y$ or $y \mid x$. ■

However, unlike the case for GCD domains, we cannot add «h is rigid» to Theorem 2.3. For take $R = \mathbb{Q} + P_P$ where $P = \{f \in \mathbb{R}[X; \mathbb{Q}_0] \mid f$ has zero constant term$\}$. Then [16], R is a pre-Schreier domain (which is not a Schreier domain) with P_P its unique maximal t-ideal. So every nonzero element of P_P is completely primal and homogeneous, but no nonzero element of P_P is rigid.

3. – Uniqueness of factorizations.

When considering factorization into elements from Figure 1 (Section 2), only factorization into primes is necessarily unique. However, it is easily seen that factorization into prime powers is unique once powers of associated primes are combined, that is, if $x = \lambda_1 p_1^{a_1} \ldots p_n^{a_n} = \lambda_2 q_1^{b_1} \ldots q_s^{b_s}$ where λ_1 and λ_2 are units, the p_i 's and q_j's are prime with p_i and p_j (resp., q_i and q_j) nonassociates for $i \neq j$ and each $a_i, b_i \geqslant 1$, then $n = s$ and after reordering, if necessary, p_i and q_i are associates and $a_i = b_i$. A similar result is also true for factorization into primary elements; see [5, page 146]. In both cases, the uniqueness comes after elements with the same radical (or which are non-v-coprime) are combined. We can also do the same thing with t-pure elements.

THEOREM 3.1. – *Let R be an integral domain.*

(1) *If x and y are P-pure where P is a maximal t-ideal, then xy is again P-pure.*

(2) *If $0 \neq x \in R$ is a product of t-pure elements, then x is a product of v-coprime t-pure elements.*

(3) *Suppose that $x = x_1 \ldots x_n = y_1 \ldots y_m$ where each x_i, y_i is t-pure and x_i, x_j (resp., y_i, y_j) are v-coprime for $i \neq j$. Then $n = m$ and after reordering, if necessary, x_i and y_i are associates.*

PROOF. – (1) Clearly $xy \in P$. If $xy \in N$ for some other maximal t-ideal $N \neq P$, then $x \in N$ or $y \in N$ and hence x or y is not P-pure (Lemma 2.1), a contradiction.

(2) Let $x = x_1 \ldots x_n$ where x_i is t-pure. Let $M_1, \ldots, M_s$ be the necessarily finite set of maximal t-ideals involved and let $y_i = \prod \{x_j \mid x_j \text{ is } M_i\text{-pure}\}$. Then $x = y_1 \ldots y_s$, each y_i is M_i-pure (and hence t-pure) and for $i \neq j$, y_i and y_j are v-coprime.

(3) Suppose that $x = x_1 \ldots x_n$ where x_i is M_i-pure (M_i a maximal t-ideal) and x_i and x_j are v-coprime for $i \neq j$. Then $\{M_1, \ldots, M_n\}$ is precisely the set of maximal t-ideals containing x. Now $xR_{M_i} = x_1 \ldots x_n R_{M_i} = x_i R_{M_i}$, so $x_i R = x_i R_{M_i} \cap R = xR_{M_i} \cap R$. This shows that (up to associates) x_i depends only on x and M_i and hence the uniqueness result follows.

COROLLARY 3.2. – *Suppose that $x \in R$ is a finite product of completely primal homogeneous elements. Then x is uniquely expressible as a finite product of mutually coprime completely primal homogeneous elements.*

We next consider the integral domains in which every nonzero nonunit is a product of t-pure elements. But first some definitions. An integral domain R has *t-finite character* (or is *t-locally finite*) if $R = \bigcap_{P \in t\text{-Max}(R)} R_P$ has finite character or equivalently if each $0 \neq x \in R$ is contained in only finitely many maximal t-ideals. We say that R is *t-independent* if for distinct $P, Q \in t\text{-Max}(R)$, $P \cap Q$ contains no nonzero prime ideal. Since a prime ideal minimal over a principal ideal is a t-ideal, R is t-independent if and only if for distinct maximal t-ideals P and Q, there is no prime t-ideal contained in $P \cap Q$. Hence R is t-independent if and only if each prime t-ideal is contained in a unique maximal t-ideal.

Now in general, if N is a maximal t-ideal of R, N_N need not be a maximal t-ideal of R_N; see [17, Proposition 4.3]. However, we next show that this cannot happen if R has t-finite character.

LEMMA 3.3. – *Suppose that R has t-finite character. If $N \in t\text{-Max}(R)$, then $N_N \in t\text{-Max}(R_N)$. Thus if R has t-finite character, R is a finite character inter-*

section of localizations of R such that the maximal ideal of each localization is a t-ideal. Conversely, if $R = \bigcap_{Q \in \mathcal{S}} R_Q$ *where the intersection has finite character and* $\mathcal{S}$ *is a set of prime ideals of R such that no two elements of* $\mathcal{S}$ *are comparable and for each* $Q \in \mathcal{S}$, Q_Q *is a maximal t-ideal of* R_Q, *then* $\mathcal{S} = t\text{-Max}(R)$ *and hence R has t-finite character.*

PROOF. – Since $R = \bigcap_{M \in t\text{-Max}(R)} R_M$ has finite character, by [1, Theorem 2(4)] the star operation given by $I^* = \bigcap_{M \in t\text{-Max}(R)} (IR_M)_t$ has finite character. Now for $N \in t\text{-Max}(R)$, $N = N^* = \bigcap_{M \in t\text{-Max}(R)} (NR_M)_t = \bigcap_{\substack{M \in t\text{-Max}(R) \\ M \neq N}} R_M \cap (NR_N)_t$. Thus $(NR_N)_t \neq R_N$, so NR_N is a maximal t-ideal of R_N.

For the last statement it suffices to prove that each $Q \in \mathcal{S}$ is a t-ideal, for then by [6] $\mathcal{S} = t\text{-Max}(R)$. We show that if Q_Q is a t-ideal, then Q is a t-ideal. Let $q_1, \ldots, q_n \in Q$. If $x \in (q_1, \ldots, q_n)_v$, then $x[R:(q_1, \ldots, q_n)] \subseteq R$. Hence $x[R_Q:(q_1, \ldots, q_n)_Q] = x[R:(q_1, \ldots, q_n)]_Q \subseteq R_Q$ so $x \in ((q_1, \ldots, q_n) R_Q)_v \subseteq Q_Q$ since Q_Q is a t-ideal. Thus $x \in R \cap Q_Q = Q$. So $(q_1, \ldots, q_n)_v \subseteq Q$ and hence Q is a t-ideal. ■

THEOREM 3.4. – *For an integral domain R the following conditions are equivalent.*

(1) *R is semi-t-pure, i.e., every nonzero nonunit of R is a product of t-pure elements.*

(2) *Every nonzero nonunit of R can be written uniquely* (*up to order and associates*) *as a product of mutually v-coprime t-pure elements.*

(3) *R has t-finite character, is t-independent and has* $\mathrm{Cl}_t(R) = 0$.

(4) *R is a finite character intersection* $\bigcap_{Q \in \mathcal{S}} R_Q$ *for some set* $\mathcal{S}$ *of independent primes* (*i.e., for distinct* $P, Q \in \mathcal{S}$, $P \cap Q$ *contains no nonzero prime ideal*), *each* R_Q *is t-pure* (*i.e., each nonzero nonunit of* R_Q *is t-pure*) *and* $\mathrm{Cl}_t(R) = 0$.

PROOF. – $(1) \Leftrightarrow (2)$ Theorem 3.1. $(1) \Leftrightarrow (3)$ [6, Corollary 4.4]. $(3) \Leftrightarrow (4)$ Lemma 3.3. ■

We next consider the integral domains in which every nonzero nonunit is a product of completely primal homogeneous elements. Such domains are of course semi-homogeneous pre-Schreier domains.

LEMMA 3.5. – *If R is pre-Schreier, then* $\mathrm{Cl}_t(R) = 0$.

PROOF. – This follows immediately from [16, Corollary 3.7] since if A is t-invertible, then both A_t and A^{-1} are finite type v-ideals. ■

Lemma 3.5 offers another proof of the result that $R[X]$ pre-Schreier $\Rightarrow R$ is integrally closed. For if $R[X]$ is pre-Schreier then $\mathrm{Cl}_t(R[X]) = 0$. Thus the natural map $\mathrm{Cl}_t(R) \to \mathrm{Cl}_t(R[X])$ is an isomorphism and hence R is integrally closed [10, Theorem 3.6].

THEOREM 3.6. – *Let R be a t-finite character, t-independent integral domain. Then R is (pre-)Schreier if and only if* $\mathrm{Cl}_t(R) = 0$ *and R_M is (pre-)Schreier for each $M \in t$-Max(R).*

PROOF. – It suffices to prove the pre-Schreier case since R is integrally closed $\Leftrightarrow$ each R_M ($M \in t$-Max(R)) is integrally closed. ($\Rightarrow$) The fact that $\mathrm{Cl}_t(R) = 0$ follows from Lemma 3.5. And if R is pre-Schreier, so is R_S for any multiplicatively closed set S [9, Theorem 2.6].

($\Leftarrow$) Let $M \in t$-Max(R) and let x be M-pure. It suffices to prove that x is primal since every nonzero nonunit of R is a product of t-pure elements. Suppose that $x \mid a_1 a_2$ in R. Then since R_M is pre-Schreier, in R_M we have $x = x_1 x_2$ where $x_i \mid a_i$. Put $x_i = c_i / d_i$ where c_i, $d_i \in R$ with $d_i \in M$. Now since R has t-finite character, is t-independent and has $\mathrm{Cl}_t(R) = 0$, every nonzero nonunit has a unique factorization (up to units and order) into mutually v-coprime t-pure elements. Let c_i' be the M-pure factor of c_i (i.e., $Rc_i' = c_i R_M \cap R$). Now since $d_1 d_2 x = c_1 c_2$ and $d_1 d_2 \notin M$, we get $x = c_1' c_2'$ (perhaps after modifying c_1' by a unit of R). Now $c_i / d_i = x_i \mid a_i$ so $(c_i / d_i)(r_i / s_i) = a_i$ where r_i, $s_i \in R$ with $s_i \notin M$. So $r_i c_i = d_i s_i a_i$. Let r_i' be the M-pure factor of r_i. Since $d_i s_i \notin M$, $r_i' c_i'$ is the M-pure factor of a_i. So $r_i' c_i' \mid a_i$. Thus $c_i' \mid a_i$ and x is primal. ■

For an alternative proof of Theorem 3.6, observe that for R t-finite character and t-independent with $\mathrm{Cl}_t(R) = 0$, $G(R)$ is order-isomorphic to the cardinal sum $\bigoplus_{P \in t\text{-Max}(R)} G(R_P)$. Thus if each R_P is pre-Schreier, then each $G(R_P)$ is a Riesz group. Hence $G(R) = \bigoplus G(R_P)$ is a Riesz group and thus R is pre-Schreier.

COROLLARY 3.7. – *For an integral domain R the following are equivalent.*

(1) *R is a pre-Schreier semi-homogeneous integral domain.*

(2) *Every nonzero nonunit of R may be written uniquely as a product of mutually coprime completely primal homogeneous elements.*

(3) *R is a t-finite character, t-independent pre-Schreier domain.*

(4) *R is a finite character intersection $\bigcap_{Q \in \mathcal{S}} R_Q$ for some set $\mathcal{S}$ of independent prime ideals such that each R_Q is a homogeneous pre-Schreier domain and* $\mathrm{Cl}_t(R) = 0$.

PROOF. – (1)$\Rightarrow$(2) Corollary 3.2. (2)$\Rightarrow$(1) This follows from the facts that products of completely primal elements are completely primal [9] and that R is by definition a pre-Schreier domain if and only if every nonzero element of R is primal. (1)⇔(3) This follows from Theorem 2.3 and Theorem 3.4. (3)⇒(4) Theorem 3.6. (4)⇒(3) Lemma 3.3 and Theorem 3.6. ∎

Suppose that R is a t-finite character, t-independent domain. Now for $M \in t\text{-Max}(R)$, $M_M \in t\text{-Max}(R_M)$. Thus R_M is a GCD domain if and only if R_M is a valuation domain. Hence if R is a GCD domain, each R_M is a valuation domain and thus R is an *independent ring of Krull type* (i.e., $R = \bigcap R_{P_\alpha}$ where the intersection has finite character, each R_{P_α} is a valuation domain, and for $\alpha \neq \beta$, $P_\alpha \cap P_\beta$ contains no nonzero prime ideal). Conversely, suppose that for each $M \in t\text{-Max}(R)$, R_M is a valuation domain. Then R is a GCD domain $\Leftrightarrow \mathrm{Cl}_t(R) = 0 \Leftrightarrow R$ is pre-Schreier. Indeed, we always have for any domain R: GCD domain ⇒ Schreier ⇒ pre-Schreier ⇒ $\mathrm{Cl}_t(R) = 0$. But if R is t-finite character, t-independent with $\mathrm{Cl}_t(R) = 0$, then $G(R)$ is order-isomorphic to $\bigoplus_{P \in t\text{-Max}(R)} G(R_P)$ [6, Corollary 4.4] and hence is lattice ordered. Thus R is a GCD domain. Also, note that if R is weakly Krull (i.e., $R = \bigcap_{P \in X^{(1)}(R)} R_P$ has finite character where $X^{(1)}(R)$ is the set of height one primes of R), then R has t-finite character and is t-independent. For if R is weakly Krull, then by [6, Lemma 2.1] $X^{(1)}(R) = t\text{-Max}(R)$.

COROLLARY 3.8. – *For an integral domain R the following are equivalent.*

(1) *R is a semi-rigid GCD domain.*

(2) *R is a semi-t-pure GCD domain.*

(3) *R is a GCD domain that is an independent ring of Krull type.*

(4) *R is an independent ring of Krull type with* $\mathrm{Cl}_t(R) = 0$.

PROOF. – $(1) \Leftrightarrow (2)$ Corollary 2.4. The equivalence of (2)-(4) follows from Theorem 3.4 and the remarks preceding Corollary 3.8. ∎

The implication (1)⇒(2) is given in [13, Theorem 5] and the implication (3)⇒(1) is given in [14, Theorem B].

Kaplansky [12, Theorem 5] proved that R is a UFD if and only if every nonzero prime ideal of R contains a (nonzero) prime element. For each of the properties ($*$) of an element given in Figure 1 of Section 2 we can ask whether every nonzero nonunit of an integral domain R is a product of elements having property ($*$) if and only if each nonzero prime ideal of R contains a (nonzero) element with property ($*$). By Kaplansky's Theorem this is the case where ($*$) is «prime» or «prime power». And by [4, Theorem 9], R is a GUFD if and only if each nonzero prime ideal contains a prime quantum. Further Kaplansky-like theorems are investigated in [7].

The example $R = k + XK[[X]]$ given in Section 2 shows that even if R has a unique nonzero prime ideal and this prime ideal contains a primal element, every nonzero nonunit need not be a product of primal elements. However, in Theorem 3.9 below, we show that you can take ($*$) to be «completely primal». Let R be a Krull domain with $\mathrm{Cl}(R)$ finite but nontrivial. For each $P \in X^{(1)}(R)$, some $P^{(n)}$ is principal. Thus each nonzero prime ideal of R contains a primary (and hence t-pure) element. But R is not semi-t-pure nor weakly factorial. Also, each prime ideal contains a packet, but each nonzero nonunit is not necessarily a product of packets. Here the problem is that $\mathrm{Cl}_t(R) = \mathrm{Cl}(R) \neq 0$. However, on the positive side, we have the following theorem.

THEOREM 3.9. – *Let R be an integral domain.*

(1) *R is semi-t-pure if and only if each nonzero prime ideal of R contains a t-pure element and* $\mathrm{Cl}_t(R) = 0$.

(2) *R is pre-Schreier if and only if each nonzero prime ideal contains a completely primal element.*

(3) *R is a semi-homogeneous pre-Schreier domain if and only if each nonzero prime ideal of R contains a completely primal homogeneous element.*

PROOF. – (1) ($\Rightarrow$) Clear. ($\Leftarrow$) By [6, Theorem 4.3] R has t-finite character and is t-independent. Since $\mathrm{Cl}_t(R) = 0$, R is semi-t-pure by [6, Corollary 4.4].

(2) Apply [7, Theorem 1] where ($*$) is the property «completely primal».

(3) ($\Rightarrow$) Clear. ($\Leftarrow$) By (2), R is pre-Schreier. Then by (1) R is semi-t-pure and hence semi-homogeneous. ∎

Let R be the semigroup ring $\mathbb{Q}[X; \langle \{1/2^n \mid n \geqslant 0\}, \mathbb{Q} \cap [1.1, \infty) \rangle]$ and let $P = \{f \in R \mid f \text{ has zero constant term}\}$. Then R_P is a one-dimensional quasilocal domain that is not atomic. However, P_P contains the atom $X^{1.1}$. It can easily be checked that $X^{1.2}$ cannot be written as a product of strongly homogeneous (and hence rigid) elements. Thus R_P is neither semi-rigid nor semi-strongly-homogeneous.

REFERENCES

[1] D. D. ANDERSON, *Star operations induced by overrings*, Comm. Algebra, **16** (1988), 2535-2553.

[2] D. D. ANDERSON - D. F. ANDERSON - M. ZAFRULLAH, *Factorization in integral domains*, J. Pure Appl. Algebra, **69** (1990), 1-19.

[3] D. D. ANDERSON - D. F. ANDERSON - M. ZAFRULLAH, *Atomic domains in which almost all atoms are prime*, Comm. Algebra, **20** (1992), 1447-1462.
[4] D. D. ANDERSON - D. F. ANDERSON - M. ZAFRULLAH, *A generalization of unique factorization*, Boll. Un. Mat. Ital. A (7), **9** (1995), 401-413.
[5] D. D. ANDERSON - L. A. MAHANEY, *On primary factorizations*, J. Pure Appl. Algebra, **54** (1988), 141-154.
[6] D. D. ANDERSON - J. L. MOTT - M. ZAFRULLAH, *Finite character representations for integral domains*, Boll. Un. Mat. Ital. B (7), **6** (1992), 613-630.
[7] D. D. ANDERSON - M. ZAFRULLAH, *On a theorem of Kaplansky*, Boll. Un. Mat. Ital. A (7), 8 (1994), 397-402.
[8] D. D. ANDERSON - M. ZAFRULLAH, *P. M. Cohn's completely primal elements*, in *Zero Dimensional Commutative Rings* (D. F. Anderson and D. E. Dobbs, eds.), Lecture Notes in Pure and Applied Mathematics, vol. 171, Marcel Dekker, Inc., New York (1995), pp. 115-123.
[9] P. M. COHN, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc., **64** (1968), 251-264.
[10] S. GABELLI, *On divisorial ideals in polynomial rings over Mori domains*, Comm. Algebra, **15** (1987), 2349-2370.
[11] R. GILMER, *Multiplicative Ideal Theory*, Marcel Dekker, New York (1972).
[12] I. KAPLANSKY, *Commutative Rings*, revised edition, The University of Chicago Press, Chicago and London (1974).
[13] M. ZAFRULLAH, *Semirigid GCD domains*, Manuscripta Math., **17** (1975), 55-66.
[14] M. ZAFRULLAH, *Rigid elements in GCD domains*, J. Natur. Sci. Math., **17** (1977), 7-14.
[15] M. ZAFRULLAH, *Unique representation domains*, J. Natur. Sci. Math., **18** (1978), 19-29.
[16] M. ZAFRULLAH, *On a property of pre-Schreier domains*, Comm. Algebra, **15** (1987), 1895-1920.
[17] M. ZAFRULLAH, *The $D + xDS[x]$ construction from GCD-domains*, J. Pure Appl. Algebra, **50** (1988), 93-107.

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Pervenuta in Redazione
il 16 aprile 1997 e, in forma revisionata, il 15 settembre 1997