Stefano Montaldo

$p$-minimising tangent maps and harmonic $k$-forms


Unione Matematica Italiana

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**p-Minimising Tangent Maps and Harmonic k-Forms.**

**STEFANO MONTALDO (*)**

**Sunto.** – Si studiano le applicazioni p-tangenti da $\mathbb{R}^m$ a $S^n$ date come estensioni omogenee di k-forme armoniche. Vengono ricavate condizioni necessarie sul grado k affinche tali applicazioni p-tangenti siano di energia minima. Una classificazione completa viene data nel caso in cui tali applicazioni tangenti di energia minima vadano da $\mathbb{R}^8$ su $S^4$.

1. – Introduction.

Let $(M^m, g), (N^n, h)$ be two Riemannian manifolds of dimensions $m$ and $n$ respectively. By J. Nash’s Theorem [10] we can always assume that $(N, h)$ is isometrically embedded in some Euclidean space $\mathbb{R}^q$. For $2 \leq p < \infty$, define (see [4])

$$\mathcal{X}^p_1(M, N) := \{ u \in \mathcal{X}^p_1(M, \mathbb{R}^q); u(x) \in N \text{ for almost all } x \in M \},$$

where $\mathcal{X}^p_1(M, \mathbb{R}^q)$ is the Sobolev space of equivalence classes of p-integrable maps whose first derivatives are $p$-integrable. Thus, an element of $\mathcal{X}^p_1(M, N)$ is in fact an equivalent class of maps defined almost everywhere on $M$, two maps being equivalent if they agree almost everywhere. Note that, if $1 > m/p$, then $\mathcal{X}^p_1(M, N) \subset C^0(M, N)$. A map $u \in \mathcal{X}^p_1(M, N)$ is called continuous if its class contains a continuous representative.

Let $\Omega \subset M$ be a compact domain. The $p$-energy of a map $u \in \mathcal{X}^p_1(M, N)$ over $\Omega$ is the number

$$E_p(u, \Omega) = \frac{1}{p} \int_{\Omega} \| du \|_p^p dV_g.$$

A map $u \in \mathcal{X}^p_1(M, N)$ is said to be weakly $p$-harmonic (see [3]) if it is a critical point of the $p$-energy over every compact domain $\Omega \subset M$, i.e. for any $\eta \in C^\infty(M, \mathbb{R}^q)$ with compact support $\Omega$ and the corresponding variation $u_t(x) = \pi \circ (u(x) + t\eta(x))$, the first variation $D_t E(u, \Omega) = (d/dt)(E(u_t, \Omega)) |_{t=0} = 0$. Here $\pi$ is the orthogonal projection of $\mathbb{R}^q$ onto $N$ (well defined for $t$ small enough).

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A weakly $p$-harmonic map $u \in \mathcal{C}_1^p(M, N)$ is called $p$-energy minimising if $E_p(u) \leq E_p(\overline{u})$ for every $\overline{u} \in \mathcal{C}_1^p(M, N)$ such that the support of $(u - \overline{u})$ is contained in a compact subset of $M$.

Given a $p$-energy minimising map $u$ a basic question is whether it is regular, i.e. continuous. If $m < p$, then every $L_1^p$-map is continuous; if $m < [p] + 1$, where $[a]$ denotes the integer part of $a$, then any $p$-energy minimising map is continuous [7]. For the case $N = S^n$, R. Schoen and K. Uhlenbeck [13] proved that if $m \leq d(n)$, where

$$d(2) = 2, \quad d(3) = 3, \quad d(n) = [1 + \min \{n/2, 5\}] \quad \text{for } n > 3,$$

then any 2-energy minimising map is continuous.

In general, a $p$-energy minimising map in not continuous. However the behaviour around singular points is understood, as follows.

Let $\mathcal{C}_1^p, \text{loc}(\mathbb{R}^m, N^n)$ the space of measurable maps whose restriction to each compact subset is $L_1^p$.

**Definition 1.1.** A map $\overline{u} \in \mathcal{C}_1^p, \text{loc}(\mathbb{R}^m, N^n)$ is called a $p$-tangent map if $\overline{u}$ is weakly $p$-harmonic and $\partial \overline{u}/\partial r = 0$, where $r$ denotes the radial coordinate, i.e.

$$\overline{u}(x) = u \left( \frac{x}{|x|} \right) = u \circ \pi_\mathbb{R},$$

where $u : S^{m-1} \rightarrow N$ is a weakly $p$-harmonic map.

Note that a tangent map $\overline{u}(x) = u(x/|x|)$ has a singularity at 0 if and only if $u$ is non-constant. A $p$-tangent map $\overline{u} : \mathbb{R}^m \rightarrow N$ is a $p$-minimising tangent map if it is $p$-energy minimising as a weakly $p$-harmonic map.

**Theorem 1.2 (1)** [7]. Let $f \in \mathcal{C}_1^p(M, N)$ be a $p$-energy minimising map, and let $B^m$ a geodesic ball centred at a singular point $x_0$. Then there exists a sequence $\{\sigma_i\}_{i=0}^\infty$, which converges to 0, such that $\overline{u}_i : B^m \rightarrow N$, defined by $\overline{u}_i(x) = f(\exp_{x_0}(\sigma_i x))$, converges to a $p$-minimising tangent map in $\mathcal{C}_1^p(B^m, N)$ as $i \rightarrow \infty$.

Theorem 1.2 shows that the study of $p$-minimising tangent maps is an essential tool in the regularity theory of weakly $p$-harmonic maps.

The aim of this note is to study a particular class of $p$-minimising tangent maps to the $n$-dimensional sphere. Part of this note is devoted to the case when $p = 2$. In this case a 2-harmonic map is just a harmonic map and we shall write minimising instead of 2-minimising etc.

(1) The Theorem was first proved for the case when $p = 2$ by Schoen–Uhlenbeck [12].
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2. – $p$-minimising tangent maps and harmonic $k$-forms.

We first prove a necessary conditions for a map $\bar{u} = u \circ \pi_R : \mathbb{R}^m \to S^n$ to be a $p$-minimising tangent map.

**Lemma 2.1** [13]. – Let $q$ be a positive integer. Then

$$\inf_{0 \to \infty} \frac{\int (\phi')^2 r^{q-1} dr}{\int_0^\infty \phi^2 r^{q-3} dr} \leq \frac{(q - 2)^2}{4},$$

where the infimum is taken over all non zero functions $\phi \in C^\infty(\mathbb{R}^m)$ with compact support.

**Lemma 2.2.** – Let $\bar{u} = u \circ \pi_R : \mathbb{R}^m \to S^n$ be a $p$-minimising tangent map. Then

$$(n + p - 2) \int_{\mathbb{R}^m} |d\bar{u}|^{p-2} |\nabla \phi|^2 dx - (n - p) \int_{\mathbb{R}^m} \phi^2 |d\bar{u}|^p dx \geq 0,$$

for any function $\phi \in C^\infty(\mathbb{R}^m)$ with compact support.

**Proof.** – The proof follows using the same argument in [13 Theorem 2.4] adapted to the $p$-harmonic case (see also [9]).

**Theorem 2.3.** – Let $m > n$ and let $\bar{u} = u \circ \pi_R : \mathbb{R}^m \to S^n$ be a $p$-minimising tangent map. Then the energy of $u$ satisfies the following inequalities:

$$E(u) \leq \frac{n}{2(n - 2)} \frac{(m - 2)^2}{4} |S^{m-1}|$$

for $n > p = 2$;

$$E_p(u) \leq \frac{(p - 2)(n + p - 2)}{p(n - p)} \frac{(m - p)^2}{4} E_{p-2}(u)$$

for $n > p > 2$;

$$E_p(u) \geq \frac{(p - 2)(n + p - 2)}{p(p - n)} \frac{(m - p)^2}{4} E_{p-2}(u)$$

for $p > n \geq 2$.

where $|S^{m-1}|$ is the volume of $S^{m-1}$. 
Proof. – By Lemma 2.2, choosing $f$ to be a function of $r^{4N} x^N$ we have
\[(n + p - 2)(p - 2)E_{p-2}(u) \int_0^\infty (\phi')^2 r^{m-p+1} dr - p(n - p) E_p(u) \int_0^\infty \phi^2 r^{m-p-1} dr \geq 0 .\]
That is,
\[E_p(u) \leq \frac{(p - 2)(n + p - 2)}{p(n - p)} \int_0^\infty (\phi')^2 r^{m-p+1} dr E_{p-2}(u) \quad \text{for } p < n ,
\]
or
\[E_p(u) \geq \frac{(p - 2)(n + p - 2)}{p(p-n)} \int_0^\infty (\phi')^2 r^{m-p+1} dr E_{p-2}(u) \quad \text{for } p > n .
\]
Finally using Lemma 2.1 (with $q = m - p + 2$) we get immediately the result.

Definition 2.4 (see [5]). – (i) We say that $f: S^{m-1} \to S^n$ is a (homogeneous) polynomial map if it is the restriction of a map $F: \mathbb{R}^m \to \mathbb{R}^{n+1}$ whose components $F^\alpha: \mathbb{R}^m \to \mathbb{R}$ are homogeneous polynomials.

(ii) A map $f: S^{m-1} \to S^n$ is a harmonic $k$-form if each $F^\alpha$ is a harmonic homogeneous polynomial of common degree $k$.

We have the following.

Proposition 2.5 (See [1] and [8]). – Let $f: S^{m-1} \to S^n$ be a harmonic $k$-form, then $f$ is a $p$-harmonic map and the $p$-energy is given by
\[E_p(u) = \left(\frac{(k(n + m - 2))^{p^2}}{p} \right) |S^{m-1}| .
\]

Remark 2.6. – A harmonic 1-form is called a harmonic linear form. If $m = 2$ and $n = 1$, then all harmonic linear forms $f: S^1 \to S^1$ are the restriction to $S^1$ of an element $F \in O(2)$. If $n > 1$, the harmonic linear forms $f: S^1 \to S^n$ are the restriction to $S^1$ of an element $F \in O(\mathbb{R}^2, \mathbb{R}^{n+1}) = \{ A \in M_{2,n+1}(\mathbb{R}) : AA^T = I \}$. The resulting harmonic linear form $f: S^1 \to S^n$ is a totally geodesic embedding of $S^1$ in $S^n$. 
In contrast if \( m > n + 1 \) there are no harmonic linear forms \( f: S^{m-1} \to S^n \). This because a linear map \( F: \mathbb{R}^m \to \mathbb{R}^{n+1} \) carrying \( S^{m-1} \) to \( S^n \) must be injective.

From Theorem 2.3 and Proposition 2.5 we have immediately

**THEOREM 2.7.** – Let \( m > n \) and \( m \geq [p] + 1 \). Let \( \overline{u} = u \circ \pi_R: \mathbb{R}^m \to S^n \) be a \( p \)-minimising tangent map, where \( u: S^{m-1} \to S^n \) is a \( p \)-harmonic \( k \)-form. Then \( k \) satisfies the following inequalities:

\[
\begin{align*}
(k + m - 2) &\leq \frac{(n + p - 2) (m - p)^2}{(n - p)} \quad \text{for } p < n; \\
(k + m - 2) &\geq \frac{(n + p - 2) (m - p)^2}{(p - n)} \quad \text{for } p > n.
\end{align*}
\]

3. – *p*-minimising tangent maps and Hopf forms.

**DEFINITION 3.1** [5]. – An orthogonal multiplication is a bilinear map

\[ f: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \]

which is norm-preserving:

\[ |f(x, y)| = |x| |y| \quad \text{for all } x \in \mathbb{R}^p, y \in \mathbb{R}^q. \]

The Hopf construction on an orthogonal multiplication \( f \) is the map

\[ F: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \]

given by

\[ F(x, y) = (|x|^2 - |y|^2, 2f(x, y)). \]

Because \( |F(x, y)|^2 = (|x|^2 + |y|^2)^2 \), its restriction defines a map

\[ H: S^{p+q-1} \to S^n \]

also called the Hopf construction on \( f \).

**EXAMPLE 3.2.** – We list three basic examples of Hopf construction.

1) (The complex Hopf map) The Hopf map \( H_2: S^3 \to S^2 \) is defined by the restriction to

\[ S^3 = \{(z, w) \in \mathbb{C}^2: |z|^2 + |w|^2 = 1\} \]

of

\[ H_2(z, w) = (|z|^2 - |w|^2, 2z \overline{w}) \in \mathbb{R} \times \mathbb{C} = \mathbb{R}^3. \]
2) (The quaternionic Hopf map) Let $\mathbb{H}$ be the skew-field of quaternions. The Hopf map $H_4: \mathbb{S}^7 \to \mathbb{S}^4$ is defined by identifying $\mathbb{R}^8$ as $\mathbb{H} \times \mathbb{H}$ and taking the restriction to

$$
\mathbb{S}^7 = \{(q_1, q_2) \in \mathbb{H}^2: |q_1|^2 + |q_2|^2 = 1\}
$$

of

$$
H_4(q_1, q_2) = (|q_1|^2 - |q_2|^2, 2q_1\overline{q}_2) \in \mathbb{R} \times \mathbb{H} = \mathbb{R}^5.
$$

3) (The Cayley Hopf map) Define the Cayley space by

$$
\text{Cay} = \{A = (q, p): q, p \in \mathbb{H}\}.
$$

The Hopf map $H_8: \mathbb{S}^{15} \to \mathbb{S}^8$ is defined by identifying $\mathbb{R}^{16}$ as $\text{Cay} \times \text{Cay}$ and taking the restriction to

$$
\mathbb{S}^{15} = \{(A_1, A_2) \in \text{Cay}^2: |A_1|^2 + |A_2|^2 = 1\}
$$

of

$$
H_8(A_1, A_2) = (|A_1|^2 - |A_2|^2, 2A_1\overline{A}_2) \in \mathbb{R} \times \text{Cay} = \mathbb{R}^9.
$$

All three Hopf maps are quadratic forms.

**Remark 3.3.** – In [11], M. Parker classified orthogonal multiplications $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ for $m = 2, 3$. In particular, for $m = 3$ and $n = 4$, there is essentially only the multiplication of purely imaginary quaternions, and the Hopf construction provides a quadratic form $f: \mathbb{S}^3 \to \mathbb{S}^4$. For general constructions of harmonic $k$-forms see also [6].

A quadratic form $f: \mathbb{S}^{p+q-1} \to \mathbb{S}^n$ is a Hopf form if, modulo orthogonal transformation, it is obtained from the Hopf construction on an orthogonal multiplication.

**Theorem 3.4** [14]. – Every quadratic form $f: \mathbb{S}^m \to \mathbb{S}^n$ is homotopy equivalent to a Hopf form.

**3.1. The case $p = 2$.**

**Theorem 3.5.** – Let

$$
m - 1 > n > 2 \quad \text{and} \quad d(n) \leq m < 2 + \frac{6(n - 2)}{n} \left(\sqrt{\frac{2n - 2}{n - 2}} + 1\right),
$$
where \( d(n) \) is given in (1.1). Let \( \bar{u} = u \circ \pi_R : \mathbb{R}^m \to S^n \) be a minimizing tangent map which is the homogeneous extension of a harmonic \( k \)-form \( u : S^{m-1} \to S^n \). Then \( u \) is homotopy equivalent to a Hopf form.

**Proof.** – From Theorem 3.4 we only have to prove that \( u \) is a quadratic form, i.e. \( k = 2 \). From the first inequality (2.1) (with \( p = 2 \)) the greatest value of \( k \) is

\[
k_{\text{max}} = \frac{m - 2}{2} \left( \sqrt{\frac{2n - 2}{n - 2}} - 1 \right).
\]

If \( m < 2 + \frac{6(n - 2)}{n} \left( \sqrt{\frac{2n - 2}{n - 2}} + 1 \right) \), then \( k_{\text{max}} < 3 \). Hence we have \( k = 1 \) or \( 2 \), and from Remark 2.6 the case \( k = 1 \) is excluded. 

Suppose \( m = 2n \), then the first inequality (2.1) becomes

\[
k(k + 2n - 2) \leq \frac{n(n - 1)^2}{(n - 2)}.
\]

Tabulating \( n \) from 3 to 8 and using (3.1) we have the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>possible ( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>2 and 3</td>
</tr>
<tr>
<td>8</td>
<td>2 and 3</td>
</tr>
</tbody>
</table>

(T)

We have immediately the following corollary of Theorem 3.5.

**Corollary 3.6.** – Let \( 3 \leq n \leq 6 \), and let \( \bar{u} = u \circ \pi_R : \mathbb{R}^{2n} \to S^n \) be a minimizing tangent map which is the homogeneous extension of a harmonic \( k \)-form. Then \( u \) is homotopy equivalent to a Hopf form.

The most interesting case is when \( n \) is a power of two. In this situation P. Yiu [15] proved that (i) if \( n \geq 16 \) is a power of 2, then any quadratic form \( f: S^{2n-1} \to S^n \) is constant, (ii) if \( n = 2, 4, 8 \), then any non constant quadratic form \( f: S^{2n-1} \to S^n \) is (up to isometries) the Hopf map. Moreover, J. M. Coron and R. Gulliver [3] proved that the homogeneous extension of the Hopf map \( H_n \) (\( n = 2, 4, 8 \)) is a minimizing tangent map.
THEOREM 3.7. – A non-constant tangent map \( \overline{u} = u \circ \pi_R: \mathbb{R}^8 \to S^4 \), with \( u \) a harmonic \( k \)-form, is a minimising tangent map if and only if \( u \) is the Hopf map \( H_4 \) (up to isometries).

PROOF. – If \( u \) is the Hopf map \( H_4 \) then \( \overline{u} \) is a minimising tangent map, this is the result of Coron and Gulliver. Conversely, assume that \( \overline{u} \) is a minimising tangent map, then from the Table (T) \( u \) is a quadratic form and from Yiu’s result \( u \) is the Hopf map \( H_4 \).

In the same way we have

PROPOSITION 3.8. – Let \( \overline{u} = u \circ \pi_R: \mathbb{R}^{16} \to S^8 \), be a minimising tangent map, with \( u \) a harmonic \( k \)-form. Then \( u \) is the the Hopf map \( H_8 \) or a harmonic 3-form.

REMARK 3.9. – It is not known whether there is any 3-form \( u: S^{15} \to S^8 \) although some general constructions of 3-forms have been developed [2].

3.2. The case \( p > 2 \)

PROPOSITION 3.10. – Let \( \overline{u} = u \circ \pi_R: \mathbb{R}^8 \to S^4 \) be a \( p \)-minimising tangent map, where \( u: S^7 \to S^4 \) is a \( p \)-harmonic \( k \)-form. Let \( \alpha \) be the real root of

\[
(p + 2)(p - 8)^2 - 108(4 - p) = 0.
\]

Then, if \( 2 < p < \alpha \approx 2.797 \), \( u \) is the Hopf map \( H_4 \).

PROOF. – If \( 2 < p < \alpha \), from the first inequality (2.1) the degree \( k < 3 \), which implies that \( u \) is the Hopf map.

PROPOSITION 3.11. – Let \( \overline{u} = H_n \circ \pi_R: \mathbb{R}^{2n} \to S^n \) \((n = 2, 4, 8)\) be the homogeneous extension of the Hopf map \( H_n: S^{2n-1} \to S^n \) with \( p > n \). If \( p \) satisfies:

\[
G_n(p) = (n + p - 2)(p - 2n)^2 - 16n(p - n) > 0,
\]

then \( \overline{u} \) is not a \( p \)-minimising tangent map.

PROOF. – If \( \overline{u} \) is a \( p \)-minimising tangent map and \( G_n(p) > 0 \), from the second inequality (2.1) the degree \( k > 2 \), which excludes the Hopf maps.
REMARK 3.12. – For $n = 2, 4, 8$, the values of $p$ such that $G_n(p) > 0$ are shown in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p$ such that $G_n(p) &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2 &lt; p \lesssim 2.219$</td>
</tr>
<tr>
<td>4</td>
<td>$4 &lt; p \lesssim 4.989$</td>
</tr>
<tr>
<td>8</td>
<td>$8 &lt; p \lesssim 11.150$</td>
</tr>
</tbody>
</table>

REFERENCES


University of Leeds, Department of Pure Mathematics
LS2 9JT Leeds, UK
e-mail: pmtms@amsta.leeds.ac.uk

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