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p-Minimising Tangent Maps and Harmonic k-Forms.

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Sunto. – *Si studiano le applicazioni p-tangenti da $\mathbb{R}^m$ a $\mathbb{S}^n$ date come estensioni omogenee di k-forme armoniche. Vengono ricavate condizioni necessarie sul grado k affinche tali applicazioni p-tangenti siano di energia minima. Una classificazione completa viene data nel caso in cui tali applicazioni tangenti di energia minima vadano da $\mathbb{R}^8$ su $\mathbb{S}^4$.*

1. – Introduction.

Let (M^m, g), (N^n, h) be two Riemannian manifolds of dimensions m and n respectively. By J. Nash's Theorem [10] we can always assume that (N, h) is isometrically embedded in some Euclidean space $\mathbb{R}^q$. For $2 \leq p < \infty$, define (see [4])

$$\mathcal{L}_1^p(M, N) := \{u \in \mathcal{L}_1^p(M, \mathbb{R}^q) \colon u(x) \in N \text{ for almost all; } x \in M\},$$

where $\mathcal{L}_1^p(M, \mathbb{R}^q)$ is the Sobolev space of equivalence classes of p-integrable maps whose first derivatives are p-integrable. Thus, an element of $\mathcal{L}_1^p(M, N)$ is in fact an equivalent class of maps defined almost everywhere on M, two maps being equivalent if they agree almost everywhere. Note that, if $1 > m/p$, then $\mathcal{L}_1^p(M, N) \subseteq C^0(M, N)$. A map $u \in \mathcal{L}_1^p(M, N)$ is called continuous if its class contains a continuous representative.

Let $\Omega \subseteq M$ be a compact domain. The *p-energy* of a map $u \in \mathcal{L}_1^p(M, N)$ over Ω is the number

$$E_p(u, \Omega) = \frac{1}{p} \int_\Omega \|du\|^p \, dv_g .$$

A map $u \in \mathcal{L}_1^p(M, N)$ is said to be *weakly p-harmonic* (see [3]) if it is a critical point of the *p-energy* over every compact domain $\Omega \subseteq M$, i.e. for any $\eta \in C^\infty(M, \mathbb{R}^q)$ with compact support Ω and the corresponding variation $u_t(x) = \pi \circ (u(x) + t\eta(x))$, the first variation $D_V E(u, \Omega) = (d/dt)(E(u_t, \Omega))|_{t=0} = 0$. Here π is the orthogonal projection of $\mathbb{R}^q$ onto N (well defined for t small enough).

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A weakly p-harmonic map $u \in \mathcal{L}_1^p(M, N)$ is called *p-energy minimising* if $E_p(u) \leqslant E_p(\overline{u})$ for every $\overline{u} \in \mathcal{L}_1^p(M, N)$ such that the support of $(u - \overline{u})$ is contained in a compact subset of M.

Given a p-energy minimising map u a basic question is whether it is regular, i.e. continuous. If $m < p$, then every L_1^p-map is continuous; if $m < [p] + 1$, where $[a]$ denotes the integer part of a, then any p-energy minimising map is continuous [7]. For the case $N = \mathbb{S}^n$ R. Schoen and K. Uhlenbeck [13] proved that if $m \leqslant d(n)$, where

$$(1.1) \quad d(2) = 2\,, \qquad d(3) = 3\,, \qquad d(n) = [1 + \min\{n/2, 5\}] \qquad \text{for } n > 3\,,$$

then any 2-energy minimising map is continuous.

In general, a p-energy minimising map in not continuous. However the behaviour around singular points is understood, as follows.

Let $\mathcal{L}_{1,\,\mathrm{loc}}^p(\mathbb{R}^m, N^n)$ the space of measurable maps whose restriction to each compact subset is L_1^p.

DEFINITION 1.1. – A map $\overline{u} \in \mathcal{L}_{1,\,\mathrm{loc}}^p(\mathbb{R}^m, N^n)$ is called a *p-tangent map* if $\overline{u}$ is weakly p-harmonic and $\partial \overline{u}/\partial r = 0$, where r denotes the radial coordinate, i.e.

$$\overline{u}(x) = u\left(\frac{x}{|x|}\right) = u \circ \pi_{\mathbb{R}}\,,$$

where $u\colon\ \mathbb{S}^{m-1} \to N$ is a weakly p-harmonic map.

Note that a tangent map $\overline{u}(x) = u(x/|x|)$ has a singularity at 0 if and only if u is non-constant. A p-tangent map $\overline{u}\colon \mathbb{R}^m \to N$ is a *p-minimising tangent map* if it is p-energy minimising as a weakly p-harmonic map.

THEOREM 1.2 (1) [7]. – *Let $f \in \mathcal{L}_1^p(M, N)$ be a p-energy minimising map, and let B^m a geodesic ball centred at a singular point x_0. Then there exists a sequence $\{\sigma_i\}_{i=o}^{\infty}$, which converges to 0, such that $\overline{u}_i\colon B^m \to N$, defined by $\overline{u}_i(x) = f(\exp_{x_0}(\sigma_i x))$, converges to a p-minimising tangent map in $\mathcal{L}_1^p(B^m, N)$ as $i \to \infty$.*

Theorem 1.2 shows that the study of p-minimising tangent maps is an essential tool in the regularity theory of weakly p-harmonic maps.

The aim of this note is to study a particular class of p-minimising tangent maps to the n-dimensional sphere. Part of this note is devoted to the case when $p = 2$. In this case a 2- harmonic map is just a harmonic map and we shall write *minimising* instead of 2-*minimising* etc.

(1) The Theorem was first proved for the case when $p = 2$ by Schoen–Uhlenbeck [12].

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2. – p-minimising tangent maps and harmonic k-forms.

We first prove a necessary conditions for a map $\overline{u} = u \circ \pi_{\mathbb{R}} : \mathbb{R}^m \to \mathbb{S}^n$ to be a p-minimising tangent map.

LEMMA 2.1 [13]. – *Let q be a positive integer. Then*

$$\inf \frac{\int_0^\infty (\phi')^2 r^{q-1}\, dr}{\int_0^\infty \phi^2 r^{q-3}\, dr} \leqslant \frac{(q-2)^2}{4} ,$$

where the infimum is taken over all non zero functions $\phi \in C^\infty(\mathbb{R}^m)$ with compact support.

LEMMA 2.2. – *Let $\overline{u} = u \circ \pi_{\mathbb{R}} : \mathbb{R}^m \to \mathbb{S}^n$ be a p-minimising tangent map. Then*

$$(n+p-2) \int_{\mathbb{R}^m} |d\overline{u}|^{p-2} |\nabla \phi|^2 \, dx - (n-p) \int_{\mathbb{R}^m} \phi^2 |d\overline{u}|^p \, dx \geqslant 0 ,$$

for any function $\phi \in C^\infty(\mathbb{R}^m)$ with compact support.

PROOF. – The proof follows using the same argument in [13 Theorem 2.4] adapted to the p-harmonic case (see also [9]). ■

THEOREM 2.3. – *Let $m > n$ and let $\overline{u} = u \circ \pi_{\mathbb{R}} : \mathbb{R}^m \to \mathbb{S}^n$ be a p-minimising tangent map. Then the energy of u satisfies the following inequalities:*

$$E(u) \leqslant \frac{n}{2(n-2)} \frac{(m-2)^2}{4} |\mathbb{S}^{m-1}| \qquad \text{for } n > p = 2 ;$$

$$E_p(u) \leqslant \frac{(p-2)(n+p-2)}{p(n-p)} \frac{(m-p)^2}{4} E_{p-2}(u) \quad \text{for } n > p > 2 ;$$

$$E_p(u) \geqslant \frac{(p-2)(n+p-2)}{p(p-n)} \frac{(m-p)^2}{4} E_{p-2}(u) \quad \text{for } p > n \geqslant 2 .$$

where $|\mathbb{S}^{m-1}|$ is the volume of $\mathbb{S}^{m-1}$.

PROOF. – By Lemma 2.2, choosing ϕ to be a function of $r = |x|$ we have

$$(n+p-2)(p-2)E_{p-2}(u)\int_0^\infty (\phi')^2 r^{m-p+1}\,dr - p(n-p)\,E_p(u)\int_0^\infty \phi^2 r^{m-p-1}\,dr \geq 0\,.$$

That is,

$$E_p(u) \leq \frac{(p-2)(n+p-2)}{p(n-p)} \frac{\int_0^\infty (\phi')^2 r^{m-p+1}\,dr}{\int_0^\infty \phi^2 r^{m-p-1}\,dr} E_{p-2}(u) \qquad \text{for } p<n\,,$$

or

$$E_p(u) \geq \frac{(p-2)(n+p-2)}{p(p-n)} \frac{\int_0^\infty (\phi')^2 r^{m-p+1}\,dr}{\int_0^\infty \phi^2 r^{m-p-1}\,dr} E_{p-2}(u) \qquad \text{for } p>n\,.$$

Finally using Lemma 2.1 (with $q = m-p+2$) we get immediately the result. ■

DEFINITION 2.4 (see [5]). – (i) We say that $f\colon \mathbb{S}^{m-1} \to \mathbb{S}^n$ is a (*homogeneous*) *polynomial map* if it is the restriction of a map $F\colon \mathbb{R}^m \to \mathbb{R}^{n+1}$ whose components $F^\alpha\colon \mathbb{R}^m \to \mathbb{R}$ are homogeneous polynomials.

(ii) A map $f\colon \mathbb{S}^{m-1} \to \mathbb{S}^n$ is a *harmonic k-form* if each F^α is a harmonic homogeneous polynomial of common degree k.

We have the following.

PROPOSITION 2.5 (See [1] and [8]). – *Let $f\colon \mathbb{S}^{m-1} \to \mathbb{S}^n$ be a harmonic k-form, then f is a p-harmonic map and the p-energy is given by*

$$E_p(u) = \frac{(k(k+m-2))^{p/2}}{p} |\mathbb{S}^{m-1}|\,.$$

REMARK 2.6. – A harmonic 1-form is called a harmonic linear form. If $m=2$ and $n=1$, then all harmonic linear forms $f\colon \mathbb{S}^1 \to S^1$ are the restriction to $\mathbb{S}^1$ of an element $F \in O(2)$. If $n>1$, the harmonic linear forms $f\colon \mathbb{S}^1 \to \mathbb{S}^n$ are the restriction to $\mathbb{S}^1$ of an element $F \in O(\mathbb{R}^2, \mathbb{R}^{n+1}) = \{A \in M_{2,n+1}(\mathbb{R})\colon AA^T = I\}$. The resulting harmonic linear form $f\colon S^1 \to \mathbb{S}^n$ is a totally geodesic embedding of $\mathbb{S}^1$ in $\mathbb{S}^n$.

In contrast if $m > n+1$ there are no harmonic linear forms $f\colon \mathbb{S}^{m-1} \to \mathbb{S}^n$. This because a linear map $F\colon \mathbb{R}^m \to \mathbb{R}^{n+1}$ carrying $\mathbb{S}^{m-1}$ to $\mathbb{S}^n$ must be injective.

From Theorem 2.3 and Proposition 2.5 we have immediately

THEOREM 2.7. – *Let $m > n$ and $m \geqslant [p]+1$. Let $\overline{u} = u \circ \pi_{\mathbb{R}}\colon \mathbb{R}^m \to \mathbb{S}^n$ be a p-minimising tangent map, where $u\colon \mathbb{S}^{m-1} \to \mathbb{S}^n$ is a p-harmonic k-form. Then k satisfies the following inequalities:*

$$
\left\{
\begin{aligned}
&k(k+m-2) \leqslant \frac{(n+p-2)}{(n-p)} \frac{(m-p)^2}{4} \quad \text{for } p < n\,;\\
&k(k+m-2) \geqslant \frac{(n+p-2)}{(p-n)} \frac{(m-p)^2}{4} \quad \text{for } p > n\,.
\end{aligned}
\right.
\tag{2.1}
$$

3. – p-minimising tangent maps and Hopf forms.

DEFINITION 3.1 [5]. – An *orthogonal multiplication* is a bilinear map

$$f\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$$

which is norm-preserving:

$$|f(x, y)| = |x|\,|y| \qquad \text{for all } x \in \mathbb{R}^p,\ y \in \mathbb{R}^q\,.$$

The *Hopf construction* on an orthogonal multiplication f is the map

$$F\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$$

given by

$$F(x, y) = (|x|^2 - |y|^2, 2f(x, y))\,.$$

Because $|F(x, y)|^2 = (|x|^2 + |y|^2)^2$, its restriction defines a map

$$H\colon \mathbb{S}^{p+q-1}\mathbb{S}^n$$

also called the *Hopf construction* on f.

EXAMPLE 3.2. – We list three basic examples of Hopf construction.

1) (The complex Hopf map) The Hopf map $H_2\colon \mathbb{S}^3 \to \mathbb{S}^2$ is defined by the restriction to

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2\colon\ |z|^2 + |w|^2 = 1\}$$

of

$$H_2(z, w) = (|z|^2 - |w|^2, 2z\overline{w}) \in \mathbb{R} \times \mathbb{C} = \mathbb{R}^3\,.$$

2) (The quaternionic Hopf map) Let $\mathbb{H}$ be the skew-field of quaternions. The Hopf map H_4: $\mathbb{S}^7 \to \mathbb{S}^4$ is defined by identifying $\mathbb{R}^8$ as $\mathbb{H} \times \mathbb{H}$ and taking the restriction to

$$\mathbb{S}^7 = \{(q_1, q_2) \in \mathbb{H}^2 : \ |q_1|^2 + |q_2|^2 = 1\}$$

of

$$H_4(q_1, q_2) = (|q_1|^2 - |q_2|^2, 2q_1\bar{q}_2) \in \mathbb{R} \times \mathbb{H} = \mathbb{R}^5 .$$

3) (The Cayley Hopf map) Define the Cayley space by

$$Cay = \{A = (q, p) : \ q, p \in \mathbb{H}\} .$$

The Hopf map H_8: $\mathbb{S}^{15} \to \mathbb{S}^8$ is defined by identifying $\mathbb{R}^{16}$ as $Cay \times Cay$ and taking the restriction to

$$\mathbb{S}^{15} = \{(A_1, A_2) \in Cay^2 : \ |A_1|^2 + |A_2|^2 = 1\}$$

of

$$H_8(A_1, A_2) = (|A_1|^2 - |A_2|^2, 2A_1\overline{A}_2) \in \mathbb{R} \times Cay = \mathbb{R}^9 .$$

All three Hopf maps are quadratic forms.

REMARK 3.3. – In [11], M. Parker classified orthogonal multiplications $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ for $m = 2, 3$. In particular, for $m = 3$ and $n = 4$, there is essentially only the multiplication of purely imaginary quaternions, and the Hopf construction provides a quadratic form f: $\mathbb{S}^5 \to S^4$. For general constructions of harmonic k-forms see also [6].

A quadratic form f: $\mathbb{S}^{p+q-1} \to S^n$ is a *Hopf form* if, modulo orthogonal transformation, it is obtained from the Hopf construction on an orthogonal multiplication.

THEOREM 3.4 [14]. – *Every quadratic form f: $\mathbb{S}^m \to S^n$ is homotopy equivalent to a Hopf form.*

3.1. *The case $p = 2$.*

THEOREM 3.5. – *Let*

$$m - 1 > n > 2 \ \textit{ and } \ d(n) \leq m < 2 + \frac{6(n-2)}{n}\left(\sqrt{\frac{2n-2}{n-2}} + 1\right),$$

where $d(n)$ *is given in* (1.1). *Let* $\overline{u} = u \circ \pi_{\mathbb{R}}$: $\mathbb{R}^m \to \mathbb{S}^n$ *be a minimising tangent map which is the homogeneous extension of a harmonic k-form* u: $\mathbb{S}^{m-1} \to S^n$. *Then* u *is homotopy equivalent to a Hopf form.*

PROOF. – From Theorem 3.4 we only have to prove that u is a quadratic form, i.e. $k = 2$. From the first inequality (2.1) (with $p = 2$) the greatest value of k is

$$k_{\max} = \frac{m-2}{2}\left(\sqrt{\frac{2n-2}{n-2}} - 1\right).$$

If $m < 2 + \dfrac{6(n-2)}{n}\left(\sqrt{\dfrac{2n-2}{n-2}} + 1\right)$, then $k_{\max} < 3$. Hence we have $k = 1$ or 2, and from Remark 2.6 the case $k = 1$ is excluded. ■

Suppose $m = 2n$, then the first inequality (2.1) becomes

$$k(k + 2n - 2) \leqslant \frac{n(n-1)^2}{(n-2)} \ . \tag{3.1}$$

Tabulating n from 3 to 8 and using (3.1) we have the following table:

(T)

n	possible k
3	2
4	2
5	2
6	2
7	2 and 3
8	2 and 3

We have immediately the following corollary of Theorem 3.5.

COROLLARY 3.6. – *Let* $3 \leqslant n \leqslant 6$, *and let* $\overline{u} = u \circ \pi_{\mathbb{R}}$: $\mathbb{R}^{2n} \to \mathbb{S}^n$ *be a minimising tangent map which is the homogeneous extension of a harmonic k-form. Then* u *is homotopy equivalent to a Hopf form.*

The most interesting case is when n is a power of two. In this situation P. Yiu [15] proved that (i) if $n \geqslant 16$ is a power of 2, then any quadratic form f: $\mathbb{S}^{2n-1} \to S^n$ is constant, (ii) if $n = 2, 4, 8$, then any non constant quadratic form f: $\mathbb{S}^{2n-1} \to S^n$ is (up to isometries) the Hopf map. Moreover, J. M. Coron and R. Gulliver [3] proved that the homogeneous extension of the Hopf map H_n ($n = 2, 4, 8$) is a minimising tangent map.

THEOREM 3.7. – *A non-constant tangent map* $\overline{u} = u \circ \pi_{\mathbb{R}}\colon \mathbb{R}^8 \to \mathbb{S}^4$, *with* u *a harmonic k-form, is a minimising tangent map if and only if* u *is the Hopf map* H_4 *(up to isometries).*

PROOF. – If u is the Hopf map H_4 then $\overline{u}$ is a minimising tangent map, this is the result of Coron and Gulliver. Conversely, assume that $\overline{u}$ is a minimising tangent map, then from the Table (T) u is a quadratic form and from Yiu's result u is the Hopf map H_4. ∎

In the same way we have

PROPOSITION 3.8. – *Let* $\overline{u} = u \circ \pi_{\mathbb{R}}\colon \mathbb{R}^{16} \to S^8$, *be a minimising tangent map, with* u *a harmonic k-form. Then* u *is the the Hopf map* H_8 *or a harmonic 3-form.*

REMARK 3.9. – It is not known whether there is any 3-form $u\colon \mathbb{S}^{15} \to S^8$ although some general constructions of 3-forms have been developed [2].

3.2. *The case* $p > 2$

PROPOSITION 3.10. – *Let* $\overline{u} = u \circ \pi_{\mathbb{R}}\colon \mathbb{R}^8 \to \mathbb{S}^4$ *be a p-minimising tangent map, where* $u\colon \mathbb{S}^7 \to \mathbb{S}^4$ *is a p-harmonic k-form. Let* α *be the real root of*

$$(p+2)(p-8)^2 - 108(4-p) = 0\,.$$

Then, if $2 < p < \alpha \approx 2.797$, *u is the Hopf map* H_4.

PROOF. – If $2 < p < \alpha$, from the first inequality (2.1) the degree $k < 3$, which implies that u is the Hopf map. ∎

PROPOSITION 3.11. – *Let* $\overline{u} = H_n \circ \pi_{\mathbb{R}}\colon \mathbb{R}^{2n} \to \mathbb{S}^n$ $(n = 2, 4, 8)$ *be the homogeneous extension of the Hopf map* $H_n\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ *with* $p > n$. *If* p *satisfies*:

$$G_n(p) = (n+p-2)(p-2n)^2 - 16n(p-n) > 0\,,$$

then $\overline{u}$ *is not a p-minimising tangent map.*

PROOF. – If $\overline{u}$ is a p-minimising tangent map and $G_n(p) > 0$, from the second inequality (2.1) the degree $k > 2$, which excludes the Hopf maps. ∎

REMARK 3.12. – For $n = 2, 4, 8$, the values of p such that $G_n(p) > 0$ are shown in the following table.

n	p such that $G_n(p) > 0$
2	$2 < p \lessapprox 2.219$
4	$4 < p \lessapprox 4.989$
8	$8 < p \lessapprox 11.150$

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