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On Interpolation of Bilinear Operators
by Methods Associated to Polygons

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Introduction.

An important result of real and complex interpolation methods is the bilinear interpolation theorem. This kind of theorems have a variety of interesting applications in Analysis, as one can see in the book by Bergh and Löfström [2] or in the papers by Lions and Peetre [12], Peetre [13], Zafran [16], Favini [9], Janson [11] and Astashkin [1].

In this paper we investigate the behaviour of bilinear operators under interpolation by the methods associated to polygons. These methods are similar to the real method, but they work with \(N\)-tuples \((N \geq 3)\) of Banach spaces instead of couples. They were introduced by Peetre and one of the present authors in [7]. The resulting theory has a clear geometrical flavour (see [7] and [6]), and gives a unified point of view for dealing with spaces studied by Sparr [14] and by Fernandez [10].

We start by recalling in Section 1 the main properties of methods defined by polygons. In Section 2 we establish a bilinear interpolation theorem for a combination of the \(K\)- and \(J\)-methods, and another one for the \(J\)-method. We also show by means of a counterexample that a similar result fails for the \(K\)-method.

Finally in Section 3, we describe an application to interpolation of operator spaces starting from Banach lattices. This formula does not hold for general operator spaces as we prove as well.

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1. – Interpolation methods defined by means of polygons.

By a Banach $N$-tuple we mean a family $\mathcal{A} = \{A_1, \ldots, A_N\}$ of $N$ Banach spaces $A_j$ which are continuously embedded in a common linear Hausdorff space. The sum $\Sigma(\mathcal{A}) = A_1 + \ldots + A_N$ and the intersection $\Delta(\mathcal{A}) = A_1 \cap \ldots \cap A_N$ are then also Banach spaces when normed by

$$
\|a\|_{\Sigma(\mathcal{A})} = \inf \left\{ \sum_{j=1}^{N} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, \ a_j \in A_j \right\},
$$

$$
\|a\|_{\Delta(\mathcal{A})} = \max_{1 \leq j \leq N} \{\|a\|_{A_j}\}.
$$

Let $\Pi = P_1 \ldots P_N$ be a convex polygon in the plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. In what follows, it will be useful to imagine each space $A_j$ from the $N$-tuple $\mathcal{A}$ as sitting in the vertex $P_j$.

By means of the polygon $\Pi$, we may equivalently renorm $\Sigma(\mathcal{A})$ by the $K$-functional

$$
K(t, s; a) = \inf \left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, \ a_j \in A_j \right\}.
$$

Here $t$ and $s$ stand for positive numbers. Similarly, the $J$-functional

$$
J(t, s; a) = \max_{1 \leq j \leq N} \{t^{x_j} s^{y_j} \|a\|_{A_j}\}
$$

defines an equivalent norm to $\|\cdot\|_{\Delta(\mathcal{A})}$.

Let $(\alpha, \beta)$ be an interior point of $\Pi$ [$(\alpha, \beta) \in \text{Int}\,\Pi$] and let $1 \leq q \leq \infty$. The $K$-space $\mathcal{A}_{(\alpha, \beta)}, q; K$ is defined as the set of all elements $a \in \Sigma(\mathcal{A})$ having a finite norm

$$
\|a\|_{(\alpha, \beta), q; K} = \left( \sum_{(m, n) \in \mathbb{Z}^2} (2^{-\alpha m} \beta^n K(2^m, 2^n; a))^q \right)^{1/q} \quad \text{(if} \ q < \infty, \text{)},
$$

$$
\|a\|_{(\alpha, \beta), \infty; K} = \sup_{(m, n) \in \mathbb{Z}^2} \{2^{-\alpha m} \beta^n K(2^m, 2^n; a)\}.
$$

The $J$-space $\mathcal{A}_{(\alpha, \beta), q; J}$ is formed by all those elements $a \in \Sigma(\mathcal{A})$ which can be represented in the form

$$
a = \sum_{(m, n) \in \mathbb{Z}^2} u_{m, n},
$$
\begin{align*}
\left( \sum_{(m, n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_m, u_n))^q \right)^{1/q} < \infty
\end{align*}

(\text{the sum should be replaced by the supremum if } q = \infty). \text{ The norm on } \mathbb{A}(\alpha, \beta; q; J) \text{ is}

\[ \|a\|_{(\alpha, \beta; q; J)} = \inf \left\{ \left( \sum_{(m, n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_m, u_n))^q \right)^{1/q} \right\} \]

where the infimum is taken over all representations \((u_m, u_n)\) as above.

\(K\)- and \(J\)-spaces can be equivalently defined by using integrals instead of sums (see [7]).

Next we show up some important cases.

\textbf{EXAMPLE 1.1. –} If \(\Pi\) coincides with the simplex \(\{(0, 0), (1, 0), (0, 1)\}\) and \((\alpha, \beta) \in \text{Int} \Pi\) (i.e., \(\alpha > 0, \beta > 0\) with \(\alpha + \beta < 1\)), we recover spaces studied by Sparr in [14].

\textbf{EXAMPLE 1.2. –} If \(\Pi\) is equal to the unit square \(\{(0, 0), (1, 0), (0, 1), (1, 1)\}\) and \(0 < \alpha, \beta < 1\), then we obtain spaces investigated by Fernandez in [10].

\textbf{EXAMPLE 1.3. –} The classical real interpolation space \((A_0, A_1)_{\theta; q}\) can be also described by a similar scheme to the one developed above, but working now in \(\mathbb{R}\), with the segment \([0, 1]\) taking the role of \(\Pi\) and \(0 < \theta < 1\) being an interior point of \([0, 1]\). In this case

\[ (A_0, A_1)_{\theta; q; K} = (A_0, A_1)_{\theta; q; J} = (A_0, A_1)_{\theta; q} \quad \text{(see [2] and [15])}. \]

Working with \(N\)-tuples \((N \geq 3), K\)- and \(J\)-spaces do not coincide in general (see [14], [8] or [6]). But the following continuous embedding still holds

\[ \mathbb{A}(\alpha, \beta; q; J) \subset \mathbb{A}(\alpha, \beta; q; K). \]

For the proof see [7], Thm. 1.3. The argument given there also shows that if \((u_m, u_n) \subset \Delta(\mathbb{A})\) with

\[ \left( \sum_{(m, n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_m, u_n))^q \right)^{1/q} < \infty, \]

then the series \(\sum_{(m, n) \in \mathbb{Z}^2} u_{m, n}\) is absolutely convergent in \(\Sigma(\mathbb{A})\). We shall use this fact in our later considerations.

Let \(\mathcal{B} = \{B_1, \ldots, B_N\}\) be another Banach \(N\)-tuple which we also imagine as sitting on the vertices of another copy of the polygon \(\Pi\). By \(T: \mathbb{A} \rightarrow \mathcal{B}\)
we denote a linear operator from $\Sigma(\mathcal{A})$ into $\Sigma(\mathcal{B})$ whose restriction to each $A_j$ defines a bounded operator from $A_j$ into $B_j$ ($T \in \mathcal{L}(A_j, B_j)$).

If $T: A \rightarrow B$ it is easy to check that the restrictions $T: \mathcal{A}_{(\alpha, \beta), q; K} \rightarrow \mathcal{B}_{(\alpha, \beta), q; K}$, $T: \mathcal{A}_{(\alpha, \beta), q; J} \rightarrow \mathcal{B}_{(\alpha, \beta), q; J}$, are also bounded. According to [6], Thm. 1.9, their norms can be estimated by

$$
\|T\|_{\mathcal{A}_{(\alpha, \beta), q; K}, \mathcal{B}_{(\alpha, \beta), q; K}} \leq C \max \left\{ \|T\|_{\mathcal{A}_{i}, \mathcal{B}_{i}, \mathcal{B}_{j}, \mathcal{B}_{k}} \right\}
$$

and a similar inequality holds for $J$-spaces. Here $C$ is a constant that only depends on $\Pi$ and $(\alpha, \beta)$, $\mathcal{P}$ is the set of all those triples $\{i, j, k\}$ such that $(\alpha, \beta)$ belongs to the triangle with vertices $P_i$, $P_j$, $P_k$, and $(c_i, c_j, c_k)$ are the barycentric coordinates of $(\alpha, \beta)$ with respect to $P_i$, $P_j$, $P_k$. If we work with the continuous description of $K$- and $J$-spaces (i.e., if we replace the sums by integrals), then $C = 1$.

2. – Interpolation of bilinear operators.

In order to extend the bilinear interpolation theorem for the real method (see [12]) to methods associated to polygons one should be careful with the role played by $K$- and $J$-constructions that now are not equal in general. Other obstruction comes from the relationship between duality and bilinear interpolation. As it was shown in [3] and [5], duality for $K$- and $J$-spaces requires a more delicate study than in the case of the real method. Some well-known duality formulae for couples are no longer valid for $N$-tuples. These problems are also reflected in the behaviour of bilinear operators as our next example announces.

Example 2.1. – An interesting application of bilinear interpolation theorem for couples reads that

$$
(\mathcal{L}(A_0, B_0), \mathcal{L}(A_1, B_1))_{(\alpha, \beta), \infty} \subset \mathcal{L}(\mathcal{A}_{\theta}, 1, \mathcal{B}_{\theta}, \infty)
$$

(see [13], page 176). Here $\mathcal{A} = (A_0, A_1)$ and $\mathcal{B} = (B_0, B_1)$ are Banach couples with $A_0 \cap A_1$ dense in $A_0$ and $A_1$. Similarly, one may expect that formula

$$
(\mathcal{L}(A_1, B_1), \ldots, \mathcal{L}(A_N, B_N))_{(\alpha, \beta), \infty} \subset \mathcal{L}(\mathcal{A}_{(\alpha, \beta), 1; K}, \mathcal{B}_{(\alpha, \beta), \infty; K})
$$

holds for $N$-tuples (and even that it would be a consequence of bilinear interpolation theorem for $N$-tuples). However (2.2) is not true in general.
Indeed, let \( \Pi = P_1 P_2 P_3 \) be a triangle and consider the 3-tuples \( \bar{A}, \bar{B} \) defined by

\[
A_1 = \left\{ \xi = (\xi_m) : \|\xi\|_{A_1} = \sum_{m = -\infty}^{\infty} |\xi_m| < \infty \text{ and } \sum_{m = -\infty}^{\infty} \xi_m = 0 \right\},
\]

\[
A_2 = \left\{ \xi = (\xi_m) : \|\xi\|_{A_2} = \sum_{m = -\infty}^{\infty} \min \{1, 2^m\} |\xi_m| < \infty \right\},
\]

\[
A_3 = \left\{ \xi = (\xi_m) : \|\xi\|_{A_3} = \sum_{m = -\infty}^{\infty} \min \{1, 2^{-m}\} |\xi_m| < \infty \right\},
\]

and \( B_1 = B_2 = B_3 = \mathbb{K} \) the scalar field. It is not hard to check that \( A_1 = \Delta(\bar{A}) \) is dense in each \( A_j \) for \( j = 1, 2, 3 \). Moreover, \( \mathcal{L}(A_j, B_j) = A_j^* \) and

\[
\mathcal{L}(\bar{A}_{(a, \beta)}, 1; \mathbb{K}, \bar{B}_{(a, \beta)}, \infty; \mathbb{K}) = \mathcal{L}(\bar{A}_{(a, \beta)}, 1; \mathbb{K}) = (\bar{A}_{(a, \beta)}, 1; \mathbb{K})^*.
\]

Take now the linear functional \( f \in A_j^* \) given by

\[
f(\xi) = \sum_{m = 0}^{\infty} \xi_m - \sum_{m = -1}^{\infty} \xi_m.
\]

It is easy to verify that \( f \) admits continuous extensions to \( A_j \) for \( j = 2, 3 \). Hence

\[
f \in \bigcap_{j=1}^{3} A_j^* \subset \mathcal{L}(\mathcal{L}(A_1, B_1), \mathcal{L}(A_2, B_2), \mathcal{L}(A_3, B_3))_{(a, \beta), \infty; \mathbb{K}}.
\]

Nevertheless \( f \) cannot be continuously extended to \( \bar{A}_{(a, \beta), 1; J} \), (see [3]). Consequently, (2.2) does not hold.

Example 2.1 points out that bilinear interpolation theorem for \( N \)-tuples will require a careful analysis. We shall return to Example 2.1 in Section 3.

We proceed now to bilinear results. Given a polygon \( \Pi = P_1 \ldots P_N \) and \( (a, \beta) \in \text{Int} \Pi \), we denote by \( \mathcal{P} \) the set of all triples \( \{i, j, k\} \) such that \( (a, \beta) \) belongs to the triangle with vertices \( P_i, P_j, P_k \).

**Theorem 2.2.** – Let \( \Pi = P_1 \ldots P_N \) be a convex polygon with \( P_j = (x_j, y_j) \), let \( (a, \beta) \in \text{Int} \Pi \) and let \( \mathcal{P} \) be as before. If \( \bar{A} = \{A_1, \ldots, A_N\} \), \( \bar{B} = \{B_1, \ldots, B_N\} \) and \( \bar{E} = \{E_1, \ldots, E_N\} \) are Banach \( N \)-tuples, \( 1 \leq p, q \leq \infty \) with \( 1/p + 1/q = 1 + 1/r \) and \( R : \Sigma(\bar{A}) \times \Sigma(\bar{B}) \to \Sigma(\bar{E}) \) is a bounded bilinear map, whose restriction to \( A_j \times B_j \) defines a bounded map \( R_j : A_j \times B_j \to E_j \) with norm \( M_j \) for \( j = 1, \ldots, N \), then the restriction

\[
R : \bar{A}_{(a, \beta), p; J} \times \bar{B}_{(a, \beta), q; K} \to \bar{E}_{(a, \beta), r; K}
\]

is also bounded, and its norm \( M \) satisfies

\[
M \leq C \max \{M_i^p, M_j^q, M_k^r : \{i, j, k\} \in \mathcal{P}\}.
\]
Here $C$ is a constant which only depends on $\Pi$ and $(\alpha, \beta)$, and $(c_i, c_j, c_k)$ are the barycentric coordinates of $(\alpha, \beta)$ with respect to $P_i, P_j, P_k$.

**Proof.** – Let $a \in A(\alpha, \beta), p; J, b \in B(\alpha, \beta), q; K$ and let $(\lambda_{m, n})$ be a sequence of positive numbers such that $\sum_{(m, n) \in Z^2} (2^{-an} 2^{-\beta n} \lambda_{m, n})^q = 1$. Given any $\varepsilon > 0$ and any numbers $m, n, s, w, u, v \in Z$, we can find a decomposition $b = \sum_{j=1}^N b_j$ so that

$$\sum_{j=1}^N 2^{(s - m + u) x_j} 2^{(w - n + v) y_j} \|b_j\|_{B_j} \leq K(2^{s - m + u}, 2^{w - n + v}; b) + \varepsilon \lambda_{s - m + u, w - n + v} (\text{the decomposition does depend on } m, n, s, w, u \text{ and } v \text{ but we do not point it out in our notation for the sake of simplicity}).$$

Given any $J$-representation $a = \sum_{(m, n) \in Z^2} a_{m, n}$ of $a$, we have

$$K(2^s, 2^w; R(a, b)) \leq \sum_{(m, n) \in Z^2} K(2^s, 2^w; R(a_{m, n}, b)) \leq$$

$$\sum_{(m, n) \in Z^2} \sum_{j=1}^N 2^{(s - m + u) x_j} 2^{(w - n + v) y_j} M_{x_j} a_{m, n} \|b_j\|_{B_j} 2^{(m - n) x_j} 2^{(n - v) y_j} \leq$$

$$\max_{1 \leq j \leq N} \{2^{-uw_j} 2^{-vy_j} M_{x_j}\} \sum_{(m, n) \in Z^2} J(2^m, 2^n; a_{m, n}) \sum_{j=1}^N 2^{(s - m + u) x_j} 2^{(w - n + v) y_j} \|b_j\|_{B_j} \leq$$

$$\max_{1 \leq j \leq N} \{2^{-uw_j} 2^{-vy_j} M_{x_j}\} \sum_{(m, n) \in Z^2} J(2^m, 2^n; a_{m, n}).$$

Thus

$$\|R(a, b)\|_{(\alpha, \beta), r; k} \leq \max_{1 \leq j \leq N} \{2^{u(\alpha - x_j)} 2^{(\beta - y_j)} M_{x_j}\}.$$
that
\[ \| R(a, b) \|_{(\alpha, \beta), r; K} \leq \max_{1 \leq j \leq N} \left\{ 2^{(\alpha - x_j)} 2^{(\beta - y_j)} M_j \right\} \| a \|_{(\alpha, \beta), p; J} \| b \|_{(\alpha, \beta), q; K}. \]

Since \((u, v) \in \mathbb{Z}^2\) is arbitrary, we derive that
\[ \| R(a, b) \|_{(\alpha, \beta), r; K} \leq \inf_{(u, v) \in \mathbb{Z}^2} \left\{ \max_{1 \leq j \leq N} \left\{ 2^{(\alpha - x_j)} 2^{(\beta - y_j)} M_j \right\} \| a \|_{(\alpha, \beta), p; J} \| b \|_{(\alpha, \beta), q; K} \right\} \]
\[ \leq C \inf_{t > 0, s > 0} \left\{ \max_{1 \leq j \leq N} \left\{ t^{x_j - \alpha} s^{y_j - \beta} M_j \right\} \| a \|_{(\alpha, \beta), p; J} \| b \|_{(\alpha, \beta), q; K} \right\} \]
where \(C\) is a constant depending only on \(\Pi\) and \((\alpha, \beta)\). According to [6], Thm. 1.9, the last infimum is equal to
\[ \max \left\{ M_i^\xi M_j^\eta M_k^\zeta : \{i, j, k\} \in \mathcal{P} \right\} \]
which gives the result.

**Remark 2.3.** – In the special case \(p = 1\) and \(q = r\), if we consider in the \(K\)-spaces the equivalent norm given by
\[ \| a \|_{(\alpha, \beta), q; K} = \left( \int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} K(t, s; \alpha))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} \]
then the same conclusion holds with \(C = 1\). Indeed, fix \(a \in \Delta(\overline{A})\) and apply the (continuous version of) interpolation theorem mentioned in Section 1 to the operator \(T_a(b) = R(a, b)\). Then conclude the result by using Minkowski’s inequality.

Next we show that bilinear maps can be also interpolated if we only work with \(J\)-spaces.

**Theorem 2.4.** – Under the same assumption as in Theorem 2.2, the map
\[ R: \overline{A}_{(\alpha, \beta), p; J} \times \overline{B}_{(\alpha, \beta), q; J} \to \overline{E}_{(\alpha, \beta), r; J} \]
is also bounded, and inequality
\[ M \leq C \max \left\{ M_i^\xi M_j^\eta M_k^\zeta : \{i, j, k\} \in \mathcal{P} \right\} \]
is still valid.

**Proof.** – Let \(a \in \overline{A}_{(\alpha, \beta), p; J}, b \in \overline{B}_{(\alpha, \beta), q; J}\) and take any \(J\)-representations
\[ a = \sum_{(m, n) \in \mathbb{Z}^2} a_{m, n}, \quad b = \sum_{(m, n) \in \mathbb{Z}^2} b_{m, n}, \]
Given any \((u, v), (s, w) \in \mathbb{Z}^2\), put
\[
R_{s, w}(a, b) = \sum_{(m, n) \in \mathbb{Z}^2} R(a_m, b, s - m + u, w - n + v).
\]

Then
\[
J(2^u, 2^v; R_{s, w}(a, b)) \leq \sum_{(m, n) \in \mathbb{Z}^2} \max_{1 \leq j \leq N} \left\{ 2^{\delta y_j} 2^{\gamma y_j} M_j \|a_m, n\|_{A_j} \|b_{s - m + u, w - n + v}\|_{B_j} \right\} \leq 
\]
\[
\max_{1 \leq j \leq N} \left\{ 2^{\delta y_j} 2^{\gamma y_j} M_j \right\} \sum_{(m, n) \in \mathbb{Z}^2} J(2^m, 2^n; a_m, n) J(2^{s - m + u}, 2^{w - n + v}; b_{s - m + u, w - n + v}).
\]

It follows that
\[
\|R(a, b)\|_{(\alpha, \beta), r; J} \leq 2^{\alpha u} 2^{\beta v} \left( \sum_{(u, w) \in \mathbb{Z}^2} (2^{-\alpha(s + u)} 2^{-\beta(w + v)} J(2^s, 2^w; R_{s, w}(a, b)))^{1/r} \right) \leq 
\]
\[
\max_{1 \leq j \leq N} \left\{ 2^{\delta y_j} 2^{\gamma y_j} M_j \right\} \left[ \sum_{(s, w) \in \mathbb{Z}^2} \left( \sum_{(m, n) \in \mathbb{Z}^2} 2^{-\alpha m} 2^{-\beta n} J(2^m, 2^n; a_m, n) \right)^{1/p} \right]^{1/q} \leq 
\]
\[
\max_{1 \leq j \leq N} \left\{ 2^{\delta y_j} 2^{\gamma y_j} M_j \right\} \left( \sum_{(s, w) \in \mathbb{Z}^2} (2^{-\alpha s} 2^{-\beta w} J(2^s, 2^w; b_{s, w}))^{1/p} \right)^{1/q}.
\]

where we have applied Young’s inequality in the last step. Taking the infimum over all \(J\)-representations and using \([6]\), Thm. 1.9, as at the end of Theorem 2.2, we conclude that
\[
\|R(a, b)\|_{(\alpha, \beta), r; J} \leq C \max_{\{i, j, k\} \in \mathcal{G}} \left\{ M_i^{\alpha} M_j^{\beta} M_k^{\gamma} \right\} \|a\|_{(\alpha, \beta), p; J} \|b\|_{(\alpha, \gamma), q; J}.
\]

If we replace \(J\)-spaces by \(K\)-spaces in the statement of Theorem 2.4, then the result is no longer valid as we show next.

**Counterexample 2.5.** – We have mentioned in Section 1 that for any Banach \(N\)-tuple \(A\), the space \(A(\alpha, \beta), q; J\) is continuously embedded in \(A(\alpha, \beta), q; K\). The norm of this inclusion can be estimated independently of \(A\) (see \([7]\), Thm. 1.3). On the other hand, embeddings \(A(\alpha, \beta), q; K \hookrightarrow \Sigma(A)\) and \(\Delta(A) \hookrightarrow A(\alpha, \beta), q; J\) have norms less than or equal to 1.
Let now $\Delta = P_1 P_2 P_3$ be a triangle whose vertices all belong to $\Pi$ and such that $(\alpha, \beta) \in \text{Int} \Delta$. Put $\tilde{\Delta} = \{A_1, A_2, A_3\}$. According to [4], Lemma 1.4, it follows that

$$\tilde{\Delta}(\alpha, \beta), q; K \hookrightarrow \tilde{\Delta}(\alpha, \beta), q; K,$$

$$\tilde{\Delta}(\alpha, \beta), q; J \hookrightarrow \tilde{\Delta}(\alpha, \beta), q; J,$$

with norms less than or equal to 1.

We shall use these general results to identify $K$-interpolation spaces in some concrete cases. Let $\Pi = P_1 \ldots P_6$ be a regular hexagon and let $(\alpha, \beta)$ be the center of $\Pi$. For any $n \in \mathbb{N}$, let $\tilde{\Pi}, \tilde{\Pi}$ be the 6-tuples given by

$$A_j = \begin{cases} \ell_1^n & \text{for } j = 1, 3, 5, \\ \ell_\infty^n & \text{for } j = 2, 4, 6. \end{cases}$$

$$B_j = \begin{cases} \ell_\infty^n & \text{for } j = 1, 3, 5, \\ \ell_1^n & \text{for } j = 2, 4, 6. \end{cases}$$

Here $\ell_1^n$ is $\mathbb{K}^n$ with the $l_1$-norm and $\ell_\infty^n$ is defined analogously. Write $\tilde{\Delta} = \{A_2, A_4, A_6\}$, then

$$\ell_\infty^n = \Delta(\tilde{\Delta}) \hookrightarrow \tilde{\Delta}(\alpha, \beta), p; J \hookrightarrow \tilde{\Delta}(\alpha, \beta), p; K \hookrightarrow \Delta(\tilde{\Delta}) = \ell_\infty^n.$$ 

Whence $\tilde{\Delta}(\alpha, \beta), p; K = \ell_\infty^n$ with equivalence of norms, being the constants in the equivalence independent of $n$. The choice $\tilde{\Delta} = \{B_1, B_3, B_5\}$, gives also that $\tilde{\Delta}(\alpha, \beta), q; K = \ell_1^n$.

Take now $\tilde{\Pi} = \{E_1, \ldots, E_6\}$ where $E_j = \mathbb{K}$ for $j = 1, \ldots, 6$. Clearly $\tilde{\Delta}(\alpha, \beta), r; K = \mathbb{K}$.

In order to see that Theorem 2.4 fails for $K$-spaces, consider the bilinear map $R$ defined by $R((\xi_j), (\eta_j)) = \sum_{j=1}^n \xi_j \eta_j$. It is clear that $R: \Sigma(\tilde{\Delta}) \times \Sigma(\tilde{\Pi}) \rightarrow \Sigma(\tilde{\Pi})$ is bounded, and that restrictions $R: A_j \times B_j \rightarrow E_j$ have norm 1. If Theorem 2.4 would be valid for $K$-spaces, then

$$R: \tilde{\Delta}(\alpha, \beta), p; K \times \tilde{\Pi}(\alpha, \beta), q; K \rightarrow \tilde{\Pi}(\alpha, \beta), r; K$$

would be bounded independently of $n$. In other words, there would exists some $M < \infty$ such that

$$n = \|R: \ell_1^n \times \ell_\infty^n \rightarrow \mathbb{K}\| \leq M$$

for every $n \in \mathbb{N}$, which is impossible.

3. – Application to interpolation of operator spaces.

In order to use bilinear results to derive formulae of the kind (2.1) and (2.2), one needs to modify assumptions of Theorem 2.2 working now with a bounded bilinear map $R: \Sigma(\tilde{\Delta}) \times \Delta(\tilde{\Pi}) \rightarrow \Sigma(\tilde{\Pi})$, whose restrictions $R: A_j \times$
($\Delta(\mathcal{B}), \| \cdot \|_{\mathcal{B}}) \to E_j$ are bounded. The problem is to determine whether or not $R$ admits a bounded extension from $\overline{A}_{(\alpha, \beta), p} : J \times \overline{B}_{(\alpha, \beta), q} : K \to \overline{E}_{(\alpha, \beta), r} : K$.

In the case of the real method for couples, the answer is positive if $q < \infty$. The argument of the proof of Theorem 2.2 can be repeated with minor modifications because, dealing with couples, if $b \in \Delta(\mathcal{B})$ and $b = b_0 + b_1$, with $b_j \in B_j$, then each $b_j$ should belongs to $\Delta(\mathcal{B})$. Moreover, $K$- and $J$-spaces coincide, so $\Delta(\mathcal{B})$ is dense in $\overline{B}_{\theta, q}$ if $q < \infty$. Formula (2.1) is a consequence of this result (see [13], pag. 176).

In the case of $N$-tuples ($N \geq 3$) the answer is negative in general as can be concluded from Example 2.1.

However, there is an interesting case where the answer is still positive. Recall that a Banach lattice $N$-tuple $\overline{B} = \{B_1, \ldots, B_N\}$ is an $N$-tuple of Banach lattices which are order ideals of a common topological Riesz space $\Xi$, with continuous inclusions (see [5]).

**Theorem 3.1.** -- Let $\Pi = P_1 \ldots P_N$ be a convex polygon, let $(\alpha, \beta) \in \text{Int} \, \Pi$ and let $\mathcal{P}$ as before. Assume that $\overline{A} = \{A_1, \ldots, A_N\}$, $\overline{E} = \{E_1, \ldots, E_N\}$ are Banach $N$-tuples, that $\overline{B} = \{B_1, \ldots, B_N\}$ is a Banach lattice $N$-tuple such that $\Delta(\mathcal{B})$ is dense in $B_j$ for $j = 1, \ldots, N$, and that $1 \leq q < \infty$, $1 \leq p, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. If $R : \Sigma(\overline{A}) \times \Delta(\mathcal{B}) \to \Sigma(\overline{E})$ is a bounded bilinear map, whose restriction $R : A_j \times (\Delta(\mathcal{B}), \| \cdot \|_{\mathcal{B}}) \to E_j$ is bounded with norm $M_j$ for $j = 1, \ldots, N$, then $R$ may be uniquely extended to a bilinear map from $\overline{A}_{(\alpha, \beta), p} : J \times \overline{B}_{(\alpha, \beta), q} : K \to \overline{E}_{(\alpha, \beta), r} : K$ with norm at most $C \max \{M_{ij}^{\varepsilon} M_{jk}^{\varepsilon} M_{ik}^{\varepsilon} : \{i, j, k\} \in \mathcal{P}\}$, where $C$ only depends on $\Pi$ and $(\alpha, \beta)$.

**Proof.** -- Since $\overline{B}$ is a Banach lattice $N$-tuple, in order to compute the $K$-functional for $b \in \Delta(\mathcal{B})$ it suffices to consider decompositions given by elements of $\Delta(\mathcal{B})$, that is,

$$K(t, s; b) = \inf \left\{ \sum_{j=1}^{N} t_j s_j \|b_j\|_{B_j} : b = \sum_{j=1}^{N} b_j, \ b_j \in \Delta(\mathcal{B}) \right\}$$

(see, for example, [5], Prop. 3.1). Hence, we can repeat the argument of the proof of Theorem 2.2 concluding that for any $a \in \overline{A}_{(\alpha, \beta), p} : J$ and $b \in \Delta(\mathcal{B})$

$$\|R(a, b)\|_{(\alpha, \beta), r} : K \leq C \max_{\{i, j, k\} \in \mathcal{P}} \{M_{ij}^{\varepsilon} M_{jk}^{\varepsilon} M_{ik}^{\varepsilon}\} \|a\|_{(\alpha, \beta), p} : J \|b\|_{(\alpha, \beta), q} : K.$$

Now the result follows taking into account that under the assumption on $\overline{B}$, the intersection $\Delta(\mathcal{B})$ is dense in $\overline{B}_{(\alpha, \beta), q} : K$ (see [5], Prop. 3.2).

We finish the paper by establishing formula (2.2) when $\overline{B}$ is a Banach lattice $N$-tuple.
**COROLLARY 3.2.** – Let \( \Pi, (\alpha, \beta), \mathcal{P}, \mathcal{A}, \mathcal{B}, p, q, \) and \( r \) as in Theorem 3.1.

Then

\[
(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J \subseteq \mathcal{L}(\mathcal{B}(\alpha, \beta), q; K, \mathcal{A}(\alpha, \beta), r; K)}.
\]

**PROOF.** – Since \( \Delta(\mathcal{B}) \) is dense in \( B_j \) for \( j = 1, \ldots, N \), each space \( \mathcal{L}(B_j, A_j) \) is continuously embedded in \( \mathcal{L}(\Delta(\mathcal{B}), \Sigma(\mathcal{A})) \). Thus \( (\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N)) \) is a Banach \( N \)-tuple.

Let \( R: \mathcal{L}(\Delta(\mathcal{B}), \Sigma(\mathcal{A})) \times \Delta(\mathcal{B}) \rightarrow \Sigma(\mathcal{A}) \) be the bounded bilinear map defined by \( R(T, b) = T(b) \). The restriction \( R: \mathcal{L}(B_j, A_j) \times (\Delta(\mathcal{B}), \|b\|_B) \rightarrow A_j \) is bounded with norm at most 1. Therefore, using Theorem 3.1, \( R \) can be boundedly extended from

\[
(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J \subseteq \mathcal{L}(\mathcal{B}(\alpha, \beta), q; K, \mathcal{A}(\alpha, \beta), r; K)}.
\]

In other words,

\[
(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J \subseteq \mathcal{L}(\mathcal{B}(\alpha, \beta), q; K, \mathcal{A}(\alpha, \beta), r; K)}.
\]

The choice \( p = \infty, q = 1, \) and \( r = \infty \) gives formula (2.2).

**REFERENCES**


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