Bollettino Unione Matematica Italiana

Fernando Cobos, José María Cordeiro, Antón Martínez

On interpolation of bilinear operators by methods associated to polygons

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.2, p. 319–330.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_2_319_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 319-330

On Interpolation of Bilinear Operators by Methods Associated to Polygons

FERNANDO COBOS (*) - JOSÉ MARÍA CORDEIRO - ANTÓN MARTÍNEZ (*)

Sunto. – *Stabiliamo teoremi di interpolazione bilineare per una combinazione dei metodi di K- e J-interpolazione associati ai poligoni, e per il J-metodo. Mostriamo che un simile risultato fallisce per il K-metodo, e diamo applicazioni all'interpolazione di spazi di operatori.*

Introduction.

An important result of real and complex interpolation methods is the bilinear interpolation theorem. This kind of theorems have a variety of interesting applications in Analysis, as one can see in the book by Bergh and Löfström [2] or in the papers by Lions and Peetre [12], Peetre [13], Zafran [16], Favini [9], Janson [11] and Astashkin [1].

In this paper we investigate the behaviour of bilinear operators under interpolation by the methods associated to polygons. These methods are similar to the real method, but they work with N-tuples ($N \geq 3$) of Banach spaces instead of couples. They were introduced by Peetre and one of the present authors in [7]. The resulting theory has a clear geometrical flavour (see [7] and [6]), and gives a unified point of view for dealing with spaces studied by Sparr [14] and by Fernandez [10].

We start by recalling in Section 1 the main properties of methods defined by polygons. In Section 2 we establish a bilinear interpolation theorem for a combination of the K- and J-methods, and another one for the J-method. We also show by means of a counterexample that a similar result fails for the K-method.

Finally in Section 3, we describe an application to interpolation of operator spaces starting from Banach lattices. This formula does not hold for general operator spaces as we prove as well.

(*) Supported in part by DGICYT (PB94-0252).

1. – Interpolation methods defined by means of polygons.

By a Banach N-tuple we mean a family $\overline{A} = \{A_1, \ldots, A_N\}$ of N Banach spaces A_j which are continuously embedded in a common linear Hausdorff space. The sum $\Sigma(\overline{A}) = A_1 + \ldots + A_N$ and the intersection $\Delta(\overline{A}) = A_1 \cap \ldots \cap A_N$ are then also Banach spaces when normed by

$$\|a\|_{\Sigma(\overline{A})} = \inf\left\{ \sum_{j=1}^{N} \|a_j\|_{A_j} \colon\ a = \sum_{j=1}^{N} a_j, \quad a_j \in A_j \right\},$$

$$\|a\|_{\Delta(\overline{A})} = \max_{1 \leqslant j \leqslant N} \{\|a\|_{A_j}\}.$$

Let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon in the plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. In what follows, it will be useful to imagine each space A_j from the N-tuple $\overline{A}$ as sitting in the vertex P_j.

By means of the polygon Π, we may equivalently renorm $\Sigma(\overline{A})$ by the K-funtional

$$K(t, s; a) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} \colon\ a = \sum_{j=1}^{N} a_j, \quad a_j \in A_j \right\}.$$

Here t and s stand for positive numbers. Similarly, the J-funtional

$$J(t, s; a) = \max_{1 \leqslant j \leqslant N} \{t^{x_j} s^{y_j} \|a\|_{A_j}\}$$

defines an equivalent norm to $\|\cdot\|_{\Delta(\overline{A})}$.

Let (α, β) be an interior point of Π [$(\alpha, \beta) \in \operatorname{Int}\Pi$] and let $1 \leqslant q \leqslant \infty$. The K-space $\overline{A}_{(\alpha, \beta), q; K}$ is defined as the set of all elements $a \in \Sigma(\overline{A})$ having a finite norm

$$\|a\|_{(\alpha, \beta), q; K} = \left(\sum_{(m, n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} K(2^m, 2^n; a))^q \right)^{1/q} \qquad (\text{if} \quad q < \infty),$$

$$\|a\|_{(\alpha, \beta), \infty; K} = \sup_{(m, n) \in \mathbb{Z}^2} \{2^{-\alpha m - \beta n} K(2^m, 2^n; a)\}.$$

The J-space $\overline{A}_{(\alpha, \beta), q; J}$ is formed by all those elements $a \in \Sigma(\overline{A})$ which can be represented in the form

$$a = \sum_{(m, n) \in \mathbb{Z}^2} u_{m, n}$$

(convergence in $\Sigma(\overline{A})$) with $(u_{m,n}) \subset \Delta(\overline{A})$ and

$$\left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} < \infty$$

(the sum should be replaced by the supremum if $q = \infty$). The norm on $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J} = \inf \left\{ \left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} \right\}$$

where the infimum is taken over all representations $(u_{m,n})$ as above.

K- and J-spaces can be equivalently defined by using integrals instead of sums (see [7]).

Next we show up some important cases.

EXAMPLE 1.1. – If Π coincides with the simplex $\{(0,0),(1,0),(0,1)\}$ and $(\alpha,\beta) \in \operatorname{Int} \Pi$ (i.e., $\alpha > 0$, $\beta > 0$ with $\alpha + \beta < 1$), we recover spaces studied by Sparr in [14].

EXAMPLE 1.2. – If Π is equal to the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ and $0 < \alpha, \beta < 1$, then we obtain spaces investigated by Fernandez in [10].

EXAMPLE 1.3. – The classical real interpolation space $(A_0, A_1)_{\theta,q}$ can be also described by a similar scheme to the one developed above, but working now in $\mathbb{R}$, with the segment $[0,1]$ taking the role of Π and $0 < \theta < 1$ being an interior point of $[0,1]$. In this case

$$(A_0, A_1)_{\theta,q;K} = (A_0, A_1)_{\theta,q;J} = (A_0, A_1)_{\theta,q} \qquad \text{(see [2] and [15])}.$$

Working with N-tuples ($N \geq 3$), K- and J-spaces do not coincide in general (see [14], [8] or [6]). But the following continuous embedding still holds

$$\overline{A}_{(\alpha,\beta),q;J} \hookrightarrow \overline{A}_{(\alpha,\beta),q;K} .$$

For the proof see [7], Thm. 1.3. The argument given there also shows that if $(u_{m,n}) \subset \Delta(\overline{A})$ with

$$\left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-\alpha m - \beta n} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} < \infty ,$$

then the series $\sum_{(m,n) \in \mathbb{Z}^2} u_{m,n}$ is absolutely convergent in $\Sigma(\overline{A})$. We shall use this fact in our later considerations.

Let $\overline{B} = \{B_1, \ldots, B_N\}$ be another Banach N-tuple which we also imagine as sitting on the vertices of another copy of the polygon Π. By $T \colon \overline{A} \to \overline{B}$

we denote a linear operator from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j ($T \in \mathcal{L}(A_j, B_j)$).

If $T \colon \overline{A} \to \overline{B}$ it is easy to check that the restrictions

$$T\colon \overline{A}_{(\alpha,\beta),q;K} \to \overline{B}_{(\alpha,\beta),q;K},$$

$$T\colon \overline{A}_{(\alpha,\beta),q;J} \to \overline{B}_{(\alpha,\beta),q;J},$$

are also bounded. According to [6], Thm. 1.9, their norms can be estimated by

$$\|T\|_{\overline{A}_{(\alpha,\beta),q;K},\overline{B}_{(\alpha,\beta),q;K}} \leq C \max_{\{i,j,k\}\in\mathcal{P}} \{\|T\|^{c_i}_{A_i,B_i}\|T\|^{c_j}_{A_j,B_j}\|T\|^{c_k}_{A_k,B_k}\}$$

and a similar inequality holds for J-spaces. Here C is a constant that only depends on Π and (α, β), $\mathcal{P}$ is the set of all those triples $\{i, j, k\}$ such that (α, β) belongs to the triangle with vertices P_i, P_j, P_k, and (c_i, c_j, c_k) are the barycentric coordinates of (α, β) with respect to P_i, P_j, P_k. If we work with the continuous description of K- and J-spaces (i.e., if we replace the sums by integrals), then $C = 1$.

2. – Interpolation of bilinear operators.

In order to extend the bilinear interpolation theorem for the real method (see [12]) to methods associated to polygons one should be careful with the role played by K- and J-constructions that now are not equal in general. Other obstruction comes from the relationship between duality and bilinear interpolation. As it was shown in [3] and [5], duality for K- and J-spaces requires a more delicate study than in the case of the real method. Some well-known duality formulae for couples are no longer valid for N-tuples. These problems are also reflected in the behaviour of bilinear operators as our next example announces.

EXAMPLE 2.1. – An interesting application of bilinear interpolation theorem for couples reads that

$$(\mathcal{L}(A_0, B_0), \mathcal{L}(A_1, B_1))_{\theta,\infty} \subseteq \mathcal{L}(\overline{A}_{\theta,1}, \overline{B}_{\theta,\infty}) \tag{2.1}$$

(see [13], page 176). Here $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ are Banach couples with $A_0 \cap A_1$ dense in A_0 and A_1. Similarly, one may expect that formula

$$(\mathcal{L}(A_1, B_1), \ldots, \mathcal{L}(A_N, B_N))_{(\alpha,\beta),\infty;J} \subseteq \mathcal{L}(\overline{A}_{(\alpha,\beta),1;K}, \overline{B}_{(\alpha,\beta),\infty;K}) \tag{2.2}$$

holds for N-tuples (and even that it would be a consequence of bilinear interpolation theorem for N-tuples). However (2.2) is not true in general.

Indeed, let $\Pi = \overline{P_1 P_2 P_3}$ be a triangle and consider the 3-tuples $\overline{A}$, $\overline{B}$ defined by

$$A_1 = \left\{ \xi = (\xi_m)\colon\ \|\xi\|_{A_1} = \sum_{m=-\infty}^{+\infty} |\xi_m| < \infty \ \text{ and } \ \sum_{m=-\infty}^{+\infty} \xi_m = 0 \right\},$$

$$A_2 = \left\{ \xi = (\xi_m)\colon\ \|\xi\|_{A_2} = \sum_{m=-\infty}^{+\infty} \min\{1, 2^m\} |\xi_m| < \infty \right\},$$

$$A_3 = \left\{ \xi = (\xi_m)\colon\ \|\xi\|_{A_3} = \sum_{m=-\infty}^{+\infty} \min\{1, 2^{-m}\} |\xi_m| < \infty \right\},$$

and $B_1 = B_2 = B_3 = \mathbb{K}$ the scalar field. It is not hard to check that $A_1 = \Delta(\overline{A})$ is dense in each A_j for $j = 1, 2, 3$. Moreover, $\mathcal{L}(A_j, B_j) = A_j^*$ and

$$\mathcal{L}(\overline{A}_{(\alpha,\beta),1;K}, \overline{B}_{(\alpha,\beta),\infty;K}) = \mathcal{L}(\overline{A}_{(\alpha,\beta),1;K}, \mathbb{K}) = (\overline{A}_{(\alpha,\beta),1;K})^* .$$

Take now the linear functional $f \in A_1^*$ given by

$$f(\xi) = \sum_{m=0}^{\infty} \xi_m = - \sum_{m=-1}^{-\infty} \xi_m .$$

It is easy to verify that f admits continuous extensions to A_j for $j = 2, 3$. Hence

$$f \in \bigcap_{j=1}^{3} A_j^* \subseteq (\mathcal{L}(A_1, B_1), \mathcal{L}(A_2, B_2), \mathcal{L}(A_3, B_3))_{(\alpha,\beta),\infty;K} .$$

Nevertheless f cannot be continuously extended to $\overline{A}_{(\alpha,\beta),1;J}$, (see [3]). Consequently, (2.2) does not hold.

Example 2.1 points out that bilinear interpolation theorem for N-tuples will require a careful analysis. We shall return to Example 2.1 in Section 3.

We proceed now to bilinear results. Given a polygon $\Pi = \overline{P_1 \dots P_N}$ and $(\alpha, \beta) \in \operatorname{Int}\Pi$, we denote by $\mathcal{P}$ the set of all triples $\{i, j, k\}$ such that (α, β) belongs to the triangle with vertices P_i, P_j, P_k.

THEOREM 2.2. – *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \operatorname{Int}\Pi$ and let $\mathcal{P}$ be as before. If $\overline{A} = \{A_1, \dots, A_N\}$, $\overline{B} = \{B_1, \dots, B_N\}$ and $\overline{E} = \{E_1, \dots, E_N\}$ are Banach N-tuples, $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$ and $R\colon \Sigma(\overline{A}) \times \Sigma(\overline{B}) \to \Sigma(\overline{E})$ is a bounded bilinear map, whose restriction to $A_j \times B_j$ defines a bounded map $R\colon A_j \times B_j \to E_j$ with norm M_j for $j = 1, \dots, N$, then the restriction*

$$R\colon \overline{A}_{(\alpha,\beta),p;J} \times \overline{B}_{(\alpha,\beta),q;K} \to \overline{E}_{(\alpha,\beta),r;K}$$

is also bounded, and its norm M satisfies

$$M \leq C \max\{M_i^{c_i} M_j^{c_j} M_k^{c_k} \colon \{i, j, k\} \in \mathcal{P}\} .$$

Here C is a constant which only depends on Π and (α, β), and (c_i, c_j, c_k) are the barycentric coordinates of (α, β) with respect to P_i, P_j, P_k.

PROOF. – Let $a \in \overline{A}_{(\alpha, \beta), p; J}$, $b \in \overline{B}_{(\alpha, \beta), q; K}$ and let $(\lambda_{m,n})$ be a sequence of positive numbers such that $\sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m} 2^{-\beta n} \lambda_{m,n})^q = 1$. Given any $\varepsilon > 0$ and any numbers $m, n, s, w, u, v \in \mathbb{Z}$, we can find a descomposition $b = \sum_{j=1}^{N} b_j$ so that

$$\sum_{j=1}^{N} 2^{(s-m+u)x_j} 2^{(w-n+v)y_j} \|b_j\|_{B_j} \leqslant K(2^{s-m+u}, 2^{w-n+v}; b) + \varepsilon\lambda_{s-m+u, w-n+v}$$

(the decomposition does depend on m, n, s, w, u and v but we do not point it out in our notation for the sake of simplicity).

Given any J-representation $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}$ of a, we have

$$K(2^s, 2^w; R(a, b)) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} K(2^s, 2^w; R(a_{m,n}, b)) \leqslant$$

$$\sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=1}^{N} 2^{(s-m+u)x_j} 2^{(w-n+v)y_j} M_j \|a_{m,n}\|_{A_j} \|b_j\|_{B_j} 2^{(m-u)x_j} 2^{(n-v)y_j} \leqslant$$

$$\max_{1\leqslant j\leqslant N} \{2^{-ux_j} 2^{-vy_j} M_j\} \sum_{(m,n)\in\mathbb{Z}^2} J(2^m, 2^n; a_{m,n}) \sum_{j=1}^{N} 2^{(s-m+u)x_j} 2^{(w-n+v)y_j} \|b_j\|_{B_j} \leqslant$$

$$\max_{1\leqslant j\leqslant N} \{2^{-ux_j} 2^{-vy_j} M_j\} \sum_{(m,n)\in\mathbb{Z}^2} J(2^m, 2^n; a_{m,n}) \cdot$$

$$(K(2^{s-m+u}, 2^{w-n+v}; b) + \varepsilon\lambda_{s-m+u, w-n+v}) .$$

Thus

$$\|R(a, b)\|_{(\alpha, \beta), r; K} \leqslant \max_{1\leqslant j\leqslant N} \{2^{u(\alpha - x_j)} 2^{v(\beta - y_j)} M_j\} \cdot$$

$$\left(\sum_{(s,w)\in\mathbb{Z}^2} \left(\sum_{(m,n)\in\mathbb{Z}^2} 2^{-\alpha m} 2^{-\beta n} J(2^m, 2^n; a_{m,n}) 2^{-\alpha(s-m+u)} 2^{-\beta(w-n+v)} \cdot \right.\right.$$

$$\left.\left. (K(2^{s-m+u}, 2^{w-n+v}; b) + \varepsilon\lambda_{s-m+u, w-n+v}) \right)^r \right)^{1/r} .$$

By Young's inequality, we obtain

$$\|R(a, b)\|_{(\alpha, \beta), r; K} \leqslant \max_{1\leqslant j\leqslant N} \{2^{u(\alpha - x_j)} 2^{v(\beta - y_j)} M_j\} \cdot$$

$$\left(\sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m} 2^{-\beta n} J(2^m, 2^n; a_{m,n}))^p \right)^{1/p} (\|b\|_{(\alpha, \beta), q; K} + \varepsilon) .$$

Taking the infimum over all J-representations of a and letting ε goes to 0, it follows that

$$\|R(a, b)\|_{(\alpha, \beta), r; K} \leq \max_{1 \leq j \leq N}\{2^{u(\alpha - x_j)} 2^{v(\beta - y_j)} M_j\} \|a\|_{(\alpha, \beta), p; J} \|b\|_{(\alpha, \beta), q; K} .$$

Since $(u, v) \in \mathbb{Z}^2$ is arbitrary, we derive that

$$\|R(a, b)\|_{(\alpha, \beta), r; K} \leq \inf_{(u, v) \in \mathbb{Z}^2}\{\max_{1 \leq j \leq N}\{2^{u(\alpha - x_j)} 2^{v(\beta - y_j)} M_j\}\} \|a\|_{(\alpha, \beta), p; J} \|b\|_{(\alpha, \beta), q; K} \leq$$

$$C \inf_{t > 0,\, s > 0}\{\max_{1 \leq j \leq N}\{t^{x_j - \alpha} s^{y_j - \beta} M_j\}\} \|a\|_{(\alpha, \beta), p; J} \|b\|_{(\alpha, \beta), q; K}$$

where C is a constant depending only on Π and (α, β). According to [6], Thm. 1.9, the last infimum is equal to

$$\max\{M_i^{c_i} M_j^{c_j} M_k^{c_k} : \{i, j, k\} \in \mathcal{P}\}$$

which gives the result. ■

REMARK 2.3. – In the special case $p = 1$ and $q = r$, if we consider in the K-spaces the equivalent norm given by

$$\|a\|_{(\alpha, \beta), q; K}^{\sim} = \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} K(t, s; a))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q}$$

then the same conclusion holds with $C = 1$. Indeed, fix $a \in \Delta(\overline{A})$ and apply the (continuous version of) interpolation theorem mentioned in Section 1 to the operator $T_a(b) = R(a, b)$. Then conclude the result by using Minkowski's inequality.

Next we show that bilinear maps can be also interpolated if we only work with J-spaces.

THEOREM 2.4. – *Under the same assumption as in Theorem* 2.2, *the map*

$$R : \overline{A}_{(\alpha, \beta), p; J} \times \overline{B}_{(\alpha, \beta), q; J} \to \overline{E}_{(\alpha, \beta), r; J}$$

is also bounded, and inequality

$$M \leq C \max\{M_i^{c_i} M_j^{c_j} M_k^{c_k} : \{i, j, k\} \in \mathcal{P}\}$$

is still valid.

PROOF. – Let $a \in \overline{A}_{(\alpha, \beta), p; J}$, $b \in \overline{B}_{(\alpha, \beta), q; J}$ and take any J-representations

$$a = \sum_{(m, n) \in \mathbb{Z}^2} a_{m, n} , \qquad b = \sum_{(m, n) \in \mathbb{Z}^2} b_{m, n} .$$

Given any $(u, v), (s, w) \in \mathbb{Z}^2$, put

$$R_{s,w}(a, b) = \sum_{(m,n)\in\mathbb{Z}^2} R(a_{m,n}, b_{s-m+u,\,w-n+v}).$$

Then

$$J(2^s, 2^w; R_{s,w}(a,b)) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} J(2^s, 2^w; R(a_{m,n}, b_{s-m+u,\,w-n+v})) \leqslant$$

$$\sum_{(m,n)\in\mathbb{Z}^2} \max_{1\leqslant j\leqslant N} \{2^{sx_j}2^{wy_j} M_j \|a_{m,n}\|_{A_j} \|b_{s-m+u,\,w-n+v}\|_{B_j}\} \leqslant$$

$$\max_{1\leqslant j\leqslant N}\{2^{-ux_j}2^{-vy_j}M_j\} \sum_{(m,n)\in\mathbb{Z}^2} J(2^m, 2^n; a_{m,n})\, J(2^{s-m+u}, 2^{w-n+v}; b_{s-m+u,w-n+v}).$$

It follows that

$$\|R(a,b)\|_{(\alpha,\beta),r;J} \leqslant 2^{\alpha u}2^{\beta v}\left(\sum_{(s,w)\in\mathbb{Z}^2}(2^{-\alpha(s+u)}2^{-\beta(w+v)}J(2^s,2^w;R_{s,w}(a,b)))^r\right)^{1/r} \leqslant$$

$$\max_{1\leqslant j\leqslant N}\{2^{(\alpha-x_j)u}2^{(\beta-y_j)v}M_j\}\left[\sum_{(s,w)\in\mathbb{Z}^2}\left(\sum_{(m,n)\in\mathbb{Z}^2}2^{-\alpha m}2^{-\beta n}J(2^m,2^n;a_{m,n})\cdot\right.\right.$$

$$\left.\left.2^{-\alpha(s-m+u)}2^{-\beta(w-n+v)}J(2^{s-m+u},2^{w-n+v};b_{s-m+u,\,w-n+v})\right)^r\right]^{1/r} \leqslant$$

$$\max_{1\leqslant j\leqslant N}\{2^{(\alpha-x_j)u}2^{(\beta-y_j)v}M_j\}\left(\sum_{(m,n)\in\mathbb{Z}^2}(2^{-\alpha m}2^{-\beta n}J(2^m,2^n;a_{m,n}))^p\right)^{1/p}\cdot$$

$$\left(\sum_{(s,w)\in\mathbb{Z}^2}(2^{-\alpha s}2^{-\beta w}J(2^s,2^w;b_{s,w}))^q\right)^{1/q}$$

where we have applied Young's inequality in the last step. Taking the infimum over all J-representations and using [6], Thm. 1.9, as at the end of Theorem 2.2, we conclude that

$$\|R(a,b)\|_{(\alpha,\beta),r;J} \leqslant C \max_{\{i,j,k\}\in\mathcal{P}}\{M_i^{c_i}M_j^{c_j}M_k^{c_k}\}\|a\|_{(\alpha,\beta),p;J}\|b\|_{(\alpha,\beta),q;J}. \qquad \blacksquare$$

If we replace J-spaces by K-spaces in the statement of Theorem 2.4, then the result is no longer valid as we show next.

COUNTEREXAMPLE 2.5. – We have mentioned in Section 1 that for any Banach N-tuple $\overline{A}$, the space $\overline{A}_{(\alpha,\beta),q;J}$ is continuously embedded in $\overline{A}_{(\alpha,\beta),q;K}$. The norm of this inclusion can be estimated independently of $\overline{A}$ (see [7], Thm. 1.3). On the other hand, embeddings $\overline{A}_{(\alpha,\beta),q;K} \hookrightarrow \Sigma(\overline{A})$ and $\Delta(\overline{A}) \hookrightarrow \overline{A}_{(\alpha,\beta),q;J}$ have norms less than or equal to 1.

Let now $\Delta = \overline{P_i P_j P_k}$ be a triangle whose vertices all belong to Π and such that $(\alpha, \beta) \in \operatorname{Int}\Delta$. Put $\widetilde{A} = \{A_i, A_j, A_k\}$. According to [4], Lemma 1.4, it follows that

$$\widetilde{A}_{(\alpha, \beta), q; K} \hookrightarrow \overline{A}_{(\alpha, \beta), q; K} ,$$

$$\overline{A}_{(\alpha, \beta), q; J} \hookrightarrow \widetilde{A}_{(\alpha, \beta), q; J} ,$$

with norms less than or equal to 1.

We shall use these general results to identify K-interpolation spaces in some concrete cases. Let $\Pi = \overline{P_1 \ldots P_6}$ be a regular hexagon and let (α, β) be the center of Π. For any $n \in \mathbb{N}$, let $\overline{A}$, $\overline{B}$ be the 6-tuples given by

$$A_j = \begin{cases} l_1^n & \text{for } j = 1, 3, 5 , \\ l_\infty^n & \text{for } j = 2, 4, 6 , \end{cases} \qquad B_j = \begin{cases} l_\infty^n & \text{for } j = 1, 3, 5 , \\ l_1^n & \text{for } j = 2, 4, 6 . \end{cases}$$

Here l_1^n is $\mathbb{K}^n$ with the l_1-norm and l_∞^n is defined analogously. Write $\widetilde{A} = \{A_2, A_4, A_6\}$, then

$$l_\infty^n = \Delta(\widetilde{A}) \hookrightarrow \widetilde{A}_{(\alpha, \beta), p; J} \hookrightarrow \widetilde{A}_{(\alpha, \beta), p; K} \hookrightarrow \overline{A}_{(\alpha, \beta), p; K} \hookrightarrow \Sigma(\overline{A}) = l_\infty^n .$$

Whence $\overline{A}_{(\alpha, \beta), p; K} = l_\infty^n$ with equivalence of norms, being the constants in the equivalence independent of n. The choice $\widetilde{B} = \{B_1, B_3, B_5\}$, gives also that $\overline{B}_{(\alpha, \beta), q; K} = l_\infty^n$.

Take now $\overline{E} = \{E_1, \ldots, E_6\}$ where $E_j = \mathbb{K}$ for $j = 1, \ldots, 6$. Clearly $\overline{E}_{(\alpha, \beta), r; K} = \mathbb{K}$.

In order to see that Theorem 2.4 fails for K-spaces, consider the bilinear map R defined by $R((\xi_j), (\eta_j)) = \sum_{j=1}^{n} \xi_j \eta_j$. It is clear that $R\colon \Sigma(\overline{A}) \times \Sigma(\overline{B}) \to \Sigma(\overline{E})$ is bounded, and that restrictions $R\colon A_j \times B_j \to E_j$ have norm 1. If Theorem 2.4 would be valid for K-spaces, then

$$R\colon \overline{A}_{(\alpha, \beta), p; K} \times \overline{B}_{(\alpha, \beta), q; K} \to \overline{E}_{(\alpha, \beta), r; K}$$

would be bounded independently of n. In other words, there would exists some $M < \infty$ such that

$$n = \|R\colon\ l_\infty^n \times l_\infty^n \to \mathbb{K}\| \leq M$$

for every $n \in \mathbb{N}$, which is impossible.

3. – Application to interpolation of operator spaces.

In order to use bilinear results to derive formulae of the kind (2.1) and (2.2), one needs to modify assumptions of Theorem 2.2 working now with a bounded bilinear map $R\colon \Sigma(\overline{A}) \times \Delta(\overline{B}) \to \Sigma(\overline{E})$, whose restrictions $R\colon A_j \times$

$(\varDelta(\overline{B}), \|\cdot\|_{B_j}) \to E_j$ are bounded. The problem is to determine whether or not R admits a bounded extension from $\overline{A}_{(\alpha,\beta),p;J} \times \overline{B}_{(\alpha,\beta),q;K}$ to $\overline{E}_{(\alpha,\beta),r;K}$.

In the case of the real method for couples, the answer is positive if $q<\infty$. The argument of the proof of Theorem 2.2 can be repeated with minor modifications because, dealing with couples, if $b \in \varDelta(B)$ and $b=b_0+b_1$, with $b_j \in B_j$, then each b_j should belongs to $\varDelta(\overline{B})$. Moreover, K- and J-spaces coincide, so $\varDelta(\overline{B})$ is dense in $\overline{B}_{\theta,q}$ if $q<\infty$. Formula (2.1) is a consequence of this result (see [13], pag. 176).

In the case of N-tuples ($N \geqslant 3$) the answer is negative in general as can be concluded from Example 2.1.

However, there is an interesting case where the answer is still positive. Recall that a Banach lattice N-tuple $\overline{B}=\{B_1, \ldots, B_N\}$ is an N-tuple of Banach lattices which are order ideals of a common tological Riesz space $\mathcal{Z}$, with continuous inclusions (see [5]).

THEOREM 3.1. – *Let $\Pi=\overline{P_1\ldots P_N}$ be a convex polygon, let $(\alpha,\beta)\in \operatorname{Int}\Pi$ and let $\mathcal{P}$ as before. Assume that $\overline{A}=\{A_1,\ldots,A_N\}$, $\overline{E}=\{E_1,\ldots,E_N\}$ are Banach N-tuples, that $\overline{B}=\{B_1,\ldots,B_N\}$ is a Banach lattice N-tuple such that $\varDelta(\overline{B})$ is dense in B_j for $j=1,\ldots,N$, and that $1\leqslant q<\infty$, $1\leqslant p, r\leqslant\infty$ with $1/p+1/q=1+1/r$. If $R\colon \Sigma(\overline{A})\times\varDelta(\overline{B})\to\Sigma(\overline{E})$ is a bounded bilinear map, whose restriction $R\colon A_j\times(\varDelta(\overline{B}), \|\cdot\|_{B_j})\to E_j$ is bounded with norm M_j for $j=1,\ldots,N$, then R may be uniquely extended to a bilinear map from $\overline{A}_{(\alpha,\beta),p;J}\times\overline{B}_{(\alpha,\beta),q;K}$ to $\overline{E}_{(\alpha,\beta),r;K}$ with norm at most $C\max\{M_i^{c_i}M_j^{c_j}M_k^{c_k}\colon \{i,j,k\}\in\mathcal{P}\}$ where C only depends on Π and (α,β).*

PROOF. – Since $\overline{B}$ is a Banach lattice N-tuple, in order to compute the K-functional for $b\in\varDelta(\overline{B})$ it suffices to consider decompositions given by elements of $\varDelta(\overline{B})$, that is,

$$K(t,s;b)=\inf\left\{\sum_{j=1}^N t^{x_j}s^{y_j}\|b_j\|_{B_j}\colon\ b=\sum_{j=1}^N b_j,\ \ b_j\in\varDelta(\overline{B})\right\}$$

(see, for example, [5], Prop. 3.1). Hence, we can repeat the argument of the proof of Theorem 2.2 concluding that for any $a\in\overline{A}_{(\alpha,\beta),p;J}$ and $b\in\varDelta(\overline{B})$

$$\|R(a,b)\|_{(\alpha,\beta),r;K}\leqslant C\max_{\{i,j,k\}\in\mathcal{P}}\{M_i^{c_i}M_j^{c_j}M_k^{c_k}\}\,\|a\|_{(\alpha,\beta),p;J}\,\|b\|_{(\alpha,\beta),q;K}\,.$$

Now the result follows taking into account that under the assumption on $\overline{B}$, the intersection $\varDelta(\overline{B})$ is dense in $\overline{B}_{(\alpha,\beta),q;K}$ (see [5], Prop. 3.2). ∎

We finish the paper by establishing formula (2.2) when $\overline{B}$ is a Banach lattice N-tuple.

COROLLARY 3.2. – *Let Π, (α, β), $\mathcal{P}$, $\overline{A}$, $\overline{B}$, p, q, and r as in Theorem* 3.1. *Then*

$$(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J} \subseteq \mathcal{L}(\overline{B}_{(\alpha, \beta), q; K}, \overline{A}_{(\alpha, \beta), r; K}) .$$

PROOF. – Since $\Delta(\overline{B})$ is dense in B_j for $j = 1, \ldots, N$, each space $\mathcal{L}(B_j, A_j)$ is continuously embedded in $\mathcal{L}(\Delta(\overline{B}), \Sigma(\overline{A}))$. Thus $(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))$ is a Banach N-tuple.

Let $R: \mathcal{L}(\Delta(\overline{B}), \Sigma(\overline{A})) \times \Delta(\overline{B}) \to \Sigma(\overline{A})$ be the bounded bilinear map defined by $R(T, b) = T(b)$. The restriction $R: \mathcal{L}(B_j, A_j) \times (\Delta(\overline{B}), \|\cdot\|_{B_j}) \to A_j$ is bounded with norm at most 1. Therefore, using Theorem 3.1, R can be boundedly extended from

$$(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J} \times \overline{B}_{(\alpha, \beta), q; K} \text{ to } \overline{A}_{(\alpha, \beta), r; K} .$$

In other words,

$$(\mathcal{L}(B_1, A_1), \ldots, \mathcal{L}(B_N, A_N))_{(\alpha, \beta), p; J} \subseteq \mathcal{L}(\overline{B}_{(\alpha, \beta), q; K}, \overline{A}_{(\alpha, \beta), r; K}) . \qquad \blacksquare$$

The choice $p = \infty$, $q = 1$, and $r = \infty$ gives formula (2.2).

REFERENCES

[1] S. V. ASTASHKIN, *On interpolation of bilinear operators with a real method*, Mat. Zametki, **52** (1992), 15-24.

[2] J. BERGH - J. LÖFSTRÖM, «*Interpolation Spaces. An Introduction*», Springer, Berlin-Heidelberg-New York (1976).

[3] F. COBOS - P. FERNÁNDEZ-MARTÍNEZ, *A duality theorem for interpolation methods associated to polygons*, Proc. Amer. Math. Soc. **121**, (1994), 1093-1101.

[4] F. COBOS - P. FERNÁNDEZ-MARTÍNEZ - A. MARTÍNEZ, *On reiteration and the behavior of weak compactness under certain interpolation methods*, Collect. Math. (to appear).

[5] F. COBOS - P. FERNÁNDEZ-MARTÍNEZ, A. MARTÍNEZ - Y. RAYNAUD, *On duality between K- and J-spaces*, Proc. Edinburgh Math. Soc. (to appear).

[6] F. COBOS - P. FERNÁNDEZ-MARTÍNEZ - T. SCHONBEK, *Norm estimates for interpolation methods defined by means of polygons*, J. Approx. Theory, **80** (1995), 321-351.

[7] F. COBOS - J. PEETRE, *Interpolation of compact operators: The multidimensional case*, Proc. London Math. Soc., **63** (1991), 371-400.

[8] M. CWIKEL - S. JANSON, *Real and complex interpolation methods for finite and infinite families of Banach spaces*, Adv. Math., **66** (1987), 234-290.

[9] A. FAVINI, *Some results on interpolation of bilinear operators*, Boll. Un. Mat. Ital., **15** (1978), 170-181.

[10] D. L. FERNANDEZ, *Interpolation of 2^d Banach spaces and the Calderón spaces $X(E)$*, Proc. London Math. Soc., **56** (1988), 143-162.

[11] S. JANSON, *On interpolation of multi-linear operators*, in *Function Spaces and Applications*, Proceedings, Lund 1986, Lect. Notes Math., **1302** (1988), 290-302.
[12] J. L. LIONS - J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math., **19** (1964), 5-68.
[13] J. PEETRE, *Paracommutators and minimal spaces*, in *Operators and Funtion Theory*, Proceedings of the NATO Advanced Study Institute on Operators and Function Theory, Lancaster, 1984, ed. S. C. Power, Reidel, Dordrecht (1985), 163-223.
[14] G. SPARR, *Interpolation of several Banach spaces*, Ann. Math. Pura Appl., **99** (1974), 247-316.
[15] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam (1978).
[16] M. ZAFRAN, *A multilinear interpolation theorem*, Studia Math., **62** (1978), 107-124.

F. Cobos: Departamento de Análisis Matemático, Facultad de Matemáticas
Universidad Complutense de Madrid - 28040 Madrid, Spain
e-mail: cobos@eucmax.sim.ucm.es

J. M. Cordeiro: Departamento de Matemática Aplicada
E.U.I. Técnica Industrial, Universidad de Vigo - 36200 Vigo, Spain
e-mail: cordeiro@uvigo.es

A. Martínez: Departamento de Matemática Aplicada
E.T.S. Ingenieros Industriales, Universidad de Vigo - 36200 Vigo, Spain
e-mail: antonmar@uvigo.es

Pervenuta in Redazione
il 7 aprile 1997