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A Nontransitive Space Based on Combinatorics (*).

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Sunto. – *Costruiamo uno spazio nontransitivo analogo al piano di Kofner. Mentre gli argomenti usati per la costruzione del piano di Kofner si fondano su riflessioni geometriche, le nostre prove si basano su idee combinatorie.*

1. – Introduction.

A binary relation V is called a *neighbornet* of a topological space X provided that $V(x) = \{y \in X : (x, y) \in V\}$ is a neighborhood of x for each $x \in X$. A neighbornet V is called *transitive* if V is a transitive relation, and a neighbornet V is called *normal* if there is a sequence $(V_n)_{n \in \omega}$ of neighbornets such that $V_{n+1}^2 \subseteq V_n$ and $V_0 \subseteq V$. A space is called *transitive* if every normal neighbornet contains a transitive neighbornet.

It is well-known that a topological space is non-archimedeanly quasi-metrizable iff there is a sequence $\{T_n : n \in \omega\}$ of transitive neighbornets on X such that for each $x \in X$, $\{T_n(x) : n \in \omega\}$ is a neighborhood basis at x and $\bigcap \{T_n(x) : n \in \omega\} = \{x\}$.

For further information about transitive spaces we refer the reader to [1, Chapter 6]. The Kofner plane was introduced in [2]. Various open problems concerning the concept of a transitive space are mentioned in [3, Section 3].

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2. – The example.

Our space X is the set $\boldsymbol{Q}^\omega$ (of sequences from ω to the rationals $\boldsymbol{Q}$) equipped with an appropriate topology defined with the help of a quasi-uniformity having a countable base $\{U_n : n \in \omega\}$.

For each $n \in \omega$ and $x \in X$ the set $U_n(x)$ is the set of all $x' \in X$ which (1) agree on the initial segment n, and (2) for which the first coordinate $\Delta := \Delta(x, x')$ in which x and x' differ satisfies $x(\Delta) \leqslant x'(\Delta) \leqslant x(\Delta) + 2^{-n}$.

First we check that X is quasi-metrizable (compare [1, Theorem 1.5]). Since clearly $\bigcap_{n \in \omega} U_n(x) = \{x\}$ whenever $x \in X$, it suffices to show that for any $n \in \omega$ and any $x, x', x'' \in X$, we have that $x' \in U_{n+1}(x)$, $x'' \in U_{n+1}(x')$ imply $x'' \in U_n(x)$.

Indeed, note first that under the hypothesis x, x', x'' agree on $n+1$. Set $\Delta_0 := \Delta(x', x)$ and $\Delta := \Delta(x'', x')$. We now distinguish three cases.

Case 1: If $\Delta < \Delta_0$, then $\Delta = \Delta(x'', x)$, $x'(\Delta) = x(\Delta)$, $x'(\Delta) \leqslant x''(\Delta) \leqslant x'(\Delta) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 2: If $\Delta > \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x''(\Delta_0) = x'(\Delta_0)$, $x(\Delta_0) \leqslant x'(\Delta_0) \leqslant x(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 3: If $\Delta = \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x(\Delta_0) \leqslant x'(\Delta_0) \leqslant x(\Delta_0) + 2^{-(n+1)}$ and $x'(\Delta_0) \leqslant x''(\Delta_0) \leqslant x'(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_n(x)$.

Let us finally prove that X is not non-archimedeanly quasi-metrizable. In particular our argument will show that X is not transitive. Suppose the contrary and let $(T_n)_{n \in \omega}$ be a sequence of transitive neighbornets witnessing non-archimedean quasi-metrizability of X.

For each $x \in X$, choose $k(x) \in \omega$ such that $T_{k(x)}(x) \subseteq U_1(x)$; furthermore fix $n_x \in \omega$ such that $U_{n_x}(x) \subseteq T_{k(x)}(x)$.

Viewing the rationals as having the discrete topology and working in the resulting space ω^ω, with the help of the Baire Category Theorem we find $k, n \in \omega$, $\sigma \in \boldsymbol{Q}^n$ and a dense $D \subseteq [\sigma]$ such that, for all $x \in D$, $k(x) = k$ and $n_x \leqslant n$. (Here, as usual, $[\sigma] = \{x' \in \boldsymbol{Q}^\omega : x' \mid n = \sigma\}$.)

Construct $x_1, \ldots, x_{2^n+1} \in D$ such that for $i = 1, \ldots, 2^n$, x_i restricted to n equals σ and such that $x_i(n) + 2^{-n} = x_{i+1}(n)$. We see that $x_{i+1} \in T_k(x_i)$, because $x_{i+1} \in U_n(x_i)$. By transitivity we conclude that $x_{2^n+1} \in T_k(x_1) \subseteq U_1(x_1)$. But now $x_{2^n+1} \in U_1(x_1)$ requires $x_{2^n+1}(\Delta) \leqslant x_1(\Delta) + 2^{-1}$ where $\Delta := \Delta(x_1, x_{2^n+1}) = n$. However $x_1(\Delta) + 1 = x_1(\Delta) + 2^{-n}(2^n) = x_{2^n+1}(\Delta)$. This contradiction completes the proof.

Observe that X is zero-dimensional. It is also separable, because functions which are eventually zero are dense.

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