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Russell Johnson, Alberto Tesi

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On the D-Stability Problem for Real Matrices

RUSSELL JOHNSON - ALBERTO TESI

Sunto. – *Vengono discusse delle condizioni sufficienti affinchè una matrice reale A delle dimensioni $n \times n$ sia diagonalmente (o D-) stabile. Esse includono delle ipotesi geometriche (condizioni degli ortanti), e un criterio che generalizza un criterio di Carlson. Inoltre si discute la D-stabilità robusta per le matrici reali delle dimensioni 4×4.*

1. – Introduction.

The purpose of this paper is to present some remarks concerning the concept of D- (or diagonal-) stability of $n \times n$ real matrices A. Recall that an $n \times n$ real matrix A is called diagonally stable if and only if for every diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with positive diagonal entries, the eigenvalues of the matrix DA all lie in the left half-plane. This concept is of importance in e.g. economics [2], [7] and control systems theory [13], [1].

The problem of characterizing the D-stable matrices has received considerable attention and much is now known, especially for restricted classes of matrices like Minkowski (or M-) matrices and sign-stable matrices. See the reviews by Johnson [12] and Hershkowitz [11]. Cain [4] gave an elegant characterization of the set of real 3×3 diagonally stable matrices. See also [3].

In the present paper, we first consider conditions which, when combined with the stability of A, are sufficient for D-stability. We derive an elementary necessary and sufficient condition for D-stability which seems not well-known. Then, we consider a geometric condition (orthant condition) which is sufficient for D-stability. Next we show that this geometric condition is implied by an analytical condition (Proposition 2.5) which is related to a condition of Carlson [5]. Finally, we sharpen the Carlson condition in a significant way in Proposition 2.9. All these matters are discussed in Section 2.

In Sections 3 and 4 we consider the characterization problem for D-stable matrices. It is well known that this is a complicated matter if the dimension n is greater than 3.

Section 3 is devoted to a proof that there is a polynomial decision procedure for describing the class of D-stable matrices. More precisely, we show that the complement in R^{n^2} of the set $\mathcal{D}$ of D-stable matrices is semi-algebraic. This implies that $\mathcal{D}$ itself has finitely many components. Our analysis uses arguments of the paper of Seidenberg [15], which elaborates the well-known Tarski-Seidenberg decision procedure.

Finally, in Section 4, we consider the problem of characterizing the interior of $\mathcal{D}$ (the class of *robustly* D-stable matrices [1]) when $n = 4$. Although the method of Section 3 can in principle be used to do this, it is as a practical matter difficult to implement. We take advantage of some simplifying features when $n = 4$ to characterize a set $\mathcal{E}$ which is intermediate between $\mathcal{D}$ and $\operatorname{int}\mathcal{D}$. The description of $\mathcal{E}$ is relatively straightforward. We then show how $\operatorname{int}\mathcal{D}$ can be obtained by removing finitely many semialgebraic sets from $\mathcal{E}$.

2. – Conditions for D-stability.

The main purpose of this section is to discuss sufficient conditions for D-stability, one of which is of a geometric nature and consists of various «orthant conditions». However, we first present a warm-up lemma which gives a necessary condition for D-stability and illustrates what we mean by the term «orthant condition».

LEMMA 2.1. – *Let $A\colon R^n \to R^n$ be D-stable and let Θ be an open orthant in R^n. Then, $A\Theta \cap \Theta = \emptyset$.*

PROOF. – Suppose not, and let $0 \neq v \in A\Theta \cap \Theta$. Since $Av \in \Theta$, we can find a diagonal matrix D with positive diagonal entries such that $DAv = v$. That is, 1 is a positive eigenvalue of DA and so A is not D-stable. ■

The converse of this lemma is false as we see from the following

EXAMPLE 2.2. – Consider the matrix

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 7 & -9 & 9 \\ -30 & -1 & -1 \end{bmatrix}.$$

Then, it can be checked that $A\Theta \cap \Theta = \emptyset$ for every open orthant $\Theta \subset R^3$, but A admits two eigenvalues with positive real part. ■

We wish to relate orthant conditions to a standard condition arising in D-stability theory [12]. For this we introduce orthants in coordinate hyperplanes as well as the open orthants considered above. Consider the coordinate hyperplane $\{x_i = 0\} \in R^{n-1} \subset R^n$ where $1 \leq i \leq n$. Let Θ be an open orthant in $\{x_i = 0\}$. The argument of Lemma 2.1 shows that a necessary condition for D-stability of A is that $A\Theta \cap \Theta = \emptyset$. In general, if Θ is an open orthant of the codimension k coordinate subspace $\{x_{i_1} = \ldots = x_{i_k} = 0\} \in R^{n-k} \subset R^n$ $(1 \leqslant k < n)$, then D-stability of A implies that $A\Theta \cap \Theta = \emptyset$.

Taking into account the equivalency of the conditions $A\Theta \cap \Theta = \emptyset$ and $A(-\Theta) \cap (-\Theta) = \emptyset$, we see that there are

$$2^{n-1} + n2^{n-2} + \binom{n}{2} 2^{n-3} + \ldots + n$$

orthant conditions which are necessary for D-stability. It is easy to see that all these conditions can be summarized in the following equivalent condition

(1) $\det(A - D) \neq 0$ for all diagonal D with positive diagonal entries.

For, the orthant conditions taken all together imply that there is no non-zero vector $v \in R^n$ and no positive diagonal matrix D with $Av = Dv$. And, condition (1) implies the orthant conditions.

Now, condition (1) is closely related to the standard property P_0 arising in the stability theory of matrices [8], [9], [12]. In fact, assume that $\det A \neq 0$. Then, direct expansion of the quantity $\det(A - D)$ shows that (1) holds if and only if, for each $1 \leqslant k \leqslant n$, all signed principal minors of order k of A are non-negative:

$$(-1)^k m_{i_1 \ldots i_k} \geqslant 0 \qquad (1 \leqslant i_1 < \ldots < i_k \leqslant n).$$

This is the P_0 property; see especially [9].

In conclusion, we have introduced a geometric (orthant) condition which is necessary for D-stability and which, for non-singular A, is equivalent to the P_0-condition. After this warm-up, we turn to sufficient conditions for D-stability. Our point of departure is the following

LEMMA 2.2. – *Suppose A is stable but not D-stable. Then, we can find a diagonal matrix D with positive diagonal entries and non-zero vectors v', $v'' \in R^n$ such that*

$$DAv' = -v'',$$

$$DAv'' = v'.$$

PROOF. – If A is stable but not D-stable, then there exists a positive diagonal D_1 such that $D_1 A$ has an eigenvalue with non-negative real part. Since zero is not an eigenvalue of DA for any positive diagonal D, there exists a positive diagonal D such that DA admits $i\beta$ as an eigenvalue for some $\beta > 0$. Multiplying D by β^{-1}, we can assume that i is an eigenvalue of DA.

Let $v' + iv''$ be an eigenvector of DA corresponding to the eigenvalue i. Since A is real, we have $v' \neq 0$, $v'' \neq 0$. It is easily seen that v' and v'' satisfy the conditions of the lemma. ■

As an immediate consequence of this lemma we have the necessary and sufficient condition for D-stability promised in the Introduction.

PROPOSITION. – 2.3. – *Necessary and sufficient conditions for D-stability of the matrix A are stability of A and*

$$\det \begin{pmatrix} A & D \\ -D & A \end{pmatrix} \neq 0 \tag{2}$$

for all diagonal $n \times n$ matrices with positive diagonal entries. ∎

Let us observe that, if $v', v'' \in R^n$ are two vectors satisfying the condition of Lemma 2.2, then

$$w' = c'v' - c''v'',$$

$$w'' = c''v' + c'v'',$$

also satisfy that condition for all real constants c', c''. Thus if the i-th component of, say, v' is not zero, then we can choose c', c'' in such a way that the i-th components of w' and w'' are both non-zero. Conversely, if the i-th components of both v' and v'' are zero, then the same is true of the i-th components of w' and w''.

Using this observation, we can formulate orthant conditions which are sufficient for D-stability. For each $0 \leqslant k \leqslant n-1$, consider the coordinate subspace of codimension k

$$S_{i_1 \dots i_k} = \{x = (x_1, \dots, x_n) \in R^n \mid x_{i_1} = \dots = x_{i_k} = 0\}$$

where $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $k = 0$ corresponds to $S_0 = R^n$. Consider the conditions $(0 \leqslant k \leqslant n-1)$:

(C_k): For each pair of open orthants $\Theta_r, \Theta_s \subset S_{i_1 \dots i_k}$, there holds

$$\underline{\text{either}}\ A\Theta_r \cap -\Theta_s = \emptyset \ \underline{\text{or}}\ A\Theta_s \cap \Theta_r = \emptyset .$$

We have

PROPOSITION 2.4. – *Condition $(C_0), \dots, (C_{n-1})$ together with the stability of A are sufficient for the D-stability of A.*

PROOF. – Suppose that (C_0) holds. Then for all positive diagonal D, the matrix DA cannot have eigenvalue i with complex eigenvector $v' + iv''$ such that all components of v' and of v'' are non-zero. For if this were so, we would have $v' \in \Theta_r$, $v'' \in \Theta_s$ for some open orthants $\Theta_r, \Theta_s \subset R^n$, and (C_0) would be violated. Similarly, (C_1) implies that v' and v'' cannot have exactly one component equal to zero, and so on. ∎

Although these conditions are many in number, they are individually easily verified.

We now consider algebraic conditions which imply the orthant conditions (C_0)-(C_{n-1}). We first prove,

PROPOSITION 2.5. – *Suppose that A is stable and that*

$$\det\begin{pmatrix} A & D_1 \\ -D_2 & A \end{pmatrix} \neq 0 \tag{3}$$

for each pair *of diagonal matrices D_1, D_2 with positive diagonal entries. Then, A is D-stable.*

PROOF. – Clearly the hypothesis is stronger than the (necessary and) sufficient condition of Proposition 2.3. ■

REMARK 2.6. – The hypothesis of Proposition 2.5 implies each of the orthant conditions $(C_0), \dots, (C_{n-1})$ is valid. Thus, suppose for contradiction that there is a coordinate subspace $S_{i_1 \dots i_k}$ $(0 \leqslant k \leqslant n-1)$ and that there are open orthants Θ_r, $\Theta_s \subset S_{i_1 \dots i_k}$ such that

$$A\Theta_r \cap -\Theta_s \neq \emptyset \quad \underline{\text{and}} \quad A\Theta_s \cap \Theta_r \neq \emptyset .$$

Let $x' \in \Theta_r$ and $w' \in \Theta_s$ satisfy $Ax' = w'$, and let $x'' \in \Theta_s$ and $w'' \in \Theta_r$ satisfy $Ax'' = w''$. There are diagonal matrices with positive diagonal entries D_1, D_2 such that

$$Ax' = -w' = -D_1 x'' ,$$

$$Ax'' = w'' = D_2 x' .$$

That is, D_1 and D_2 satisfy

$$\det\begin{pmatrix} A & D_1 \\ -D_2 & A \end{pmatrix} = 0 .$$

This contradicts the hypothesis of Proposition 2.5. ■

The condition of Proposition 2.5 can be reformulated in the following elegant algebraic way.

Write

$$D_1 = \operatorname{diag}(d_1, \dots, d_n), \qquad D_2 = \operatorname{diag}(e_1, \dots, e_n).$$

Then the quantity

$$P(d_1, \dots, d_n; e_1, \dots, e_n) = \det\begin{pmatrix} A & D_1 \\ -D_2 & A \end{pmatrix}$$

is a polynomial in its arguments. The terms of the polynomial P are of the form

$$cd_{i_1}\dots d_{i_k}e_{j_1}\dots e_{j_l}, \qquad (i_1, \dots, j_l \in \{1, 2, \dots, n\}),$$

where the coefficient $c = c(i_1, \dots, i_k, j_1, \dots, j_l)$ depends only on the entries of the matrix A. We verify the following three claims.

CLAIM 1. – The constant term in the polynomial P equals $(\det A)^2 > 0$.

The proof of Claim 1 is trivial: set $d_1 = \dots = d_n = e_1 = \dots = e_n = 0$.

We see that the non-vanishing of the polynomial P for all positive values of $d_1, \dots, e_n$ is equivalent to its positivity for all such values of $d_1, \dots, e_n$. Thus the criterion of Proposition 2.5 is equivalent to the stability of A together with strict positivity of P for $d_1 > 0, \dots, e_n > 0$.

CLAIM 2. – The coefficient $c(i_1, \dots, i_k, j_1, \dots, j_l) = 0$ if $k \neq l$ and $k, l < n$.

To prove this claim, note that the coefficient in question is obtained by expanding the determinant

$$\text{(4)} \qquad \left|\begin{array}{ccc|ccc} & & & d_1 & \dots & 0 \\ & A & & \vdots & \dots & \vdots \\ & & & 0 & \dots & d_n \\ -e_1 & \dots & 0 & & & \\ \vdots & \dots & \vdots & & A & \\ 0 & \dots & -e_n & & & \end{array}\right|$$

and retaining only those terms containing $d_{i_1}, \dots, d_{i_k}, e_{j_1}, \dots, e_{j_l}$. Thus without loss of generality we can set $d_\alpha = 0$ if $\alpha \dots \in \{i_1, \dots, i_k\}$ and $e_\beta = 0$ if $\beta \notin \{j_1, \dots, j_l\}$.

Write $r = n - k$, $s = n - l$, and let $m_1, \dots, m_r; n_1, \dots, n_s$ be the increasing enumerations of $\{1, 2, \dots, n\} \setminus \{i_1, \dots, i_k\}$ and $\{1, 2, \dots, n\} \setminus \{j_1, \dots, j_l\}$ respectively. By direct calculation from (4), one finds that

$$\text{(5)} \qquad c(i_1, \dots, i_k, j_1, \dots, j_l) = \left|\begin{array}{cccccc} a_{m_1 n_1} & \dots & a_{m_1 n_s} & & & \\ \vdots & \dots & \vdots & & 0 & \\ a_{m_r n_1} & \dots & a_{m_r n_s} & & & \\ & & & a_{n_1 m_1} & \dots & a_{n_1 m_r} \\ & 0 & & \vdots & \dots & \vdots \\ & & & a_{n_s m_1} & \dots & a_{n_s m_r} \end{array}\right|.$$

The blocks of zeroes are respectively $s \times s$ and $r \times r$, and since $r \neq s$ the determinant is zero.

CLAIM 3. – If $k=l$, then

$$c(i_1, \ldots, i_k, j_1, \ldots, j_k) = \begin{vmatrix} a_{m_1 n_1} & \cdots & a_{m_1 n_r} \\ \vdots & \cdots & \vdots \\ a_{m_r n_1} & \cdots & a_{m_r n_r} \end{vmatrix} \begin{vmatrix} a_{n_1 m_1} & \cdots & a_{n_1 m_r} \\ \vdots & \cdots & \vdots \\ a_{n_r m_1} & \cdots & a_{n_r m_r} \end{vmatrix}$$

where again $r=n-k$ and $m_1, \ldots, m_r$; $n_1, \ldots, n_r$ are the increasing enumerations of $\{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ resp. $\{1, 2, \ldots, n\} \setminus \{j_1, \ldots, j_l\}$.

Claim 3 follow from Claim 2 and a verification of sign.

Now, one verifies without difficulty that P is positive for all positive values of $d_1, \ldots, e_n$ if and only if each coefficient $c(i_1, \ldots, i_k, j_1, \ldots, j_k)$ is non-negative, for each $1 \leqslant k \leqslant n$. Thus we have (almost) proved

PROPOSITION 2.7. – *The non-negativity of each of the following products is sufficient for a stable matrix A to be D-stable*

$$\begin{vmatrix} a_{m_1 n_1} & \cdots & a_{m_1 n_r} \\ \vdots & \cdots & \vdots \\ a_{m_r n_1} & \cdots & a_{m_r n_r} \end{vmatrix} \begin{vmatrix} a_{n_1 m_1} & \cdots & a_{n_1 m_r} \\ \vdots & \cdots & \vdots \\ a_{n_r m_1} & \cdots & a_{n_r m_r} \end{vmatrix} | \geqslant 0 \tag{6}$$

where $1 \leqslant r \leqslant n-1$ *and* $1 \leqslant m_1 < \ldots < m_r \leqslant n,\ 1 \leqslant n_1 < \ldots < n_r \leqslant n.$

PROOF. – The proof is complete if we note the (obvious) fact that the coefficient of $d_1 \ldots d_n e_1 \ldots e_n$ is $+1$, and that, if k or l equals n in Claim 2, then the corresponding coeffcient vanishes (eventhough formula (5) no longer holds). ■

EXAMPLE 2.8. – We apply Proposition 2.7 to 2×2 matrices. It turns out that a 2×2 matrix A is D-stable if

$$a_{11} < 0; \qquad a_{22} < 0; \qquad a_{11} a_{22} > a_{12} a_{21}; \qquad a_{12} a_{21} > 0 \ .$$

The last inequality is not necessary for D-stability [12]. ■

The determinants appearing in Proposition 2.7 are exactly those occurring in Carlson's Theorem [5], [11]. Indeed our Proposition 2.7 is equivalent to Carlson's Theorem because stability together with (6) imply and are implied by the P_0-property together with (6).

Let us note, however, that Proposition 2.7 can be significantly strengthened, as follows. By Proposition 2.3, the D-stability of A is equivalent to stability together with the positivity of

$$F(d_1, \ldots, d_n) = \begin{vmatrix} A & D \\ -D & A \end{vmatrix} = P(d_1, \ldots, d_n; d_1, \ldots, d_n)$$

for all positive numbers $d_1, \ldots, d_n$. Now $F(d_1, \ldots, d_n) = \sum f(k_1, \ldots, k_p) d_{k_1}^{l_1} \ldots d_{k_p}^{l_p}$

where $1 \leqslant k_1 < \ldots < k_p \leqslant n$ and $1 \leqslant l_1, \ldots, l_p \leqslant 2$. The coefficient $f(k_1, \ldots, k_p)$ can be calculated using Theorem 2.7. The result is zero if $l_1 + \ldots + l_p$ is odd. If $l_1 + \ldots + l_p = 2s$ is even, let $Q(k_1, \ldots, k_p) = \{(I, J) \mid I \subset \{1, 2, \ldots, n\}, J \subset \{1, 2, \ldots, n\}, |I| = |J| = s$ and $d_{i_1} \ldots d_{i_s} d_{j_1} \ldots d_{j_s} = d_{k_1}^{l_1} \ldots d_{k_p}^{l_p}\}$. Then

$$f(k_1, \ldots, k_p) = \sum_{(I, J) \in Q} c(I, J).$$

Clearly, if A is stable and each coefficient $f(k_1, \ldots, k_p) \geqslant 0$, then A is D-stable.

Summarizing, we have proved

THEOREM 2.9. – *If A is stable and if each coefficient $f(k_1, \ldots, k_p)$ is non-negative, then A is D-stable.* ■

This condition is considerably less restrictive that the Carlson's Theorem. Moreover its proof does not make use of determinantal inequalities. It may be that more careful analysis of the polynomial F will allow a resolution of Carlson's conjecture (weakly sign symmetric P-matrices are D-stable).

3. – On characterizing D-stability.

Our goal is to show that the set of D-stable matrices is the complement of a semi-algebraic set in R^{n^2}, i.e., the complement of a set describable by a finite collection of sets of polynomial equalities and inequalities.

As a starting point we take the characterization of D-stable matrices of Proposition 2.3, namely that A is D-stable if and only if A is stable and

$$\det \begin{pmatrix} A & -D \\ D & A \end{pmatrix} \neq 0 . \tag{7}$$

for all positive diagonal D. The stability of A can be determined by finitely many polynomial inequalities in the coefficients $a_{11}, a_{12}, \ldots, a_{nn}$ of A (the Hurwitz criterion).

The relation (7) is a polynomial P_1 in the coefficients of A and the diagonal elements $d_1, \ldots, d_n$ of D:

$$P_1(A; d_1, \ldots, d_n) \neq 0 .$$

Write $d_1 = t_1^2, d_2 = t_2^2, \ldots, d_n = t_n^2$ and note that $P_1(A; 0, \ldots, 0) = (\det A)^2 > 0$. We see that (7) is equivalent to the relation

$$P(A; t_1, \ldots, t_n) \equiv P_1(A; t_1^2, \ldots, t_n^2) > 0 \tag{8}$$

for all real values of $t_1, \ldots, t_n$.

Now we apply Theorem 2 of Seidenberg [15] to conclude that there exists a finite number N of polynomials $\sigma_i(a_{11}, a_{12}, \ldots, a_{nn})$ and an equal number of polynomials $g_i(a_{11}, a_{12}, \dots, a_{nn}; t)$ such that the relation

$$P(A; t_1, \dots, t_n) = 0 \tag{9}$$

has a solution $(t_1, \dots, t_n) \in R^n$ if and only if for at least one i, $1 \leq i \leq N$, the system

$$\begin{cases} \sigma_i(a_{11}, a_{12}, \dots, a_{nn}) \neq 0\,, \\ g_i(a_{11}, a_{12}, \dots, a_{nn}; t) = 0\,, \end{cases} \tag{10}$$

has a solution $t \in R$. We have written $t_1 = t$.

Now, solvability of (10) can be determined by constructing a Sturm chain for

$$\sigma_i^2(\cdot)\,\frac{(\partial g_i/\partial t)(\cdot, t)}{g(\cdot, t)}$$

and considering the number of variations in sign of this chain between $t = -\infty$ and $t = \infty$. See Gantmacher [10, Vol. II, p. 175]. The result is a finite set $\Gamma_1, \dots, \Gamma_{N'}$ $(N' = N'(i))$ of collections of polynomial equalities and inequalities in $a_{11}, a_{12}, \dots, a_{nn}$ such that (10) has a solution if and only if at least one collection Γ_j is satisfied.

We conclude, then, that the set of D-stable matrices is the complement of a semi-algebraic subset of R^{n^2}. This property has an important corollary which we include in the following

THEOREM 3.4. – *The set of D-stable matrices is the complement in R^{n^2} of a semi-algebraic set and as a consequence has a finite number of connected components.* ■

Finally, we wish to note that, in a recent paper [6], the question of D-stability is related to an NP-hard problem (that of the exact computation of the real structured singular value of a complex matrix).

4. – Robust D-stability in dimension $n = 4$.

As stated in the Introduction, a matrix is robustly D-stable if it lies in $\mathcal{D} = \{A \in R^{n^2} \mid A \text{ is } D-\text{stable}\}$ and if all sufficiently near matrices A' are also in $\mathcal{D}$. Even in four dimensions, it seems that it is unwieldy to write down a complete set of algebraic defining relations for $\operatorname{int}\mathcal{D}$ (and that it is even more unwieldy to describe $\mathcal{D}$ itself). Nevertheless, we will describe a set $\mathcal{E}$ satisfying $\operatorname{int}\mathcal{D} \subset \mathcal{E} \subset \mathcal{D}$, then state how a semialgebraic set can be removed from $\mathcal{E}$ so as to obtain $\operatorname{int}\mathcal{D}$.

It will be convenient to begin from the classical Hurwitz criterion for the stability of DA, where D is a positive diagonal matrix. First let $p(\lambda)$ be the characteristic polynomial of DA:

(11) $$\det(\lambda I - DA) = p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0 .$$

The coeffcients p_{n-k} equals $(-1)^k$ time the sum of the k-th order principal minors of the matrix A [10]. Each coefficient p_{n-k} is a function of the diagonal entries $d_1, \ldots, d_n$ of D.

Let us assume that A itself is stable; this can be verified by the Hurwitz criterion. Then using Orlando's Theorem [10, Vol. II, pp. 196], we can conclude that A is D-stable if and only if the third Hurwitz determinant H_3 of DA is positive for all positive diagonal D:

(12) $$H_3(DA) = \begin{vmatrix} p_3 & 1 & 0 \\ p_1 & p_2 & p_3 \\ 0 & p_0 & p_1 \end{vmatrix} > 0$$

for all positive values of d_1, d_2, d_3, d_4.

Next observe that $H_3(DA)$ is homogeneous of degree 6 in $d_1, \ldots, d_4$. In fact, in general, if A is an $n \times n$ matrix and H_k is the k-th Hurwitz determinant of DA, then H_k is homogeneous in $d_1, \ldots, d_n$ of degree $1+2+\ldots+k=k(k+1)/2$.

It is therefore natural to study (12) by first eliminating a variable through projectivization. Concretely, divide $H_3(DA) = H_3(A; d_1, \ldots, d_4)$ by d_4^6. Write $x = d_1/d_4$, $y = d_2/d_4$, $z = d_3/d_4$ and define

$$b(x, y, z) = \frac{H_3(d_1, \ldots, d_4)}{d_4^6} .$$

We suppress the A-dependence unless it is useful to display it explicitly. We see that (12) is equivalent to

(13) $$b(x, y, z) > 0 \qquad \text{for all positive } x, y, z .$$

We study the polynomial inequality (13). First consider $b(x) \equiv b(x; y, z)$ as a polynomial in x and write

$$b(x) = B_3(y, z)x^3 + B_2(y, z)\,x^2 + B_1(y, z)x + B_0(y, z) .$$

Observe that a necessary condition for robust D-stability of the matrix A is

(14) $$B_3(y, z) > 0 , \qquad B_0(y, z) > 0 \qquad \text{for all } y > 0 , \quad z > 0 .$$

For if, for example, $B_3(\bar{y}, \bar{z}) \leq 0$ for some values of $\bar{y}$, $\bar{z}$, then an arbitrarily small perturbation of A will yield a matrix A' such that $B_3(A'; \bar{y}, \bar{z}) < 0$. Then, $b(A'; x, y, z)$ is negative for large x, and A' is not D-stable. Similarly, letting $x \to 0$ we see that robust D-stability of A implies positivity of B_0 for all $y > 0$, $z > 0$.

Observe further that a necessary condition for D-stability (and a fortiori for robust D-stability) of A is

$$b(x, 1, 1) > 0 \quad \text{for all } x > 0 . \tag{15}$$

We assume until further notice that the matrix A satisfies (14) and (15), and proceed to derive more necessary conditions for D-stability of A.

First of all, we have for each $y > 0$, $z > 0$:

$$b(0) > 0 , \qquad \lim_{x \to -\infty} b(x) = -\infty , \qquad \lim_{x \to \infty} b(x) = \infty .$$

Combining these observations with (14) and (15), we see that D-stability of A implies that there does not exist a pair (y, z) for which $b(x)$ has a positive double zero x_0, and conversely if there does not exist a pair (y, z) for which $b(x)$ has a positive double zero, then A is D-stable.

We are led to study the existence of double zeroes of $b(x)$, i.e. points $x_0 > 0$ such that $0 = b(x_0) = b'(x_0)$. In general, $b(x)$ admits a complex double zero x_0 if and only if the discriminant $\Delta = \Delta(y, z)$ of the coefficients of b is zero:

$$\Delta(y, z) = \Delta(B_3, B_2, B_1, B_0) = 0 . \tag{16}$$

Explicitly [14, p. 141]:

$$\Delta(B_3, B_2, B_1, B_0) = B_1^2 B_2^2 - 4B_0 B_2^3 - 4B_1^3 B_3 - 27 B_0^2 B_3^2 + 18 B_0 B_1 B_2 B_3 .$$

We study the solutions (y, z) of (16).

LEMMA 4.6. – *Let (y, z) be a solution of $\Delta(y, z) = 0$, and let $x_0 = \alpha + i\beta$ be a double root of $b(x) = 0$ with $\beta \neq 0$. Then, $B_2(y, z) > 0$ and $B_1(y, z) > 0$.*

PROOF. – By (14), $b(x)$ has a negative root $-a$ where $a > 0$. Since x_0 and $\overline{x}_0 = \alpha - i\beta$ are roots of b and of b' we can conclude that

$$3\, b(x) = (x + a)\, b'(x) . \tag{17}$$

The condition (17) implies that

$$\begin{cases} 3B_2 = 2B_2 + 3aB_3 , \\ 3B_1 = B_1 + 2aB_2 , \\ 3B_0 = aB_1 . \end{cases} \tag{18}$$

The first relation implies that $B_2 > 0$, and the third implies that $B_1 > 0$. This is the assertion of the lemma. ■

LEMMA 4.7. – *Let (y, z) be a solution of $\Delta(y, z) = 0$, and suppose that $x_0 \leq 0$ is a double root of b. Then, $B_2(y, z) > 0$ and $B_1(y, z) > 0$.*

PROOF. – First of all, $b(0) > 0$ so we must have $x_0 < 0$. It is then easy to see that $b'(0) > 0$, that is $B_1(y, z) > 0$. It can further be seen that $b'(x) = 3B_3 x^2 + 2B_2 x + B_1$ has two negative real roots, and this forces $B_2(y, z) > 0$. The lemma is proved. ■

We now combine these lemmas to prove the following

PROPOSITION 4.8. – *Suppose that the matrix A satisfies* (14) *and* (15). *Then A is D-stable if and only if* neither *of the following relations admits a solutiony* > 0, $z > 0$:

$$\Delta(y, z) = 0\,, \qquad B_1(y, z) < 0\,, \tag{19}$$

$$\Delta(y, z) = 0\,, \qquad B_2(y, z) < 0\,. \tag{20}$$

PROOF. – ($\to$) If A is D-stable and (y, z) is a solution of $\Delta(y, z) = 0$, then Lemmas 4.6 and 4.7 show that $B_1(y, z)$ and $B_2(y, z)$ are both positive. So neither (18) nor (19) admits a solution.

($\leftarrow$) Suppose that neither (19) nor (20) admits a solution, and let $y > 0$, $z > 0$ be a solution of $\Delta(y, z) = 0$. We must consider several special cases.

(i) If $\Delta(y, z) = 0 = B_1(y, z)$ and if $b(x, y, z)$ has a positive double root x_0, then $B_2(y, z) < 0$, a contradiction.

(ii) If $B_2(y, z) = 0$, then $b(x)$ and $b'(x)$ cannot have a common positive root x_0.

(iii) If $B_1(y, z) > 0$ and $B_2(y, z) > 0$, then $b(x)$ has no positive root.

Putting these observations together with the assumption that neither (19) nor (20) has a solution, we conclude that there is no pair $y > 0$, $z > 0$ for which $b(x, y, z)$ admits a double root x_0. This implies that A is D-stable and proves the proposition. ■

REMARK 4.9. – A brief examination of the proof of Proposition 4.8 shows that the following assertion is valid. Suppose that A satisfies (19) and (20); then A is D-stable if and only if neither of the following relations admits a solution $y > 0$, $z > 0$:

$$\Delta(y, z) = 0\,, \qquad B_1(y, z) \leqslant 0\,, \tag{21}$$

$$\Delta(y, z) = 0\,, \qquad B_2(y, z) \leqslant 0\,, \tag{22}$$

We now consider how to determine whether (19)-(20) have solutions. There are at least two ways to handle such problems. The first, convenient in special cases, is to explicitly determine those points $y > 0$, $z > 0$ for which $B_1(y, z) > 0$ resp. $B_2(y, z) > 0$, then to solve the quadratic equation $b'(x) = 0$ for such points (y, z), and finally to plug any real positive solutions x_0 into the equation $b(x) = 0$.

The second way is to use the method of Sturm chains. It is somewhat more convenient to use relations (21)-(22) as the point of departure here. Consider for example $\Delta(y, z)$ and $B_1(y, z)$ as functions of y. Construct a Sturm chain for the rational function

$$\frac{\Delta'(y)}{\Delta(y)}\ .$$

where the prime $'$ of course denotes the derivative with respect to y. Let $V_0^\infty(\Delta'/\Delta)$ denote the difference in the number of sign changes in this chain between $y=0$ and $y=\infty$. Then $V_0^\infty(\Delta'/\Delta)$ is the number of positive zeroes of $\Delta(y)$. See Gantmacher [10, Vol. II, p. 175].

Next construct a Sturm chain for

$$B_1(y)\,\frac{\Delta'(y)}{\Delta(y)}\ .$$

Using [10], we see that (21) does *not* admit a solution $y>0$ (for the fixed z in question) if and only if

$$V_0^\infty\left(\frac{\Delta'}{\Delta}\right)=V_0^\infty\left(B_1\,\frac{\Delta'}{\Delta}\right). \tag{23}$$

Now the verification of (23) involves the examination of finitely many collections of equalities and inequalities involving polynomials in the variable z. The investigation of these sets of equalities and inequalities, and their reduction to sets of equalities and inequalities which only involve the coefficient $a_{11}, a_{12}, \dots, a_{nn}$ of the matrix A, is in principle an elementary task.

Since relation (22) can be studied in the same manner, it is natural to make the following

DEFINITION 4.11. – *Let* $\mathcal{E}$ *be the set of all* 4×4 *real matrices which satisfy* (14) *and* (15) *and such that neither of relations* (21)-(22) *admits a solution* $y>0,\ z>0$.

This is the set of $\mathcal{E}$ referred to in the Introduction. Clearly we have $\operatorname{int}\mathcal{D}\subset\mathcal{E}\subset\mathcal{D}$, so $\mathcal{E}$ contains the robustly D-stable matrices and is itself contained in the set of D-stable matrices.

Let us remark that (14) and (15) are verifiable by elementary (and simple) means. In fact, $B_0(y,z)$ is quadratic in y and z, so it is straightforward to determine conditions on the coefficients of B_0 which are necessary and sufficient for its positivity for $y>0$, $z>0$. The function $B_3(y,z)$ is cubic in y and z. One can fix z and look for roots $y>0$ of $B_0(y,z)$ by Sturm's method. One obtains in this way finitely many sets of equalities and inequalities involving polynomials in the variable z, which can be studied by elementary means. Finally, (15) can be studied directly by examination of a Sturm chain.

We finish the paper with comments on how the class of robustly D-stable matrices A can be picked out of $\mathcal{E}$. The problem can be posed as follows: determine «robust» conditions on A under which $B_3(y,z)>0$ and $B_0(y,z)>0$ for all $y > 0$, $z > 0$; determine robust conditions on A under which neither (19) nor (20) admits a solution $y > 0$, $z > 0$. [Observe that, if (14) is true for all matrices A' in an open neighborhood V of some matrix A, and if (15) holds at A, then (15) also holds for all $A' \in V$. Thus the condition (15) need not be considered further.]

This task can in principle be carried out for relations (19)-(20) by examining the collections of equalities and inequalities involving polynomials in z which result from studying (23). Since one has to do with polynomials in a single variable, the «robustification» of (19)-(20) presents no theoretical difficulties (though it may be unappealing to carry out in detail ...). The same remarks apply to the polynomials B_3 and B_0. However, for $B_3(y, z)$ at least, a direct analysis is easier. We can write

$$B_3(y, z) = (\alpha z + \alpha^*)\, y^2 + Q(z)\, y + (\gamma z + \gamma^* z^2)$$

where

$$Q(z) = \eta_1 z^2 + \sigma z + \eta_2 \, .$$

The coefficients $\alpha, \alpha^*, \gamma, \gamma^*, \eta_1, \sigma, \eta_2$ are function of $a_{11}, a_{12}, \ldots, a_{nn}$. It is easily seen that positivity of B_3 for all $y > 0$, $z > 0$ implies that $\alpha, \alpha^*, \gamma, \gamma^*$ and η_1 are all positive. Assume that conditions have been written down which ensure that $B_3(A; y, z) > 0$ for all $y > 0$, $z > 0$, where A is fixed. Then one can check, if $\eta_2 > 0$, then $B_3(A'; y, z) > 0$ for all $y > 0$, $z > 0$, for all A' in some neighborhood of A. Thus *robust* positivity of B_3 can be expressed in terms of polynomial relations involving the coefficients of A.

EXAMPLE 4.12. – In this example we apply the discussed method to the following matrix

$$A = \begin{bmatrix} -1 & 0 & a & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

where a is a real parameter. It is easily checked that A is stable for $a > -8/3$. On the other hand our procedure can be used to prove that A is D-stable if $a \geqslant -1$. First, we get that the coefficients of the characteristic polynomial of DA (see (11)) are

$$\begin{aligned} p_3(d_1, d_2, d_3, d_4) &= d_1 + d_2 + d_3 + d_4 \, , \\ p_2(d_1, d_2, d_3, d_4) &= d_1 d_2 + d_1 d_3 (a+1) + d_2 d_3 + d_1 d_4 + d_2 d_4 + d_3 d_4 \, , \\ p_1(d_1, d_2, d_3, d_4) &= d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 (a+1) + d_2 d_3 d_4 \, , \\ p_0(d_1, d_2, d_3, d_4) &= d_1 d_2 d_3 d_4 \, . \end{aligned}$$

Thus, a necessary condition for D-stability is $a \geqslant -1$. To prove that A is actually D-stable for $a \geqslant -1$ we find it not convenient to view $H_3(DA)/d_4^6$ as a cubic polynomial in $x = d_1/d_4$ with coefficients which are functions of $y = d_2/d_4$ and $z = d_3/d_4$. It is better to write instead

$$\frac{H_3(DA)}{d_4^6} = B_2(1+a)^2 + B_1(1+a) + B_0\,.$$

It can be checked that B_2, B_1, and B_0 are all positive if $x, y, z > 0$. Indeed a calculation shows that

$$B_2 = x^2 z^2 (x+y+z)\,,$$

$$B_1 = xz\{x^2[y(z+2)+1] + x[y^2(z+2) + y(z^2+3z+2) + 2z + 1] + y^2(2z+1) + y(2z^2+2z+1) + z(z+1)\}\,,$$

$$B_0 = y\{x^3[y(z+1)+1] + x^2[y^2(z+1) + y(z^2+2z+2) + 2z+1] + x[y^2(z+1)^2 + y(z^3+2z^2+3z+1) + z(2z+1)] + z(z+1)[y^2 + y(z+1) + z]\}\,,$$

and these expressions are positive if $x, y, z > 0$.

As a final comment we note that D-stability of A cannot be checked by the well-known Lyapunov condition [12], at least if $a > 8(1+\sqrt{2})$. In fact, suppose for contradiction that there exists a diagonal matrix

$$X = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & t \end{bmatrix}$$

with positive entries such that $XA + A'X < 0$ ($'$ denotes transpose). Clearly, such an X exists if and only if the matrices

$$X_1 = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 & a \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}$$

satisfy $M = -(X_1A_1 + A_1'X_1) > 0$. But

$$M = \begin{bmatrix} 2x & y & -ax+z \\ y & 2y & z \\ -ax+z & z & 2z \end{bmatrix}$$

is not positive definite for any $x, y, z > 0$ if, say, $a > 8(1+\sqrt{2})$. ∎

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Dipartimento di Sistemi e Informatica, Università di Firenze
Via di Santa Marta 3 - 50139 Firenze

e-mail: johnson@ingfi1.ing.unifi.it; atesi@ingfi1.ing.unifi.it

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