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On the Fiber of the Compound of a Real Analytic Function by a Projection (*).

ALAIN JACQUEMARD

In Memoriam Mario Raimondo

Sunto. – *Sia f: $\mathbb{R}^m \to \mathbb{R}^k$ con $m \geq k \geq 1$ una funzione analitica. Se il luogo critico di f è compatto, esiste una fibrazione $\mathcal{C}^\infty$ localmente triviale associata ai livelli f. Supponiamo $k \geq 2$ e sia π_k la proiezione $(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1})$. Sotto una condizione sul luogo critico di $\tilde{f} = \pi_k \circ f$ esiste anche una fibrazione $\mathcal{C}^\infty$ localmente triviale associata ai livelli di $\tilde{f}$. Siano F e $\widetilde{F}$ le fibre rispettitive, e I l'intervallo unità reale. Dimostriamo qui che $\widetilde{F}$ è omeomorfa al prodotto $F \times I$. Nel caso di polinomi studiamo criteri effettivi. Diamo inoltre un'applicazione del risultato principale.*

1. – Introduction and statements of the results.

In his book [M. p. 100] J.Milnor asks some questions in the context of real polynomial functions with isolated singularity at the origin. The first one is a question about the fiber of a polynomial mapping $\mathbb{R}^m \mapsto \mathbb{R}^k$ with isolated singularity at the origin and the fiber of its compound by the canonical projection $\mathbb{R}^k \mapsto \mathbb{R}^{k-1}$ which forgets the last coordinate: is the fiber of the latter homeomorphic to the product of the fiber of the former mapping by the unit interval? We investigate this question in a larger context, allowing f to have a non isolated singularity.

Let us consider an analytic function $f = (f_1, f_2, \ldots, f_k)$: $\mathbb{R}^m \to \mathbb{R}^k$, with $m \geq k \geq 1$.

We recall that $x \in \mathbb{R}^m$ is a *singular point* of f if and only if the rank of the Jacobian matrix of f at x is not maximal. We call *singular locus* of f the set of the singular points of f.

In the following, we consider functions satisfying:

HYPOTHESIS $\mathcal{C}$ *The singular locus Σ_f is compact and $\Sigma_f \subset f^{-1}(0)$.*

Moreover, we may assume that Σ_f is given as the zero set of a positive analytic function $\theta\colon \mathbb{R}^m \mapsto \mathbb{R}^+$ (for instance, one may take θ defined by $\theta(x) = \sum_{i=0}^{n_k} m_i^2(x)$, where m_i, $1 \leqslant i \leqslant n_k$ are the $k \times k$ minors of the Jacobian matrix of f).

NOTATIONS 1. – *For all* $\varepsilon > 0$, $m \geqslant 1$, $n \geqslant 2$, $\eta > 0$, *let us denote*:

$$\mathbb{B}_\varepsilon = \{x \in \mathbb{R}^m / \theta(x) \leqslant \varepsilon\},$$

$$\mathbb{S}_\varepsilon = \{x \in \mathbb{R}^m / \theta(x) = \varepsilon\},$$

$$S_\eta^{n-1} = \left\{(u_1, u_2, \ldots, u_l) \in \mathbb{R}^n / \sum_{i=1}^{l} u_i^2 = \eta^2\right\}.$$

Since $\Sigma_f = \theta^{-1}(0)$ is compact, for all neighbourhood $\mathcal{V}_0$ of Σ_f, there exists $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon \subset \mathcal{V}_0$.

We first prove (Lemma 2) that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\mathbb{B}_\varepsilon$ is a compact manifold with smooth boundary. We then show:

THEOREM 1. – *Let* $f = (f_1, f_2, \ldots, f_k)\colon \mathbb{R}^m \to \mathbb{R}^k$, *with* $m \geqslant k \geqslant 1$ *be an analytic function. Under Hypothesis* $\mathcal{C}$, *there exists* $\varepsilon_1 \in]0, \varepsilon_0]$ *such that: for all* $\varepsilon \in]0, \varepsilon_1]$ *there exists* $\eta_\varepsilon \geqslant 0$ *such that for all* $\eta \in]0, \eta_\varepsilon]$, *the restriction of* f

$$f_{|\mathbb{B}_\varepsilon \cap f^{-1}(S_\eta^{k-1})}\colon \mathbb{B}_\varepsilon \cap f^{-1}(S_\eta^{k-1}) \to S_\eta^{k-1}$$

is a $\mathcal{C}^\infty$ *locally trivial fibration onto* S_η^{k-1}.

Notice (cf. Remark 2) that this fibration can be pushed on $\mathbb{S}_\varepsilon \setminus f^{-1}(0)$.

When Σ_f is not compact, but has at least one compact connected component, one may prove the same kind of result, just replacing $\mathbb{B}_\varepsilon$ by its intersection with a suitable neighbourhood of the union of the connected compact components of Σ_f (cf. Remark 2).

In the following, we assume $k \geqslant 2$.

NOTATION 2. – *We denote by* π_k *the projection*

$$\begin{array}{ccc} \mathbb{R}^k & \to & \mathbb{R}^{k-1} \\ (x_1, \ldots, x_{k-1}, x_k) & \mapsto & (x_1, \ldots, x_{k-1}) \end{array}$$

and we set up the following

HYPOTHESIS $\mathcal{H}_k$. – *The singular locus* $\Sigma_{\tilde f}$ *of* $\tilde f = \pi_k \circ f$ *is such that* $\Sigma_f = \Sigma_{\tilde f}$.

Let us suppose now that $k \geqslant 2$ and that f satisfies both Hypotheses $\mathcal{C}$ and $\mathcal{H}_k$. We can apply Theorem 1 to both f and $\pi_k \circ f$. We denote by F and $\widetilde{F}$ the corresponding fibers. The following result establishes the relationship between F and $\widetilde{F}$:

THEOREM 2. – *Let* $f=(f_1, f_2, \ldots, f_k)\colon \mathbb{R}^m \to \mathbb{R}^k$, *with* $m \geq k \geq 2$ *be an analytic function. Under Hypotheses* $\mathcal{C}$ *and* $\mathcal{H}_k$, *there exist* $\alpha > 0$, $\eta_\alpha > 0$, *such that*: $\forall \varepsilon \in]0, \alpha]$, $\forall \eta \in]0, \eta_\alpha]$, $\forall \tilde{c} \in S_\eta^{k-2}$, $\forall c \in S_\eta^{k-1}$,

the fiber $\tilde{f}^{-1}(\tilde{c}) \cap \mathbb{B}_\varepsilon$ *is homeomorphic to the product* $(f^{-1}(c) \cap \mathbb{B}_\varepsilon) \times [0, 1]$.

Notice that a similar result holds (cf. Remarks 2 and 3) if Σ_f is not compact but has at least one compact component.

In the case f has an isolated singularity, both Hypotheses $\mathcal{C}$ and $\mathcal{H}_k$ hold, and this answers the particular question of J. Milnor. We should mention that in the isolated singularity case, the question has been first answered by H. C. King [K. Prop. 2.8, unpublished]: it is even proven that the fibers are diffeomorphic (with corners rounded off) as long as analytic functions are involved.

We then investigate (section 4) the problem of giving effective criteria (that is to say to be verified by a computer) for Hypotheses $\mathcal{C}$ and $\mathcal{H}_k$. These criteria are sufficient ones and rely upon the Theory of Gröbner bases.

At the end is given an application of Theorem 2 in the context of an isolated singularity at the origin. We investigate a subsidiary question of J. Milnor about the construction of non-trivial examples of such fibrations. We mention that there are answers to that question in E. Looijenga, P. T. Church and K. Lamotke [Loo, CL] who gave different classes of non-trivial examples. In section 5 we show as an application of Theorem 2 that it is impossible to construct non-trivial examples of mappings $\mathbb{R}^{2n} \mapsto \mathbb{R}^3$ by starting from holomorphic functions $\mathbb{C}^n \mapsto \mathbb{R}^2$ and simply adding a new coordinate function.

2. – Fibrations for real analytic functions.

NOTATION 3. – *We will denote by* $\langle\!\langle\, , \rangle\!\rangle$ *the inner scalar product in* $\mathbb{R}^m$.

We recall also the famous (see [Loj])

LEMMA 1 (curve selection lemma). – *Let* W *a real semi-analytic set of* $\mathbb{R}^m$ *such that* $0 \in \overline{W}$. *Then there exist* $\xi \in \mathbb{R}$ *and a real analytic curve* $r\colon [0, \xi] \to \mathbb{R}^m$ *such that* $r(0) = 0$ *and* $r(t) \in W$ *for* $t > 0$.

Let $f = (f_1, f_2, \ldots, f_k)\colon \mathbb{R}^m \to \mathbb{R}^k$, $k \geq 1$, $m \geq k$, be an analytic function satisfying the Hypothesis $\mathcal{C}$. We fix ε_0 such that, in $\mathbb{B}_{\varepsilon_0}^m \backslash \Sigma_f$, f has no singular point.

LEMMA 2. – *There exists* $\varepsilon_0 > 0$ *such that for all* ε *such that* $0 < \varepsilon < \varepsilon_0$, $\mathbb{B}_\varepsilon$ *is a compact manifold with smooth boundary* $\mathbb{S}_\varepsilon$.

PROOF. – The fact that $\mathbb{B}_\varepsilon$ is compact provided ε is sufficiently small is an obvious consequence of the compactness of Σ_f. Assume that for all ε sufficiently small there exists a point p_ε such that $p_\varepsilon \in \mathbb{S}_\varepsilon$ and θ is singular at p_ε. By compactness, there exists an accumulation point p_0 such that $\theta(p_0) = 0$. Hence $p_0 \in \Sigma_f$. Let us consider the following subset:

$$S = \{x \in \mathbb{B}_\varepsilon \setminus \Sigma_f / \overrightarrow{\mathrm{grad}}_x\, \theta = 0\}\,.$$

S is a semi-analytic subset of $\mathbb{R}^m$. Suppose there is a point $p_0 \in \overline{S} \cap \Sigma_f$. Then there exists a path $r:[0,\xi] \to \mathbb{R}^m$ such that:

$$\begin{cases} \theta(r(0)) = 0\,, \\ \forall t \in]0,\xi], \qquad r(t) \in S\,, \end{cases}$$

and by multiplication by $r'(t)$ we get:

$$\langle\!\langle\, \overrightarrow{\mathrm{grad}}_{r(t)}\, \theta,\, r'(t) \,\rangle\!\rangle = \theta'(r(t)) = 0\,.$$

Hence for all $t \in [0,\xi]$, $\theta(r(t)) = \theta(r(0)) = 0$. Contradiction. ■

2.1. *Proof of Theorem* 1.

PROOF. – We have to check that for $\varepsilon > 0$ sufficiently small $f^{-1}(0)$ is transversal to $\mathbb{S}_\varepsilon$.

For $x \in f^{-1}(0) \cap \mathbb{S}_\varepsilon$, transversality means exactly linear independance between $\overrightarrow{\mathrm{grad}}_x\, \theta$, $\overrightarrow{\mathrm{grad}}_x f_1, \ldots, \overrightarrow{\mathrm{grad}}_x f_k$. Then, denote:

$$Z = \left\{x \in f^{-1}(0) \setminus \Sigma_f / \exists (\lambda_0, \lambda_1, \ldots, \lambda_k) \in \mathbb{R}^{k+1} \setminus \{0\} / \lambda_0\, \overrightarrow{\mathrm{grad}}_x\, \theta + \sum_{i=1}^{k} \lambda_i\, \overrightarrow{\mathrm{grad}}_x f_i = 0\right\}.$$

It is a semi-analytic subset of $\mathbb{R}^m$. Assume that for all $\varepsilon > 0$ there exists a point $p_\varepsilon \in Z \cap \mathbb{B}_\varepsilon$. By compactness of $\mathbb{B}_\varepsilon$, there exists an accumulation point $p_0 \in \Sigma_f \cap \overline{Z}$. So, there would exist an analytic path: $r:[0,\xi] \to \mathbb{R}^m$ such that:

$$\begin{cases} \theta(r(0)) = 0\,, \\ \forall t \in]0,\xi], \qquad r(t) \in Z\,. \end{cases}$$

Then, for $t > 0$,

$$\lambda_0(t)\, \overrightarrow{\mathrm{grad}}_{r(t)}\, \theta + \sum_{i=1}^{k} \lambda_i(t)\, \overrightarrow{\mathrm{grad}}_{r(t)} f_i = 0\,.$$

By multiplication with $r'(t)$, and using $\langle\!\langle\, \overrightarrow{\mathrm{grad}}_{r(t)} f_i,\, r'(t) \,\rangle\!\rangle = [f_i(r)]'(t) = 0$, we obtain:

$$\forall t > 0 \qquad \lambda_0(t)\, \langle\!\langle\, \overrightarrow{\mathrm{grad}}_{r(t)}\, \theta,\, r'(t) \,\rangle\!\rangle = 0\,.$$

But $\lambda_0(t) \neq 0$, since $\overrightarrow{\mathrm{grad}}_{r(t)} f_1, \ldots, \overrightarrow{\mathrm{grad}}_{r(t)} f_k$ are independent when $\theta(r(t)) \neq 0$.

Therefore: $\forall t > 0 \ \langle\!\langle \overrightarrow{\mathrm{grad}_{r(t)}}\,\theta, r'(t) \rangle\!\rangle = 0$, so $\theta(r(t))$ is constant, and $\theta(r(t)) = \theta(r(0)) = 0$. Contradiction. ■

REMARK 1. – *When Σ_f is not compact, but has at least one compact connected component, the same kind of result holds.*

Let us denote by $C \subset \Sigma_f$ the union of the compact connected components of Σ_f. There exists a neighbourhood C^* of C such that $C^* \cap \Sigma_f = C$. By the curve selection lemma it is straightforward to prove that there exists ε_0 sufficiently small such that for all $\varepsilon \in]0, \varepsilon_0]$, $\mathbb{B}_\varepsilon \cap C^*$ is a compact manifold with smooth boundary $\mathbb{S}_\varepsilon \cap C^*$. We then can repeat the same proof as in Theorem 1 just replacing $\mathbb{B}_\varepsilon$ by $\mathbb{B}_\varepsilon \cap C^*$.

REMARK 2. – *The fibration on $\mathbb{B}_\varepsilon \cap f_{|\mathbb{B}_\varepsilon \cap f^{-1}(S_\eta^{k-1})}$ can be pushed on $\mathbb{S}_\varepsilon \backslash f^{-1}(0)$.*

To achieve this, one proceeds in the same way as in some of the following proofs (for instance the Proof of Proposition 1) first by constructing a vector field $\vec{\omega}$ on $\mathbb{B}_\varepsilon \backslash f^{-1}(0)$ such that, for all $x \in \mathbb{B}_\varepsilon \backslash f^{-1}(0)$:

1) $\langle\!\langle \vec{\omega}(x), \overrightarrow{\mathrm{grad}_x}\,\theta \rangle\!\rangle > 0$.

2) $\langle\!\langle \vec{\omega}(x), \sum_{i=1}^{n} f_i(x)\,\overrightarrow{\mathrm{grad}_x f_i} \rangle\!\rangle = 0$.

It suffices then to integrate this vector field.

2.2. *Fibrations for f and $\pi_k \circ f$.*

Let $f = (f_1, f_2, \ldots, f_k)\colon \mathbb{R}^m \to \mathbb{R}^k$, with $m \geq k \geq 2$ be an analytic function, satisfying both Hypotheses $\mathcal{C}$ and $\mathcal{H}_k$. Let us consider the mapping $\tilde{f} = \pi_k \circ f$:

$$\begin{array}{ccc} \mathbb{R}^m & \to & \mathbb{R}^{k-1}, \\ x & \mapsto & (f_1(x), \ldots, f_{k-1}(x)), \end{array}$$

$\tilde{f}$ is an analytic function. We have $\Sigma_{\tilde{f}} = \Sigma_f \subset f^{-1}(0) \subset \tilde{f}^{-1}(0)$. Hence Hypothesis $\mathcal{C}$ holds for $\tilde{f}$.

Applying then Theorem 1 to $\tilde{f}$:

COROLLARY 1. – *There exists $\tilde{\varepsilon}_1 \in]0, \varepsilon_0]$ such that, for all $\varepsilon \in]0, \tilde{\varepsilon}_1]$, there exists $\tilde{\eta}_\varepsilon$ such that for all $\eta \in]0, \tilde{\eta}_\varepsilon]$*

$$\tilde{f}_{|\mathbb{B}_\varepsilon \cap f^{-1}(S_\eta^{k-2})}\colon \mathbb{B}_\varepsilon \cap f^{-1}(S_\eta^{k-2}) \to S_\eta^{k-2}$$

is a $\mathcal{C}^\infty$ locally trivial fibration.

Furthermore, we can substitute $\varepsilon_1^* = \inf\{\varepsilon_1, \tilde{\varepsilon}_1\}$ for ε_1 and $\tilde{\varepsilon}_1$, and $\eta_\varepsilon^* =$

$\inf\{\eta_\varepsilon, \tilde{\eta}_\varepsilon\}$ for η_ε and $\tilde{\eta}_\varepsilon$, to be sure that both the restrictions of f and $\tilde{f}$ are already fibrations with given $\varepsilon \in]0, \varepsilon_1^*]$ and $\eta \in]0, \eta_\varepsilon^*]$.

Let now $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}) \in S_\eta^{k-2}$, $\eta \leqslant \eta_\varepsilon^*$. We can consider the fiber

$$\widetilde{F}_{(\alpha_1, \alpha_2, \ldots, \alpha_{k-1})} = \tilde{f}^{-1}((\alpha_1, \alpha_2, \ldots \alpha_{k-1})) \cap \mathbb{B}_\varepsilon .$$

Furthermore, $(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, 0) \in S_\eta^{k-1}$, so we can also consider the fiber

$$F_{(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, 0)} = f^{-1}((\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, 0)) \cap \mathbb{B}_\varepsilon .$$

Notice the obvious inclusion $F_{(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, 0)} \subset \widetilde{F}_{(\alpha_1, \alpha_2, \ldots, \alpha_{k-1})}$.

We will now prove that there exists a close topological relationship between these two fibers.

3. – Fibers of f and $\pi_k \circ f$.

First, let us set up the following notations.

NOTATIONS 4. – *Let* $\varepsilon \in]0, \varepsilon_1^*]$ *and* $\eta \in]0, \eta_\varepsilon^*]$.

$$F_0^\varepsilon = f^{-1}(0) \cap \mathbb{B}_\varepsilon , \qquad F_\eta^\varepsilon = f^{-1}((\eta, 0, \ldots, 0, 0)) \cap \mathbb{B}_\varepsilon ,$$

$$\widetilde{F}_0^\varepsilon = \tilde{f}^{-1}(0) \cap \mathbb{B}_\varepsilon , \qquad \widetilde{F}_\eta^\varepsilon = \tilde{f}^{-1}((\eta, 0, \ldots, 0, 0)) \cap \mathbb{B}_\varepsilon ,$$

$$\widetilde{F}_0^{\varepsilon +} = \{x \in \widetilde{F}_0^\varepsilon / f_k(x) > 0\}, \quad \widetilde{F}_\eta^{\varepsilon +} = \{x \in \widetilde{F}_\eta^\varepsilon / f_k(x) > 0\}.$$

In a first step, we will be interested in the study of the singular fiber $\widetilde{F}_0^\varepsilon$, which is transversal to $\mathbb{S}_\varepsilon$ provided that $0 < \varepsilon < \varepsilon' \leqslant \varepsilon_1^*$.

In fact, we deal only with the half fiber $\widetilde{F}_0^{\varepsilon +}$ and we claim:

PROPOSITION 1. – *There exists* $\varepsilon_2 \in]0, \varepsilon_1^*]$ *such that* $\forall \varepsilon \in]0, \varepsilon_2]$, $\forall (\eta, \eta')$ *such that* $0 < \eta' < \eta \leqslant \eta_\varepsilon^*$,

$$\widetilde{F}_0^{\varepsilon +} \setminus \{x \in \mathbb{R}^m / f_k(x) \leqslant \eta'\} \text{ is homeomorphic to } (f^{-1}((0, 0, \ldots, \eta)) \cap \mathbb{B}_\varepsilon) \times [0, 1]$$

At first, we construct a vector field on $\widetilde{F}_0^{\varepsilon +}$.

3.1. *A vector field on* $\widetilde{F}_0^{\varepsilon +}$.

PROPOSITION 2. – *There exists* $\varepsilon_2 \in]0, \varepsilon_1^*]$ *such that, for all* $\varepsilon \in]0, \varepsilon_2]$: *there exists a* $\mathcal{C}^\infty$ *vector field* $\vec{\omega}_1$ *on* $\widetilde{F}_0^{\varepsilon +}$, *such that*:

1) $\forall x \in \widetilde{F}_0^{\varepsilon +}$, $\langle\!\langle \vec{\omega}_1(x), \overrightarrow{\mathrm{grad}_x}\, \theta \rangle\!\rangle = 1$,

2) $\forall x \in \widetilde{F}_0^{\varepsilon +}$, $\langle\!\langle \vec{\omega}_1(x), \overrightarrow{\mathrm{grad}_x}\, f_k \rangle\!\rangle > 0$.

PROOF. – In order to be a vector field on $\widetilde{F}_0^{\varepsilon +}$, $\vec{\omega}_1$ has to satisfy the additional conditions: $\forall x \in \widetilde{F}_0^{\varepsilon +}$,

$$\langle\!\langle \vec{\omega}_1(x), \overrightarrow{\mathrm{grad}_x} f_1 \rangle\!\rangle = \langle\!\langle \vec{\omega}_1(x), \overrightarrow{\mathrm{grad}_x} f_2 \rangle\!\rangle = \ldots = \langle\!\langle \vec{\omega}_1(x), \overrightarrow{\mathrm{grad}_x} f_{k-1} \rangle\!\rangle = 0$$

because the tangent space to $\widetilde{F}_0^{\varepsilon +}$ at x is $\{\overrightarrow{\mathrm{grad}_x} f_1, \overrightarrow{\mathrm{grad}_x} f_2, \ldots, \overrightarrow{\mathrm{grad}_x} f_{k-1}\}^{\perp}$.

Let us denote by $\vec{u}(x)$ and $\vec{v}(x)$ the orthogonal projections of $\overrightarrow{\mathrm{grad}_x}\, \theta$ and $\overrightarrow{\mathrm{grad}_x} f_k$ on

$$\{\overrightarrow{\mathrm{grad}_x} f_1, \overrightarrow{\mathrm{grad}_x} f_2, \ldots, \overrightarrow{\mathrm{grad}_x} f_{k-1}\}^{\perp} .$$

We claim:

LEMMA 3. – *There exists $\varepsilon_2 \in]0, \varepsilon_1^*]$ such that for all $x \in \mathbb{B}^m_{\varepsilon_2}$, $\vec{u}(x)$ and $\vec{v}(x)$ are not linearly dependent with opposite directions.*

We postpone the proof of this lemma to the end of the section.

Now, we construct the vector field. Locally, in a point $x_0 \in \widetilde{F}_0^{\varepsilon +}$ it is possible to find a vector $\vec{w}_{x_0}$ such that:

$$\forall i \in [1, k-1]: \langle\!\langle \vec{w}_{x_0}, \overrightarrow{\mathrm{grad}_{x_0}} f_i \rangle\!\rangle = 0 ,$$
$$\langle\!\langle \vec{w}_{x_0}, \overrightarrow{\mathrm{grad}_{x_0}} f_k \rangle\!\rangle > 0 ,$$
$$\langle\!\langle \vec{w}_{x_0}, \overrightarrow{\mathrm{grad}_{x_0}}\, \theta \rangle\!\rangle > 0 .$$

On a small neighbourhood of x_0, the projection $\vec{w}_0(x)$ of $\vec{w}_{x_0}$ on the space $\{\overrightarrow{\mathrm{grad}_x} f_1, \overrightarrow{\mathrm{grad}_x} f_2, \ldots, \overrightarrow{\mathrm{grad}_x} f_{k-1}\}^{\perp}$ has the same properties. We finish the construction of $\vec{\omega}_1$ on the whole half-fiber, by using a $\mathcal{C}^\infty$ partition of unity, getting first a vector field $\vec{\omega}$ satisfying

$$\forall i \in [1, k-1]: \langle\!\langle \vec{w}, \overrightarrow{\mathrm{grad}_x} f_i \rangle\!\rangle = 0 ,$$
$$\langle\!\langle \vec{w}, \overrightarrow{\mathrm{grad}_x} f_k \rangle\!\rangle > 0 ,$$
$$\langle\!\langle \vec{w}, \overrightarrow{\mathrm{grad}_x}\, \theta \rangle\!\rangle > 0 .$$

Then, we get $\vec{\omega}_1$ by division by $\langle\!\langle \vec{w}, \overrightarrow{\mathrm{grad}_x}\, \theta \rangle\!\rangle$. ■

3.2. *Proof of Proposition* 1.

PROOF. – Let $x_0 \in f^{-1}((0, 0, \ldots, \eta)) \cap \mathbb{B}_\varepsilon$, and denote by φ_{x_0} the trajectory of $\vec{\omega}_1$ such that $\varphi_{x_0}(0) = x_0$.

By the properties of $\vec{\omega}_1$, the functions $t \mapsto f_i(\varphi_{x_0}(t))$ are always 0, for all $i \in [1, k-1]$.

So, $\varphi_{x_0}(t)$ stays in $\widetilde{F}_0^{\varepsilon +}$. Furthermore, the function $t \mapsto f_k(\varphi_{x_0}(t))$ is strictly increasing. And:

$$\theta(\varphi_{x_0}(t)) = t + \theta(x_0)$$

The result follows. ■

3.3. *Proof of Lemma* 3.

PROOF. – Consider the following set:

$$T = \{x \in \mathbb{B}_\varepsilon / \tilde{f}(x) = 0,\ f_k(x) > 0,\ \exists \sigma \leqslant 0 / \vec{v}(x) = \sigma \vec{u}(x)\}.$$

For $\tilde{f}^{-1}(0)$ is transversal to $\mathbb{S}_\varepsilon$ provided that ε is sufficiently small, the vector $\vec{u}(x)$ is never zero, and:

$$\vec{u}(x) = \overrightarrow{\mathrm{grad}_x}\,\theta + \sum_{i=1}^{k-1} \lambda_i\, \overrightarrow{\mathrm{grad}_x f_i}\,,$$

$$\vec{v}(x) = \overrightarrow{\mathrm{grad}_x f_k} + \sum_{i=1}^{k-1} \mu_i\, \overrightarrow{\mathrm{grad}_x f_i}\,,$$

with:

$$(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \in \mathbb{R}^{k-1} \backslash \{(0, 0, \ldots, 0)\}\,,$$

$$(\mu_1, \mu_2, \ldots, \mu_{k-1}) \in \mathbb{R}^{k-1} \backslash \{(0, 0, \ldots, 0)\}\,.$$

Using:

$$\forall i \in [1, k-1] \qquad \langle\!\langle \vec{u}(x), \overrightarrow{\mathrm{grad}_x f_i} \rangle\!\rangle = \langle\!\langle \vec{v}(x), \overrightarrow{\mathrm{grad}_x f_i} \rangle\!\rangle = 0$$

the coefficients $(\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ satisfy the following linear system:

$$\begin{cases} \displaystyle\sum_{i=1}^{k-1} \lambda_i \langle\!\langle \overrightarrow{\mathrm{grad}_x f_i}, \overrightarrow{\mathrm{grad}_x f_1} \rangle\!\rangle = -\langle\!\langle \overrightarrow{\mathrm{grad}_x}\,\theta, \overrightarrow{\mathrm{grad}_x f_1} \rangle\!\rangle\,, \\ \qquad\vdots \\ \displaystyle\sum_{i=1}^{k-1} \lambda_i \langle\!\langle \overrightarrow{\mathrm{grad}_x f_i}, \overrightarrow{\mathrm{grad}_x f_{k-1}} \rangle\!\rangle = -\langle\!\langle \overrightarrow{\mathrm{grad}_x}\,\theta, \overrightarrow{\mathrm{grad}_x f_{k-1}} \rangle\!\rangle\,, \end{cases}$$

and then, each λ_i is expressed as a quotient of two determinants involving the products $\langle\!\langle \overrightarrow{\mathrm{grad}_x f_i}, \overrightarrow{\mathrm{grad}_x f_j} \rangle\!\rangle$ or $\langle\!\langle \overrightarrow{\mathrm{grad}_x}\,\theta, \overrightarrow{\mathrm{grad}_x f_j} \rangle\!\rangle$. As a result, each λ_i is a meromorphic function. The same occurrs for the μ_i's, hence T is a real semi-analytic set.

Suppose a point $p_0 \in \Sigma_f$ belongs to its adherence $\overline{T}$. Then there would exist an analytic path $r: [0, \xi] \to \mathbb{R}^m$ such that:

$$\begin{cases} \theta(r(0)) = 0\,, \\ \forall t \in]0, \xi], \qquad r(t) \in T\,. \end{cases}$$

So, for all $t \in]0, \xi]$:

$$\vec{v}\,(r(t)) = \sigma(t)\, \vec{u}\,(r(t))$$

with

$$\sigma(t) \leqslant 0\,.$$

Then:

$$\langle\!\langle\, \vec{v}\,(r(t)),\, r'(t) \,\rangle\!\rangle = \sigma(t)\, \langle\!\langle\, \vec{u}(r(t)),\, r'(t) \,\rangle\!\rangle\,.$$

But, for all $t \in]0, \xi]$, $f_1(r(t)) = f_2(r(t)) = \ldots = f_{k-1}(r(t)) = 0$, so,

$$\forall i \in [1, k-1], \qquad \langle\!\langle\, \overrightarrow{\mathrm{grad}_{r(t)}} f_i,\, r'(t) \,\rangle\!\rangle = 0\,.$$

Finally, we obtain:

$$\langle\!\langle\, \overrightarrow{\mathrm{grad}_{r(t)}} f_k,\, r'(t) \,\rangle\!\rangle = \sigma(t)\, \langle\!\langle\, \overrightarrow{\mathrm{grad}_{r(t)}}\, \theta,\, r'(t) \,\rangle\!\rangle$$

that is:

$$\frac{d}{dt}\,(f_k(r)) = \sigma\, \frac{d}{dt}\,(\theta(r))\,.$$

Developping in power series,

$$\begin{aligned} f_k(r(t)) &= l_1 t^{m_1} + \ldots\,, \\ \theta(r(t)) &= \varrho_1 t^{k_1} + \ldots\,, \\ \sigma(t) &= \sigma_1 t^{\mu_1} + \ldots\,, \end{aligned} \qquad \text{with} \begin{cases} l_1 \in (\mathbb{R}^+ \setminus \{0\})\,, & m_1 \in \mathbb{N}\,, \\ \varrho_1 \in (\mathbb{R}^+ \setminus \{0\})\,, & k_1 \in \mathbb{N}\,, \\ \sigma_1 \in (\mathbb{R} \setminus \{0\})\,, & \mu_1 \in \mathbb{Z}\,. \end{cases}$$

We find:

$$m_1 l_1 t^{m_1 - 1} + \ldots = \sigma_1 \varrho_1 t^{k_1 + \mu_1 - 1} + \ldots$$

and so:

$$\sigma_1 = \frac{m_1 l_1}{\varrho_1} > 0$$

which contradicts $\sigma(r(t)) \leqslant 0$ for all $t \in]0, \xi]$.

In conclusion, p_0 is not adherent to T, and there exists $\varepsilon_2 \in]0, \varepsilon_1^*]$ such that on $\mathbb{B}^m_{\varepsilon_2}$, $\vec{u}(x)$ and $\vec{v}(x)$ are never dependent with opposite directions. ■

3.4. *The half-fibers $\tilde{F}^{\varepsilon+}_{\eta}$ and $\tilde{F}^{\varepsilon+}_{0}$.*

Now, we compare the parts $\{f_k \geqslant \eta'\}$ of the singular fiber of $\tilde{f}$ and of a regular fiber of $\tilde{f}$. Our aim is:

Proposition 3. – *$\forall \varepsilon \in]0, \varepsilon_2]$, $\forall (\eta, \eta', \eta'')/0 < \eta' < \eta < \eta'' < \eta^*_\varepsilon$ with $\eta' < \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$:*

$\tilde{F}^{\varepsilon +}_0 \cap \{x \in \mathbb{R}^m / f_k(x) \geqslant \eta'\}$ is diffeomorphic to $\tilde{F}^{\varepsilon +}_\eta \cap \{x \in \mathbb{R}^m / f_k(x) \geqslant \eta'\}$.

The way is the same as previously, constructing first a vector field. We have to distinguish the cases $k > 2$ and $k = 2$.

Proposition 4. – *Let $k > 2$. $\forall \varepsilon \in]0, \varepsilon_2]$, $\forall \eta, \eta''/0 < \eta < \eta'' < \eta^*_\varepsilon$, there exists a $\mathcal{C}^\infty$ vector field $\vec{\omega}_2$ on:*

$$(x \in \mathbb{B}_\varepsilon \backslash \Sigma_f / f_1(x) < \eta'',\ f_2(x) = 0,\ \ldots,\ f_{k-1}(x) = 0, f_k(x) \geqslant 0\}$$

such that:

1) $\forall x \in \{x \in \mathbb{B}_\varepsilon / f_2(x) = \ldots = f_{k-1}(x) = 0\}$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x f_1 \rangle\!\rangle > 0$;

2) $\forall x \in \{x \in \mathbb{B}_\varepsilon / f_2(x) = \ldots = f_{k-1}(x) = 0,\ f_k(x) \leqslant \sqrt{\eta^{*2}_\varepsilon - \eta''^2}\}$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x f_k \rangle\!\rangle = 0$;

3) $\forall x \in \{x \in \mathbb{S}_\varepsilon / f_2(x) = \ldots = f_{k-1}(x) = 0\}$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x\, \theta \rangle\!\rangle = 0$.

Proof. – Let $x_0 \in \mathbb{B}_\varepsilon$ such that $f_1(x_0) < \eta''$, $f_2(x_0) = \ldots = f_{k-1}(x_0) = 0$. We will examine three cases:

1) If $x_0 \in \mathbb{B}_\varepsilon \backslash \mathbb{S}_\varepsilon \backslash \Sigma_f$. We know that the vectors $\{\overrightarrow{\mathrm{grad}}_x f_i,\ i \in [1, k]\}$ are independent in a neighbourhood $\mathcal{V}_{x_0} \subset \mathbb{B}_\varepsilon \backslash \mathbb{S}_\varepsilon \backslash \Sigma_f$. Therefore $\{\overrightarrow{\mathrm{grad}}_x f_1\}^\perp$ is not included in $\{\overrightarrow{\mathrm{grad}}_x f_2, \ldots, \overrightarrow{\mathrm{grad}}_x f_k\}^\perp$, and it is possible to find $\vec{w}_0(x)$ satisfying the conditions in $\mathcal{V}_{x_0}$.

2) If $x_0 \in \mathbb{S}_\varepsilon$ and $f_k(x) > \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$. Since $\sum_{i=1}^{k-1} f_i^2(x_0) \leqslant \eta^{*2}_\varepsilon$, the fiber $\tilde{f}^{-1}(\tilde{f}(x_0))$ is transversal to $\mathbb{S}_\varepsilon$. So the vectors $\overrightarrow{\mathrm{grad}}_{x_0} \theta, \overrightarrow{\mathrm{grad}}_{x_0} f_1, \ldots, \overrightarrow{\mathrm{grad}}_{x_0} f_{k-1}$ are independent. We can find a vector $\vec{w}_0$ in $\{\overrightarrow{\mathrm{grad}}_{x_0} \theta, \overrightarrow{\mathrm{grad}}_{x_0} f_2, \ldots, \overrightarrow{\mathrm{grad}}_{x_0} f_{k-1}\}^\perp$ such that $\langle\!\langle \vec{w}_0, \overrightarrow{\mathrm{grad}}_{x_0} f_1 \rangle\!\rangle > 0$. In a small neighbourhood $\mathcal{V}_{x_0}$ of x_0, the projection $\vec{w}_0(x)$ of $\vec{w}_0$ on $\{\overrightarrow{\mathrm{grad}}_x \theta, \overrightarrow{\mathrm{grad}}_x f_2, \ldots, \overrightarrow{\mathrm{grad}}_x f_{k-1}\}^\perp$ has the requested properties.

3) If $x_0 \in \mathbb{S}_\varepsilon$ and $f_k(x) \leqslant \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$ then $\sum_{i=1}^{k} f_i^2(x_0) \leqslant \eta^{*2}_\varepsilon$. Hence the fiber $f^{-1}(f(x_0))$ is transversal to $\mathbb{S}_\varepsilon$, and the vectors $\overrightarrow{\mathrm{grad}}_{x_0} \theta$, $\overrightarrow{\mathrm{grad}}_{x_0} f_1, \ldots, \overrightarrow{\mathrm{grad}}_{x_0} f_k$ are independent. We can choose a vector $\vec{w}_0$ in $\{\overrightarrow{\mathrm{grad}}_{x_0} \theta, \overrightarrow{\mathrm{grad}}_{x_0} f_2, \ldots, \overrightarrow{\mathrm{grad}}_{x_0} f_k\}^\perp$ such that

$$\langle\!\langle \vec{w}_0, \overrightarrow{\mathrm{grad}}_{x_0} f_1 \rangle\!\rangle > 0\,.$$

The projection $\vec{w}_0(x)$ of $\vec{w}_0$ on $\{\overrightarrow{\mathrm{grad}}_x \theta, \overrightarrow{\mathrm{grad}}_x f_2, \ldots, \overrightarrow{\mathrm{grad}}_x f_k\}^\perp$ has the requested properties on $\mathcal{V}_{x_0}$.

We end the construction of $\vec{\omega}_2$, starting from the local fields $\vec{\omega}_0(x)$ and using a $\mathcal{C}^\infty$ partition of unity.

PROPOSITION 5. – *Let* $k=2$. $\forall \varepsilon \in]0, \varepsilon_2]$, $\forall \eta, \eta''/0<\eta<\eta''<\eta^*_\varepsilon$, *there exists a* $\mathcal{C}^\infty$ *vector field* $\vec{\omega}_2$ *on*:

$$\{x \in \mathbb{B}_\varepsilon \setminus \Sigma_f / f_1(x) < \eta'', f_2(x) \geqslant 0\}$$

such that:

1) $\forall x \in \mathbb{B}_\varepsilon$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x f_1 \rangle\!\rangle > 0$;
2) $\forall x \in \{x \in \mathbb{B}_\varepsilon / f_2(x) \leqslant \sqrt{\eta^{*2}_\varepsilon - \eta''^2}\}$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x f_2 \rangle\!\rangle = 0$;
3) $\forall x \in \mathbb{S}_\varepsilon$, $\langle\!\langle \vec{\omega}_2(x), \overrightarrow{\mathrm{grad}}_x \theta \rangle\!\rangle = 0$.

PROOF. – The proof is similar to the previous one.
Let $x_0 \in \mathbb{B}_\varepsilon$ such that $f_1(x_0) < \eta''$.

1) If $x_0 \in \mathbb{B}_\varepsilon \setminus \mathbb{S}_\varepsilon \setminus \Sigma_f$, use the fact that $\overrightarrow{\mathrm{grad}}_x f_1$ and $\overrightarrow{\mathrm{grad}}_x f_2$ are independent in a neighbourhood $\mathcal{V}_{x_0} \subset \mathbb{B}_\varepsilon \setminus \mathbb{S}_\varepsilon \setminus \Sigma_f$.

2) If $x_0 \in \mathbb{S}_\varepsilon$ and $f_2(x) > \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$. Since $f_1^2(x_0) \leqslant \eta^{*2}_\varepsilon$, the fiber $\tilde{f}^{-1}(\tilde{f}(x_0))$ is transversal to $\mathbb{S}_\varepsilon$, so the vectors $\overrightarrow{\mathrm{grad}}_{x_0} \theta, \overrightarrow{\mathrm{grad}}_{x_0} f_1$ are independent.

3) If $x_0 \in \mathbb{S}_\varepsilon$ and $f_2(x) \leqslant \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$ then $f_1^2(x_0) + f_2^2(x_0) \leqslant \eta^{*2}_\varepsilon$, and so, the fiber $f^{-1}(f(x_0))$ is transversal to $\mathbb{S}_\varepsilon$, so the vectors $\overrightarrow{\mathrm{grad}}_{x_0} \theta, \overrightarrow{\mathrm{grad}}_{x_0} f_1, \overrightarrow{\mathrm{grad}}_{x_0} f_2$ are independent. We conclude as in Proposition 4. ■

We prove now Proposition 3.

PROOF. – Let us integrate $\vec{\omega}_2$. We fix η, η', η'' with the conditions:

$$0 < \eta' < \eta < \eta'' < \eta^*_\varepsilon \quad \text{and} \quad \eta' < \sqrt{\eta^{*2}_\varepsilon - \eta''^2}\,.$$

Let $x_0 \in \widetilde{F}^{\varepsilon+}_0 \cap \{x \in \mathbb{R}^m / f_k(x) \geqslant \eta'\}$ and φ_{x_0} the trajectory defined by $\vec{\omega}_2$ such that $\varphi_{x_0}(0) = x_0$.

The properties of $\vec{\omega}_2$ induce the following for φ_{x_0}:

- the function $t \mapsto f_1(\varphi_{x_0}(t))$ is strictly increasing;
- if $k>2$, for all $i \in [2, k-1]$, the functions $t \mapsto f_i(\varphi_{x_0}(t))$ are always zero;
- $f_k(\varphi_{x_0}(t)) \geqslant \eta'$;
- $\varphi_{x_0}(t) \in \mathbb{B}_\varepsilon$.

The consequence is that there exists $\tau \in \mathbb{R}^+$ such that

- $f_1(\varphi_{x_0}(\tau)) = \eta$;
- if $k>2$, $f_2(\varphi_{x_0}(\tau)) = \ldots = f_{k-1}(\varphi_{x_0}(\tau)) = 0$;
- $f_k(\varphi_{x_0}(\tau)) \geqslant \eta'$;
- $\varphi_{x_0}(\tau) \in \mathbb{B}_\varepsilon$.

This mapping $x_0 \mapsto \varphi_{x_0}(\tau)$ constructs the diffeomorphism. ∎

3.5. *Proof of Theorem 2.*

We are now able to show the main result: $\forall \varepsilon \in]0, \varepsilon_2]$, $\forall \eta \in]0, \eta^*_\varepsilon]$,

$$\text{the fiber } \widetilde{F}^\varepsilon_\eta \text{ is homeomorphic to } F^\varepsilon_\eta \times [-1, +1].$$

PROOF. – For all $0 < \eta' < \eta < \eta'' < \eta^*_\varepsilon$ with $\eta' < \sqrt{\eta^{*2}_\varepsilon - \eta''^2}$, we have:

$$\widetilde{F}^\varepsilon_\eta = \begin{cases} \widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / |f_k(x)| < \eta'\}, \\ \cup \\ \widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / f_k(x) \geq \eta'\}, \\ \cup \\ \widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / f_k(x) \leq -\eta'\}. \end{cases}$$

But $\widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / f_k(x) \geq \eta'\}$ is homeomorphic to $f^{-1}((0, 0, \ldots, \eta)) \times [0, 1]$, which is the same as $f^{-1}((0, 0, \ldots, \eta)) \times [1/2, 1]$.

On the other hand, $\widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / f_k(x) \leq -\eta'\}$ is homeomorphic to $f^{-1}((0, 0, \ldots, \eta)) \times [-1, -1/2]$.

To end this proof, we claim that there exists η'_ε such that $\forall \alpha \in]0, \eta'_\varepsilon]$, $\widetilde{F}^\varepsilon_\eta \cap \{x \in \mathbb{R}^m / |f_k(x)| < \alpha\}$ is homeomorphic to $f^{-1}((0, 0, \ldots, \eta)) \times]-1/2, 1/2[$. This is clear, since the fiber $\{x \in F^\varepsilon_\eta / f_k(x) = 0\}$ is transversal to $\widetilde{F}^\varepsilon_\eta$.

We take η' such that $\eta' < \eta'_\varepsilon$ and this concludes the proof. ∎

REMARK 3. – *When Σ_f is not compact, but has at least one compact connected component, the same result holds for the fibers that arise in Remark* 1.

(The statements of the previous Propositions are similar, and their proofs are performed in the same way.)

4. – Some effective sufficient criteria.

We consider here the case of polynomial functions. Our aim is to give some sufficient effective criteria for the Hypotheses $\mathcal{C}$ and $\mathcal{H}_k$, via the Gröbner bases Theory. We mention that these criteria are only sufficient, because they deal with the complex structure of the algebraic varieties they involve. We suggest the reader to consult [E, CLO] for the exposition of the Gröbner basis construction algorithm and of the associated division process. We only recall the following facts about Gröbner bases. Assume that a monomial ordering (compatible with the multiplication) is put on $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Let $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, $f \neq 0$. We define the initial part of f, denoted by $in(f)$, as the highest monomial of f. Then, for any given ideal I of $\mathbb{R}[x_1, x_2, \ldots, x_n]$, there exists a finite set of polynomials $\{g_1, g_2, \ldots, g_n\}$ (a Gröbner basis) such that

1) $I = \langle g_1, g_2, \ldots, g_n \rangle$;
2) for all $f \in I$, $in(f) \in \langle in(g_1), in(g_2), \ldots, in(g_n) \rangle$;
3) $\{g_1, g_2, \ldots, g_n\}$ can be constructed by an algorithm;
4) for all $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, one can perform an algorithmic division, that is to write: $f = \sum_{i=1}^{n} h_i g_i + h_0$ with $h_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, $0 \leq i \leq n$, and h_0 such that if $h_0 \neq 0$ then $in(h_0) \notin \langle in(g_1), in(g_2), \ldots, in(g_n) \rangle$. Moreover, h_0 does not depend upon the particular construction of the basis $\{g_1, g_2, \ldots, g_n\}$ and is called the rest of the division of f by I. We shall denote $h_0 = R(f, I)$.

We first recall how to test the inclusion of one algebraic variety into another. Let $V = (f_1, f_2, \ldots, f_r)^{-1}(0)$ and $V' = (f_1', f_2', \ldots, f_{r'}')^{-1}(0)$ two algebraic varieties of $\mathbb{R}^m$. Let us denote $I = \langle f_1, f_2, \ldots, f_r \rangle$ and $I' = \langle f_1', f_2', \ldots, f_{r'}' \rangle$. We have the classical (see for instance [CLO]) effective *sufficient* criterion for the inclusion:

SUFFICIENT CRITERION 0. – *If for all* $i \in [1, r']$, $R(f_i', I) = 0$, *then* $V \subset V'$.

Notice that this criterion involves in fact the complex structure of the varieties.

We then investigate how to test Hypothesis $\mathcal{C}$. Σ_f is defined as the vanishing set in $\mathbb{R}^m$ of the $k \times k$ minors of the Jacobian matrix of f. These minors are polynomials, they belong to $\mathbb{R}[x_1, x_2, \ldots, x_n]$, and generate an ideal $M(f)$. We assume that a monomial ordering is fixed, and that G is a Gröbner basis for $M(f)$. Hence, we can perform algorithmically the division of any polynomial by $M(f)$, just dividing it by G. Applying the sufficient criterion 0 gives:

SUFFICIENT CRITERION 1. – *If for all* $i \in [1, k]$, $R(f_i, M(f)) = 0$, *then* $\Sigma_f \subset f^{-1}(0)$.

The second problem is to determine if Σ_f is compact. The strategy consists in considering the homogeneous part of highest homogeneous degree of every polynomial of a set of generators of $M(f)$ (for instance G), and to consider the ideal M_∞ generated by these polynomials. Then, compute a Gröbner basis. At this point, we may examine if the complex locus at infinity is void or not: it depends upon the fact that the Gröbner basis for M_∞ is reduced to 1 or not.

SUFFICIENT CRITERION 2. – *If* $M_\infty = \mathbb{R}[x_1, x_2, \ldots, x_n]$, *then* Σ_f *is compact.*

In the case there are complex roots, the Gröbner basis for M_∞ is not reduced to 1 and one has to test the possibility of real roots. In order to prove that they are no real root at infinity, one may try other algorithmic methods such as Cylindrical Algebraic Decomposition. However we should stress that this method is of great complexity, and strongly depend upon the number of variables, the degree, as well as the singularities at infinity. We suggest the reader to refer to [JKM] for the difficulties that may arise in practical Cylindrical Algebraic Decomposition. This kind of method also holds for testing the inclusion of algebraic real varieties.

It remains to give a test for the Hypothesis $\mathcal{H}_k$. Let us define $M(\tilde{f})$ as the ideal generated by the $(k-1)\times(k-1)$ minors of the Jacobian matrix of $\tilde{f} = (f_1, f_2, \ldots, f_{k-1})$. Of course $\Sigma_{\tilde{f}}$ is the zero set in $\mathbb{R}^m$ of $M(\tilde{f})$, and we have $M_{\tilde{f}} \subset M_f$. Hypothesis $\mathcal{H}_k$ is nothing but the converse inclusion, and for that we can use Sufficient criterion 0. Let us denote by $m_1, m_2, \ldots m_k$ the $k \times k$ minors of the Jacobian matrix of f involving the derivatives of f_k.

SUFFICIENT CRITERION 3. – *If for all $i \in [1, k]$, $R(m_i, M(\tilde{f})) = 0$, then Hypothesis $\mathcal{H}_k$ holds.*

(This criterion implies that $M_f \subset M_{\tilde{f}}$, and so $\Sigma_{\tilde{f}} = \Sigma_f$.)

5. – An application of Theorem 2.

In [M. p. 100], J. Milnor asks for non-trivial examples of such fibrations. By trivial examples he means fibrations with fiber homeomorphic to a disk. One could think of constructing such examples starting from the well known examples of fibrations associated to holomorphic functions $\mathbb{C}^n \to \mathbb{R}^2$, and then increase the dimension of the target space, completing them to get functions $\mathbb{R}^{2n} \to \mathbb{R}^3$. We show here, using our previous result, that this is infortunately impossible to do so.

THEOREM 3. – *Let π_x be the projection $\mathbb{C} \times \mathbb{R} \to \mathbb{C} : (z, x) \mapsto z$.*
Let $\tilde{f}$: $\mathbb{C}^n \to \mathbb{C}$ be an analytic function with isolated singularity at the origin, μ its Milnor number.

- *If $\mu = 0$ or $\mu = 1$, there could exist f: $\mathbb{C}^n \to \mathbb{C} \times \mathbb{R}$ with isolated singularity at the origin, such that $\tilde{f} = \pi_x \circ f$.*
- *If $\mu \neq 0$ and $\mu \neq 1$ this construction is impossible.*

PROOF.

- First, let us consider the case $\mu = 0$: one example is

$$\tilde{f}: \mathbb{C}^n \to \mathbb{C} \qquad \text{and we can construct} \qquad f: \mathbb{C}^n \to \mathbb{C} \times \mathbb{R}$$
$$(z_1, \ldots, z_n) \mapsto z_1 \qquad\qquad (z_1, \ldots, z_n) \mapsto (z_1, Re(z_2))\,.$$

- For $\mu = 1$, let $\tilde{f}$: $\mathbb{C}^n \to \mathbb{C}$

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1, z_2 + z_3^2 + \ldots + z_n^2)$$

and we can take f: $\mathbb{C}^n \to \mathbb{C} \times \mathbb{R}$

$$(z_1, \ldots, z_n) \mapsto (z_1, z_2 + z_3^2 + \ldots + z_n^2, |z_1|^2 - |z_2|^2).$$

- If $\mu \neq 0$ and $\mu \neq 1$, we will prove the theorem by reductio ad absurdum. Suppose $\tilde{f}$ completed in f: $\mathbb{C}^n \to \mathbb{C} \times \mathbb{R}$ with isolated singularity, and denote by $\tilde{F}$ and F the fibers $\tilde{f}^{-1}(\tilde{c}) \cap B_\varepsilon^{2n}$ and $f^{-1}(c) \cap B_\varepsilon^{2n}$ for ε, c, $\tilde{c}$ chosen conveniently with respect to Theorem 1 and Corollary 2 (B_ε^{2n} denotes the euclidian ball of radius ε centered at $0 \in \mathbb{R}^{2n}$). It is well known that the fibration (for ε, η sufficiently small)

$$\tilde{f}_{|\tilde{f}^{-1}(S_\eta^1) \cap B_\varepsilon^{2n}}\colon \tilde{f}^{-1}(S_\eta^1) \cap B_\varepsilon^{2n} \to S_\eta^1$$

is equivalent to the Milnor fibration:

$$\frac{\tilde{f}}{|\tilde{f}|} : S_\varepsilon^{2n-1} \backslash K_\varepsilon \to S^1$$

where K_ε denotes $\tilde{f}^{-1}(0) \cap S_\varepsilon^{2n-1}$.

Let us consider the closed fiber

$$G = K_\varepsilon \cup \left(\tilde{f}^{-1}\left(\frac{\tilde{c}}{|\tilde{c}|}\right) \cap S_\varepsilon^{2n-1}\right)$$

G and $\tilde{F}$ are diffeomorphic, so G is homeomorphic to $F \times [0, 1]$. Let h_{n-1}^*: $H_{n-1}(G) \to H_{n-1}(G)$ be the monodromy. The boundary of G is homeomorphic to that of $\tilde{F}$, so to:

$$H = F \times \{0\} \cup F \times \{1\} \cup \partial F \times [0, 1]$$

In order to simplify, identify ∂G with H.

It is well known that the restriction of the monodromy to $H_{n-1}(\partial G)$ is the identity, so it is the same for the restriction of h_{n-1}^* to $H_{n-1}(F \times \{0\})$.

But $H_{n-1}(G) \simeq H_{n-1}(F \times [0, 1]) \simeq H_{n-1}(F \times \{0\})$ and we have the following commutative diagram:

$$\begin{tikzcd}
H_{n-1}(G) \arrow[r, "h_{n-1}^*"] \arrow[d, "\simeq"'] & H_{n-1}(G) \arrow[d, "\simeq"] \\
H_{n-1}(F \times \{0\}) \arrow[r, "id"] & H_{n-1}(F \times \{0\}).
\end{tikzcd}$$

We deduce from this: $h_{n-1}^* = id_{H_{n-1}(G)}$.

But, from [A'C] we know ($\mu \neq 0$) that:

$$\sum_{i=0}^{n-1} (-1)^i \text{ trace } (h_i^*) = 0$$

so:

$$(-1)^0 + (-1)^{n-1} \mu = 0$$

and this is impossible in the case we consider. ■

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